TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 225, 1977

STABILITY IN WITT RINGS

BY

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ABSTRACT. An abstract Witt ring R is defined to be a certain quotient of an integral group ring for a group of exponent 2. The ring R has a unique maximal ideal M containing 2. A variety of results are obtained concerning n-stability, the condition that $M^{n+1} = 2M^n$, especially its relationship to the ring of continuous functions from the space of minimal prime ideals of R to the integers. For finite groups, a characterization of integral group rings is obtained in terms of n-stability. For Witt rings of formally real fields, conditions equivalent to n-stability are given in terms of the real places defined on the field.

1. **Introduction.** In recent years, a fruitful way of studying quadratic or bilinear forms over a field F has been to look at the Witt ring $W(F)$ of equivalence classes of nondegenerate symmetric bilinear forms over the field. This has been generalized in various ways, in particular to equivalence classes of nondegenerate hermitian forms over a commutative ring C with involution [10]. If C is a connected semilocal ring, then most of the abstract structure theory for the Witt ring of a field still holds [10]. Knebusch, Rosenberg and Ware have defined the notion of an abstract Witt ring for an abelian q-group G [10, Definition 3.12], a ring of the form $R = \mathbf{Z}G/K$ where $\mathbf{Z}G$ is the integral group ring of G and K is an ideal such that R has only q-torsion. Witt rings of fields and commutative semilocal rings always have this form where G is a group of exponent 2 (in the case of a field, G is the square factor group of the field). In this paper the term Witt ring will always mean Witt ring for a group of exponent 2.

An abstract Witt ring R has a unique maximal ideal containing 2 [10, Lemma 2.13] and we shall always denote this ideal by M. In fact, M is the image of the maximal ideal of $\mathbf{Z}G$ which is the kernel of the homomorphism defined by composing the augmentation map $\mathbf{Z}G \to \mathbf{Z}$ with the projection $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$. The main purpose of this paper is to investigate the following condition on R.

Presented to the Society, January 22, 1976; received by the editors September 8, 1975.

AMS (MOS) subject classifications (1970). Primary 13A15, 12D15, 54H10; Secondary 15A63, 16A26.

Key words and phrases. Witt ring, n-stability, group ring, real place, Boolean space.

(1) Partially supported by NSF Grant MPS75-06987.

DEFINITION 1.1. A Witt ring R is *n-stable* if $M^{n+1} = 2M^n$, where n is any integer greater than or equal to 0 and by M^0 we mean the ring R.

The concept of n-stability was first introduced in [5] for Witt rings of formally real fields where it can be related to the theory of Pfister forms over the field. Note that if R is n-stable, then R is m-stable for all $m \geqslant n$.

In general, Witt rings can be divided into two large classes, those for which the torsion subgroup R_t equals R and those for which R_t equals the nilradical Nil R [10, §3]. For Witt rings of fields, the latter case contains precisely those for which the field is formally real. If $R_t = R$, then R is a local ring with M as its unique prime ideal and $2^m = 0$ for some m [10, Proposition 3.16]. Thus if R is n-stable, then $M^{m+n} = 2^m M^n = 0$.

Consequently, our main interest is in the case where $R_t = \text{Nil } R$, and this will be the only case considered in §§2, 3 and 4. In this case, the reduced Witt ring $R_{\text{red}} = R/\text{Nil } R$ is still a Witt ring for the same group [10, Remark 3.13] and can be embedded naturally in $\mathcal{C}(X, \mathbf{Z})$, the ring of continuous functions from the Boolean (compact, Hausdorff, totally disconnected) topological space X or $X(R)$ of all minimal prime ideals of R (with the induced Zariski topology) to the integers endowed with the discrete topology [9, §3]. The points of X also correspond to the homomorphisms $R \to \mathbf{Z}$, so the homomorphism $R \to \mathcal{C}(X, \mathbf{Z})$ takes an element $r \in R$ to "evaluation at r". If $f \in R$, we shall write $\bar{f}$ for the image of f in R_{red} and identify it with the corresponding function on X. For a field F the set X can be identified with the set of all (total) orderings of F. The map $W(F) \to \mathcal{C}(X, \mathbf{Z})$ is defined by taking a representative $\sum a_i x_i^2$ for a class in $W(F)$ and computing its signature at each ordering (the number of positive a_i minus the number of negative a_i). This can be generalized to semilocal rings [9, §2].

Corresponding to R there is a natural subbasis of clopen (both closed and open) sets $\mathcal{K}(R)$ for the topology on $X(R)$, and $R_{\text{red}} = \mathbf{Z} + \sum_{U \in \mathcal{K}} 2\mathbf{Z}\chi_U$ where χ_U is the characteristic function of U. The family $\mathcal{K}$ consists of all sets $W(g) = \{x \in X \mid \psi(g)(x) = -1\}$ and their complements where $g \in G$ and ψ is the homomorphism $\mathbf{Z}G \to R_{\text{red}}$ [9, §3]. For the complement, we shall write $W(-g)$ and use the notation $\pm G$ for the set $\{\pm g \mid g \in G\}$, a subset of $\mathbf{Z}G$. The family $\mathcal{K}$ contains X and is closed under the operation of symmetric difference (denoted by +) since $W(g) + W(h) = W(gh)$. Thus $\mathcal{K}$ can be thought of as an $\mathbf{F}_2$-vector space where $\mathbf{F}_2$ is the field of two elements. In §2 we investigate the relationship between n-stability and conditions on $\mathcal{K}$. In §3, we look specifically at finite spaces and find that a special role is played by the Witt rings R which are isomorphic to group rings. These rings have previously been studied from a different point of view for Witt rings of fields [5] and semilocal rings [15]. In §4 we apply our results to Witt rings of fields. Our main theorem gives a characterization of n-stability of the reduced Witt ring of a formally real field in terms of the real places on the field, thus generalizing results in [5], [6], [14].

We close this section with two results which are valid even if $R_t = R$.

PROPOSITION 1.2. *Let R be an abstract Witt ring. Then M^n is additively generated by elements of the form $\prod_{i=1}^n (1 + g_i)$ where $g_i^2 = 1$, $g_i \in R$.*

PROOF. Let $G = \{g \in R | g^2 = 1\}$. Then R is a Witt ring for G [9, p. 219], and M is the image in R of $M_0 = \ker(\mathbf{Z}G + \mathbf{Z}/2\mathbf{Z}) = \{\sum n_i g_i \in \mathbf{Z}G | \sum n_i$ is even$\}$. Thus any element of M can be written in the form

$$\sum_{i=1}^{2m} g_i = \sum_{i=1}^{m} (g_i + 1) - \sum_{i=m+1}^{2m} (-g_i + 1) \qquad (g_i \in G, -g_i \in G)$$

and the proposition is true for $n = 1$. For $n > 1$, the ideal M^n is generated by n-fold products of the generators of M.

REMARK 1.3. If $R_t = \mathrm{Nil}\, R$, then as a function on $X(R)$, the element $\prod (1 - g_i)$ becomes

$$2^n \prod \chi_{W(g_i)} = 2^n \chi_{\cap W(g_i)}.$$

THEOREM 1.4. *Let R be a Witt ring. If M^n is a finitely generated abelian group for some $n \geqslant 1$ (with say r generators of the form $\prod_{i=1}^n (1 + g_i)$), then R is i-stable for some i (any $i \geqslant n(r + 1) - 1$).*

PROOF. Let $f_1, \ldots, f_r$ be the generators of M^n, each of the form $\prod_{i=1}^n (1 + g_i)$. Then $M^{n(r+1)}$ is generated by all $(r + 1)$-fold products of the elements f_i. Since $f_1^2 = 2^n f_i$ –because

$$\prod_{i=1}^n (1 + g_i)^2 = \prod_{i=1}^n (2 + 2g_i) = 2^n \prod_{i=1}^n (1 + g_i),$$

we see that

$$M^{n(r+1)} \subseteq 2^n M^{nr} \subseteq 2M^{n(r+1)-1} \subseteq M^{n(r+1)},$$

so that R is i-stable for $i = n(r + 1) - 1$.

2. **Relationship of n-stability to $\mathcal{K}$.** From now on we shall assume without explicitly stating it that $R_t = \mathrm{Nil}\, R$ so that $X(R) \neq \varnothing$. In this section we present several theorems relating the condition of n-stability for R to the structure of the subbasis $\mathcal{K}(R)$. We shall see that in some sense n-stability measures how far the additive group $\mathcal{K}(R)$ is from being closed under the operation of intersection (the multiplication in the Boolean ring of all clopen subsets of X). Our first three results generalize results in [5] from Witt rings of fields to our abstract situation.

LEMMA 2.1. *Let R be a Witt ring for G. Let B be a closed subset of $X = X(R)$ and let α be any point of X not in B. Then there exist $g_1, \ldots, g_n$ elements of $\pm G$, such that*

$$\alpha \in D = \bigcap_{i=1}^{n} W(-g_i) \quad \text{and} \quad B \subseteq D^c = \bigcup_{i=1}^{n} W(g_i).$$

PROOF. Let β be any point in B. Since $\mathcal{H}$ is a subbasis and X is Hausdorff, there exists an element g_β in $\pm G$ such that $\alpha \in W(-g_\beta)$ and $\beta \in W(g_\beta)$. Doing this for each point β in B we obtain an open cover $\{W(g_\beta) | \beta \in B\}$ of the compact set B. Hence there exist $g_1, \ldots, g_n$ in $\{g_\beta | \beta \in B\}$ such that $B \subseteq \bigcup_{i=1}^n W(g_i)$. By our choice of the elements g_β, we have $\alpha \in \bigcap_{i=1}^n W(-g_i)$.

PROPOSITION 2.2. *Let R be a Witt ring for G and let A, B be disjoint closed subsets of $X(R)$. Then there exist an integer n and an element $f \in M^n$ such that $\bar{f}(\beta) = 0$ for $\beta \in B$ and $\bar{f}(\alpha) = 2^n$ for $\alpha \in A$.*

PROOF. Let α be in A. Apply Lemma 2.1 and set f_α equal to the image in R of the group ring element $(1 + g_1)(1 + g_2) \cdots (1 + g_n)$ and $C_\alpha = \bigcap W(-g_i)$. Doing this for each $\alpha \in A$, the sets C_α form an open cover of A with $(\bigcup C_\alpha) \cap B = \varnothing$. Since A is compact there is a finite family $C_1, \ldots, C_m$ such that $A \subseteq \bigcup C_i$. Multiply the corresponding elements f_i by powers of 2 if necessary so that we may assume they each have r factors of the form $1 + g$; since $W(-1) = X(R)$, the sets C_i are unchanged. We now have

$$\bar{f}_i(x) = \begin{cases} 2^r & \text{if } x \in C_i, \\ 0 & \text{if } x \in B. \end{cases}$$

Set

$$f = 2^{(m-1)r} \sum_{i=1}^{m} f_i - 2^{(m-2)r} \sum_{i<j} f_i f_j + \cdots + (-1)^{m-1} f_1 \cdots f_m.$$

Then $f \in M^n$ where $n = mr$ and $\bar{f}(x) = 2^n$ for $x \in \bigcup C_i \supseteq A$ and $\bar{f}(x) = 0$ for $x \in (\bigcup C_i)^c \supseteq B$.

THEOREM 2.3. *Let R be a Witt ring for G. Assume R is n-stable, $n \geqslant 0$. Then the following condition holds:*

(2.4) *If A, B are disjoint closed subsets of $X(R)$, then there exists an element $f \in M^n$ such that $\bar{f}(\beta) = 0$ for $\beta \in B$ and $\bar{f}(\alpha) = 2^n$ for $\alpha \in A$.*

Furthermore, if M^{n+1} is torsion free, then (2.4) *implies n-stability.*

PROOF. First assume that R is n-stable, i.e., $M^{n+1} = 2M^n$. Let A, B be disjoint closed subsets of $X(R)$. By Proposition 2.2 there exist an integer r and an element $f_0 \in M^r$ such that $\bar{f}_0(\beta) = 0$ for $\beta \in B$ and $\bar{f}_0(\alpha) = 2^r$ for α

$\in A$. If $r \leqslant n$, set $f = 2^{n-r}f_0$. Then $f \in M^n$ and satisfies (2.4). If $r > n$, then n-stability implies that $f_0 = 2^{r-n}f$ for some f in M^n. But then again f satisfies (2.4).

Conversely, we must show $M^{n+1} = 2M^n$. Let f_0 be the image in R of the group ring element $\prod_{i=1}^{n+1}(1 + g_i)$, g_i in $\pm G$. By Proposition 1.2 it will suffice to show that $f_0 \in 2M^n$. Let $A = \{x \in X(R) | \bar{f}_0(x) \neq 0\}$, a clopen set equal to $\bigcap W(-g_i)$, and let B be the complement of A. Then (2.4) implies that there exists an element $f \in M^n$ such that $\bar{f}(x) = 2^n$ for $x \in A$ and $\bar{f}(x) = 0$ for $x \notin A$. But then $2\bar{f} = \bar{f}_0$ as elements of $\mathcal{C}(X, \mathbf{Z})$ and $2f, f_0 \in M^{n+1}$. Since M^{n+1} is torsion free we have $f_0 = 2f \in 2M^n$.

It follows from Theorem 2.3 that R_{red} is n-stable for some n if and only if there is a bound on the minimum choices for n in Proposition 2.2. This is related to Theorem 1.4 in that if some power M^r is finitely generated, then there does exist such a bound. We next look at the applications of Theorem 2.3 when $n = 1$.

Definition 2.5. We say that R satisfies the *weak approximation property* (WAP) if $\mathcal{H}(R)$ is a basis, and R satisfies the *strong approximation property* (SAP) if $\mathcal{H}(R)$ is the entire Boolean algebra of clopen sets in $X(R)$.

The SAP condition was first introduced in [11] with more details appearing in [9, §3]. The WAP condition was introduced for Witt rings of fields in [5] in which context it was shown to be equivalent to SAP. For Witt rings of fields there are many equivalent formulations of WAP and SAP [5], [6], many of which have no analogue in the context in which we are working. In our situation, both conditions can be stated in terms of separating sets of points in $X(R)$ by elements of G, essentially as in the case for the Witt ring of a field $W(F)$ where G is the square factor group of the field. It is also known that WAP is not equivalent to SAP in general [4, §3], although it is possible to give a topological argument that they are equivalent if $X(R)$ has only countably many points. Our interest in SAP stems from the fact that R satisfies SAP if and only if $R_{\text{red}} = \mathbf{Z} + \mathcal{C}(X, 2\mathbf{Z})$ [9, Theorem 3.20]. The importance of SAP is evident from the fact that the Witt ring of any formally real algebraic extension of the rational numbers has the property [13, Chapter 3, Example 2.10].

Corollary 2.6. *Let R be a Witt ring for G. If R is* 1*-stable then R satisfies SAP. Furthermore, R_{red} is* 1*-stable if and only if R satisfies SAP.*

Proof. We apply Theorem 2.3 with $n = 1$. Condition (2.4) says that for any clopen set A, the function $2\chi_A$ is in the image of M under the canonical map $R \to \mathcal{C}(X, \mathbf{Z})$. This implies $2\chi_A = \sum n_i 2\chi_{A_i}$ where $n_i \in \mathbf{Z}$, $A_i \in \mathcal{H}(R)$. Dividing by 2 and reducing modulo 2 we obtain $A = \sum A_i \in \mathcal{H}(R)$, where the sum is over all i for which $n_i \equiv 1 \pmod 2$. Thus SAP holds. For the converse it suffices to note that R_{red} is torsion free and apply the second part of Theorem 2.3.

REMARK 2.7. For the second part of the above corollary it is interesting to know when R is torsion free. For Witt rings of fields, this is equivalent to the field being formally real and pythagorean (any sum of squares is again a square). More generally, for a connected semilocal ring A with 2 a unit, the Witt ring of A is torsion free if and only if every unit in A which is a sum of squares is already a square and -1 is not a square [9, Corollary 4.20]. In general, R_{red} being 1-stable does not imply that R is 1-stable as shown by the following example.

EXAMPLE 2.8. Let $\mathbf{Q}$ denote the field of rational numbers. Then $W_{red}(\mathbf{Q})$ is 1-stable since $\mathbf{Q}$ satisfies SAP (the space of orderings X has only one point), but we shall see that $W(\mathbf{Q})$ is not 1-stable. Using the notation and terminology of [5], consider the bilinear form $\langle 1,2,5,-10\rangle$. Since the Legendre symbol $(-2/5) = -1$, the form is anisotropic over $\mathbf{Q}$ [7, Corollary 27c]; since $\langle 1,2,5,-10\rangle$ is a four-dimensional form of determinant -1 (modulo squares), the ring $W(\mathbf{Q})$ is not 1-stable [5, Proposition 3.9]. It is true, however, that $W(\mathbf{Q})$ is 2-stable. Indeed [13, Chapter IV, Corollary 2.5] says that M^3 is additively generated by $8 = 8\langle 1\rangle$. Thus we have $M^3 = 8W(\mathbf{Q}) \subseteq 2M^2 \subseteq M^3$, which implies $M^3 = 2M^2$.

As another application of Theorem 2.3, we shall determine what it means to be 0-stable. It is quite easy to see that the Witt ring of a formally real field is 0-stable if and only if the field has only two square classes, and this is equivalent to the Witt ring being $\mathbf{Z}$ [5, p. 1176]. In fact, this is the case in general.

THEOREM 2.9. *Let R be a Witt ring for G. Then R is 0-stable if and only if $R = \mathbf{Z}$.*

PROOF. We first note that $X = X(R)$ has only one point. For, Theorem 2.3 implies that if there exist two disjoint nonempty subsets of X, then there exists an element f in R which induces a function which is 0 on one set and 1 on the other, a contradiction of [9, Proposition 3.8] which states that the values of $\tilde{f}$ must all be congruent modulo 2. Thus $\mathcal{C}(X,\mathbf{Z}) = \mathbf{Z}$ and so $R_{red} = \mathbf{Z}$. Assume Nil $R \neq 0$.

Without loss of generality we may assume G is the group $\{g \in R \mid g^2 = 1\}$. Let K be the kernel of the homomorphism

$$\mathbf{Z}G \to R.$$

If G is infinite, let H be a finite subgroup of G with at least four elements. Replace R by $\mathbf{Z}H/(\mathbf{Z}H \cap K)$; then R is a Witt ring for H, $2R$ is a maximal ideal (i.e., R is still 0-stable), $R/\text{Nil } R = \mathbf{Z}$ and Nil $R \neq 0$ (since H has at least four elements, the ring R has at least four units, hence is not equal to $\mathbf{Z}$).

But now R is finitely generated as a $\mathbf{Z}$-module. We now get a contradiction by showing that Nil R is 2-divisible and thus not finitely generated. Let $a \in \text{Nil } R \subseteq M = 2R$. Then $a = 2b$ for some $b \in R$. Since a is nilpotent, $a^n = 0$ for some n; hence $2^n b^n = 0$. Thus b^n is torsion; but $R_t = \text{Nil } R$, so b^n is nilpotent and $b \in \text{Nil } R$.

The converse is clear since $\mathbf{Z}$ is certainly 0-stable.

The remainder of this section will be spent looking carefully at the relationship between n-stability and intersections of elements of $\mathcal{H}$.

Theorem 2.10. *If R is n-stable, $n \geqslant 0$, then any intersection of elements of $\mathcal{H}(R)$ can be written as the symmetric difference of a finite number of n-fold intersections of elements of $\mathcal{H}(R)$. If $n = 1$, the converse holds for R_{red} since $\mathcal{H}(R)$ is closed under symmetric difference.*

Proof. If $n = 0$, Theorem 2.9 implies that $X(R)$ has only one point, and so the conclusion holds since an empty intersection is the whole space. If $n > 0$, let R be a Witt ring for the group G. Consider an intersection $A = W(g_1) \cap \cdots \cap W(g_m)$, $g_i \in \pm G$. If $m \leqslant n$ we are done, using $W(-1) = X$ if $m < n$; so assume $m > n$.

Assume R is a Witt ring for G with ψ: $\mathbf{Z}G \to R$ the corresponding surjection. Consider $\prod_{i=1}^{m} (1 - \psi(g_i)) \in M^m$; since R is n-stable, we have $M^m = 2^{m-n} M^n$. Then by Proposition 1.2, we can write

$$\prod (1 - \psi(g_i)) = 2^{m-n} \sum_j n_j \prod_{i=1}^{n} (1 - \psi(h_{ij})),$$

where $n_j \in \mathbf{Z}$, $h_{ij} \in \pm G$. Modulo Nil R, these elements become functions from $X(R)$ to $\mathbf{Z}$ giving us the equation

$$2^m \chi_A = 2^{m-n} \sum n_j 2^n \chi_{\cap W(h_{ij})} = 2^m \sum n_j \chi_{\cap W(h_{ij})}.$$

Since 2 is not a zero divisor, we have $\chi_A = \sum n_j \chi_{\cap W(h_{ij})}$. Reducing this modulo 2 we obtain $\chi_A = \chi_{\sum \cap W(h_{ij})}$ where the sum is over all j for which $n_j \equiv 1 \pmod 2$. Therefore $A = \sum \bigcap_{i=1}^{n} W(h_{ij})$ and the theorem is proved.

Lemma 2.11. *For any sets A_{ij}, the set $\bigcup_{i=1}^{r} \bigcap_{j=1}^{n_i} A_{ij}$ can be written as a finite disjoint union of $(\sum_{i=1}^{r} n_i)$-fold intersections of the sets A_{ij} and their complements.*

Proof. We begin by assuming $r = 2$,

$$A = \bigcap_{i=1}^{n} A_i, \quad B = \bigcap_{i=1}^{m} B_i.$$

Then $A \cup B$ is the disjoint union of $A \cap B$, $A \cap B^c$ and $B \cap A^c$. We also have $A \cap B^c$ equal to a $(2^m - 1)$-fold disjoint union of $(m + n)$-fold intersections where we write B^c as the union over all possible ways of choosing

$B_i^* \in \{B_i, B_i^c\}$, not all $B_i^* = B_i$, of the intersections $\bigcap_{i=1}^m B_i^*$. Similarly, $B \cap A^c$ can be written as such a disjoint union, and thus $A \cup B$ can be written in the prescribed form. For $r > 2$, a simple induction argument completes the proof.

THEOREM 2.12. *Let R be a Witt ring for G. If there exists a number n such that every clopen subset of $X(R)$ can be written as a disjoint union of n-fold intersections of elements of $\mathcal{H}(R)$, then R_{red} is n-stable. If $n = 1$, the converse is also true.*

PROOF. Without loss of generality we may assume $R = R_{\text{red}}$, a subring of $\mathcal{C}(X(R), \mathbf{Z})$. Let ψ: $\mathbf{Z}G \to R$ be the usual map, and consider $\prod_{i=1}^{n+1} (1 - \psi(g_i)) = 2^{n+1}\chi_A \in M^{n+1}$ where $A = \bigcap W(g_i)$. By Proposition 1.2, it suffices to show that this lies in $2M^n$. By hypothesis, A can be written as a disjoint union of sets $B_k = \bigcap_{j=1}^n W(h_{jk})$, h_{jk} in $\pm G$. Then $\chi_A = \sum \chi_{B_k}$ in $\mathcal{C}(X(R), \mathbf{Z})$. Multiplying by 2^{n+1} to obtain elements of R, we get

$$\prod (1 - \psi(g_i)) = 2 \sum 2^n \chi_{B_k} = 2 \sum \prod_{j=1}^{n} (1 - \psi(h_{jk})) \in 2M^n.$$

If $n = 1$, Corollary 2.6 implies that $\mathcal{H}(R)$ contains all clopen sets, and thus the converse holds.

COROLLARY 2.13. *If there exist numbers m and n such that every clopen subset of $X(R)$ can be written as an m-fold union of n-fold intersections of elements of $\mathcal{H}(R)$, then R_{red} is (mn)-stable.*

PROOF. Immediate from Lemma 2.11 and Theorem 2.12.

3. **Finite spaces and group rings.** In this section we are primarily concerned with the case where $X(R)$ is finite. This is not as special as one might think since for any reduced Witt ring R, we can take a finite number of elements in $\mathcal{H}(R)$. The additive subgroup they generate corresponds to a subring S of R with $X(S)$ a finite space homeomorphic to the quotient space of $X(R)$ defined by identifying points of $X(R)$ which are not separated by the chosen subgroup of $\mathcal{H}(R)$.

We assume throughout this section that R is torsion free (i.e., $\operatorname{Nil} R = 0$) and we identify R with its image in $\mathcal{C}(X, \mathbf{Z})$. We shall show that when $X(R)$ is finite, R is always n-stable for some n depending on the cardinality of X which we shall denote by $|X|$.

Our main application is to integral group rings. In Theorem 3.8 we give several conditions equivalent to the condition that R be a group ring. Group rings are important for several reasons. They have the property that the subbasis $\mathcal{H}(R)$ is as small as possible for the given number of points in X, the precise opposite of SAP. Furthermore, they illustrate the worst that can happen in terms of stability. For these reasons Theorem 3.8 will play a central role in our characterization of n-stability in fields in §4. Also, Witt rings of fields (or semilocal rings) are group rings if and only if the field (or ring) has a particularly nice and well-understood structure [3], [5], [15].

We begin our study by noting a trivial but often-used fact.

LEMMA 3.1. *Let X be a Boolean space, Y a closed subset of X. If $\mathcal{H}$ is an additive subbasis of clopen sets for X, then $\mathcal{H}_Y = \{H \cap Y | H \in \mathcal{H}\}$ is an additive subbasis of clopen sets for Y.*

PROOF. $(H_1 \cap Y) + (H_2 \cap Y) = (H_1 + H_2) \cap Y$ for $H_1, H_2 \in \mathcal{H}$.

LEMMA 3.2. *Let X be a finite discrete space with at least two points; let $\mathcal{H}$ be an additive subbasis of clopen sets containing X. Then for any $x \in X$, there exists a set $H \in \mathcal{H}$ such that $x \in H$ and $|H| \leqslant \frac{1}{2}|X|$.*

PROOF. First note that the lemma is true if $|X| = 2$ since $\mathcal{H}$ contains all subsets of X. Assume X, $\mathcal{H}$ give a counterexample to the lemma with $n = |X|$ minimal. Take $x \in X$ and any $y \neq x$; set $Y = \{y\}^c$ and let $\mathcal{H}_Y$ be as in Lemma 3.1. Let $x \in H' \in \mathcal{H}_Y$ with say $H' = H \cap Y$, $H \in \mathcal{H}$. Since $x \in H$, we have $|H| > \frac{1}{2}|X|$. If $y \notin H$, then $|H'| = |H| > \frac{1}{2}|X| > \frac{1}{2}|Y|$. If $y \in H$, then $|H'| = |H| - 1 > \frac{1}{2}|X| - 1 = \frac{1}{2}|Y| - \frac{1}{2}$; thus if $|X|$ is even, $|H'| \geqslant \frac{1}{2}|Y| + \frac{1}{2} > \frac{1}{2}|Y|$, a contradiction of the minimality of n since H' is an arbitrary element of $\mathcal{H}_Y$ containing x. Thus we may assume n is odd. Also, minimality implies there is some H' containing x such that $|H'| \leqslant \frac{1}{2}|Y| = \frac{1}{2}(n - 1)$. If S is chosen in $\mathcal{H}$ so that $H' = S \cap Y$, then $|S| = \frac{1}{2}(n + 1)$. Now consider the subbasis $\mathcal{H}_S$ for the subspace S. Again the minimality of n implies there exists a set $H_0 \in \mathcal{H}$ containing x such that $|S \cap H_0| \leqslant \frac{1}{2}|S|$. Also $|H_0| \geqslant \frac{1}{2}(n + 1)$ since $x \in H_0$ and $|S + H_0| \leqslant \frac{1}{2}(n - 1)$ since $x \in (S + H_0)^c \in \mathcal{H}$. Therefore

$$\frac{1}{2}(n - 1) \geqslant |S + H_0| = |S| + |H_0| - 2|S \cap H_0| \geqslant n + 1 - 2(\tfrac{1}{2}|S|)$$
$$= \tfrac{1}{2}(n + 1),$$

a contradiction.

THEOREM 3.3. *If $1 < |X(R)| < \infty$, then R is n-stable for $n \geqslant r$ where r is the largest integer such that $2^r \leqslant |X(R)|$.*

PROOF. For any $x \in X$, Lemma 3.2 implies that $\{x\}$ can be written as an intersection of r or less elements of $\mathcal{H}(R)$; hence any subset of X can be written as a disjoint union of r-fold intersections of elements of $\mathcal{H}(R)$. Therefore R is r-stable by Theorem 2.12.

REMARK 3.4. We shall see in Theorem 3.8 that this is the best result possible in the sense that, given X as in the theorem, there exists a ring R which is not

$(r-1)$-stable for which $X = X(R)$. In fact, if $|X| = 2^r$ then R is a group ring. If $|X| > 2^r$, one can choose the appropriate subbasis for a group ring (see below) for 2^r of the points and any subbasis for the remaining points; together these will correspond to a ring which has a group ring as a direct factor.

The next two lemmas are aimed at obtaining a stronger version of Lemma 3.2.

LEMMA 3.5. *Let X be a finite discrete space with additive subbasis $\mathcal{H}$ containing X. Let x be an element of X with the property that any $H \in \mathcal{H}$ containing x satisfies $|H| \geqslant \frac{1}{2}|X|$. Then*

(a) *for any $H_1, H_2 \in \mathcal{H}$ with $x \in H_1 \cap H_2$, we have $|H_1 \cap H_2| \geqslant \frac{1}{2}|H_1|$;*
(b) *for any $H \in \mathcal{H}$ containing x, we have $|H|$ is a power of 2.*

PROOF. For (a), we note that

$$\begin{aligned} |H_1 \cap H_2| &= \tfrac{1}{2}(|H_1| + |H_2| - |H_1 + H_2|) \\ &\geqslant \tfrac{1}{2}(|H_1| + \tfrac{1}{2}|X| - \tfrac{1}{2}|X|) = \tfrac{1}{2}|H_1| \end{aligned}$$

since $x \in (H_1 + H_2)^c \in \mathcal{H}$.

To prove (b), let $E_1 \in \mathcal{H}$ be any set containing x and set $\mathcal{H}_1 = \{H \cap E_1 | H \in \mathcal{H}\}$. By Lemma 3.2 there exists $E_2 \in \mathcal{H}_1$ such that $x \in E_2$ and $|E_2| \leqslant \frac{1}{2}|E_1|$; so by (a), $|E_2| = \frac{1}{2}|E_1|$. Set $\mathcal{H}_2 = \{H \cap E_2 | H \in \mathcal{H}_1\}$. Again Lemma 3.2 implies there exists $E_3 \in \mathcal{H}_2$ such that $x \in E_3$ and $|E_3| \leqslant \frac{1}{2}|E_2|$. If possible, choose E_3 so that $|E_3| < \frac{1}{2}|E_2|$; otherwise $|E_3| = \frac{1}{2}|E_2|$, and we set $\mathcal{H}_3 = \{H \cap E_3 | H \in \mathcal{H}_2\}$ and continue. If $|E_1|$ is not a power of 2, then there exists a least integer k such that $|E_{k+1}| < \frac{1}{2}|E_k|$; then $E_{k+1} \in \mathcal{H}_k$ so there exists $J \in \mathcal{H}_{k-1}$ such that $J \cap E_k = E_{k+1}$. Now apply (a) with $X = E_{k-1}$, $\mathcal{H} = \mathcal{H}_{k-1}$, $H_1 = E_k$ and $H_2 = J$: it says that $|J \cap E_k| \geqslant \frac{1}{2}|E_k|$, a contradiction. Therefore $|E_1|$ must be a power of 2 and the lemma is proved.

LEMMA 3.6. *Let X be a finite discrete space with additive subbasis $\mathcal{H}$ containing X. If $|\mathcal{H}| > 2|X|$, then for each $x \in X$ there exists $H \in \mathcal{H}$ containing x such that $|H| < \frac{1}{2}|X|$.*

PROOF. The lemma is trivially true if $|X| = 1$ or 2. Assume X, $\mathcal{H}$ give a counterexample with X of minimal cardinality n. Then there exists an $x \in X$ such that $|H| \geqslant \frac{1}{2}n$ for all $H \in \mathcal{H}$ containing x. By Lemma 3.2 there exists a set $Y \in \mathcal{H}$ containing x such that $|Y| = \frac{1}{2}n$. Also, there exists a set $E \in \mathcal{H}$ such that $E \subseteq Y^c$ and $E \neq \varnothing, Y^c$: for if not, then for all $H_1, H_2 \in \mathcal{H}$ such that $H_1 \cap Y = H_2 \cap Y$, we have $H_1 + H_2 = \varnothing$ or Y^c; in this case the set $\mathcal{H}_Y = \{H \cap Y | H \in \mathcal{H}\}$ has cardinality $\frac{1}{2}|\mathcal{H}|$, so by the minimality of n, there exists a set $H \in \mathcal{H}$ such that $|H \cap Y| < \frac{1}{4}n$, $x \in H \cap Y$, a contradiction of

Lemma 3.5(a). So let $S = Y \cup E = Y + E \in \mathcal{H}$. Then $x \in S$ and $\frac{1}{2}n < |S| < n$. Since Y and S cannot both have cardinality equal to a power of 2, we have a contradiction of Lemma 3.5(b).

We would like to thank Roy Olson for his valuable assistance in proving Lemmas 3.2 and 3.6.

LEMMA 3.7. *Let X be a Boolean space and $\mathcal{H}$ an additive subbasis of clopen sets containing X. Then*

(a) *if A is a nonempty subset of X and $A = \bigcap_{i=1}^{r} H_i$ with $H_i \in \mathcal{H}$ and r minimal, then $H_1, \ldots, H_r, X$ are linearly independent in the $\mathbf{F}_2$-vector space $\mathcal{H}$.*

For (b) *and* (c) *assume also that $|X| = 2^n$ and $|H| = 2^{n-1}$ for all $H \in \mathcal{H}$; then*

(b) *if $H_1, \ldots, H_r, X$ are $\mathbf{F}_2$-linearly independent, then $|\bigcap_{i=1}^{r} H_i| = 2^{n-r}$;*

(c) *any nonempty r-fold intersection of elements of $\mathcal{H}$ has cardinality 2^{n-j} for some j, $0 \leqslant j \leqslant r$.*

PROOF. (a) Assume $\sum_{i=1}^{p} H_i = \varnothing$ or X, renumbering the sets if necessary. In the first case, we have $H_p = \sum_{i=1}^{p-1} H_i$. But then $\bigcap_{i=1}^{p} H_i = \bigcap_{i=1}^{p-1} H_i$ if p is even and is empty if p is odd, a contradiction of either $A \neq \varnothing$ or r minimal. Similarly, if $H_p + X = \sum_{i=1}^{p-1} H_i$, then $\bigcap_{i=1}^{p} H_i = \bigcap_{i=1}^{p-1} H_i$ if p is odd and is empty if p is even, again a contradiction.

(b) We prove this by induction on r. It is true by our hypothesis on $\mathcal{H}$ if $r = 1$. Assume (b) holds for intersections of less than r sets, and consider $\bigcap_{i=1}^{r} H_i$ where $H_1, \ldots, H_r, X$ are linearly independent. It is easy to check that the following equation holds (for any family of sets):

$$\left|\sum_{i=1}^{r} H_i\right| = \sum_{i=1}^{r} |H_i| - 2\sum_{i<j} |H_i \cap H_j| + \cdots + (-1)^{r-1} 2^{r-1} \left|\bigcap_{i=1}^{r} H_i\right|.$$

Since $\sum H_i \in \mathcal{H}$ and is not eaual to $\varnothing$ or X by linear independence, we have $|\sum H_i| = 2^{n-1}$. Applying the induction hypothesis for the intersections of less than r sets, we obtain $|\bigcap_{i=1}^{r} H_i| = 2^{n-r}$.

(c) follows immediately from (a) and (b).

THEOREM 3.8. *Let R be a Witt ring, $R = R_{\text{red}}$ and $|X(R)| = 2^n$, $n \geqslant 0$. Then the following are equivalent:*

(a) *$R = \mathbf{Z}G$ (and G has order 2^n);*

(b) *if $H \in \mathcal{H}(R)$, $H \neq \varnothing, X$, then $|H| = 2^{n-1}$;*

(c) *R is not $(n-1)$-stable;*

(d) *R is i-stable if and only if $i \geqslant n$;*

(e) $|\mathcal{H}(R)| = 2^{n+1}$.

PROOF. If $n = 0$, the space X has only one point, so $R = \mathcal{C}(X, \mathbf{Z}) = \mathbf{Z}$ and (a)–(e) all hold. Henceforth we assume $n \geqslant 1$.

(a) $\Rightarrow$ (b). We assume $R = \mathbf{Z}G$. Then X is the set of all ring homomorphisms $\mathbf{Z}G \to \mathbf{Z}$. Since G has exponent 2, these homomorphisms are in one-to-one correspondence with elements of Hom $(G, \{\pm 1\})$; and for any $g \in G$ other than the identity, the element g will be mapped to -1 by exactly half of the homomorphisms. Since the elements of $\mathcal{H}$ are the sets $W(g)$ and their complements, we see that (b) holds.

(b) $\Rightarrow$ (c). Assume R is $(n-1)$-stable. Then Theorem 2.10 states that any singleton $\{x\}$ can be written as the symmetric difference of a finite number of $(n-1)$-fold intersections of elements of $\mathcal{H}$. If (b) holds, then Lemma 3.7(c) implies that any $(n-1)$-fold intersection has an even number of elements; but symmetric differences of such sets must again have an even number of elements, a contradiction.

(c) $\Rightarrow$ (d). Condition (c) implies that R is not i-stable for $i \leqslant n-1$. On the other hand, Theorem 3.3 states that R is i-stable for $i \geqslant n$.

(d) $\Rightarrow$ (e). Assume (d) holds and $|\mathcal{H}| < 2^{n+1}$, i.e., $|\mathcal{H}| \leqslant 2^n$. Let $H_1, \ldots, H_r = X$ be an $\mathbf{F}_2$-basis for $\mathcal{H}$, $r \leqslant n$. Since X is discrete and $\mathcal{H}$ is a subbasis, each element of X can be written as the intersection of all elements of $\{H_1, \ldots, H_{r-1}, H_1^c, \ldots, H_{r-1}^c\}$ which contain it. But there are only $2^{r-1} < 2^n$ such intersections and $|X| = 2^n$. Therefore $|\mathcal{H}| \geqslant 2^{n+1}$; assume $|\mathcal{H}| > 2^{n+1}$. By Lemma 3.6, every singleton $\{X\}$ can be written as the intersection of at most $n-1$ sets in $\mathcal{H}$; Theorem 2.12 then implies the R is $(n-1)$-stable, a contradiction.

(e) $\Rightarrow$ (a). Assume $|\mathcal{H}| = 2^{n+1}$. By [9, Proposition 3.8], $R = \sum_{H \in \mathcal{H}} \mathbf{Z} g_H$ where $g_H \in \mathcal{C}(X, \mathbf{Z})$ is defined by $g_H(x) = -1$ if $x \in H$ and equals 1 if $x \in H^c$. Let $H_1, \ldots, H_n, X$ be an $\mathbf{F}_2$-basis for $\mathcal{H}$, and let $\mathcal{S}$ be the subgroup of $\mathcal{H}$ generated by $H_1, \ldots, H_n$. Then $|\mathcal{S}| = 2^n$. Since $g_I g_J = g_{I+J}$ for $I, J \in \mathcal{H}$ and $g_H = -g_{H+X}$, we have $R = \sum_{H \in \mathcal{S}} \mathbf{Z} g_H$. Let G be the group $\{g_H | H \in \mathcal{S}\}$. Then we claim $\sum_{H \in \mathcal{S}} \mathbf{Z} g_H$ is the group ring $\mathbf{Z}G$. Indeed, we have a surjection ψ: $\mathbf{Z}G \to R$ which is a ring homomorphism. Let K be the kernel of ψ. For any ring homomorphism φ: $R \to \mathbf{Z}$, we certainly have $K \subseteq \ker(\varphi\psi)$. But $\varphi\psi \in X(\mathbf{Z}G)$ and $|X(R)| = 2^n = |X(\mathbf{Z}G)|$. Thus K is contained in the kernel of every map $\mathbf{Z}G \to \mathbf{Z}$ and hence is in every minimal prime ideal of $\mathbf{Z}G$ [10, Lemma 3.1]. But Nil $\mathbf{Z}G = 0$, hence $K = 0$ and the theorem is proved.

We shall now complete our study of n-stability in group rings by looking at infinite groups. This provides an example of how our results for finite spaces $X(R)$ can be used to obtain information in the infinite case.

LEMMA 3.9. *Let R be an n-stable abstract Witt ring. Then any Witt ring which is a quotient ring of R is also n-stable.*

PROOF. Since every abstract Witt ring has a unique maximal ideal containing 2, the conclusion is obtained by reducing the equation $M^{n+1} = 2M^n$ modulo the kernel of the homomorphism from R to its quotient ring.

THEOREM 3.10. *Let $R = \mathbf{Z}G$ where G is an infinite group of exponent 2. Then R is not n-stable for any integer n.*

PROOF. Let G_0 be any finite subgroup of G of cardinality at least 2^{n+1} and let R_0 be the group ring $\mathbf{Z}G_0$. Then Theorem 3.8 says that R_0 is not n-stable. Since there is an obvious surjection of R onto R_0, the previous lemma implies that R is not n-stable.

4. **Stability for formally real fields.** In this section we apply our previous results to $W_{\text{red}}(F)$, the reduced Witt ring of a formally real field F. Our main theorem will be a complete characterization of the fields for which the reduced Witt ring is n-stable in terms of the real places of the field. The importance of n-stability for fields can be seen in [5] where it is related to K-theory of fields. We begin with some definitions and notation.

By a *formally real place* on a field F, we mean a place into a real closed field in the sense of [8] and [12]. By a *real place*, we shall mean a formally real place into the field of real numbers.

DEFINITION 4.1. Given a Witt ring R, we shall call a subset Y of X a *2^n-box* if $|Y| = 2^n$ and the quotient ring obtained by restricting the functions in R to the set Y is an integral group ring; that is, it satisfies the equivalent conditions of Theorem 3.8.

For any formally real place σ on a field F, we shall let Ord (σ) be the set of orderings of F compatible with σ (orderings for which any positive $a \in F$ has value $\sigma(a) \geqslant 0$ or $\sigma(a) = \infty$). By [2] or [8, Theorem 2.5], the set Ord (σ) is always homeomorphic to a product of two point spaces and is a 2^n-box in $X(F) = X(W_{\text{red}}(F))$ whenever $|\text{Ord}(\sigma)| = 2^n$. For any formally real place σ, we shall use Λ_σ to denote the value group of σ reduced modulo 2. The (nonidentity) elements of this group correspond to (nontrivial) intersections of sets in $\mathcal{H}(F) = \mathcal{H}(W_{\text{red}}(F))$ with Ord (σ), and $|\text{Ord}(\sigma)| = |\Lambda_\sigma|$ [2], [8]. Given two real places σ, τ, the field $K_{\sigma,\tau}$ will be the residue field of the finest (formally real) place through which both σ and τ factor, and $\Lambda_{\sigma,\tau}$ will denote the reduced value group of this place; note that this place has valuation ring generated by the valuation rings of σ and τ.

PROPOSITION 4.2. *Let F be a formally real field with a finite number of real places. If $|\text{Ord}(\sigma)| \leqslant 2^n$ for each real place σ, then $W_{\text{red}}(F)$ is $(n + 1)$-stable.*

PROOF. Let $x \in X(F)$, say $x \in \text{Ord}(\sigma)$ (every ordering is associated with some real place). Ord(σ) is a 2^m-box for $m \leqslant n$, so by Lemma 3.7 there exist sets $H_1, \ldots, H_m \in \mathcal{H}(F)$ such that $\bigcap_{i=1}^m H_i \cap \text{Ord}(\sigma) = \{x\}$. By [1, Theorem

2.1(B)], there exists an element $a \in F$ such that $\mathrm{Ord}(\sigma) = W(a) = \{x \in X(F) | a$ is negative in the ordering corresponding to $x\}$. Thus $\mathrm{Ord}(\sigma) \in \mathcal{K}(F)$. Since x was arbitrary, Theorem 2.12 implies that $W_{\mathrm{red}}(F)$ is $(n+1)$-stable.

In general, under the hypotheses of Proposition 4.2, $W_{\mathrm{red}}(F)$ may or may not be n-stable. The following theorem shows precisely what added condition is needed to force n-stability.

THEOREM 4.3. *Let F be a formally real field with finitely*(2) *many real places. Then the following statements are equivalent.*

(a) $W_{\mathrm{red}}(F)$ *is n-stable.*

(b) *Any group ring* $\mathbf{Z}G$ *which is a quotient ring of* $W_{\mathrm{red}}(F)$ *has the order of G less than or equal to* 2^n.

(c) *The space of orderings* $X(F)$ *has no* 2^{n+1}*-box.*

(d) *For all formally real places* σ *on F, we have* $|\mathrm{Ord}(\sigma)| \leqslant 2^n$; *and if* $|\mathrm{Ord}(\sigma)| = 2^n$, *the residue class field of* σ *has a unique ordering.*

(e) *For all real places* σ, *we have* $|\mathrm{Ord}(\sigma)| \leqslant 2^n$; *and if* $|\mathrm{Ord}(\sigma)| = 2^n$, *then for all real places* $\tau \neq \sigma$, *the kernel of the canonical homomorphism* $\Lambda_\sigma \to \Lambda_{\sigma,\tau}$ *is nontrivial.*

PROOF. (a) $\Rightarrow$ (b). By Lemma 3.9, any group ring which is a quotient ring is n-stable. By Theorems 3.8 and 3.10, the group G must have order no greater than 2^n.

(b) $\Rightarrow$ (c). For any subset Y of $X(F)$, we obtain a quotient ring of $W_{\mathrm{red}}(F) \subseteq \mathcal{C}(X(F), \mathbf{Z})$ by restricting the functions from $X(F)$ to Y. If Y is a 2^{n+1}-box, Theorem 3.8 implies that the corresponding quotient ring is a group ring where the group has order larger than 2^n.

(c) $\Rightarrow$ (d). Since $\mathrm{Ord}(\sigma)$ is a 2^m-box in $X(F)$ for some m, we have $|\mathrm{Ord}(\sigma)| \leqslant 2^n$. Assume $|\mathrm{Ord}(\sigma)| = 2^n$ and the residue class field K_σ has two orderings (or more). Corresponding to these two orderings are two real places σ_1, σ_2 on K_σ. Composing these with σ gives two real places on F, $\tau_i = \sigma_i\sigma$, $i = 1, 2$. Since $|\Lambda_\sigma| = 2^n$ is the maximum allowed, we must have $|\Lambda_{\sigma_i}| = 1$ $(i = 1,2)$. So there exists only one ordering of K_σ per place. Since the corresponding orderings are distinct we must have $\sigma_1 \neq \sigma_2$, and so the two real places τ_1, τ_2 on F are distinct. Since they agree as places into K_σ, the canonical homomorphisms $\Lambda_{\tau_i} \to \Lambda_{\tau_1,\tau_2}$ $(i = 1,2)$ are isomorphisms. This implies that $\mathrm{Ord}(\tau_1) \cup \mathrm{Ord}(\tau_2)$ is a 2^{n+1}-box [8, Theorem 2.5] which contradicts (c).

(d) $\Rightarrow$ (e). Since every real place is a formally real place, we have $|\mathrm{Ord}(\sigma)| \leqslant 2^n$ for all real places σ. Assume there exist real places σ, τ such that

(2) NOTE ADDED IN PROOF. R. Brown has pointed out that the finiteness hypothesis can be removed by using results in [L. Brocker, Math. Ann. **210** (1974), 233–256].

$|\text{Ord}(\sigma)| = 2^n$ and $\Lambda_\sigma \to \Lambda_{\sigma,\tau}$ is injective. Since $|\Lambda_\tau| \leqslant 2^n$, it must equal 2^n and thus both $\Lambda_\sigma \to \Lambda_{\sigma,\tau}$ and $\Lambda_\tau \to \Lambda_{\sigma,\tau}$ are isomorphisms. This implies that the reduced value groups, for the induced real places σ_*, τ_* on $K_{\sigma,\tau}$, are trivial. Thus σ_* and τ_* correspond uniquely to orderings on $K_{\sigma,\tau}$ [2]. Since (d) implies that $K_{\sigma,\tau}$ has a unique ordering, we must have $\sigma_* = \tau_*$, and so $\sigma = \tau$.

(e) $\Rightarrow$ (a). Let $x \in X(F)$ and let σ be the associated real place. If Ord (σ) is a 2^m-box, then by Lemma 3.7 we can find $H_1, \ldots, H_m \in \mathcal{H}(F)$ such that $\{x\} = \bigcap_{i=1}^m H_i \cap \text{Ord}(\sigma)$. If $m < n$, this is an intersection of at most n elements of $\mathcal{H}(F)$. Now assume $m = n$. Since Ord(σ) has the maximum possible size, the valuation ring corresponding to σ is minimal among the valuation rings of F corresponding to real places. Also, the subrings of F containing a valuation ring are linearly ordered. Thus the condition that $\Lambda_\sigma \to \Lambda_{\sigma,\tau}$ have nontrivial kernel for each $\tau \neq \sigma$ implies that there is some element λ of Λ_σ which maps to 1 in $\Lambda_{\sigma,\tau}$ for all $\tau \neq \sigma$. Let $b \in F$ be in the inverse image of λ under the canonical homomorphism from the multiplicative group of nonzero elements of F to Λ_σ. Then we can apply [1, Theorem 2.1(B)] to obtain an element $a \in F$ which is close to b under the place σ and close to 1 under each of the finite number of other real places. This means that we have found an element a such that $W(a) \subseteq \text{Ord}(\sigma)$ and $W(a) \neq \varnothing$, Ord (σ). Again applying Lemma 3.7, we can choose our family $H_1, \ldots, H_m$ with $H_1 = W(a)$ or $H_1 = W(a) + \text{Ord}(\sigma)$, whichever contains x. Therefore $\{x\} = \bigcap_{i=1}^n H_i$, an n-fold intersection of elements of $\mathcal{H}(F)$. By Theorem 2.12, the $W_{\text{red}}(F)$ is n-stable. Thus the theorem is proved.

We shall conclude this section by exploring some of the implications of this theorem.

Remark 4.4. For the case $n = 1$, but without the restriction to a finite number of real places, the equivalence of (a) and (d) is the valuation theoretic characterization of SAP due to Elman, Lam and Prestel [5], [6], [14, Satz 2.2]. Our proof is much more direct, however, as theirs depends on work in all three of the above cited papers and requires a great deal of work with quadratic forms.

Remark 4.5. It is easy to see that condition (c) of the theorem is equivalent to the following:

(f) Given any 2^{n+1} points in $X(F)$, there exists a set $H \in \mathcal{H}(F)$ such that H contains at least one but less than 2^n of the points.

If we take $n = 1$, we obtain the statement that (under the hypotheses of the theorem) SAP is equivalent to being able to separate one point from any three others by an element of $\mathcal{H}(F)$. Using [14, Satz 2.2], it is possible to improve [5, Theorem 3.5] to give this result in general.

Example 4.6. It is not possible to generalize Theorem 4.3 to abstract Witt rings. Indeed, we shall now give an example where X has no 4-box but the ring is not 1-stable. It is constructed by taking X to be a set of six points and $\mathcal{H}(R)$

to be all subsets of X containing an even number of points. Then (f) clearly holds with $n = 1$; hence there is no 4-box. On the other hand, R is not 1-stable by Corollary 2.6. This is, in fact, the only torsion free abstract Witt ring with $|X(R)| \leqslant 6$ which cannot be the reduced Witt ring of a field.

The author wishes to thank Ron Brown for many useful discussions regarding real places and applications of his approximation theorem [1].

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