CONTINUOUSLY PERFECTLY NORMAL SPACES
AND SOME GENERALIZATIONS

BY

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ABSTRACT. In this work we continue the study of continuously perfectly normal, continuously normal, and continuously completely regular spaces which was begun by Phillip Zenor. Among other results, we prove that separable continuously completely regular spaces are metrizable, and provide an example of a nonmetrizable continuously perfectly normal space.

I. **Introduction.** The classes of continuously perfectly normal spaces, continuously normal spaces, and continuously completely regular spaces were first studied by Phillip Zenor in [16]. In this paper we continue this study, obtaining extensions and improvements of a number of Zenor's results, as well as new results.

In §II, we introduce two new classes of spaces and use them to obtain results concerning certain local properties of the spaces which are the subject of this paper. For example, we show that every continuously perfectly normal space is countably bisequential, and then extend this result under various assumptions to continuously normal and continuously completely regular spaces.

In §III, we turn our attention to metrization theorems. We show that every separable continously completely regular space is metrizable. We obtain an even better result for continuously perfectly normal spaces. §IV is devoted to examples, including an example of a nonmetrizable continuously perfectly normal space (which answers a question of Zenor).

The concept of continuously perfectly normal spaces had its origins in a characterization of stratifiable spaces due to Carlos Borges [1, Theorem 5.2]. Let 2^X denote the space of closed subsets of a space X, with the finite topology([2]), and let $C^+(X)$ be the space of continuous nonnegative real-valued functions defined on X, with the compact-open topology. By Borges' theorem, a space X is

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([2]) Let $\{U_1, U_2, \ldots, U_n\}$ be a set of open subsets of the topological space X. Let $\langle U_1, U_2, \ldots, U_n \rangle = \{H \subset \bigcup_{i=1}^{n} U_i |$ for each i, $H \cap U_i \neq \phi \}$. Sets of this form are a basis for a topology on 2^X. This topology has been called the finite topology, the Vietoris topology, and the exponential topology by various authors. For a study of this topology, see K. Kuratowski [6] or E. Michael [8].

stratifiable if and only if there is a function ϕ: $2^X \longrightarrow C^+(X)$ satisfying the following requirements:

(a) for each $H \in 2^X$, $H = \{x \in X \mid \phi(H)(x) = 0\}$,

(b) if H and K are elements of 2^X such that $H \subset K$, then $\phi(K)(x) \leqslant \phi(H)(x)$ for all $x \in X$.

In [15], Zenor proved that if we add the property

(c) ϕ is a continuous map,

then X is metrizable. His attempts to improve his metrization theorem by getting rid of a property implied by (b) led to the notion of a continuously perfectly normal space. Every continuously perfectly normal space admits a function satisfying (a) and (c).

We shall now define continuously perfectly normal, continuously normal, and continuously completely regular spaces. All our spaces are assumed to be T_1. Let $M(X) = \{(H, K) \in 2^X \times 2^X \mid H \cap K = \varnothing\}$. Let $\mathcal{D}(X) = \{(x, K) \in X \times 2^X \mid x \notin K\}$.

A function T: $X \times 2^X \longrightarrow [0, 1]$ is called a *perfect normality operator* (abbreviated PN-operator) if, for each $H \in 2^X$, $H = \{x \in X \mid T(x, H) = 0\}$. A space is said to be *continuously perfectly normal* (abbreviated CPN) if it admits a continuous PN-operator.

A function T: $X \times M(X) \longrightarrow [0, 1]$ is called a *normality operator* (N-operator) if for every pair $(H, K) \in M(X)$, we have $H \subset \{x \in X \mid T(x, (H, K))\} = 0$, and $K \subset \{x \in X \mid T(x, (H, K))\} = 1$. A space is *continuously normal* (CN) if it admits a continuous N-operator.

Similarly, a function T: $X \times \mathcal{D}(X) \longrightarrow [0, 1]$ is a *complete regularity operator* (CR-operator) if, for each $(x, H) \in X \times \mathcal{D}(X)$, $T(x, (x, H)) = 0$ and $H \subset \{y \in X \mid T(y, (x, H))\} = 1$. A space is *continuously completely regular* (CCR) if it admits a continuous CR-operator. Clearly every CN-space is a CCR-space.

It is easily seen from the definition that a space X is a CPN-space only if it admits a function ϕ: $2^X \longrightarrow C^+(X)$ satisfying (a) and (c) above. However, it is not known whether every CPN-space is stratifiable; in fact, it is not known if every CPN-space is paracompact.

We now list some of the results obtained by Zenor, many of which will be referred to later in this paper.

(i) Every metric space is a CPN-space and every CPN-space is a CN-space. Every subspace of a CPN-space is a CPN-space.

(ii) Every CPN-space is a Fréchet space.

(iii) Every CPN-space is collectionwise normal.

(iv) A CPN-space is metrizable if and only if it is a $w\Delta$-space [3]. A separable CCR-space is metrizable if and only if it is a $w\Delta$-space.

(v) A space X is metrizable if and only if it admits a continuous PN-operator T such that if K is a finite subset of X and $x \in K$, then $T(y, K) \leqslant T(y, \{x\})$ for all $y \in X$. (Note that this is a generalization of property (b) above.) An analogous theorem is proved for separable CCR-spaces.

II. Local properties. As we noted above, Phillip Zenor has shown that every CPN-space is a Fréchet space. In this section, we introduce two new classes of spaces, called W-spaces and W_k-spaces, which are contained in the class of Fréchet spaces. With the help of these classes of spaces, we shall improve upon the aforementioned result of Zenor, and obtain some results concerning the local properties of CN-spaces and CCR-spaces as well.

Let S be a subset of a topological space X, and consider the following two-person infinite game: player I chooses an open set U_1 containing S, and then player II chooses a point $x_1 \in U_1$; player I chooses another open set U_2 containing S, and player II chooses some point $x_2 \in U_2$, and so on. We shall say that player I wins the game if the sequence $\{x_1, x_2, \dots\}$ converges to S (i.e., if S is contained in the open set U, then there exists $n_0 \in N$ such that $x_n \in U$ for all $n > n_0$). More formally, a *strategy at S* for player I is a map $\sigma \colon \mathsf{F}(X) \longrightarrow \tau_S(X)$, where $\mathsf{F}(X)$ is the set of finite subsets of X and $\tau_S(X)$ is the set of open subsets of X containing S. (Here $\sigma(\{x_1, x_2, \dots, x_n\})$ represents the open set player I would choose if $\{x_1, x_2, \dots, x_n\}$ have been the first n choices of player II.) A strategy σ at S is called a *winning strategy* at S if every sequence $\{x_1, x_2, \dots\}$ such that $x_{n+1} \in \sigma(\{x_1, x_2, \dots, x_n\})$ for all $n \in N$ converges to S. We shall call a space X a W_k-*space* (W-*space*) if there exists a winning strategy at each compact subset (point) of X. Certainly every W_k-space is also a W-space. Our next theorem relates these spaces to known classes of spaces. (W-spaces are studied in detail in [4]. We include here only those results which are useful in our study of CPN-spaces.)

THEOREM 2.1. *Every first countable space is a W-space and every W-space is countably bisequential*[3]; *furthermore, every subspace of a W_k-space (W-space) is a W_k-space (W-space).*

PROOF. Let $\{U_n\}_{n=1}^{\infty}$ be a countable local basis at $x_0 \in X$. Then the strategy $\sigma \colon \mathsf{F}(X) \longrightarrow \tau_{x_0}(X)$ given by $\sigma(x_1, \dots, x_n) = U_n$ is obviously a winning strategy at x_0. Thus a first countable space is a W-space.

Let $x_0 \in X$ and let $\{F_n\}_{n=1}^{\infty}$ be a sequence of subsets of X such that $F_n \supset$

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[3] A space X is said to be *countably bisequential* if whenever $\{F_1, F_2, \dots\}$ is a sequence of subsets of X such that $F_n \supset F_{n+1}$ and $x_0 \in \bar{F}_n$ for all n, there exists $x_n \in F_n$ such that $\{x_n\}_{n=1}^{\infty} \longrightarrow x_0$. Every countably bisequential space is a Fréchet space. For a study of countably bisequential spaces, see E. Michael [9] and R. C. Olsen [11].

F_{n+1} and $x_0 \in F_n$ for $n = 1, 2, \ldots$. Player II can always choose some $x_n \in F_n$, so if there is a winning strategy for player I at x_0, there must also be a sequence $\{x_n\}_{n=1}^{\infty} \rightarrow x_0$ such that $x_n \in F_n$ for all n. Thus every W-space is countably bisequential.

Finally, let $S \subset X$ be a subset of a W_k-space X. Let K be a compact subset of S and let $\sigma \colon F(X) \rightarrow \tau_K(X)$ be a winning strategy at K in the space X. Then it is easy to see that the strategy $\sigma' \colon F(S) \rightarrow \tau_K(S)$ defined by $\sigma'(F) = \sigma(F) \cap S$ is a winning strategy at K in the subspace S. The same proof shows that every subspace of a W-space is a W-space.

The following theorem is an improvement of Theorem 2.2 in [16]:

THEOREM 2.2. *Every CPN-space is a W_k-space.*

PROOF. Let T be a continuous PN-operator for the CPN-space X, and let C be a compact subset of X. Define a strategy σ at C as follows: $\sigma(x_1, \ldots, x_n)$ is the set of all $y \in X$ such that there exists a point $c(y) \in C$ for which $T(c(y), H \cup \{y\}) < 1/n$ for all $H \subset \{x_1, \ldots, x_n\}$. Clearly $C \subset \sigma(x_1, \ldots, x_n)$, and since T is continuous, $\sigma(x_1, \ldots, x_n)$ is open.

Let $\{z_n\}_{n=1}^{\infty}$ be such that $z_{n+1} \in \sigma(z_1, \ldots, z_n)$ for all n, and suppose $\{z_n\}_{n=1}^{\infty} \not\rightarrow C$. Then there is a subsequence $\{z_{n(k)}\}_{k=1}^{\infty}$ which has no cluster point in C. Let $Z = (\bigcup_{k=1}^{\infty} \{z_{n(k)}\})^-$. Let $c_k = c(z_{n(k)})$, and let $c_0 \in C$ be a cluster point of $\{c_1, c_2, \ldots\}$. Since $z_n(m) \in \sigma(z_1, \ldots, z_{n(m-1)}, \ldots, z_{n(m)-1})$, we have $T(c_m, \bigcup_{k=1}^{m} \{z_{n(k)}\}) < 1/(n(m) - 1)$, which goes to zero as $m \rightarrow \infty$. However, $\bigcup_{k=1}^{m} \{z_{n(k)}\} \rightarrow Z$ in 2^X as $m \rightarrow \infty$, and c_0 is a cluster point of $\{c_1, c_2, \ldots\}$. Thus $T(c_0, Z) = 0$, and so $\{z_{n(k)}\}_{k=1}^{\infty}$ has a cluster point in C, namely c_0. This contradiction proves the theorem.

COROLLARY 2.1. *Every CPN-space is countably bisequential.*

PROOF. Immediate, in view of Theorems 2.1 and 2.2.

Theorem 2.3 does not generalize to CCR-spaces, as is seen by Example 4.3. However, we do have the following positive result:

THEOREM 2.3. *Let X be a CCR-space and let $x_0 \in X$. If there is a countable subset C of X such that $x_0 \in \bar{C} - C$, then X is W at x_0 (i.e., there exists a winning strategy at x_0).*

PROOF. Let $\{c_1, c_2, \ldots\}$ be an enumeration of C. Define $\sigma \colon F(X) \rightarrow \tau_{x_0}(X)$ as follows: $z \in \sigma(x_1, \ldots, x_n)$ if and only if $T(x_0, (C_m, H \cup \{z\})) > 1 - 1/(n + 1)$ for all $m \leqslant n$ and $H \subset \{x_1, \ldots, x_n\}$, subject to the condition that $c_m \notin H$.

Let $\{z_n\}_{n=1}^{\infty}$ be such that $z_{n+1} \in \sigma(z_1, \ldots, z_n)$ for all n, and suppose $\{z_n\}_{n=1}^{\infty} \not\rightarrow x_0$. Then there is a subsequence $\{z_{n(k)}\}_{k=1}^{\infty}$ which does not cluster at x_0. We may assume $n(k) < n(k + 1)$ for all $k \in N$. Then let $Z = (\bigcup_{k=1}^{\infty} \{z_{n(k)}\})^-$.

Since $T(x_0, (x_0, Z)) = 0$, there are open sets U in X containing x_0 and V in 2^X containing Z such that if $y \in U$ and $K \in V$, then $T(x_0, (y, K)) < 1/2$. Choose c_M in $U - Z$, and M' so large that $n(M') \geqslant M$ and $\{z_{n(1)}, \dots, z_{n(M')}\}$ $\in V$. Since $z_{n(M'+1)} \in \sigma(z_1, \dots, z_{n(M')}, \dots, z_{n(M'+1)-1})$,

$$T(x_0, (c_M, \{z_{n(1)}, \dots, z_{n(M'+1)}\})) > 1 - 1/n(M' + 1) > 1/2.$$

However, $c_M \in U$ and $\{z_{n(1)}, \dots, z_{n(M'+1)}\} \in V$, so

$$T(x_0, (c_M, \{z_{n(1)}, \dots, z_{n(M'+1)}\})) < 1/2,$$

a contradiction. Thus σ is a winning strategy at x_0.

E. Michael [9] has shown that if the closed continuous image of a metric space is countably bisequential, it is metrizable. As a corollary to this and Theorem 2.3, we have an easy proof of the following result of Zenor:

COROLLARY 2.2 (ZENOR). *If $f: M \rightarrow X$ is a closed continuous map of the metric space M onto the CCR-space X, then X is metrizable.*

PROOF. The closed continuous image of a metric space is a Fréchet space. Thus each point of x is either isolated, or there is a nondegenerate sequence of points converging to it. By Theorem 2.3, X is a W-space, and hence countably bisequential. By Michael's result, then, X is metrizable.

To complete this section, we have a result concerning continuously normal spaces.

THEOREM 2.4. *A CN-space X is W at each point that is G_δ in X.*

PROOF. Let $x_0 \in X$, $x_0 \in \bigcap_{n=1}^{\infty} G_n$, G_n open, with $\overline{G}_{n+1} \subset G_n$ for all n. Let $H_n = \overline{G}_{2n-1} - G_{2n}$ and $K_n = \overline{G}_{2n} - G_{2n+1}$, $n = 1, 2, \dots$. If x_0 is discrete, there is nothing to prove, so assume x_0 is not discrete. Then either $x_0 \in (\bigcup_{n=1}^{\infty} H_n)^-$ or $x_0 \in (\bigcup_{n=1}^{\infty} K_n)^-$. Without loss of generality, we may assume $x_0 \in (\bigcup_{n=1}^{\infty} H_n)^-$. It is easy to see that $(\bigcup_{n=1}^{\infty} H_n) \cup \{x_0\} = (\bigcup_{n=1}^{\infty} H_n)^-$.

We claim that there is an infinite co-infinite set $A^* \subset N$ such that $x_0 \in (\bigcup \{H_n | n \in A^*\})^-$ and $x_0 \in (\bigcup \{H_n | n \in N - A^*\})^-$. Suppose not. Let T be a continuous N-operator for X, and let A be an infinite co-infinite subset of N such that $x_0 \in (\bigcup_{n \in A} H_n)^-$. Let

$$K_{m,n}^0 = \bigcup \{H_i | m < i \leqslant n, i \in A\},$$

and let

$$K_{m,n}^1 = \bigcup \{H_i | m < i \leqslant n, i \in N - A\}.$$

Then $(K_{0,\infty}^0)^- = K_{0,\infty}^0 \cup \{x_0\}$, and since we have assumed that $x_0 \notin (\bigcup_{n \notin A} H_n)^-$, $K_{0,\infty}^1$ is a closed set. Thus $T(x_0, (K_{0,\infty}^0 \cup \{x_0\}, K_{0,\infty}^1)) = 0$. Since $K_{0,n}^1$

converges to $(K_{0,\infty}^i)^-$ in 2^X, for $i = 1$ or 2, there is $n(1) \in N$ such that $T(x_0, (K_{0,n(1)}^0)) < 1/4$.

$$T(x_0, (K_{0,n(1)}^0 \cup K_{n(1),\infty}^1, K_{0,n(1)}^1 \cup K_{n(1),\infty}^0 \cup \{x_0\})) = 1.$$

Choose $n(2) > n(1)$ so that

$$T(x_0, (K_{0,n(1)}^0 \cup K_{n(1),n(2)}^1, K_{0,n(1)}^1 \cup K_{n(1),n(2)}^0)) > 3/4.$$

We continue in like manner. Let $\delta(m) = 0$ if m is even, and $\delta(m) = 1$ if m is odd. Let $\epsilon(m) = 1 - \delta(m)$. If $n(m)$ has been chosen, choose $n(m + 1) > n(m)$ so that

$$T\left(x_0, \left(\bigcup_{i=0}^m K_{n(i),n(i+1)}^{\delta(i)}, \bigcup_{i=0}^m K_{n(i),n(i+1)}^{\epsilon(i)}\right)\right)$$

is less than $1/4$ if m is even and is greater than $3/4$ if m is odd. (We let $n(0) = 0$.)

$T(x_0, ((\bigcup_{i=0}^\infty K_{n(i),n(i+1)}^{\delta(i)})^-, (\bigcup_{i=0}^\infty K_{n(i),n(i+1)}^{\epsilon(i)})^-))$ is equal to 0 or 1. Thus there is $m_0 \in N$ such that if $m > m_0$, then

$$T\left(x_0, \left(\bigcup_{i=0}^m K_{n(i),n(i+1)}^{\delta(i)}, \bigcup_{i=0}^m K_{n(i),n(i+1)}^{\epsilon(i)}\right)\right)$$

is always less than $1/4$ or always greater than $3/4$. This gives us the contradiction which proves our claim.

Let A^*, then, be an infinite co-infinite subset of N such that

$$x_0 \in \left(\bigcup \{H_n \mid n \in A^*\}\right) \cap \left(\bigcup \{H_n \mid n \in N - A^*\}\right).$$

Let $A^* = \{p(1), p(2), \dots\}$, and let $H^* = \bigcup \{H_n \mid n \in N - A^*\}$.

Choose $x_1 \in H^*$. If x_i has been chosen for all $i \leqslant m$, choose $x_{m+1} \in U \cap H^*$, where U is an open set containing x_0 such that if $x \in U$ and $K \subset \{x_1, \dots, x_m\}$, then $T(x_0, (\bigcup_{i=1}^m H_{p(i)}, K \cup \{x\})) > 1 - 1/m$. This is possible since we always have $T(x_0, (\bigcup_{i=1}^m H_{p(i)}, K \cup \{x_0\})) = 1$ for each $K \subset \{x_1, \dots, x_m\}$.

Suppose $\{x_1, x_2, \dots\}$ does not cluster at x_0. Then as $m \to \infty$,

$$T\left(x_0, \left(\bigcup_{i=1}^m H_{p(i)}, \bigcup_{i=1}^{m+1} \{x_i\}\right)\right)$$

$$\to T\left(x_0, \left(\bigcup_{i=1}^\infty H_{p(i)} \cup \{x_0\}, \left(\bigcup_{i=1}^\infty \{x_i\}\right)^-\right)\right) = 0.$$

However, $T(x_0, (\bigcup_{i=1}^m H_{p(i)}, \bigcup_{i=1}^{m+1} \{x_i\})) > 1 - 1/m \to 1$ as $m \to \infty$. Therefore $\{x_1, x_2, \dots\}$ does cluster at x_0, and it follows from Theorem 2.3 that X is W at x_0.

III. **Metrization theorems.** Suppose $T: X \times 2^X \to [0, 1]$ is a continuous
PN-operator for a space X. A base for the topology of X may be written in terms
of T in various ways. If X has a countable dense subset C, then $F(C)$, the set of
all finite subsets of C, is dense in 2^X. Thus the values of T are determined by
its values on the countable set $C \times F(C)$. Using this fact in the right way, we
can actually get a countable base for X. Indeed we can extend the idea to
CCR-spaces to get our next theorem, which substantially improves several results
of Zenor [16].

THEOREM 3.1. *A separable CCR-space is metrizable.*

PROOF. Let $T: X \times M_D(x) \to [0, 1]$ be a continuous CR-operator for the
separable space X with countable dense subset $Q = \{r_1, r_2, \ldots\}$. For each
$x \in X$ and $n \in N$, let $U_{x,n}$ be the set of all $z \in X$ such that

$$T(x, (r, H \cup \{z\})) > 1 - 1/n$$

for all pairs $(r, H) \in M_D(x)$ such that r and H are contained in $\{r_1, r_2, \ldots, r_n\}$.
Since for each n, there are only a finite number of such pairs, and since T is
continuous, it is easy to see that $U_{x,n}$ is an open set containing x. We shall show
that $\{U_{r,n} | r \in Q, n \in N\}$ is a countable basis for X.

Given $x_0 \in X$, we claim that there is a sequence $\{s_n\}_{n=1}^{\infty} \to x_0$ such that
$s_n \in Q$ and $x_0 \in U_{s_n, n}$ for all $n \in N$. If $x_0 \in Q$, we can take $x_0 = s_n$ for all n.

If $x_0 \notin Q$, let $\{q_n\}_{n=1}^{\infty}$ be a sequence converging to x_0 such that $q_n \in Q$
for all n. By Theorem 2.3, X is a Fréchet space; hence such a sequence must
exist. Let $k \in N$, and let $\{r\} \cup H \subset \{r_1, \ldots, r_k\}$ with $r \notin H$. Since
$T(x_0, (r, H \cup \{x_0\})) = 1$, there is an open set $U_{r,H,k}$ containing x_0 such that if
x and y are in $U_{r,H,k}$, then $T(x, (r, H \cup \{y\})) > 1 - 1/k$. Let

$$V_k = \bigcap \{U_{r,H,k} | \{r\} \cup H \subset \{r_1, \ldots, r_k\}, r \notin H\},$$

and choose $q_{n(k)} \in V_k$. Then $T(q_{n(k)}, (r, H \cup \{x_0\})) > 1 - 1/k$ for all r and
H such that $\{r\} \cup H \subset \{r_1, \ldots, r_k\}$, provided $r \notin H$. Thus $x_0 \in U_{q_{n(k)}, k}$. If
we now let $s_k = q_{n(k)}$, we have the desired sequence.

Let $x_0 \in U, X, U$ open. We shall show that for some $n \in N$, $x_0 \in U_{s_n, n} \subset U$.

Suppose not; then there exists an open set V such that $x_0 \in \overline{V} \subset U$ and
$U_{s_n, n} - \overline{V} \neq \emptyset$ for any n. Choose $p(1) \in N$ such that $n \geq p(1)$ implies $s_n \in V$.

Choose $t_1 \in (U_{s_{p(1)}, p(1)} - \overline{V}) \cap Q$. Let $p(2) = \max \{m(1), n(1), p(1)\} +$
1, where $m(1)$ and $n(1)$ are such that $s_{p(1)} = t_1 = r_{n(1)}$.

If $p(i)$ has been chosen for all $i \leq k$, choose $p(k + 1)$ as follows: choose
$t_k \in (U_{s_{p(k)}, p(k)} - \overline{V}) \cap Q$ and let $p(k + 1) = \max \{m(k), n(k), p(k)\} + 1$,

where $m(k)$ and $n(k)$ are such that $s_{p(k)} = r_{m(k)}$ and $t_k = r_{n(k)}$.

We have $s_{p(k-1)} = r_{m(k-1)}$ and $m(k-1) < p(k)$; also, for all $j < k$, $t_j = r_{n(j)}$ and $n(j) < p(j+1) \leqslant p(k)$. Thus

$$\{s_{p(k-1)}\} \cup \left(\bigcup_{j=1}^{k-1} \{t_j\} \right) \subset \{r_1, \ldots, r_{p(k)}\},$$

and $s_{p(k-1)} \notin \bigcup_{j=1}^{k-1} \{t_j\}$ since $s_n \in V$ for all $n \geqslant p(1)$, while $t_j \in X - \overline{V}$ for all $j \in N$. Thus $t_k \in U_{s_{p(k)}, p(k)}$ implies

$$T\left(s_{p(k)}, \left(s_{p(k-1)}, \bigcup_{j=1}^{k} \{t_j\} \right) \right) > 1 - 1/p(k),$$

which converges to 1 as $k \longrightarrow \infty$.

But $s_{p(k)} \longrightarrow x_0$ and $\bigcup_{j=1}^{k} \{t_j\} \longrightarrow (\bigcup_{j=1}^{\infty} \{t_j\})^-$ as $k \longrightarrow \infty$. Thus

$$T\left(s_{p(k)}, \left(s_{p(k-1)}, \bigcup_{j=1}^{k} \{t_j\} \right) \right) \longrightarrow T\left(x_0, \left(x_0, \left(\bigcup_{j=1}^{\infty} \{t_j\} \right)^- \right) \right) = 0,$$

a contradiction.

Therefore, for some $n \in N$, $x_0 \in U_{s_n, n} \subset U$; hence $\{U_{r,n} | r \in Q, n \in N\}$ is a basis for X. Since a regular space with a countable basis is metrizable, the theorem is proved.

COROLLARY 3.1. *A paracompact locally separable CCR-space X is metrizable.*

PROOF. Let $x_0 \in X$ and let U be an open subset of X containing x_0 with a countable dense subset. Then $\overline{U}$ is a separable CCR-space, hence metrizable. Thus X is locally metrizable, and so by a well-known theorem of Nagata [10], is metrizable.

Example 5.2 shows that the hypothesis of separability in Theorem 3.1 cannot be replaced by Lindelöf. However, if we restrict ourselves to CPN-spaces, we can improve Theorem 3.1 as follows:

THEOREM 3.2. *For a topological space X, the following are equivalent*:
(i) *X is a separable metric space.*
(ii) *X is a CPN-space satisfying the countable chain condition*[4].
(iii) *X is a Lindelöf CPN-space.*
(iv) *X is a separable CPN-space.*

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[4] A space X satisfies the *countable chain condition* if every family of disjoint open subsets of X is countable.

PROOF. (iv) $\Rightarrow$ (i). This follows immediately from Theorem 3.1.

(i) $\Rightarrow$ (ii). This follows from the fact that every metric space is CPN and every separable space satisfies the countable chain condition.

(ii) $\Rightarrow$ (iii). Let X be a CPN-space satisfying the countable chain condition, and suppose X is not Lindelöf. Since a CPN-space is hereditarily CPN, every separable subspace is metrizable, and hence Lindelöf. By [13, Theorem 3], a non-Lindelöf space in which the closure of every countable discrete subspace is Lindelöf must contain an uncountable discrete subset. However, since X is hereditarily collectionwise normal, there must then exist an uncountable family of disjoint open subsets of X. This contradiction shows that X must be Lindelöf.

(iii) $\Rightarrow$ (iv). Let X be a Lindelöf CPN-space with continuous PN-operator T. We shall prove that X is separable.

For each $x \in X$, let $U_x = \{y \in X \mid T(y, \{x\}) < 1\}$. By the continuity of T and the fact that $T(x, \{x\}) = 0$, U_x is an open set containing x. Let $\{U_{x(i)}\}_{i=1}^{\infty}$ be a countable subcover of $\{U_x \mid x \in X\}$.

For each $i \in N$ and $x \in X$, let $U_x^i = \{y \in X \mid T(y, \{x(i), x\}) < 1/2\}$. Let $\{U_{x(i,j)}^i\}_{j=1}^{\infty}$ be a countable subcover of $\{U_x^i \mid x \in X\}$.

We continue the process inductively. Suppose $x(i_1, i_2, \ldots, i_k)$ have been chosen for all $(i_1, \ldots, i_k) \in N^k$, $k < n$. Then for each element $(i_1, \ldots, i_{n-1}) \in N^{n-1}$ and $x \in X$, we define

$$U_x^{i_1, \ldots, i_{n-1}} = \left\{ y \in X \,\middle|\, T\left(y, \{x\} \cup \bigcup_{k=1}^{n-1} \{x(i_1, \ldots, i_k)\}\right) < 1/n \right\}.$$

Let

$$\{U_{x(i_1, \ldots, i_n)}^{i_1, \ldots, i_{n-1}}\}_{i_n=1}^{\infty}$$

be a countable subcover of $\{U_x^{i_1, \ldots, i_{n-1}} \mid x \in X\}$.

We shall show that $Q = \{x(i_1, \ldots, i_k) \mid (i_1, \ldots, i_k) \in N^k, k \in N\}$ is a countable dense subset of X. To this end, let x_0 be an arbitrary element of X. $x_0 \in U_{x(i_1)}$ for some $i_1 \in N$; we have $T(x_0, x(i_1)) < 1$. Choose $i_2 \in N$ such that $x_0 \in U_{x(i_1, i_2)}^{i_1}$; then

$$T(x_0, \{x(i_1), x(i_1, i_2)\}) < 1/2.$$

If $i_1, i_2, \ldots, i_k$ have been chosen, choose $i_{k+1} \in N$ so that

$$T\left(x_0, \bigcup_{j=1}^{k+1} \{x(i_1, \ldots, i_j)\}\right) < \frac{1}{k} + 1.$$

Then $T(x_0, (\bigcup_{j=1}^{\infty} \{x(i_1, \ldots, i_j)\})^-) = 0$; thus $x_0 \in \overline{Q}$ and the proof is finished.

Our next theorem is somewhat technical, but we shall use it in the next section to get an example of a first countable nonmetrizable CPN-space.

THEOREM 3.3. *Let $f\colon X \longrightarrow Y$ be a perfect map of a space X onto a CPN-space Y. If X condenses onto a metric space M with $\dim M = 0$, then X is a CPN-space.*

PROOF. Following Borges [1, Theorem 8.1], we get a homeomorphism of X into the space $M \times Y$. Since $\dim M = 0$, M is homeomorphic to a subspace of the countable product of discrete spaces. Thus by Zenor [16, Theorem 2.8], $M \times Y$ is a CPN-space. Therefore X is a CPN-space.

It is interesting to note that it is unknown whether the product of a CPN-space with a metric space is CPN. If it is, then the condition in the above theorem that $\dim M = 0$ can be removed.

IV. Examples. Our first example is the most important, answering a question of Zenor which was raised in an earlier version of [16]; namely, whether every CN-space is metrizable.

EXAMPLE 4.1. There exists a CPN-space which is not first countable.

PROOF. Let A be an uncountable set and let $Z^+ \cup \{\infty\}$ be the space of positive integers, together with the point ∞, with the discrete topology.

Let X' be the subspace of $\Pi_{\alpha \in A} (Z^+ \cup \{\infty\})_\alpha$ consisting of all functions f such that if $f(\alpha) = n \in Z^+$, then $f(\beta) = \infty$ for all $\beta \neq \alpha$. Denote by n_α that element of X' for which $n_\alpha(\alpha) = n$, and by ω that element for which $\omega(\alpha) = \infty$ for all $\alpha \in A$.

Let X be the space obtained from X' by adding all singletons except ω to the topology, together with all sets of the form

$$O_n = \{\omega\} \cup \{m_\alpha \mid m > n\}.$$

For every open set U containing ω, the set $A_U = \{\alpha \in A \mid$ there does not exist $n_\alpha \in U\}$ is finite. Suppose $\{U(n)\}_{n=1}^{\infty}$ is a countable local basis at ω. Since A is uncountable, there exists $\alpha \in \bigcap_{n=1}^{\infty} (A - A_{U(n)})$. But then $X - \{n_\alpha \mid n \in Z^+\}$ is an open set containing ω which does not contain $U(n)$ for any $n \in Z^+$. Thus X is not first countable.

We claim that X is a CPN-space. To prove this, we shall construct a function $\gamma\colon 2^X \longrightarrow Z^+ \cup \{\infty\}$ with the following properties:

(a) $\gamma(F) = \infty$ if any only if $\omega \in F$;

(b) if $\gamma(F) = n$, then there is an open set U in 2^X containing F such that $\gamma(K) = n$ for all $K \in U$;

(c) if $\omega \in F$, then for each $n \in Z^+$, there is an open set V in 2^X containing F such that $\gamma(K) > n$ for all $K \in V$.

Suppose we have such a function γ. Define $T\colon X \times 2^X \longrightarrow [0, 1]$ as follows:

$$T(\omega, F) = 1/\gamma(F),$$

$$T(n_\alpha, F) = \begin{cases} 0 & \text{if } n_\alpha \in F, \\ \max\{1/\gamma(F), 1/n\} & \text{if } n_\alpha \notin F. \end{cases}$$

(If $\gamma(F) = \infty$, we shall say $1/\gamma(F) = 0$.)

We shall first prove that T is continuous at every point $(x, F) \in X \times 2^X$; there are several cases to consider:

(i) If $x = n_\alpha \in F$, then $T(n_\alpha, F) = 0$. Clearly, $\mathcal{W} = (\{n_\alpha\}, \langle\{n_\alpha\}, X\rangle)$ is an open subset of $X \times 2^X$ containing (n_α, F) such that if $(y, H) \in U$, then $T(y, H) = 0$.

(ii) If $x = n_\alpha \notin F$, let $\mathcal{U}$ be open in 2^X such that $F \in \mathcal{U}, n_\alpha \notin \bigcup \mathcal{U}$, and if (a) $\omega \notin F$, then $H \in \mathcal{U}$ implies $\gamma(H) = \gamma(F)$; if (b) $\omega \in F$, then $H \in \mathcal{U}$ implies $\gamma(H) \geqslant n$. Let $\mathcal{W} = (\{n_\alpha\}, \mathcal{U})$. Then it is easy to see that $(y, H) \in \mathcal{W}$ implies $T(y, H) = T(n_\alpha, F)$.

(iii) If $x = \omega \in F$, $T(x, F) = 0$. Choose $\epsilon > 0$. There is $n \in Z^+$ such that $1/n < \epsilon$. Let $\mathcal{U}$ be an open subset of 2^X containing F such that $H \in \mathcal{U}$ implies $\gamma(H) \geqslant n$. Let $\mathcal{W} = (O_n, \mathcal{U})$. Then $(y, H) \in \mathcal{W}$ implies $T(y, H) < \epsilon$.

(iv) If $x = \omega \notin F$, let $\mathcal{U}$ be an open subset of 2^X containing F such that if $H \in \mathcal{U}, \gamma(\mathrm{H}) = \gamma(F)$. Let $n \geqslant \gamma(F)$, and let $\mathcal{W} = ((X - F) \cap O_n, \mathcal{U} \cap \langle F\rangle)$. Then if $(y, H) \in \mathcal{W}$, we see that $T(y, H) = 1/\gamma(F) = T(x, F)$. This takes care of all the cases; hence T is continuous.

Clearly, for each $F \in 2^X$, we have $F = \{x \in X | T(x, F) = 0\}$. Thus T is a continuous PN-operator for X, and X is a CPN-space.

Construction of the function γ. Define $C(\alpha) = \{n_\alpha | n \in Z^+\}$. Sets of the form $O_n \cap (X - \bigcup_{i=1}^m C(\alpha_i))$, where $\{\alpha_1, \ldots, \alpha_n\}$ is a finite subset of A, are a local basis at ω. Thus a set F is closed if and only if $\omega \in F$, or there exists an integer n and a finite subset $\{\alpha_1, \ldots, \alpha_m\}$ of A such that $F \subset (X - O_n) \cup \bigcup_{i=1}^m C(\alpha_i)$. For a closed set F where $\omega \notin F$, define

$$\mathrm{cov}\, F = \min\left\{ m + n \,\middle|\, F \subset (X - O_n) \cup \bigcup_{i=1}^m C(\alpha_i)\right\}$$

for some $\{\alpha_1, \ldots, \alpha_m\} \subset A$.

LEMMA 1. *For each closed set F with $\omega \notin F$, there is a finite set $H \subset F$ such that* $\mathrm{cov}\, H = \mathrm{cov}\, F$.

PROOF. Let $F \subset (X - O_N) \cup \bigcup_{i=1}^M C(\alpha_i)$, where $M + N = \mathrm{cov}\, F$. For each $i = 1, \ldots, M$, choose $x_i \in F \cap C(\alpha_i)$ such that $x_i(\alpha_i) = \max\{n | n_{\alpha_i} \in F\}$ if this maximum exists; otherwise choose $x_i \in F \cap C(\alpha_i)$ such that $x_i(\alpha_i) > \mathrm{cov}\, F$.

Since $F \not\subset (X - O_{N-1}) \cup \bigcup_{i=1}^{M} C(\alpha_i)$, there is $y_1 \in F \cap O_{N-1} \cap (X - \bigcup_{i=1}^{M} C(\alpha_i))$. Since $F \not\subset (X - O_{N-2}) \cup \bigcup_{i=1}^{M+1} C(\alpha_i)$, where $y_1 \in C(\alpha_{M+1})$, there is $y_2 \in F \cap O_{N-2} \cap (X - \bigcup_{i=1}^{M+1} C(\alpha_i))$. Continue in like manner, defining $y_3, y_4, \ldots, y_N$.

Let $H = \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} \{x_i, y_j\}$. Since $H \subset F$, cov $H \leqslant M + N =$ cov F. Suppose $H \subset (X - O_{N'}) \cup \bigcup_{i=1}^{M'} C(\beta_i)$, where cov $H = M' + N'$. If $N' < N$, then $\{y_1, \ldots, y_{N-N'}, x_1, \ldots, x_M\} \subset O_{N'}$. Therefore, since all elements of H are in different $C(\beta)$'s, $M' \geqslant M + (N - N')$. Thus $M' + N' \geqslant M + N$, and so $M' + N' = M + N$.

Suppose $N' \geqslant N$. If $X - O_{N'}$ contains more than $N' - N$ of the x_i's, there would be at most $M - (N' - N + 1)$ of the x_i's left. Then, by the way the x_i's were chosen, $X - O_{N'}$, together with the corresponding $M - N' + N - 1$ of the $C(\alpha)$'s, would cover F, contradicting cov $F = M + N$. We have, then, that $X - O_{N'}$ contains no more than $N' - N$ of the x_i's. Hence $M' \geqslant M - (N' - N)$, and so $M' + N' \geqslant M + N$. Thus cov $H = M + N$, and the lemma is proved.

Let F be closed, $\omega \notin F$. It is now clear that we can find a finite set $H^* \subset F$ with card$(H^*) =$ cov $H^* =$ cov F, such that if $m_\alpha \in F$ and $n_\alpha \in H^*$ with $m < n$, then cov$((H^* - \{n_\alpha\}) \cup \{m_\alpha\}) <$ cov H^*. Call such a set *minimal for F*. (H^* is obtained by a finite process from the H of the previous lemma. If $n_\alpha \in H$, we replace n_α by m_α, where m is the smallest integer such that $m_\alpha \in F$ and cov$((H - \{n_\alpha\}) \cup \{m_\alpha\}) =$ cov F.)

For H finite, define max $H = \max \{n \mid n_\alpha \in H$ for some $\alpha \in A\}$. For F closed with $\omega \notin F$, define

$$\gamma(F) = \max \{\max H \mid H \text{ is minimal for } F\}.$$

If $\omega \in F$, let $\gamma(F) = \infty$.

LEMMA 2. *$\gamma(F)$ is well defined, i.e., if F is closed and $\omega \notin F$, then the set $\{\max H \mid H$ is minimal for $F\}$ is bounded.*

PROOF. Let $F \subset (X - O_N) \cup \bigcup_{i=1}^{M} C(\alpha_i)$. Suppose $\{\max H_n\}_{n=1}^{\infty} \to \infty$, where each H_n is minimal for F. Then there are $p(n)_{\alpha(n)} \in H_n$ such that $\{p(n)\}_{n=1}^{\infty} \to \infty$. There must be an infinite number of the $p(n)_{\alpha(n)}$'s in one of the $C(\alpha_i)$'s, $1 \leqslant i \leqslant M$. Thus there are integers n' and n'' such that cov $F < p(n') < p(n'')$ and $\alpha(n') = \alpha(n'')$. But clearly cov$((H_{n''} - \{p(n'')_{\alpha(n'')}\}) \cup \{p(n')_{\alpha(n')}\}) =$ cov $H_{n''}$, contradicting minimality of $H_{n''}$. Thus Lemma 2 is proved.

It is clear from Lemma 2 and the definition of γ that γ has property (a).

Proof that γ has property (b). Let $\gamma(F) = N$. There is a set H minimal for F such that max $H = N$. Let $\hat{H} = \{n_\alpha \in F \mid n \leqslant N$ and $H \cap C(\alpha) \neq \varnothing\} = \{h_1, \ldots, h_k\}$.

Let $U = \langle \{h_1\}, \ldots, \{h_k\}, F \rangle$ and suppose $K \in U$. Since $H \subset \hat{H} \subset K \subset F$, H is minimal for K; hence $\gamma(K) \geqslant N$.

Let L be minimal for K and suppose $\max L = N' > N$, where $N'_\beta \in L$. If $K \subset (X - O_n) \cup \bigcup_{i=1}^m C(\alpha_i)$ where $m + n = \text{cov } K$, then $\beta = \alpha_i$ for some i, $1 \leqslant i \leqslant m$, since $N'_\beta \in K$ and $N' > N = \gamma(F) \geqslant \text{cov } F = \text{cov } K$. Thus, since $H \subset K$, $C(\beta) \cap H \neq \varnothing$, for otherwise $\text{cov } H < \text{cov } K$.

Let p be the least integer such that $p_\beta \in F$ and $\text{cov } L = \text{cov}((L - \{N'_\beta\}) \cup \{p_\beta\})$. Since L is minimal for K and $\hat{H} \subset K$, $p_\beta \notin \hat{H}$. Thus $p > N$.

Let $L_1 = ((L - \{N'_\beta\}) \cup \{p_\beta\})$. Pick $n_\alpha \in L$, $\alpha \neq \beta$. Let q be the least integer such that $q_\alpha \in F$ and $\text{cov } L_1 = \text{cov}((L_1 - \{n_\alpha\}) \cup \{q_\alpha\})$. Let $L_2 = (L_1 - \{n_\alpha\}) \cup \{q_\alpha\}$. Pick $m_\delta \in L$, $\delta \notin \{\alpha, \beta\}$, and continue in like manner. It is easy to see that if $\text{card}(L) = l$, then L_l is minimal for F. But $\max L = p > N$, contradicting $\gamma(F) = N$. Thus $\max L \leqslant N$ and we have $\gamma(K) = N$.

Proof that γ has property (c). Let F be closed with $\omega \in F$, and choose $N \in Z^+$.

Case I. Suppose $\omega \in (F - \{\omega\})^-$. Then either $\{\alpha \in A \mid C(\alpha) \cap F \text{ is infinite}\}$ is an infinite set, or $\{n \mid n_\alpha \in F \text{ and } C(\alpha) \cap F \text{ is finite}\}$ is not bounded. In either case, we can easily find $p(n)_{\alpha(n)} \in F$, $n = 1, 2, \ldots$, such that $p(n) < p(n + 1)$ and $\alpha(n) \neq \alpha(n')$ for $n \neq n'$.

Let $H = \bigcup_{i=1}^N \{p(i)_{\alpha(i)}\}$. Clearly $\text{cov } H = N$. Let $U = \langle \{p(1)_{\alpha(1)}\}, \ldots, \{p(N)_{\alpha(n)}\}, X \rangle$. If $K \in U$, then $\text{cov } K \geqslant \text{cov } H = N$, so $\gamma(K) \geqslant N$.

Case II. Suppose $F = F' \cup \{\omega\}$, F' closed and $\omega \notin F'$. Choose $N \in Z^+$, $N > \text{cov } F'$. Let $H = \{h_1, \ldots, h_k\}$ be minimal for F', and let $A_0 = \{\alpha \in A \mid C(\alpha) \cap H \neq \varnothing\}$. Let $V = (X - \bigcup_{\alpha \in A_0} C(\alpha)) \cap O_N$. Let $U = \langle \{h_1\}, \ldots, \{h_k\}, F', V \rangle$, and suppose $K \in U$. Since $\hat{H} \subset K$ and $K \cap V \neq \varnothing$, it is easy to see that $\text{cov } K > \text{cov } H = \text{cov } F'$. Thus if L is minimal for K, $L \cap V \neq \varnothing$, for otherwise $L \subset F'$ and $\text{cov } L \leqslant \text{cov } F'$. Thus $\max L > N$, and so $\gamma(K) > N$ and the proof is finished.

EXAMPLE 4.2. There is a first countable CPN-space which is not metrizable.

PROOF. Let $K \subset [0, 1]$ be the Cantor set. Let $M = (\bigcup_{n=1}^\infty (K \times \{1/n\})) \cup (K \times \{0\})$. Let Y be the set M with the following topology: the points of $\bigcup_{n=1}^\infty (K \times \{1/n\})$ are discrete, and a basic neighborhood of $(x, 0)$ is the set

$$U_n(x, 0) = \left\{ y \in Y \,\middle|\, d(y, x) < \frac{1}{n} \right\} - \left\{ \left(x, \frac{1}{m}\right) \,\middle|\, m \in Z^+ \right\}.$$

Let X be the space of Example 4.1, using K as the uncountable set A. It is easy to see that X is homeomorphic to the factor space $Y/K \times \{0\}$. Since Y condenses onto the 0-dimensional metric space M, where M is viewed as a subspace of $[0, 1] \times [0, 1]$, Y is a CPN-space by Theorem 3.3. Thus Y is a first countable CPN-space, and is not metrizable because there is a perfect map from Y onto the nonmetrizable space X.

According to Zenor [16], a CR-operator T for a space X is said to be *monotone* if whenever (y, H) and (y, K) are in $M_D(X)$ such that $H \subset K$, we have $T(x, (y, H)) \leqslant T(x, (y, K))$ for every $x \in X$. X is said to be *continuously monotonically completely regular* (CMCR) if X admits a monotone CR-operator.

EXAMPLE 4.3.   There exists a Lindelöf CMCR-space which is not a CN-space.

PROOF.   Let Ω be the first uncountable ordinal. Let X be the space obtained from ordinal space $[0, \Omega]$ with the usual interval topology by prescribing all ordinals $\alpha < \Omega$ to be discrete. (This space was shown by H. Tamano and J. E. Vaughn in [14] to be elastic; hence by a theorem of Borges [2], it is monotonically normal.)

If H is a closed subset of X, define

$$\sup H = \inf \{\alpha \in [0, \Omega] \mid h \leqslant \alpha \text{ for all } h \in H\}.$$

Define a function $T: X \times M_D(x) \longrightarrow [0, 1]$ as follows:

(i) $T(x, (y, H)) = 0$ if $x = y$, or if $y \geqslant \sup H$ and $x \notin H$;

(ii) $T(x, (y, H)) = 1$ if $x \in H$, or if $y < \sup H$ and $x \neq y$.

It is easily checked that T is well defined for each $(x, (y, H)) \in X \times M_D(X)$.

We shall prove that T is continuous by showing that $T^{-1}(0)$ and $T^{-1}(1)$ are open sets. Let $(x, (y, H)) \in T^{-1}(0)$. If $x = y \neq \Omega$, then $(\{x\}, (\{y\}, X))$ is an open set in $X \times M_D(X)$ containing $(x, (y, H))$ and contained in $T^{-1}(0)$. If $x = y = \Omega$, then $\sup H < \Omega$, and $((\sup H, \Omega], ((\sup H, \Omega], [0, \sup H]))$ contains $(\Omega, (\Omega, H))$ and is contained in $T^{-1}(0)$. If $x \neq y$, but $y \geqslant \sup H$ and $x \notin H$, then there are the following three possibilities:

(i) $x < \Omega$ and $y < \Omega$,

(ii) $x = \Omega$ and $y < \Omega$,

(iii) $x < \Omega$ and $y = \Omega$.

It is an easy matter to check that $(x, (y, H))$ is in the interior of $T^{-1}(0)$ in each of these three cases. The proof that $T^{-1}(1)$ is open is similar. Thus T is a continuous CR-operator.

Suppose (y, H) and (y, K) are elements of $M_D(X)$ such that $H \subset K$. If x is such that $T(x, (y, K)) = 1$, then certainly $T(x, (y, H)) \leqslant T(x, (y, K))$. If $T(x, (y, K)) = 0$, then either

(i) $x = y$, in which case $T(x, (y, H)) = 0$, or

(ii) $y \geqslant \sup K$ and $x \notin K$,

in which case $y \geqslant \sup H$ and $x \notin H$, so $T(x, (y, H)) = 0$. Thus $T(x, (y, H)) \leqslant T(x, (y, K))$ for every $x \in X$. Hence T is a monotone CR-operator, and so X is CMCR.

We shall prove by contradiction that X is not a CN-space. Suppose there exists a continuous N-operator T for X. Let $\alpha < \Omega$. For every pair of finite subsets

F and G contained in $[0, \alpha]$ such that $F \cap G = \varnothing$, $T(\Omega, (F \cup \{\Omega\}, G)) = 0$ and $T(\Omega, (F, G \cup \{\Omega\})) = 1$. Thus for each $n \in Z^+$ there exists an ordinal $\beta_n(\alpha, F, G)$ $< \Omega$ such that if $\gamma \geq \beta_n(\alpha, F, G)$ and $H \subset [\beta_n(\alpha, F, G), \Omega]$, $H \in 2^X$, then $T(\gamma, (F \cup H, G)) < 1/n$ and $T(\gamma, (F, G \cup H)) > 1 - 1/n$. Let

$$\beta(\alpha) = \sup\{\beta_n(\alpha, F, G) | n \in Z^+, F \text{ and } G \text{ finite disjoint subsets of } [0, \alpha]\}.$$

Then whenever $\gamma \geq \beta(\alpha)$, $H \subset [\beta(\alpha), \Omega]$, $H \in 2^X$, and F and G are finite disjoint subsets of $[0, \alpha]$, we must have $T(\gamma, (F \cup H, G)) = 0$ and $T(\gamma, (F, G \cup H)) = 1$.

Now let $\alpha_1 \neq \delta_1$, with $\alpha_1 < \Omega$ and $\delta_1 < \Omega$, and proceed as follows: if α_i and δ_i have been chosen for all $i < n$, choose $\alpha_n \neq \delta_n$ not yet chosen such that

$$\{\alpha_n, \delta_n\} \subset \left(\sup_{i<n} \{\beta(\alpha_i), \beta(\delta_i)\}, \Omega \right).$$

Consider $(\Omega, (\bigcup_{i=1}^{\infty} \{\alpha_i\}, \bigcup_{i=1}^{\infty} \{\delta_i\})) \in X \times M(X)$.

$$T\left(\Omega, \left(\bigcup_{i=1}^{n+1} \{\alpha_i\}, \bigcup_{i=1}^{n} \{\delta_i\} \right)\right) = 0$$

since $\alpha_{n+1} > \sup\{\beta(\alpha_i), \beta(\delta_i)\} \geq \beta(\sup_{i \leq n} \{\alpha_i, \delta_i\})$. Hence

$$T\left(\Omega, \left(\bigcup_{i=1}^{\infty} \{\alpha_i\}, \bigcup_{i=1}^{\infty} \{\delta_i\} \right)\right) = 0.$$

Similarly, $T(\Omega, (\bigcup_{i=1}^{n} \{\alpha_i\}, \bigcup_{i=1}^{n+1} \{\delta_i\})) = 1$, and so $T(\Omega, (\bigcup_{i=1}^{\infty} \{\alpha_i\}, \bigcup_{i=1}^{\infty} \{\delta_i\})) = 1$, a contradiction, which proves that X is not CN.

EXAMPLE 4.4. There exists a CMCR-space Y and a compact (in fact, finite) subset A of Y such that the factor space Y/A is not CCR.[5]

PROOF. Let X be the space of Example 4.3, and let Y be the disjoint union of X and a convergent sequence with limit point y_0. Let $A = \{\Omega, y_0\}$. In Y/A, there is a nondegenerate sequence of points converging to $\hat{A}$, the image of A under the projection. Thus by Theorem 2.3, if Y/A is a CCR-space, it must be W at $\hat{A}$. But Y/A is not even Fréchet at $\hat{A}$, since $[0, \Omega)$ clusters at A. Thus Y/A is not a CCR-space.

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[5] We have been able to prove that if A is a compact subset of a CPN-space X, then X/A is CPN. It is not known whether the perfect image of a CPN-space is again CPN.

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