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CRITICAL GROUPS HAVING CENTRAL MONOLITHS OF A NILPOTENT BY ABELIAN PRODUCT VARIETY OF GROUPS

BY

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ABSTRACT. Let $\mathfrak{N}$ be a variety of groups which has nilpotency class two and finite odd exponent. Let $\mathfrak{A}$ be an abelian variety of groups with finite exponent relatively prime to the exponent of $\mathfrak{N}$. The existence in the product variety $\mathfrak{N}\mathfrak{A}$ of nonnilpotent critical groups having central monoliths is established. The structure of these critical groups is studied. This structure is shown to depend on an invariant, k. The join-irreducible subvariety of $\mathfrak{N}\mathfrak{A}$ generated by the nonnilpotent critical groups of $\mathfrak{N}\mathfrak{A}$ having central monoliths is determined, in particular, for k odd.

1. **Introduction.** (Throughout any terminology or notation not explained is described in Hanna Neumann's book [4].) Any nilpotent variety of nilpotency class two and finite exponent is of the form $\mathfrak{B}_n \wedge \mathfrak{A}_n\mathfrak{A}_{n'} \wedge \mathfrak{N}_2$ (where n' divides n); let $\mathfrak{N}_{(n,n')}$ denote this variety. The variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ is a subvariety of the product variety $\mathfrak{N}_{(n,n')}\mathfrak{A}_m$, if q divides n'. Thus we may assume that n equals n' and that n is a power of a prime. In [7], the product variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ was shown to be join-irreducible for q a nontrivial power of an odd prime not dividing m. The critical groups generating these product varieties were also characterized in that paper. For distinct primes p and r, satisfying r dividing $p-1$, R. G. Burns [2] has obtained a complete description of the lattice of subvarieties of $\mathfrak{N}_{(p,p)}\mathfrak{A}_r$.

Let k be the smallest positive integer such that m divides $p^k - 1$ where p is an odd prime not dividing the positive integer m, and let q equal p^e. The join-irreducible subvariety of $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ generated by the nonnilpotent critical groups in this variety having central monoliths is determined in the case that m does not divide $p^{k/2}+1$.

The critical groups of the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having central monoliths are contained in the subvariety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m \wedge \mathfrak{C}$, where $\mathfrak{C}$ denotes the variety of groups satisfying the law,

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$$[[x,y],[u,t]]^z[[u,t],[x,y]] = 1.$$

Thus the third subgroup in the derived series of a group in $\mathfrak{C}$ is central. The following result is obtained in §4:

THEOREM I. *The variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m \wedge \mathfrak{C}$ is join-irreducible, if k is odd or if k is even with m not dividing $p^{k/2}+1$, where k is the smallest positive integer such that m divides p^k-1.*

The results of §4 are not sufficient to resolve the case for k even and m dividing $p^{k/2}+1$.

To establish Theorem I it is necessary to study in §2 the structure of the nonnilpotent critical groups in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having central monoliths. A description of this structure is given in Theorems II and III.

THEOREM II. *If C is a nonnilpotent critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith, then*

(i) *the p'-complement of the Fitting subgroup,* Fit C, *is cyclic, and*

(ii) *the exponent of the commutator subgroup,* (Fit C)$'$, *of the Fitting subgroup equals the exponent of the Fitting subgroup.*

The results of R. G. Burns [1] show that if C is a critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ with the exponent of $C/\text{Fit } C$ equaling m, then the Fitting subgroup is generated by k or $2k$ elements, where k is the invariant defined above. The following theorem is also established in §2:

THEOREM III. *Let C be a nonnilpotent critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith. If* Fit C *is generated by k elements and $C/\text{Fit } C$ has exponent m, then k is even and m divides $p^{k/2}+1$.*

It should be noted that if the word "critical" is replaced by "monolithic", the theorem remains true.

A nonnilpotent critical group in the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith is constructed in §3. The method of construction depends on the invariant k.

2. Structure of critical groups having central monoliths. Let C be a nonnilpotent critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$. Since C is a monolithic group, the Fitting subgroup, Fit C, is a p-group where $q=p^e$. Thus C is the semidirect product of Fit C and the p'-complement D of Fit C in C; that is, $C = (\text{Fit } C)\cdot D$. Set $F = \text{Fit } C$ and $\Phi = \Phi(\text{Fit } C)$, the Frattini subgroup of Fit C.

By Maschke's theorem [3], the Frattini quotient group F/Φ viewed as a $GF(p)D$-module may be decomposed into the direct product of irreducible $GF(p)D$-modules, and by Burns' results there are at most two such irreducible $GF(p)D$-modules in this decomposition. Thus either F/Φ is a minimal normal subgroup of C/Φ or

$$F/\Phi = N_1/\Phi \times N_2/\Phi \tag{2.1}$$

where N_1/Φ and N_2/Φ are minimal normal subgroups of C/Φ. Since C is a critical group, D is represented faithfully by conjugation on F/Φ, and thus D is embeddable in the direct product of two m-cycles, $C_m \times C_m$; see Burns [1].

PROOF OF THEOREM II. For this proof set $m' = \exp(D)$, $t = \exp(F)$, and $t' = \exp(F')$. If F/Φ is a minimal normal subgroup in C/Φ, then D is cyclic because it is faithfully and irreducibly represented on F/Φ. On the other hand, if F/Φ has (2.1) as its decomposition into irreducible $GF(p)D$-modules, then the centralizer of N_1/Φ, $C_D(N_1/\Phi)$, in D is trivial, because the monolith, $[N_1, N_2]^{t/p}$, of C is centralized by D. Thus again D is faithfully and irreducibly represented on N_1/Φ, and so D is cyclic; hence part (i) has been established.

Part (ii) is proved by contradiction. Assume that $t' < t$. There are elements z_1 and z_2 in F such that F is the normal closure of the subgroup generated by z_1 and z_2 in C (in some cases, one element would suffice). The subgroup F is a regular p-group since p is an odd prime. Thus either the order of z_1 or the order of z_2 is t. However, the powers $z_1^{t/p}$ and $z_2^{t/p}$ are central in F as the following calculation shows: $[z_i^{t/p}, x] = [z_i, x]^{t/p} = 1$ for any x in F. Since C is monolithic, $z_1^{t/p}$ and $z_2^{t/p}$ must be in the monolith. Hence $z_1^{t/p}$ and $z_2^{t/p}$ are centralized by D. Thus, $(z_1 z_1^{-\alpha})^{t/p}$ and $(z_2 z_2^{-\alpha})^{t/p}$ are trivial for α in D. However, if α is a generator of D, F is the normal closure of the subgroup generated by $(z_1 z_1^{-\alpha})$ and $(z_2 z_2^{-\alpha})$ in C; that is, F is generated by elements of order t/p or less. This is a contradiction since F is a regular p-group. □

Theorem III shows that the structure of the nonnilpotent critical groups in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having central monoliths depends on the invariant k. If the exponent of the p'-complement, D, of F in C is m, then the dimension of the irreducible $GF(p)D$-modules in the decomposition of F/Φ is k; this follows from M. F. Newman's results in [5].

PROOF OF THEOREM III. Let the critical group $C = F \cdot D$ as above with t the exponent of F. The p'-group D has exponent m by hypothesis and is cyclic by Theorem II, part (i); let α generate D. Also, F/Φ is an irreducible $GF(p)D$-module by hypothesis.

Suppose there is a y in F but not in $\Phi(F)$ such that some nontrivial pth power, g, of y is in the central monolith. This implies that $(yy^{-\alpha})^g$ is trivial, but the normal closure of the subgroup generated by $yy^{-\alpha}$ in C is F; contradiction. Therefore F/F^p is a nonabelian monolithic p-group; let F^* denote the quotient group F/F^p. (Note that, F is regular since $p > 2$.)

Clearly $\Phi(F^*) = (F^*)'$. The center, $Z(F^*)$, of F^* also equals $(F^*)'$ since $Z(F^*)$ is a proper characteristic subgroup of F^*. The order of $(F^*)'$ is p since it is the central monolith of $C/(F)^p$. Thus F^* is an extra-special p-group [3, p. 183].

By [3, Theorem 6.5, p. 213], the order, m, of α divides $(p^{2r} - 1)$ with $0 < 2r \leqslant k$, which contradicts the choice of k unless $2r = k$. Therefore k is even and m divides $(p^{k/2} + 1)$. □

In the above proof it is sufficient to assume that C is monolithic; the remaining hypothesis implies that D is cyclic. The author has been able to establish the converse of Theorem III; however, the proof of this is too lengthy to include here.

3. The existence of critical groups having central monoliths. Let F_k be a relatively free group of rank k of the variety $\mathfrak{N}_{(q,q)}$. In [7], the existence of an automorphism β and a generating set $\{y_1,\ldots,y_k\}$ of F_k such that $y_1\beta = y_2, \ldots, y_{k-1}\beta = y_k$ was established. Now let F_{2k} be a relatively free group of $\mathfrak{N}_{(q,q)}$ freely generated by $\{y_{11},\ldots,y_{1k},y_{21},\ldots,y_{2k}\}$. There are automorphisms β_1 and β_2 of F_{2k} each of order m such that β_i fixes each y_{jn} with $j \neq i$ and acts on $\{y_{i1},\ldots,y_{ik}\}$ as β acts on $\{y_1,\ldots,y_k\}$ [7]. Let B be the abelian subgroup of the automorphism group of F_{2k} generated by $\{\beta_1,\beta_2\}$, and let V be the splitting extension of F_{2k} by B. The group V generates the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ [7].

$$T = gp\langle [y_{ij},y_{in}], ([y_{1j},y_{2j}][y_{2j}^{\beta_2^{-1}},y_{1j}^{\beta_1}])^{\beta} | i = 1,2; 1 \leqslant j,n \leqslant k; \beta \in B\rangle$$

defines a normal subgroup of V contained in $(F_{2k})'$. (It turns out that V/T is a critical group generating the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ [7].) Let A denote the subgroup of V/T generated by $\alpha = \beta_1\beta_2^{-1}T$; $\mathfrak{Q}$ denote F_{2k}/T; and R denote the semidirect product of $\mathfrak{Q}$ by A. The commutator subgroup $\mathfrak{Q}'$ of $\mathfrak{Q}$ is homocyclic and is centralized by A. If $m = 2$, then $\mathfrak{Q}'$ is cyclic and R is monolithic.

Now, assuming $m \neq 2$, let N be the complement of

$$gp\langle [y_{11},y_{21}^{\beta_2^{-1}}][y_{21},y_{11}^{\beta_1}]T\rangle$$

in $\mathfrak{Q}'$ which is normal in R given by D. R. Taunt's results [6]. The quotient group $C^* = R/N$ is monolithic with monolith $(\mathfrak{Q}'/N)^{q/p}$, since $\mathfrak{Q}/N$ is a regular p-group. For $m = 2$, set $C^* = R$. If k is odd, or if k is even with m not dividing $p^{k/2}+1$, C^* is a nonnilpotent critical group of $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith. If k is even with m dividing $p^{k/2}+1$, then the subgroup of C^* generated by α and $y_{11}y_{21}N$ is the desired critical group having a central monolith; denote this group by C^{**}. That is, $C^{**} = gp\langle\alpha, y_{11}y_{21}N\rangle$.

We now define inductively the word ω which is used in showing that the groups C^* and C^{**} are critical groups. First let

$$\mu_2(x_1,x_2) = [x_1,x_2,(x_2^{-1}x_1)],$$

and for n in $\{3,\ldots,m\}$, put

$$\mu_n(x_1,\ldots,x_n) = [\mu_{n-1},x_n,(x_n^{-1}x_1),\ldots,(x_n^{-1}x_{n-1})].$$

Finally, set

$$\omega = ([\mu_m(x_1,\ldots,x_m),\mu_m(z_1,\ldots,z_m)])^{q/p}.$$

The following substitution shows that ω is not a law of C^* or C^{**}: $x_j \mapsto \alpha^{(j-1)}$, $z_j \mapsto \alpha^{(j-1)}$, $j = 2, \ldots, m$, and $x_1 \mapsto y_{11}y_{21}N$, $z_1 \mapsto (y_{11}y_{21}N)^{\alpha}$.

THEOREM 3.1. *The groups C^* and C^{**} are critical.*

PROOF. Since the groups C^* and C^{**} are monolithic and since ω is not a law for C^* or C^{**}, to establish that C^* and C^{**} are critical groups, we need only show by a result of Kovács and Newman (see 53.41 in [4]) that ω is a law in their proper subgroups. Only the proof for C^* is given; the proof for C^{**} is similar. Let $\mathfrak{Q}^*$ denote the Fitting subgroup of C^*; that is, $\mathfrak{Q}^* = \mathfrak{Q}/N$.

Suppose H is a subgroup of C^* which does not have ω as a law. Since ω is not a law in H, there is a substitution for the $x_1, \ldots, x_m, z_1, \ldots, z_m$ in ω with values in H such that ω is not the identity. Under this substitution $\mu_m(x_1, \ldots, x_m)$ and $\mu_m(z_1, \ldots, z_m)$ have values in $\mathfrak{Q}^*$ outside of the Frattini subgroup, $\Phi(\mathfrak{Q}^*)$. It is also clear that each of the values substituted for $x_1, \ldots, x_m$ (or for $z_1, \ldots, z_m$) lie in distinct cosets of $\mathfrak{Q}^*$ in C^*. Hence the order of H is divisible by m. Set $\mathfrak{Q}_H = \operatorname{Fit} H$. The p-group $P = \mathfrak{Q}_H(\mathfrak{Q}^*)^p/(\mathfrak{Q}^*)^p$ is an extra-special p-group. Since k is odd or k is even with m not dividing $p^{k/2} + 1$, P/P' cannot be an irreducible $GF(p)A$-module [3, p. 213]. Thus $\mathfrak{Q}_H\Phi(\mathfrak{Q}^*)/\Phi(\mathfrak{Q}^*)$ is not an irreducible $GF(p)A$-module; hence $\mathfrak{Q}_H = \mathfrak{Q}^*$. Therefore H is not a proper subgroup of C^*. □

4. **Proof of Theorem I.** For notational convenience set $\mathfrak{U} = \mathfrak{N}_{(q,q)}\mathfrak{A}_m \wedge \mathfrak{C}$ and from §3, $R = gp\langle \beta_1\beta_2^{-1}T, F_{2k}\rangle/T$. The proof of Theorem I is accomplished by establishing the following three propositions:

(i) The group R generates the variety $\mathfrak{U}$.

(ii) There is a proper subvariety of $\mathfrak{U}$ which contains any critical group C of $\mathfrak{U}$ in which either (a) the exponent of the commutator subgroup of Fit C is less than q, or (b) the order of $C/\operatorname{Fit} C$ is less than m.

(iii) For k odd or k even with m not dividing $p^{k/2} + 1$, a nonnilpotent critical group C having a central monolith generates the variety $\mathfrak{U}$ if the exponent of Fit C is q and $C/\operatorname{Fit} C$ has order m.

Proposition (i) is established by showing that any critical group C of $\mathfrak{U}$ is a factor of a direct power of R. If C is nilpotent this result is obvious. For any nontrivial divisor m^* of m and corresponding smallest positive integer k^* such that m^* divides $p^{k^*} - 1$, there is, by Clifford's theorem [3], a subgroup of R which has a corresponding structure like R for m^* and k^*. Hence we may assume that C/F has order m, where as before $F = \operatorname{Fit} C$. It was shown in [7] that any critical group C in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ is a factor of V, the splitting extension of F_{2k} by B. There are three cases: $\dim(F/\Phi) = k$, $\dim(F/\Phi) = 2k$ with m not dividing $p^{k/2} + 1$, and $\dim(F/\Phi) = 2k$ with m dividing $p^{k/2} + 1$, where $\Phi = \Phi(F)$.

Let F/Φ have dimension k. In this case, it can be shown that C is a factor of V^*, the splitting extension of F_k by the m-cycle β of §3; in the same way that C was shown to be a factor of V in [7]. The group V^* is isomorphic to the subgroup $\mathcal{V}^*$ of V generated by $y_{11}y_{21}$ and $\beta_1\beta_2^{-1}$. Hence C is a homomorphic image of $\mathcal{V}^*$, and clearly the intersection of $\mathcal{V}^*$ and T is contained in the kernel of this homomorphism of $\mathcal{V}^*$ onto C. Thus C is a factor of R.

Now suppose $\dim(F/\Phi)$ is $2k$ and m does not divide $p^{k/2} + 1$. Recall the minimal normal subgroups N_1/Φ and N_2/Φ of C/Φ in the decomposition (2.1) of F/Φ and the p'-subgroup D of C in §2. The quotient group

$$(gp\langle N_i, D\rangle)F^p/F^p$$

is monolithic, and so by Theorem III (see the note at the end of the proof of Theorem III), it must be metabelian. Thus N_1 and N_2 can be chosen so that $(N_1)'$ and $(N_2)'$ are trivial. The critical group C is a homomorphic image of the subgroup $\mathcal{V}$ of V generated by F_{2k} and $\beta_1\beta_2^{-1}$. The kernel of this homomorphism contains T, since $(N_1)'$ and $(N_2)'$ are trivial and since the monolith of C is central. Therefore C is a factor of R.

In the third case where $\dim(F/\Phi)$ is $2k$ but k is even with m dividing $p^{k/2}+1$, N_1 and N_2 need not be abelian. The critical group C is again a homomorphic image of $\mathcal{V}$. The group $\mathcal{V}$ can be embedded in the direct product $V^* \times \mathcal{V}/T_1 \times V^*$, where T_1 is the normal closure of the subgroup of V generated by commutators of the form $[y_{1j}, y_{1j'}]$ and $[y_{2j}, y_{2j'}]$ for $1 \leqslant j, j' \leqslant k$ in V. It is easy to verify that V^* is isomorphic to the subgroup of $\mathcal{V}/T_1$ generated by $\beta_1\beta_2^{-1}$ and $y_{11}y_{21}T_1$; hence C is a factor of the direct power $\mathcal{V}/T_1 \times \mathcal{V}/T_1 \times \mathcal{V}/T_1$. Since the monolith of C is central, C is a factor of the direct power $R \times R \times R$.

The proper subvariety of proposition (ii) is the subvariety of $\mathfrak{U}$ determined by ω. To establish (ii) all that needs to be shown is that if C is a critical group of $\mathfrak{U}$ which does not have ω as a law, then the exponent of (Fit C)$'$ is q and $C/$Fit C has order m. That the exponent of the commutator subgroup is q is obvious. The argument that $C/$Fit C has order m is essentially the argument in the proof of Theorem 3.1, that the order of H is divisible by m. Therefore proposition (ii) is established.

Let C be a critical group of $\mathfrak{U}$ satisfying the hypothesis of proposition (iii). The critical group C is a homomorphic image of R by the proof of (i). If $\mathcal{K}$ is the kernel of this homomorphism, then $\mathcal{K}$ is contained in the commutator subgroup of F_{2k}/T since $C/$Fit C has order m. The kernel $\mathcal{K}$ cannot contain the socle, S, of R because the exponent of (Fit C)$'$ is q by Theorem II(ii). Thus, R must lie in the variety generated by C, if the socle S of R is minimal characteristic. By proposition (i), this implies that C generates the variety $\mathfrak{U}$. Therefore the final step in the proof is to show that the socle, S, of R is minimal characteristic in R.

To establish that S is minimal characteristic, it is necessary to study the action of conjugation by β_2 on the group R. First, the subgroup $\mathcal{V}$ of V generated by $\beta_1\beta_2^{-1}$ and F_{2k} is clearly normalized by β_2, and the subgroup T of F_{2k} is also normalized by β_2. Thus conjugation by β_2 induces an automorphism on R. Now consider the action of conjugation by β_2 on the element $[y_{11}, y_{21}]^{q/p}T$ in the socle of R. Clearly the elements $([y_{11}, y_{21}]^{q/p}T)^{\beta}$ for $\beta \in gp\langle\beta_1\beta_2^{-1}, \beta_2\rangle$ generate the socle, S, of R. Therefore S is minimal characteristic in R. □

REFERENCES

1. R. G. Burns, *Verbal wreath products and certain product varieties of groups*, J. Austral. Math. Soc. **7** (1967), 356–374. MR **36** #249.

2. ———, *Some applications of wreath products of groups*, Ph.D. Thesis, Austral. National Univ., Canberra, Australia, 1966.

3. Daniel Gorenstein, *Finite groups*, Harper and Row, New York, 1968. MR **38** #229.

4. Hanna Neumann, *Varieties of groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, New York, 1967. MR **35** #6734.

5. M. F. Newman, *On a class of metabelian groups*, Proc. London Math. Soc. (3) **10** (1960), 354–364. MR **22** #8074.

6. D. R. Taunt, *On A-groups*, Proc. Cambridge Philos. Soc. 45 (1949), 24–42. MR **10**, 351.

7. James J. Woeppel, *Join-irreducible Cross product varieties of groups*, Trans. Amer. Math. Soc. **200** (1974), 141–148.

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