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ON THE BORDISM OF ALMOST FREE Z_{2k} ACTIONS

BY

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ABSTRACT. An "almost free" Z_{2k} action on a manifold is one in which only the included Z_2 may possibly fix points of the manifold. For $k = 2$, these are the stationary-point free actions. It is shown that almost free Z_{2k} bordism is generated by three subalgebras: the extension from Z_2 actions, a coset of Z_2 extensions being the restrictions of circle actions and a certain ideal of elements which annihilate the whole ring. The additive structure is determined. Free Z_{2k} bordism is shown to split as an algebra. It is shown that the kernel of the extension homomorphism from Z_2 to Z_{2k} bordism is equal to the image of the corresponding restriction homomorphism.

1. Introduction. An "almost free" Z_{2k} action on a manifold is one for which only the $Z_2 \subset Z_{2k}$ may possibly fix points on the manifold. We will determine the additive structure of the unoriented bordism of such actions along with the algebra structure of the special case of free actions. We also get some results on the general product structure.

In §2, we define the various notions and reduce the additive problem to a homology problem. We also introduce the various extension homomorphisms on bordism and show that they are given by maps on spaces.

In §3 we solve the homology problem and construct generating sets for the various bordism modules. This is done by evaluating various spectral sequences for which we are, in general, unable to compute the extensions.

In §4 we show that free Z_{2k} bordism is the tensor product of two subalgebras: the "extension of Z_2 actions" and the subalgebra generated by the circle with standard free Z_{2k} action.

In §5 we prove various algebraic relations but do not determine the algebra. In particular, we show that the relative bordism algebra of almost free manifolds with free boundaries contains as a subalgebra a direct sum of a zero algebra and another algebra, the zero algebra being the extensions of Z_2 actions.

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2. **Preliminaries.** Let G_k denote the cyclic group of order 2^k, $k \geqslant 2$ (usually denoted by Z_{2^k}) and t its generator. We will let $\mathfrak{F}_k$ be the family $\{Z_2, \{1\}\}$ of subgroups of G_k and say that a G_k action

$$\varphi\colon G_k \times M \to M$$

by smooth maps on a smooth compact manifold M is *almost free* if for each point x in M,

$$\{g \in G_k\colon \varphi(g, x) = x\} \in \mathfrak{F}_k.$$

We recall some notions of bordism. We say that two almost free actions (M_1, φ_1) and (M_2, φ_2) on closed manifolds M_i are *bordant* if there is an almost free action (V, Φ) with ∂V equal to the disjoint union of M_1 and M_2 and Φ restricting to φ_i on M_i. The resulting equivalence classes for n dimensional manifolds forms a Z_2 module $N_n(G_k; \mathfrak{F}_k)$. We let

$$N_*(G_k; \mathfrak{F}_k) = \bigoplus_{n=0}^{\infty} N_n(G_k; \mathfrak{F}_k),$$

an algebra over the unoriented bordism ring, N_*. See [8] for details.

Similarly, we can define $N_*(G_k;$ Free), the bordism of free G_k actions and $N_*(G_k; \mathfrak{F}_k,$ Free), the bordism of almost free actions on manifolds with boundary such that the action is free on the boundary.

There is an exact sequence

$$\begin{tikzcd} N_*(G_k; \mathfrak{F}_k) \arrow[r] & N_*(G_k; \mathfrak{F}_k, \text{Free}) \arrow[r, "\partial_*"] & N_*(G_k, \text{Free}) \arrow[ll, bend left] \end{tikzcd}$$

of $N_*(G_k;$ Free) modules and homomorphisms. (The action of $N_*(G_k;$ Free) is via the twisted product. See [1].) The sequence is short exact, since the identity is in the image of ∂_*. In fact, denoting the interval $[-1, 1]$ by I and using multiplication by -1 as a generator of a Z_2 action on I, we have that

$$1 = (G_k, G_k) = (\partial(G_k \times_{Z_2} I), G_k \times 1)$$

and it is clear that $G_k \times_{Z_2} I$ is almost free.

The above sequence is then

$$0 \to N_*(G_k; \mathfrak{F}_k) \xrightarrow{i_*} N_*(G_k; \mathfrak{F}_k, \text{Free}) \underset{h_*}{\overset{\partial_*}{\rightleftarrows}} N_*(G_k, \text{Free}) \to 0$$

where h_* is defined as an $N_*(G_k;$ Free) morphism by

$$h(1) = (G_k \times_{Z_2} I, G_k \times 1) = \delta_1.$$

It is well known that $N_*(G_k;$ Free) is freely generated, as an N_* module by

(i) extensions of the antipodal action on even dimensional spheres: $(S^{2n} \times_{Z_2} G_k, 1 \times G_k) = y_{2n}$;

(ii) restrictions of circle actions on odd dimensional spheres: $(S^{2n+1}, t = \exp(\pi i/2^{k-1})) = y_{2n+1}$.

See [3] for a proof of this as well as for the product on these classes. One sees easily that $y_1^2 = 0$ and therefore that J_*, the N_* submodule generated by 1 and y_1 is a subalgebra. Let $K_*(\text{Free})$ denote the algebra of Z_2 extensions. We will show

$$N_*(G_k; \text{Free}) \cong J_* \otimes_{N_*} K_*(\text{Free})$$

as N_* algebras (see Proposition 4.1).

Since $(S^{2n+1}$, antipodal map) is a boundary, $K_*(\text{Free})$ is freely generated by $\{y_{2n}: n = 0, 1, 2, \ldots,\}$ and the image of h_* is freely generated by

$$\{\delta_{2n+1} = y_{2n}\delta_1, \delta_{2n+2} = y_1\delta_{2n+1}: n = 0, 1, \ldots\}.$$

Next, we rewrite the relative group $N_*(G_k; \mathfrak{F}_k, \text{Free})$ using a standard fixed point construction. Let $BO_j(G_k)$ denote the classifying space for j-dimensional G_k vector bundles (see [8]). There is the usual classifying bundle $\gamma^j(G_k)$ with G_k action. The fixed set of Z_2 in G_k for $BO_j(G_k)$ has a component over which $\gamma^j(G_k)$ has no summand with trivial Z_2 representation. This component is a homotopy BO_j and, following [6], we will denote it by $BO_j(C^\infty, k)$. This space has a G_k action (with Z_2 in G_k acting trivially) covered by a G_k action on the pullback of $\gamma^j(G_k)$, call it $\gamma^j(C^\infty, k)$, and Z_2 acts by multiplication by -1 in the fibers of $\gamma^j(C^\infty, k)$. The pair $(BO_j(C^\infty, k), \gamma^j(C^\infty, k))$ is a classifying object for such spaces and j-plane bundles. Note that the $G_{k-1} = G_k/Z_2$ action is free on $BO_j(C^\infty, k)$ if j is *odd*.

Suppose that (M, φ) is a $(\mathfrak{F}, \text{Free}) - G_k$ action. The fixed set of $Z_2 \subset G_k$ in M is a disjoint union of closed manifolds F^{n-j} for $0 \leqslant j \leqslant n$, where $n = \dim M$. These manifolds, with their normal bundles, satisfy the above properties. We also know that $G_{k-1} = G_k/Z_2$ acts freely on the F^{n-j}, since (M, φ) is almost free. Hence, we get a homomorphism

$$F\colon N_*(G_k; \mathfrak{F}, \text{Free}) \to \bigoplus_{j=0}^{*} N_{*-j}(G_{k-1}; \text{Free})(BO_j(C^\infty, k)).$$

In fact, F is an isomorphism with inverse induced by the correspondence which assigns to a bundle, its disk bundle.

We know that (see [4])

$$N_{*-j}(G_{k-1}; \text{Free})(BO_j(C^\infty, k)) \cong N_{*-j}(BO_j(C^\infty, k) \times_{G_{k-1}} EG_{k-1}).$$

(We will take EG_k to be the infinite sphere of C^∞ with the generator of G_k acting by multiplication by $\exp(\pi i/2^{k-1})$.)

Let $X_{j,k} = BO_j(C^\infty, k) \times_{G_{k-1}} EG_{k-1}$. Then [2],

$$\bigoplus_{j=0}^{*} N_{*-j}(X_{j,k}) \cong \bigoplus_{j=0}^{*} \bigoplus_{r+s+j=*} N_r \otimes H_s(X_{j,k}; Z_2)$$

$$\cong \bigoplus_{r=0}^{*} N_r \otimes \left(\bigoplus_{j=0}^{*-r} H_j(X_{*-r-j,k}; Z_2) \right)$$

$$\cong N_* \otimes \left(\bigoplus_{j=0}^{*} H_{*-j}(X_{j,k}; Z_2) \right).$$

The isomorphims are as N_* modules and the tensor products are over Z_2.

Hence, we need to compute the Z_2 homology (or cohomology) of the spaces $X_{j,k}$.

3. $H^*(X_{j,k}; Z_2)$ and generators for $N_*(G_k$; almost free).

Case 1. $k = 2$. Let $X_j = BO_j(C^\infty, 2) \times_{Z_2} EZ_2$.

LEMMA 2.1. $X_{2j+1} \approx BSO_{2j+1} \times BZ_4$.

PROOF (R. E. STONG). Let $T\colon \gamma^{2j+1}(C^\infty) \to \gamma^{2j+1}(C^\infty)$ denote the generator of the Z_4 action. Then $T^2 = -1$ in the fibers of γ. Then $R = \det(T)$ generates a Z_4 action on η, the determinant bundle of γ. $R^2 = -1$. Hence $R \otimes T$ generates a Z_2 action on $\eta \otimes \gamma$ covering the free Z_2 action on the base. Dividing this out, we get a $2j + 1$ plane bundle ν over X_{2j+1}.

Now T gives a Z_4 action on the sphere of η, which is a BSO_{2j+1}, the quotient of which is X_{2j+1}.

Let $\varphi\colon X_{2j+1} \to BZ_4$ classify this cover and $\psi\colon X_{2j+1} \to BSO_{2j+1}$ classify $\det \nu \otimes \nu$. Since φ is covered by an equivariant map $\varphi'\colon BSO_{2j+1} \to EZ_4$, we have that

$$\begin{tikzcd}
BSO_{2j+1} = S(\det \eta) \arrow[r, "1 \times \varphi'"] \arrow[d, "\pi_\eta"] & BSO_{2j+1} \times EZ_4 \arrow[d, "1 \times \pi"] \\
X_{2j+1} \arrow[r, "\psi \times \varphi"'] & BSO_{2j+1} \times BZ_4
\end{tikzcd}$$

commutes (since $\det \nu \otimes \nu$ pulls back to the universal bundle over BSO_{2j+1}). Hence $\psi \times \varphi$ is a homotopy equivalence.

We do not know of such a splitting for X_{2j} but we will compute the Z_2-vector space structure of $H_*(X_{2j}; Z_2)$ (Proposition 3.6).

Case 2. $k > 2$. Note that there is a map

$$F\colon BO_j(\mathbf{C}^\infty, k) \to BO_j(\mathbf{C}^\infty, 2)$$

which classifies the $Z_4 \subset G_k$ action on $\gamma^j(\mathbf{C}^\infty, k)$. This is a homotopy equivalence. Crossing with EZ_2 and dividing out gives a homotopy equivalence

$$BO_j(\mathbf{C}^\infty, k) \times_{Z_2} S^\infty \approx X_j.$$

Making this identification gives X_j a free G_{k-2} action. Dividing this out, we get $X_{j,k} = X_j/G_{k-2}$.

LEMMA 2.2. $X_{2j+1,k} \approx BSO_{2j+1} \times BG_k$.

PROOF. Looking at the proof of 2.1, we see that the identification there takes the above G_{k-2} action to the standard G_{k-2} action on $BZ_4 = EG_k/Z_4$.

More generally, we have fibrations

$$\pi_{j,k}\colon X_{j,k} \twoheadrightarrow BG_{k-1} \quad \text{and} \quad \tau_{j,k}\colon X_{j,k} \twoheadrightarrow BZ_2$$

with fibers BO_j and $X_{j,k-1}$ respectively, giving a diagram:

(2.3)

$$\begin{tikzcd}
BO_j \arrow[r, tail] \arrow[d, "="'] & X_{j,k-1} \arrow[rr, two heads, "\pi_{j,k-1}"'] \arrow[d, tail, "\theta_{j,k}"] & & BG_{k-2} \arrow[d, tail, "\theta'_{j,k}"] \\
BO_j \arrow[r] & X_{j,k} \arrow[rr, two heads, "\pi_{j,k}"'] \arrow[d, two heads, "\tau_{j,k}"] & & BG_{k-1} \arrow[d, two heads] \\
 & BZ_2 & = & BZ_2
\end{tikzcd}$$

where $\theta'_{j,k}$ classifies the principal G_{k-1} bundle

$$EG_{k-2} \times_{G_{k-2}} G_{k-1} \to BG_{k-1}.$$

We first consider the case $k = 2$.

Consider the Serre cohomology spectral sequence of the fibration $BO_2 \xrightarrow{i_2} X_2 \overset{\pi_2}{\twoheadrightarrow} BZ_2$. This has a trivial coefficient system and

$$E_2^{*,*}(\pi_2) \cong E^{0,*} \otimes E^{*,0} = Z_2[\omega_1, \omega_2] \otimes Z_2[\alpha],$$

where degree ω_1 = degree $\alpha = 1$ and degree $\omega_2 = 2$. Since BU_1 is the fixed set of the involution on $BO_2(\mathbf{C}^\infty)$, nothing transgresses and $d_2(\omega_1) = d_2(\alpha) = 0$.

LEMMA 3.1. $E_3(\pi_2) \cong Z_2[\alpha, \omega_1, \omega_2^2]/(\alpha^2\omega_1)$.

PROOF. If $d_2(\omega_2) = 0$, then $\omega_2 = i_2^*(x)$, in Z_2 cohomology. Let $\gamma \to RP(1)$ be the canonical line bundle over the projective line. There is a Z_4 action on γ with Z_2 in Z_4 acting as -1 in the fibers and which covers a free involution on $RP(1)$. Classifying $\gamma \times \gamma$ over $RP(1) \times RP(1)$, we get a diagram:

$$\begin{tikzcd}
RP(1) \times RP(1) \arrow[r, "g"] \arrow[d, "f"] & BO_2(\mathbf{C}^\infty, 2) \times S^\infty \arrow[d, "i_2"] \\
RP(1) \times_{Z_2} RP(1) \arrow[r, "h"] & X_2
\end{tikzcd}$$

If $i^*(x) = \omega_2$, then

$$f^*h^*(x) = g^*i^*(x) = g^*(\omega_2) = \omega_2(\gamma \times \gamma).$$

Hence $f^*h^*(x)$ evaluates on the fundamental class of $RP(1) \times RP(1)$ to 1. But

$$\begin{aligned}
\langle f^*h^*(x), [RP(1) \times RP(1)] \rangle &= \langle h^*(x), f_*[RP(1) \times RP(1)] \rangle \\
&= \langle h^*(x), 0 \rangle \\
&= 0.
\end{aligned}$$

Let $X_2^{(n)} = (BO_2(\mathbf{C}^\infty, 2) \times \cdots \times BO_2(\mathbf{C}^\infty, 2)) \times_{Z_2} S^\infty$ (n-fold product), where we give $BO_2^{(n)} = BO_2(\mathbf{C}^\infty, 2) \times \cdots \times BO_2(\mathbf{C}^\infty, 2)$ the diagonal involution. There is a fibration $X_2^{(n)} \to BZ_2$ with fiber $BO_2^{(n)}$. The inclusion of the jth factor BO_2 and projection onto the jth factor gives a diagram:

$$X_2 = BO_2(\mathbf{C}^\infty, 2) \times_{Z_2} S^\infty \to X_2^{(n)} \xrightarrow{\pi^j} X_2,$$

the composite being a homeomorphism.

Let $Z_2[\alpha]$ be the image of $H^*(BZ_2)$ in $E_2^{*,0}(\pi_2^{(n)})$ and $Z_2[x_1^{(i)}, x_2^{(i)}]$ the cohomology of the ith factor BO_2. Then

$$\begin{aligned}
E_2^{*,*}(\pi_2^{(n)}) &\cong E^{*,0}(\pi_2^{(n)}) \otimes E^{0,*}(\pi_2^{(n)}) \\
&\cong Z_2[\alpha] \otimes \left\{ \bigotimes_{j=1}^{n} Z_2[x_1^{(j)}, x_2^{(j)}] \right\}.
\end{aligned}$$

As before, $d_2(x_1^{(j)}) = d_2(\alpha) = 0$ in $E_2^{*,*}(\pi_2^{(n)})$. Also

$$0 = (\pi^j)^*(d_2\omega_2 + \alpha^2\omega_1) = d_2(\pi^j)^*(\omega_2) + (\pi^j)^*(\alpha^2\omega_1) = d_2x_2^{(j)} + \alpha^2x_1^{(j)},$$

so that $d_2x_2^{(j)} = \alpha^2x_1^{(j)}$.

Next we classify the $2n$ plane bundle

$$\gamma^2(\mathbf{C}^\infty, 2) \times \cdots \times \gamma^2(\mathbf{C}^\infty, 2) \to BO_2^{(n)},$$

giving a map of fibrations:

$$\begin{tikzcd}
BO_2^{(n)} \arrow[r, tail] \arrow[d, "f"'] & X_2^{(n)} \arrow[r, two heads] \arrow[d, "\hat{f}"'] & BZ_2 \arrow[d, "="] \\
BO_{2n} \arrow[r, tail] & X_{2n} \arrow[r, two heads] & BZ_2
\end{tikzcd}$$

Recall that f^* takes the Stiefel-Whitney class ω_i to the ith elementary symmetric polynomial in the variables $x_1^{(1)} + x_2^{(1)}, \dots, x_1^{(n)} + x_2^{(n)}$. Denote these by τ_i, so that $f^*(w_i) = \tau_i$. Since τ_i are algebraically independent in $H^*(BO_2^{(n)})$, the homomorphism

$$\hat{f}^*\colon E_2^{*,*}(\pi_{2n}) \to E_2^{*,*}(\pi_2^{(n)})$$

is a monomorphism and the monomials $\tau_{i_1} \cdots \tau_{i_r}\alpha^s$ are linearly independent in $E_2^{*,*}(\pi_2^{(n)})$.

Proposition 3.2. *$d_2\tau_{2i} = \alpha^2\tau_{2i-1}$ and $d_2\tau_{2i-1} = 0$.*

Proof.

$$\begin{aligned}
\tau_{2i} = {} & \sum_{(n)} x_2^{(1)} \cdots x_2^{(i)} + \sum_{(n)} x_1^{(1)} x_1^{(2)} x_2^{(3)} \cdots x_2^{(i+1)} + \cdots \\
& + \sum_{(n)} x_1^{(1)} \cdots x_1^{(2i-2)} x_2^{(2i-1)} + \sum_{(n)} x_1^{(1)} \cdots x_1^{(2i)}
\end{aligned}$$

and

$$\begin{aligned}
\tau_{2i-1} = {} & \sum_{(n)} x_1^{(1)} x_2^{(2)} \cdots x_2^{(i)} + \sum_{(n)} x_1^{(1)} x_1^{(2)} x_1^{(3)} x_2^{(4)} \cdots x_2^{(i+1)} + \cdots \\
& + \sum_{(n)} x_1^{(1)} \cdots x_1^{(2i-1)},
\end{aligned}$$

where $\sum_{(n)}$ denotes symmetric sum using $x_1^{(1)}, \dots, x_1^{(n)}, x_2^{(1)}, \dots, x_2^{(n)}$. Note that,

$$\begin{aligned}
& \sum_{(n)} x_1^{(1)} \cdots x_1^{(r)} x_2^{(r+1)} \cdots x_2^{(r+s)} \\
& \quad = \sum_{(n-1)} x_1^{(1)} \cdots x_1^{(r)} x_2^{(r+1)} \cdots x_2^{(r+s)} \\
(\#) \qquad & \quad + x_2^{(n)} \sum_{(n-1)} x_1^{(1)} \cdots x_1^{(r)} x_2^{(r+1)} \cdots x_2^{(r+s-1)} \\
& \quad + x_1^{(n)} \sum_{(n-1)} x_1^{(1)} \cdots x_1^{(r-1)} x_2^{(r)} \cdots x_2^{(r+s-1)}
\end{aligned}$$

and, in this case,

$$(*)\qquad d\Big(\sum_{(n)} x_1^{(1)}\cdots x_1^{(r)}x_2^{(r+1)}\cdots x_2^{(r+2)}\Big) = \begin{cases} 0, & \text{if } r \text{ is } \textit{odd}, \\ \alpha^2 \sum_{(n)} x_1^{(1)}\cdots x_1^{(r+1)}x_2^{(r+2)}\cdots x_2^{(r+s)}, & \text{if } r \text{ is } \textit{even}. \end{cases}$$

We prove this by induction on n. If $n = 1$, $(*)$ is clearly true. Assume $(*)$ is true for $n-1$, then by $(\#)$,

$$\begin{aligned} d\Big(\sum_{(n)} x_1^{(1)}\cdots x_2^{(r+s)}\Big) = {} & d\Big(\sum_{(n-1)} x_1^{(1)}\cdots x_2^{(r+s)}\Big) \\ & + d\Big(x_2^{(n)} \sum_{(n-1)} x_1^{(1)}\cdots x_2^{(r+s-1)}\Big) \\ & + d\Big(x_1^{(n)} \sum_{(n-1)} x_1^{(1)}\cdots x_1^{(r-1)}x_2^{(r)}\cdots x_2^{(r+s-1)}\Big). \end{aligned}$$

If r is even, then according to the induction hypothesis, the last term is zero, while the second term is

$$\begin{aligned} & x_2^{(n)}\alpha^2 \sum_{(n-1)} x_1^{(1)}\cdots x_1^{(r+1)}x_2^{(r+2)}\cdots x_2^{(r+s-1)} \\ & \qquad + \alpha^2 x_1^{(n)} \sum_{(n-1)} x_1^{(1)}\cdots x_1^{(r)}x_2^{(r+1)}\cdots x_2^{(r+s-1)}. \end{aligned}$$

The first term is $\alpha^2 \sum_{(n-1)} x_1^{(1)}\cdots x_1^{(r+1)}x_2^{(r+2)}\cdots x_2^{(r+s-1)}$. Adding and using $(\#)$ again gives the result for even r.

If r is odd, the induction hypothesis says that the first term is zero and the second and third terms are both equal to

$$\alpha^2 x_1^{(n)} \sum_{(n-1)} x_1^{(1)}\cdots x_1^{(r)}x_2^{(r+1)}\cdots x_2^{(r+s-1)}.$$

Hence, $(*)$ is true.

Similarly, we can show that

$$d\Big(\sum_{(n)} x_2^{(1)}\cdots x_2^{(i)}\Big) = \Big(\sum_{(n)} x_1^{(1)}x_2^{(2)}\cdots x_2^{(i)}\Big)\alpha^2.$$

The proposition follows.

COROLLARY 3.3. *In* $E_2^{*,*}(\pi_{2n})$, $d_2(w_{2i}) = \alpha^2 w_{2i-1}$ *and* $d_2(w_{2i-1}) = 0$.

PROPOSITION 3.4. $E_3(\pi_{2n})$ *is the* $Z_2[\alpha, w_{2i-1}, w_{2i}^2\colon i = 1, \ldots, n]$ *module on a basis* $\{1, \rho_S\colon S = (i_1, \ldots, i_k)$ *with* $i_1 < \cdots < i_k \leqslant n\}$, *with relations* $\alpha^2 w_{2i-1} = \alpha^2 \rho_S = 0$ *for all* S. (ρ_S *is represented in* E_2 *by*

$$\sum_{j=1}^{k} w_{2i_1} \cdots w_{2i_{j-1}} w_{2i_j - 1} w_{2i_{j+1}} \cdots w_{2i_k} \cdot \Big)$$

PROOF. Induction on n. We know it is true for $n = 1$ and assume it true for $n - 1$. Recall that

$$E_2(\pi_{2n}) \cong Z_2[\alpha, w_1, \ldots, w_{2n}],$$

$d(w_{2i}) = \alpha^2 w_{2i-1}$ and $d(w_{2i-1}) = d(\alpha) = 0$. Call this differential graded algebra, $E_{2n}(\alpha)$. Let $\hat{E}_{2n} = E_{2n-2}(\alpha_1) \otimes Z_2[w_{2n-1}, w_{2n}, \alpha_2]$; $d(\alpha_2) = dw_{2n-1} = 0$ and $dw_{2n} = \alpha_2^2 w_{2n-1}$. Let $A_* = Z_2[w_1^2, \ldots, w_{2n}^2]$ and note that both $E_{2n}(\alpha)$ and $\hat{E}_{2n}$ are naturally A_* modules. There is a homomorphism of DGA's and A_* modules which sends α_i to α and w_j to w_j. Call it p_*. Then there is an exact sequence of A_* modules

$$\cdots \xrightarrow{\delta_*} H_*(\ker p_*) \xrightarrow{i_*} H_*(\hat{E}_{2n}) \xrightarrow{p_*} H_*(E_{2n}) \xrightarrow{\delta_*} \cdots,$$

Inductively,

$$H_*(\hat{E}_{2n}) \cong K_*^{n-1}(\alpha_1) \otimes (Z_2[w_{2n-1}, w_{2n}^2, \alpha_2]/\alpha_2^2 w_{2n-1} = 0),$$

where $K_*^r(\beta)$ is the $Z_2[w_1, \ldots, w_{2r-1}, w_2^2, \ldots, w_{2r}^2, \beta]$ module on a base $\{1, \rho_S\}$ etc. (as above). We must show that $H_*(E_{2n}) \cong K_*^n(\alpha)$.

The A_* module generators of $H_*(\hat{E}_{2n})$ are:

$$\{1, w_{2n-1}, \rho_S, \alpha_i \rho_S, \alpha_i w_{2n-1} \rho_S, \alpha_1^j, \alpha_2^j\}$$

together with products of these with the monomials $w_{j_1} \cdots w_{j_l}$ (all j_k odd and less than $2n - 1$).

There is a commutative diagram (the top row not necessarily exact):

$$\begin{tikzcd}
\cdots \arrow[r, "\hat{i}"] & K^{n-1}(\alpha_1) \otimes Z_2[w_{2n-1}, w_{2n}^2, \alpha_2]/(\alpha_2^2 w_{2n-1}) \arrow[r, "\hat{p}"] \arrow[d, "\cong"'] & K^n(\alpha) \arrow[r, "\hat{\delta}"] \arrow[d, "j_*"'] & H_*(\ker p_*) \arrow[r, "\hat{i}"] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r, "i_*"] & H_*(\hat{E}_{2n}) \arrow[r, "p_*"] & H_*(E_{2n}) \arrow[r, "\delta_*"] & H_*(\ker p_*) \arrow[r, "i_*"] & \cdots
\end{tikzcd}$$

where j_* is an inclusion, $\hat{\delta} = \delta_* \circ j_*$ and p_* is easily seen to have image in $K^n(\alpha)$.

Hence, we want to show that the kernel of i_* is contained in the image of $\hat{\delta}$.

But $H_*(\ker p_*) \cong (\alpha_1 + \alpha_2) \cdot H_{*-1}(\hat{E}_{2n})$ and $\ker i_* \cong (\alpha_1 + \alpha_2) \cdot$ (annihilator of $(\alpha_1 + \alpha_2)$ in $H_{*-1}(\hat{E}_{2n})$). Hence, we want the kernel of multiplication by the class of $\alpha_1 + \alpha_2$ in $H_*(\hat{E}_{2n})$. But checking the result of multiplying $\alpha_1 + \alpha_2$ by the various A_* module generators of $H_*(\hat{E}_{2n})$ gives the only relation

$$(\alpha_1 + \alpha_2)\alpha_i \rho_S w_{2n-1} w_\omega = \alpha_1 \alpha_2 \rho_S w_{2n-1} w_\omega; \quad i = 1, 2,$$

where $w_\omega = w_{j_1} \cdots w_{j_l}$, as before, for all $S = (i_1, \ldots, i_k \leqslant n - 1)$.

It follows that ker i_* is the A_* module generated by the

$$(\alpha_1^2 + \alpha_2^2) w_{2n-1} \rho_S w_\omega$$

in $H_*(\ker p_*)$. But note that in E_{2n},

$$\rho_{\{S,n\}} = \rho_S w_{2n} + (w_{2i_1} \cdots w_{2i_k}) w_{2n-1}$$

and hence also in $\hat{E}_{2n}$ and goes (via the differential) to $\alpha_2^2 \rho_S w_{2n-1} + \alpha_1^2 \rho_S w_{2n-1}$. Since d is trivial on w_ω,

$$\hat{\delta}(\rho_{\{S,n\}} w_\omega) = (\alpha_1^2 + \alpha_2^2) w_{2n-1} \rho_S w_\omega .$$

The result follows.

COROLLARY 3.5. $E_3(\pi_{2n}) = E_\infty(\pi_{2n})$.

PROOF. Note that $\oplus_{p \geqslant 2} E_3^{p,q}(\pi_{2n}) = \alpha^2 Z_2[\alpha, w_{2j}^2]$. Also, there is a diagram of fibrations:

$$\begin{tikzcd}
BU_n \arrow[r, tail] \arrow[d, "\tau"] & BU_n \times BZ_2 \arrow[r, two heads, "\pi"] \arrow[d, "\theta"] & BZ_2 \arrow[d, "="] \\
BO_{2n} \arrow[r, tail] & X_{2n} \arrow[r, two heads, "\pi_{2n}"'] & BZ_2
\end{tikzcd}$$

where τ includes the fixed set of the involution. Since π is a product, $E_*^{*,*}(\pi)$ collapses. There are homomorphisms

$$\theta_r^*: E_r^{*,*}(\pi_{2n}) \to E_r^{*,*}(\pi) = Z_2[\alpha, c_i : 1 \leqslant i \leqslant n]$$

such that $\theta_2^*(\alpha) = \alpha$, $\theta_2^*(w_{2i-1}) = 0$ and $\theta_2^*(w_{2i}) = c_i$. Hence, θ_3, when restricted to $\oplus_{p \geqslant 2} E_3^{p,q}(\pi_{2n})$, is a monomorphism.

It follows that $\oplus_{p \geqslant 2} E_3^{p,q}(\pi_{2n})$, is in the kernel of d_3 and, moreover, if $x = d_3 y$ then x is in $\oplus_{p \geqslant 2} E_3^{p,q}(\pi_{2n})$ and $\theta_3(x) = d_3(\theta_3(y))$, so that $x = 0$. Hence, $d_3 = 0$. Inducting on r shows that θ_r is monic on $\oplus_{p \geqslant 2} E_r^{p,q}(\pi_{2n})$ and that $d_r = 0$ for all $r \geqslant 3$.

Recall that there is a diagram of fibrations ($k > 2$):

$$\begin{tikzcd}
BO_{2n} \arrow[r, tail] \arrow[d, "="'] & X_{2n} \arrow[r, two heads] \arrow[d, "\theta_{2n}"] & BZ_2 \arrow[d, "\theta'_{2n}"] \\
BO_{2n} \arrow[r, tail] & X_{2n,k} \arrow[r, two heads] & BG_{k-1}
\end{tikzcd}$$

We see that

$$E_2^{*,*}(\pi'_{2n}) \cong Z_2[w_1,\dots,w_{2n}] \otimes Z_2[\alpha,\beta]/(\alpha^2),$$

where the degree of w_i is i, of α is 1 and of β is 2. Clearly θ_{2n}^* is monic on $E_2^{2*,*}$. Hence

$$\theta_{2n}^*(d_2 w_{2i} + \beta w_{2i-1}) = d_2(w_{2i}) + \alpha^2 w_{2i-1} = 0$$

implies that $d_2 w_{2i} = \alpha^2 w_{2i-1}$. Also, $d_2 w_{2i-1} = 0$, so that we have:

PROPOSITION 3.6. *$E_\infty^{*,*}(\pi'_{2n}) \cong Z_2[w_1,\dots,w_{2n-1},w_2^2,\dots,w_{2n}^2,\alpha,\beta]$ module on $\{1,\rho_S\}$, $S = (1 \leqslant i_1 < \cdots < i_k \leqslant n)$ with relations $\alpha^2 = \beta w_{2i-1} = \beta\rho_S = 0$.*

PROOF. Similar to the above.

This computes (the vector space structure of) the Z_2 homology of $X_{n,k}$ and therefore $(\mathfrak{F}$, Free$)$-G_k bordism.

Suppose now that (M,φ) is an almost free G_j action, where $1 \leqslant j < k$. The pair $(M \times_{G_j} G_k, 1 \times G_k)$ is called the extension of (M, φ) to a G_k action and will be denoted by $e_j^k(M,\varphi)$.

LEMMA 3.7. *$e_j^k(M,\varphi)$ is almost free.*

PROOF. Recall that t is the generator of G_k. Suppose that (m, g) is fixed by t^l. Then there is an i such that

$$(m,gt^l) = (mt^{i(2^{k-j})}, gt^{i(2^{k-j})}).$$

Since (M, φ) is an almost free G_j action, i is 2^j or 2^{j-1} so that $i(2^{k-j})$ is 2^k or 2^{k-1}. Since G_k acts freely on G_k, l is 2^k or 2^{k-1}.

Extension is well defined on bordism and gives a homomorphism

$$e_{j*}^k\colon N_*(G_j;\mathfrak{F}, \text{Free}) \to N_*(G_k;\mathfrak{F}, \text{Free}).$$

The following diagram commutes, for $j \leqslant l \leqslant k$,

$$\begin{tikzcd}
N_*(G_j;\mathfrak{F},\text{Free}) \arrow[r, "e_{j*}^l"] \arrow[rr, bend right, "e_{j*}^k"'] & N_*(G_l;\mathfrak{F},\text{Free}) \arrow[r, "e_{l*}^k"] & N_*(G_k;\mathfrak{F},\text{Free})
\end{tikzcd}$$

PROPOSITION 3.8. *The following diagram commutes*:

$$\begin{tikzcd}
N_n(G_{k-1};\mathfrak{F},\text{Free}) \arrow[r, "(e_{k-1}^k)_*"] \arrow[d, "F_{k-1}", "\cong"'] & N_n(G_k;\mathfrak{F},\text{Free}) \arrow[d, "F_k", "\cong"'] \\
\bigoplus_{j=0}^{n} N_{n-j}(X_{j,k-1}) \arrow[r, "\bigoplus(\theta_{j,k})_*"'] & \bigoplus_{j=0}^{n} N_{n-j}(X_{j,k})
\end{tikzcd}$$

where F_k denotes the Z_2 fixed point construction of §2.

PROOF. Recall that F_k is the composite of

$$F\colon N_n(G_k; \mathfrak{F}, \text{Free}) \xrightarrow{\cong} \bigoplus_{j=0}^{n} N_{n-j}(G_{k-1}, \text{Free})(BO_j(\mathbf{C}^\infty, k))$$

and

$$q\colon N_{n-j}(G_{k-1}, \text{Free})(BO_j(\mathbf{C}^\infty, k)) \xrightarrow{\cong} N_{n-j}(X_{j,k}),$$

where F sends an almost free G_k action (M, φ) to the normal bundle of its Z_2 fixed set and where q divides out the resulting free G_{k-1} action. Extension factors through F giving

$$e_F\colon \bigoplus_{j=0}^{n} N_{n-j}(G_{k-2}; \text{Free})(BO_j(\mathbf{C}^\infty, k-1))$$
$$\to \bigoplus_{j=0}^{n} N_{n-j}(G_{k-1}; \text{Free})(BO(\mathbf{C}^\infty, k))$$

where an element of the left-hand side can be considered a (G_{k-1}, j)-plane bundle η covering a G_{k-1} manifold M such that if s generates this action on M then the 2^{k-3} power of s acts freely and the 2^{k-2} power of s is trivial, while if S generates the action on η, the 2^{k-2} power of S acts as -1 in the fibers of η.

If (M, η, G_{k-1}) is such an action, then

$$e_F(M, \eta, G_{k-1}) = (M \times_{G_{k-1}} G_k, \eta \times_{G_{k-1}} G_k, 1 \times G_k).$$

(Note that $t^{2^{k-1}}[m,g] = [m, (t^2)^{2^{k-2}} g] = [s^{2^{k-2}} m, g] = [m,g]$.)

Note that if $f\colon M \to BO_j(\mathbf{C}^\infty, k-1)$ classifies η and $\hat{f}\colon M \times_{G_{k-1}} G_k \to BO_j(\mathbf{C}^\infty, k)$ classifes $\eta \times_{G_{k-1}} G_k$, then, since $\eta \times_{G_{k-1}} G_k$ pulls back via the map

$$M = M \times_{G_{k-1}} G_{k-1} \subset M \times_{G_{k-1}} G_k$$

to η, we know that for g in G_{k-1} in G_k, $\hat{f}(m,g) = f(gm)$.

Hence, $\hat{f}(t(m,g)) = \hat{f}(m, tg) = tf(mg)$.

Let $h\colon (M, G_{k-1}) \to EG_{k-2}$ classify the free action so that $h(sm) = rh(m)$, r being the usual generator of G_{k-2} on S^∞. Let $\hat{h}$ classify the free G_{k-1} action on the extension of M. Then

$$l\colon M \cong M \times_{G_{k-1}} G_{k-1} \subset M \times_{G_{k-1}} G_k \xrightarrow{\hat{h}} EG_{k-1}$$

has the property that

$$l(t^2 m) = \hat{h}(m, t^2) = \hat{h}(t^2(m, 1)) = s^2 \hat{h}(m, 1) = rl(m).$$

Hence $\hat{h}$ restricted to M is h and for g in G_{k-1} in G_k, $\hat{h}(m, tg) = sh(gm)$. Therefore, for g in G_{k-1} in G_k,

$$[\hat{f} \times \hat{h}][m, tg] = [tf(gm), sh(gm)] = [f(gm), h(gm)]$$

in $X_{j,k}$; this is $q \circ e_F$ on (M, η, G_{k-1}) and is a map from

$$g\colon M/G_{k-1} \cong (M \times_{G_{k-1}} G_k)/G_{k-1} \to BO_j(\mathrm{C}^\infty, k) \times_{G_{k-1}} EG_{k-1} \cong X_{j,k}$$

where the map becomes $g[m] = [f(m), h(m)]$. This is just q followed by the projection of $X_{j,k-1}$ to $X_{j,k}$.

Recall that there is an action [1], via twisted products, of free G_k bordism on almost free G_k bordism, which after a Z_2 fixed set construction, is given by sending $(M, G_k) \otimes (N, \eta, G_k)$ to $(M \times_{G_k} N, M \times_{G_k} \eta, G_k \times 1)$. In particular, multiplication by the class of $y_1 = [S^1, \exp(i\pi/2^{k-1}) = t]$ gives an N_* homomorphism

$$y_1\colon N_*(G_{k-1}, \mathrm{Free})(BO_j(\mathrm{C}^\infty, k)) \to N_{*+1}(G_{k-1}, \mathrm{Free})(BO_j(\mathrm{C}^\infty, k)).$$

We have that

$$y_1 \circ e_F(M, \eta, G_{k-1}) = (S^1 \times_{G_{k-1}} M, S^1 \times_{G_{k-1}} \eta, t \times 1).$$

LEMMA 3.9. *If f classifies η into $\gamma^j(\mathrm{C}^\infty, k-1)$ and $\hat{f}$ classifies $S^1 \times_{G_{k-1}} \eta$, then $\hat{f}(s, m) = sf(m)$ in $BO_j(\mathrm{C}^\infty, k)$, for s in S^1.*

PROOF. We must show that

$$\hat{f}^*(\gamma^j(\mathrm{C}^\infty, k)) \cong S^1 \times_{G_{k-1}} \eta = \xi.$$

But

$$E\hat{f}^*(\gamma^j) = \{([s; m], e, x) \text{ in } (E\xi) \times \mathrm{C}^\infty \times BO_j\colon e \text{ is in } x = sf(m)\}.$$

Since $\xi = S^1 \times_{G_{k-1}} f^*(\gamma^j)$,

$$E(\xi) = \{(s, m, e, x) \text{ in } S^1 \times M \times \mathrm{C}^\infty \times BO_j\colon e \text{ is in } x = f(m)\}$$

modulo the relation $(s, m, e, x) \cong (gs, g^{-1}m, g^{-1}e, g^{-1}x)$, for g in G_{k-1}. The map $\Phi\colon \xi \to \hat{f}^*(\gamma^j(\mathrm{C}^\infty, k))$ given by $\Phi(s, m, e, x) = (s, m, se, sx)$ induces the isomorphism.

At the universal level and for $k = 2$, we have

$$\begin{tikzcd} S^1 \times_{Z_2} \gamma^j \arrow[r] \arrow[d] & \gamma^j(\mathrm{C}^\infty, 2) \arrow[d] \\ RP(1) \times BO_j \arrow[r, "\hat{f}"] & BO_j(\mathrm{C}, 2), \end{tikzcd}$$

where $\hat{f}([s],x) = sx$ in $BO_j(\mathrm{C}^\infty, 2)$. Let g classify the free involution on $RP(1) \times BO_j$; we get a map

$$\hat{f} \times g \colon RP(1) \times BO_j \to BO_j(\mathrm{C}^\infty, k) \times EZ_2$$

and a diagram

$$\begin{tikzcd}
BO_j \arrow[r, tail] \arrow[d, "\approx"'] & (RP(1)/Z_2) \times BO_j \arrow[r, two heads] \arrow[d] & RP(1)/Z_2 \arrow[d] \\
BO_j \arrow[r, tail] & X_j \arrow[r, two heads] & BZ_2
\end{tikzcd}$$

We can do a similar construction for $k > 2$. In fact, there is the map

$$S^1 \times_{G_{k-1}} (BO_j(\mathrm{C}^\infty, k-1) \times EG_{k-2}) \to BO_j(\mathrm{C}^\infty, k) \times EG_{k-1}$$

which sends (s, x, e) to $(sx, s^2 e)$, inducing a map

$$\Psi_{jk} \colon (S^1/G_k) \times X_{j,k-1} \to X_{j,k}$$

and a diagram of fibrations:

$$\begin{tikzcd}
S^1 \times_{G_{k-1}} (BO_j(\mathrm{C}^\infty, k-1) \times EG_{k-2}) \arrow[r, tail] \arrow[d] & (S^1/G_k) \times X_{j,k-1} \arrow[r, two heads] \arrow[d, "\Psi_{jk}"] & BG_{k-1} \arrow[d, "="] \\
BO_j(\mathrm{C}^\infty, k) \times EG_{k-1} \arrow[r, tail] & X_{j,k} \arrow[r, two heads] & BG_{k-1}
\end{tikzcd}$$

Our lemma then shows:

Proposition 3.10. *The image of* $y_1 \circ e_F$ *in* $N_*(X_{j,k})$ *is the image of the homomorphism induced on bordism by* Ψ.

Note that there is also a map of double covers:

$$\begin{tikzcd}
X_{j,k-1} \times (S^1/G_{k-1}) \arrow[r, "\Psi'"] \arrow[d] & X_{j,k-1} \arrow[d] \\
X_{j,k-1} \times (S^1/G_k) \arrow[r, "\Psi_{jk}"] & X_{j,k}
\end{tikzcd}$$

Since the j-plane bundle $\eta^j = S^1 \times_{G_{k-1}} (\gamma^j(\mathrm{C}^\infty, k-1))$ covers $S^1 \times_{G_{k-1}} (BO_j(\mathrm{C}^\infty, k-1) \times EG_{k-2})$, this twisted product is homotopy equivalent to the product $(S^1/G_{k-1}) \times BO_j$, where the map is (projection) × (classify η^j) and where the inclusion in $(S^1/G_k) \times X_{j,k-1}$ is the product of $S^1/G_{k-1} \to S^1/G_k$ and the usual map of $BO_j \times EG_{k-2}$ into $X_{j,k-1}$. Our diagram is then just

$$\begin{tikzcd}
(S^1/G_{k-1}) \times (BO_j(\mathbf{C}^\infty, k-1) \times EG_{k-2}) \arrow[r, tail] \arrow[d] & (S^1/G_k) \times X_{j,k-1} \arrow[r, two heads] \arrow[d, "\Psi_{jk}"] & BG_{k-1} \arrow[d, "="] \\
(BO_j(\mathbf{C}^\infty, k) \times EG_{k-1}) \arrow[r, tail] & X_{j,k} \arrow[r, two heads] & BG_{k-1}
\end{tikzcd}$$

where the map on the fiber is just projection.

Moreover, there is a diagram (for j even):

$$\begin{tikzcd}
S^1/G_k \arrow[r, "\psi_{jk}"] \arrow[d, "1 \times (\mathrm{pt})"] & BG_{k-1} \\
(S^1/G_k) \times X_{j,k-1} \arrow[r, "\Psi_{jk}"] & X_{j,k} \arrow[u]
\end{tikzcd}$$

where ψ_{jk} is the quotient of the composite:

$$S^1/Z_2 \to (S^1/Z_2) \times (\mathrm{pt}, 1) \to S^1 \times_{G_{k-1}} (BO_j \times EG_{k-2})$$

$$\to BO_j \times EG_{k-1} \to EG_{k-1}.$$

There are in $H^*(X_{2n,k}; Z_2)$ the following classes:
α in H^1, coming from α in $H^1(BG_{k-1})$, where $\alpha^2 = 0$ if $k > 2$;
τ in H^2, coming from τ in $H^2(BG_{k-1})$, where $\alpha^2 = \tau$ if $k = 2$;
β_{2j-1} in H^{2j-1}, restricting to w_{2j-1} in $H^{2j-1}(BO_{2n})$, $1 \leqslant j \leqslant n$;
θ_{4j} in H^{4j}, restricting to w_{2j}^2 in $H^{4j}(BO_{2n})$, $1 \leqslant j \leqslant n$;
$\rho_{\{i_1 \cdots i_k\}}$ in $H^{2(i_1 + \cdots + i_k) - 1}$, restricting to ρ_S in $H^{2(i_1 + \cdots + i_k) - 1}(BO_{2n})$.

We know that $\tau\beta_{2j-1}$ and $\tau\rho_S$ are given by relations with other terms. Moreover, we can pick β_1 and θ_{4j} unambiguously by taking β_1 to be the first Stiefel-Whitney class of the quotient line bundle

$$\{\det(\gamma^{2n}(\mathbf{C}^\infty, k) \times S^\infty)\}/G_{k-1}$$

over $X_{2n,k}$ and by taking θ_{4j} to be the (Z_2 reduction of) $2j$th Chern class of the quotient $4n$-plane bundle

$$\{(\gamma^{2n}(\mathbf{C}^\infty, k) \otimes_{\mathbf{R}} \mathbf{C}) \times S^\infty\}/G_{k-1}$$

where G_{k-1} acts freely via $(t \otimes t) \times (\exp(i\pi/2^{k-2}))$.

Moreover, we have in the Z_2 cohomology of $X_{2n+1,k} \approx BSO_{2n+1} \times BG_k$, the following classes:
α in H^1 coming from $H^1(BG_k)$, $\alpha^2 = 0$;
τ in H^2 coming from $H^2(BG_k)$;
δ_j in H^j coming from the indecomposable generator in $H^j(BSO_{2n+1})$.

Consider the following classes in $H^*(X_{2n,k}; Z_2)$:

$$\sigma(i,j,k',l',S) = \alpha^i\tau^j\beta_1^{k_1}\cdots\beta_{2n-1}^{k_{2n-1}}\theta_4^{l_1}\cdots\theta_{4n}^{l_n}\rho_S,$$

where $0 \leqslant i$ and j, $k' = (k_1,\ldots,k_{2n-1})$, $l' = (l_1,\ldots,l_n)$ and S is either 0 with $\rho_0 = 1$ or $S = (i_1,\ldots,i_r)$, with $i_1 < \cdots < i_r \leqslant n$. These classes are a Z_2 generating set.

Case 1. $k = 2$. The dual classes $\sigma_*(0,0,k',l',S)$ are a Z_2 basis for the image of $H_*(BO_{2n})$ in $H_*(X_{2n})$. Also, the homomorphism Ψ^* from $H^*(X_{2n})$ to $H^*((RP_1/Z_2) \times BO_{2n})$ sends α to α' in $H^1(RP_1/Z_2)$ and $\sigma(0, 0, k', l', S)$ to $(i_{2n})^*\sigma(0, 0, k', l', S) + \alpha'\omega$, where $i_{2n}: BO_{2n} \longrightarrow X_{2n}$ is the fiber inclusion of π_{2n}. Hence

$$\Psi^*(\sigma(1,0,k',l',S)) = \alpha'((i_{2n})^*(\sigma(0,0,k',l',S))),$$

so that the dual classes $\sigma_*(1,0,k',l',S)$ are in the image of the homology homomorphism, Ψ_*.

Similarly, in $H^*(X_{2n+1,k}; Z_2)$ we have the Z_2 basis:

$$\mu(i,j,k') = \alpha^i\tau^j\delta_2^{k_2}\cdots\delta_{2n+1}^{k_{2n+1}},$$

and, for $k = 2$, the classes $\mu_*(0,j,k')$ generate the image of extension while the classes $\mu_*(1,j,k')$ are in the image of Ψ_*.

If we then pick extensions of Z_2 actions to represent the classes

$$\sigma_*(0,0,k',1',S) \quad \text{and} \quad \mu_*(0,j,k'),$$

say

$$M(0,0,k',l',S) \times_{Z_2} Z_4 \quad \text{and} \quad N(0,j,k') \times_{Z_2} Z_4,$$

$\sigma_*(1,0,k',l',S)$ and $\mu_*(1,j,k')$ can be represented by

$$M(0,0,k',l',S) \times_{Z_2} S^1 \quad \text{and} \quad N(0,j,k') \times_{Z_2} S^1.$$

Moreover, the classes $\sigma_*(i,0,0',l',0)$ can be obtained as follows. Let $l' = (l_1 \leqslant \cdots \leqslant l_n)$ and $M(l') = CP(2l_1) \times \cdots \times CP(2l_n)$. $M(l')$ maps naturally into $BO_{2n}(C^\infty,2)$, classifying the bundle $\lambda(l') = \lambda_{l_1} \times \cdots \times \lambda_{l_n}$ as a $(Z_4 - 2n)$-plane bundle, where λ_{l_j} is the usual complex line bundle over $CP(2l_j)$. Then we get a map of $M(l') \times S^i$ into $BO_{2n} \times EZ_2$ and therefore a map $f(l',i)$: $M(l') \times RP(i) \to X_{2n}$. Then $f(l',i)^*(\sigma(r,0,0',t',0)) = 1$ if and only if $(r,t') = (i,l')$ so that the classes (Disk $(\lambda(l')) \times S^i$, $Z_4 \times (-1)$) generate $N_*(Z_4$; Proper, Free) modulo (image of extension) $\oplus\, y_1 \cdot$ (image of extension).

Let $Q(l',i)$ denote the Z_4 manifold $D(\lambda(l')) \times S^i$. Then $Q(l',2i + 1)$ and S^1

$\times_{Z_4} Q(l', 2i)$ differ by N_* decomposable elements, so that we may also take the set

$$\{[Q(l',2i)], y_1[Q(l',2i)]: l' = (l_1 \leqslant \cdots \leqslant l_n), 0 \leqslant i\}$$

for our remaining generators.

Let K_* denote the image of (All, Free)-Z_2 bordism in (almost free, Free)-Z_4 bordism and let L_* denote the N_* submodule of $N_*(Z_4; \mathfrak{F}, \text{Free})$ generated by the classes $[Q(1', 2i)]$. Then we have shown:

PROPOSITION 3.11. $N_*(Z_4; \mathfrak{F}, \text{Free})$ *is generated by* $K_* \oplus y_1 K_*$ *and* $L_* \oplus y_1 L_*$.

Case 2. $k > 2$. There is the homomorphism

$$\theta^*_{2n,k}: H^*(X_{2n,k}) \to H^*(X_{2n,k-1})$$

with $\theta^*_{2n,k}(\alpha) = 0$, $\theta^*_{2n,k}(\tau) = \tau$ and

$$\theta^*_{2n,k}(\sigma(0,0,k',l',S)) = \sigma(0,0,k',l',S) + \alpha\omega,$$

so that the dual classes, $\sigma_*(0,0,k',l',S)$, are in the image of the homology homomorphism.

Also, $\Psi^*_{2n,k}$ sends α to α' in $H^1(S^1/G_k)$, so that the classes $\sigma_*(1,0,k',l',S)$ are in the image of Ψ_*. We see that, since the G_{k-1} action $(D(\lambda(l')) \times S^{2i+1}, G_{k-1} \times G_{k-2})$ is a reduction of a G_k action, it extends to zero. On the other hand,

$$\begin{aligned}
&((D(\lambda(l')) \times S^{2i}, Z_4 \times (-1)) \times_{Z_4} G_k) \times_{G_k} S^1 \\
&\quad \cong (D(\lambda(l')) \times S^{2i}) \times_{Z_4} S^1 \\
&\quad \cong (D(\lambda(l')) \times ((S^{2i}, -1) \times_{Z_2} (S^1, -1)), G_k \times G_{k-1})
\end{aligned}$$

(via the identification:

$$[(x,y),s] \to (sx,[y,s]);\ x \text{ in } D(\lambda(l')), y \text{ in } S^{2i}, s \text{ in } S^1),$$

which differs from $(D(\lambda(l')) \times S^{2i+1}, G_k \times G_{k-1})$ by N_* decomposables. Hence we have

PROPOSITION 3.12. $N_*(G_k; \mathfrak{F}, \text{Free})$ *is generated by*

(i) K_*, *the image of* $N_*(Z_2; \text{All, Free})$;

(ii) $y_1 K_*$;

(iii) L_*, *the submodule generated by the classes*

$$[(D(\lambda(l')) \times S^{2i}; Z_4 \times -1) \times_{Z_4} G_k, 1 \times G_k];$$

(iv) $y_1 L_*$.

PROOF. We need only check (iii). The Z_2 fixed set information for the class $(D(\lambda(l')) \times S^{2i}) \times_{Z_4} G_k$ is the usual map of the product

$$CP(2l_1) \times \cdots \times CP(2l_n) \to BU_1 \times \cdots \times BU_1 \to BU_n \to BO_{2n}$$

with the principal G_{k-1} bundle $S^{2i} \times_{Z_2} G_{k-1} \to S^{2i}/Z_2$. The double cover of BG_{k-1} by BG_{k-2} pulls back to S^{2i} over $RP(2i)$ and the complex line bundle over BG_{k-1} pulls back to twice the twisted real line bundle over $RP(2i)$. Hence, if $f_{l',2i}$ denotes the map of $CP(l') \times RP(2i)$ into $X_{2n,k}$, then

$$f^*_{r',2j}(\sigma(0,i,0',l',0)) = 1 \quad \text{if and only if } (r',j) = (l',i),$$

so that these classes do, in fact, give the remaining generators.

4. **Free G_k bordism.** In [3] it is shown that free unoriented G_k bordism is freely generated as an N_* module by extensions of Z_2 actions on even spheres and the restrictions of circle actions on odd spheres. Let y_i in $N_i(G_k;\text{ Free})$ be defined as in §2. Let J_* be the algebra generated by y_1, so that J_* is the direct sum of $N_* \cdot 1$ and $N_* \cdot y_1$. Let K_* be the image of the extension homomorphism, an algebra morphism,

$$e\colon N_*(Z_2;\text{ Free}) \to N_*(G_k;\text{ Free}).$$

(Recall that if $\theta = [M, Z_2]$, then $e(\theta) = [M \times_{Z_2} G_k, 1 \times G_k]$.)

PROPOSITION 4.1. *$N_*(G_k;$ Free$) \cong J_* \otimes_{N_*} K_*$, as algebras.*

PROOF. Let Δ denote the N_* epimorphism,

$$\Delta\colon N_*(G_k;\text{ Free}) \to N_{*-2}(G_k;\text{ Free})$$

defined in [3, 1.11] or in [5]. Rowlett (in [5]) shows that J_* is the kernel of Δ so that

$$0 \to J_* \xrightarrow{i^*} N_*(G_k;\text{ Free}) \to N_{*-2}(G_k;\text{ Free}) \to 0$$

is exact, where i^* is the inclusion of algebras. Lemmas 1.12, 1.13 and Theorem 1.22 of [3] show that Δ is a J_* module homomorphism.

An explicit splitting for Δ can be described as follows. Let ρ denote the forgetful homomorphism from G_k to Z_2 bordism and let x_1 denote the class of the circle with antipodal involution. Clearly, $\rho(y_1 e(w)) = x_1 w$. Recall that there are classes x_i in free Z_2 bordism defined by

(i) $x_0 = 1$;
(ii) $[S^k, -1] = \sum_{j=0}^{k} [RP(k-j)]x_j$; let $y'_k = [S^k, -1]$.

They are also characterized by:

(i) $x_0 = 1$;
(ii) $\Delta' x_{j+1} = x_j$, where Δ' is the Smith homomorphism of [2];
(iii) x_j augments to zero in N_*.

Then $x_{2j+1} = x_1 y'_{2j}$ (see [7]), so that

$$\rho(y_{2k+1}) = y'_{2k+1} = x_1\left(\sum_{j=0}^{k} [RP(2k-2j)]y'_{2i}\right)$$

$$= \rho\left(y_1 \sum_{j=0}^{k} [RP(2k-2j)]y_{2j}\right) \quad [3, 1.17].$$

Hence, $y_{2k+1} + y_1 y_{2k}$ is N_* decomposable. Let $b_{2k} = e(x_{2k})$. Then the set $\{b_{2i}, y_1 b_{2i}: i = 0, 1, \dots\}$ is a free N_* basis for $N_*(G_k$; Free). Define μ by $\mu(b_{2i}) = b_{2i+2}$ and $\mu(y_1 b_{2i}) = y_1 b_{2i+2}$ and extending N_* linearly. Then μ is J_* linear, and since

$$\left(\sum_{j=0}^{\infty} [RP(2j)]\right)\left(\sum_{k=0}^{\infty} x_{2k}\right) = \left(\sum_{i=0}^{\infty} y_{2i}\right),$$

μ is a right inverse for Δ. Using a Five-Lemma argument, we see that $N_*(G_k$; Free) is freely generated over J_* by the b_{2i}'s.

Since K_* is also freely generated (over N_*) by the b_{2i}'s, we see that $J_* \otimes_{N_*} K_*$ and $N_*(G_k$; Free) have the same Z_2 rank, degree by degree. Moreover, the evident algebra map from $J_* \otimes_{N_*} K_*$ to $N_*(G_k$; Free) is clearly an epimorphism and therefore an isomorphism.

5. **Algebraic relations.** We know that, as algebras,

$$N_*(Z_2; \text{All, Free}) \cong N_*(\alpha_1, \beta_2, \alpha_3, \beta_4, \dots),$$

where α_{i+1} is the class of the manifold $S^i \times_{Z_2} D^1$, D^1 being the unit interval, with -1×1 involution and $\beta_{2i+2} = x_1 \alpha_{2i+1}$, so that

$$\beta_{2i+2} = [(S^1 \times_{Z_2} S^{2i}) \times_{Z_2} D^1, -1 \times 1 \times 1].$$

There is the restriction homomorphism

$$\rho_k: N_*(G_k; -, -) \to N_*(Z_2, -, -)$$

and the extension homomorphism (as before). The restriction map remembers only the Z_2 in G_k; both ρ and e are defined for arbitrary families of subgroups.

Note that extension preserves a twisted product but not a cartesian product and therefore is an algebra morphism only on free bordism (as well as an $N_*(Z_2$; Free) module morphism on all the groups). Similarly, restriction is an algebra morphism only off of free bordism. We do however have the following:

LEMMA 5.1

(i) $e(x(\rho(y))) = y(e(x))$ *for all* x *in* $N_*(Z_2, \text{All}, -)$ *and* y *in* $N_*(G_k, \mathfrak{F}, -)$.

(ii) $\rho(y_1(e(x))) = x_1 x$ *for* x *in* $N_*(Z_2: \text{All}, -)$, $y_1 = [S^1, t]$ *and* $x_1 = [S^1, -1]$.

PROOF. Let $(M,\ T)$ be a Z_2 action and (N, G_k) a G_k action. Define $f\colon (M \times N) \times_{Z_2} G_k \to (M \times_{Z_2} G_k) \times N$ by $f[(m,n),g] = ([m,g],gn)$. f is an equivariant diffeomorphism. This gives i.

To get (ii), just note $S^1 \times_{G_k} (G_k \times_{Z_2} M) = S^1 \times_{Z_2} M$.

Let $\delta_{i_1} \dots {}_{i_l} = e(\alpha_{i_1} \cdots \alpha_{i_l})$, for $1 \leqslant i_1 \leqslant \cdots \leqslant i_l$ and all i_j odd. Let $\varepsilon_{i_1} \dots {}_{i_l} = y_1 \delta_{i_1} \dots {}_{i_l}$ and $\delta_{i+1} = y_1 \delta_i$, so that $\varepsilon_i = \delta_{i+1}$.

LEMMA 5.2. *$e(\alpha_{i_1} \cdots \alpha_{i_l} \beta_{j_1} \cdots \beta_{j_k}) = \delta_{i_1} \dots {}_{i_l} \delta_{j_1} \cdots \delta_{j_k}$, for all j_r even and i_r odd.*

PROOF. We can assume $j_r \geqslant 2$. Then

$$\begin{aligned} \rho(\delta_{j_1} \cdots \delta_{j_k}) &= \rho(\delta_{j_1}) \cdots \rho(\delta_{j_k}) \\ &= \rho(y_1 e(\alpha_{j_1 - 1})) \cdots \rho(y_1 e(\alpha_{j_k - 1})) \\ &= (x_1 \alpha_{j_1 - 1}) \cdots (x_1 \alpha_{j_k - 1}) = \beta_{j_1} \cdots \beta_{j_k}, \end{aligned}$$

so that

$$\begin{aligned} e(\alpha_{i_1} \cdots \alpha_{i_l} \beta_{j_1} \cdots \beta_{j_k}) &= e(\alpha_{i_1} \cdots \alpha_{i_l} \rho(\delta_{j_1} \cdots \delta_{j_k})) \\ &= e(\alpha_{i_1} \cdots \alpha_{i_l}) \delta_{j_1} \cdots \delta_{j_k} \\ &= (\delta_{i_1} \dots {}_{i_l}) \delta_{j_1} \cdots \delta_{j_k}. \end{aligned}$$

Let K_* and L_* be as in 3.12.

PROPOSITION 5.3.
(i) *The product in K_* is zero*;
(ii) *$y_1 K_*$ is an algebra*;
(iii) *If x is in $L_* \oplus y_1 L_*$, then the product of x with any other class is zero.*

PROOF.
(i) $e(z)e(w) = e(z\rho(e(w))) = e(z \cdot 0) = 0$.
(ii) $(y_1 e(z))(y_1 e(w)) = y_1 e(z\rho(y_1 e(w)))$.
(iii) Let (M, G_k) be a $(\mathfrak{F}$, Free) action and consider

$$N = ((D(\lambda(l')) \times S^{2i}, Z_4 \times -1) \times_{Z_4} G_k) \times (M, G_k).$$

We note that the boundary of

$$((D(\lambda(l')) \times D^{2i+1}, Z_4 \times -1) \times_{Z_4} G_k) \times (M, G_k)$$

is a smooth union of N,

$$P = ((S(\lambda(l')) \times D^{2i+1}) \times_{Z_4} G_k) \times M$$

and

$$Q = ((D(\lambda(l')) \times D^{2i+1}) \times_{Z_4} G_k) \times (\text{boundary of } M).$$

Since Z_4 acts freely on $S(\lambda(l'))$, G_k acts freely on P. Since G_k acts freely on the boundary of M, Q also has a free action. Therefore, $((D(\lambda(l')) \times D^{2i+1}) \times_{Z_4} G_k) \times M$ provides a ($\mathfrak{F}$, Free) bordism of N to zero.

If x is in L_* and y is any class, then $xy = 0$. Hence,

$$0 = y_1(xy) = (y_1 x)y + x(y_1 y) = (y_1 x)y.$$

We have used here the fact that y_1 is primitive in $N_*(G_k$, Free), see [1].

PROPOSITION 5.4. *The kernel of the extension homomorphism from Z_2 bordism to G_k bordism is the image of multiplication by x_1, the class of the circle with antipodal involution.*

PROOF. Since e is a $N_*(Z_2$; Free) module morphism, the kernel of e contains the image of (multiplication by the class of) x_1. If $e(x) = 0$, then

$$x_1 x = \rho(y_1 e(x)) = 0,$$

so that the kernel of e is in the kernel of x_1. But in $N_*(Z_2$; All, Free) any x in the kernel of x_1 is of the form $a + x_1 b$ where a is an N_* linear combination of squares $(\alpha_{i_1} \cdots \alpha_{i_r})^2$, see [1]. But these squares extend to N_* linearly independent elements $((\alpha_{i_1} \cdots \alpha_{i_r})^2$ extends to $D(\lambda_{i_1-1} \times \cdots \times \lambda_{i_r-1}) \times Z_2, Z_4 \times -1)$, so that if $e(x) = 0$, then $x = x_1 b$ in $N_*(Z_2$; All, Free). But there is an $N_*(Z_2$; Free) splitting homomorphism for the inclusion of $N_*(Z_2$; All) in $N_*(Z_2$; All, Free), so that the result holds in $N_*(Z_2$; All) also.

In particular then, there is the "relation":

$$\delta_{i_1 i_1 \ldots i_l i_l} \delta_{j_1} \cdots \delta_{j_k} = 0.$$

Less trivially, we also have relations

$$\sum_{j=1}^{l} \delta_{i_j - 1} \delta_{i_1 \ldots \hat{i}_j \ldots i_l} = 0,$$

obtained by expanding $e(\rho(y_1(\alpha_{i_1} \cdots \alpha_{i_l})))$. Also, since $(\delta_{i_1 \ldots i_k})(\delta_{j_1 \ldots j_l}) = 0$, we have

$$(y_1 \delta_{i_1 \ldots i_k})(\delta_{j_1 \ldots j_l}) = (y_1 \delta_{j_1 \ldots j_l})(\delta_{i_1 \ldots i_k})$$

or

$$\varepsilon_{i_1 \ldots i_k} \delta_{j_1 \ldots j_l} = \varepsilon_{j_1 \ldots j_l} \delta_{i_1 \ldots i_k}.$$

Since

$$\begin{aligned}\varepsilon_{i_1\cdots i_k}\delta_{j_1\cdots j_l} &= e(\alpha_{i_1}\cdots\alpha_{i_k}x_1(\alpha_{j_1}\cdots\alpha_{j_l}))\\ &= \sum_{r=1}^{l}\delta_{j_r-1}\delta_{i_1\cdots i_kj_1\cdots\hat{j}_r\cdots j_l},\end{aligned}$$

we also have that

$$\sum_{r=1}^{l}\delta_{j_r-1}\delta_{i_1\cdots i_kj_1\cdots\hat{j}_r\cdots j_l} = \sum_{s=1}^{k}\delta_{i_s-1}\delta_{i_1\cdots\hat{i}_s\cdots i_kj_1\cdots j_l}.$$

These are all relations in K_*. In fact the product

$$K_*\otimes y_1K_*\xrightarrow{\text{multiply}}N_*(G_k;\mathfrak{F},\text{Free})$$

has image in K_*, since $(y_1e(x))e(z) = e(z\rho(y_1e(x)))$.

Note that there is a decomposition:

$$N_*(G_k;\mathfrak{F},\text{Free})\cong(K_*+L_*)\oplus(y_1K_*+y_1L_*)$$

and that ρ is zero on K_*, L_* and y_1L_*. Hence

PROPOSITION 5.5. *The image of restriction from almost free G_k bordism to Z_2 bordism is the image of multiplication by the class of the circle with antipodal involution, x_1, and hence the sequence*

$$N_*(G_k;\mathfrak{F})\xrightarrow[\rho]{}N_*(Z_2,\text{All})\xrightarrow[e]{}N_*(G_k;\mathfrak{F})$$

is exact.

PROOF. $\rho(y_1e(x)) = x_1x$ proves the first statement while the second follows from the existence of a diagram:

$$\begin{tikzcd}
0 \arrow[r] & N_*(G_k;\mathfrak{F}) \arrow[r, "i'"] \arrow[d] & N_*(G_k;\mathfrak{F},\text{Free}) \arrow[r] \arrow[d] & N_*(G_k;\text{Free}) \arrow[r] & 0\\
0 \arrow[r] & N_*(Z_2;\text{All}) \arrow[r, shift left, "i"] & N_*(Z_2;\text{All},\text{Free}) \arrow[l, shift left, "s"] \arrow[r] & N_*(Z_2;\text{Free}) \arrow[r] & 0
\end{tikzcd}$$

where s is an $N_*(Z_2;\text{Free})$ module splitting homomorphism for i.

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