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ON THE SEQUENCE SPACES $l_{(p_n)}$ AND $\lambda_{(p_n)}$, $0 < p_n \leqslant 1$

BY

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ABSTRACT. Let (p_n) and (q_n) be sequences in the interval $(0, 1]$, let $l_{(p_n)}$ be the set of all real sequences (x_n) such that $\sum |x_n|^{p_n} < \infty$, and let $\lambda_{(q_n)}$ be the set of all real sequences (y_n) such that $\sup_\pi \sum |y_n|^{q_{\pi(n)}} < \infty$ where the sup is taken over all permutations π of the positive integers. The purpose of this paper is to investigate some of the properties of these spaces. Our results are primarily concerned with (1) conditions which are necessary and/or sufficient for $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) to equal $l_{(q_n)}$ (resp., $\lambda_{(q_n)}$), and (2) isomorphic and topological properties of the subspaces of these spaces.

In connection with (1), we show that the following four conditions are equivalent for any sequence (ε_n) which decreases to zero and has $\varepsilon_1 < 1$. (a) There exists a number $K > 1$ such that the series $\sum 1/K^{1/\varepsilon_n}$ converges; (b) the elements ε_n of the sequence satisfy the condition $\varepsilon_n = O(1/\ln n)$; (c) the sequence $((\ln n)((1/n)\sum_1^n \varepsilon_j))$ is bounded; and (d) $l_{(1-\varepsilon_n)}$ equals l_1. In connection with (2), we show that the following are true when (p_n) increases to one. (a) $\lambda_{(p_n)}$ contains an infinite-dimensional closed subspace where the $l_{(p_n)}$-topology and the $\lambda_{(p_n)}$-topology agree; (b) $l_{(p_n)}$ and $\lambda_{(p_n)}$ contain closed subspaces isomorphic to l_1; and (c) $\lambda_{(p_n)}$ contains no infinite-dimensional subspace where the $\lambda_{(p_n)}$-topology agrees with the l_1-topology if and only if

$$\lim((1/n)^{p_1} + (1/n)^{p_2} + \cdots + (1/n)^{p_n}) = \infty.$$

1. Introduction and summary. If (p_n) is a sequence of numbers in the interval $(0, 1]$, the space $l_{(p_n)}$ is the set of all real sequences $x = (x_n)$ such that $\|x\|_{(p_n)} = \sum |x_n|^{p_n}$ is finite, and the space $\lambda_{(p_n)}$ is the set of all real sequences $y = (y_n)$ such that $|y|_{(p_n)} = \sup_\pi \sum |y_{\pi(n)}|^{p_n}$ is finite where the supremum is taken over the set of all permutations π of the natural numbers. The $l_{(p_n)}$-spaces have been used or studied in many places, e.g., in [2], [7], [8], [10] and [11]; and the $\lambda_{(p_n)}$-spaces, which are nonlocally convex analogues of the symmetric sequence spaces studied in [1], [5], and [6], have been used in [8] and [9].

The purpose of this paper is to investigate a few of the properties of $l_{(p_n)}$ and $\lambda_{(p_n)}$, and we summarize now some of our results. Assume for the

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remainder of this section that $0 < p_n < q_n \leqslant 1$. We first generalize some of the results obtained in [10]. In particular, we show that $l_{(p_n)}$ equals $l_{(q_n)}$ if and only if there exists a positive number K such that the series $\sum 1/K^{p_n/(q_n-p_n)} < \infty$. We then use this result to help us show that $l_{(p_n)}$ equals $l_{(q_n)}$ when the sequence $(n(1/n)^{p_n/q_n})$ is bounded, and $l_{(p_n)}$ does not equal $l_{(q_n)}$ when $\lim n(1/n)^{p_n/q_n} = \infty$. If (p_n) increases to p, and $j(n) < k(n) < j(n+1)$, then $l_{(p_{j(n)})}$ equals $l_{(p_{k(n)})}$ (while it is not necessarily true that $l_{(p_n)}$ equals $l_{(p_{2n})}$), and $l_{(p_n)}$ equals l_p if and only if the sequence $(n(1/n)^{a_n/p})$ is bounded where a_n is the arithmetic mean given by $a_n = (1/n)(p_1 + \cdots + p_n)$.

When (p_n) increases to p, we write $l_{(p_n)} \neq^s l_p$ if and only if

$$\lim((1/n)^{p_1/p} + (1/n)^{p_2/p} + \cdots + (1/n)^{p_n/p}) = \infty.$$

Using this definition, we show that when $p = 1$, $l_{(p_n)} \neq^s l_1$ if and only if there does not exist any infinite-dimensional subspace of $\lambda_{(p_n)}$ on which the $\lambda_{(p_n)}$-topology and the l_1-topology agree. We show that every closed infinite-dimensional subspace of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) contains an isomorphic copy of l_p when (p_n) increases to p, and $\lambda_{(p_n)}$ always contains an infinite-dimensional subspace where the $\lambda_{(p_n)}$ and the $l_{(p_n)}$ topologies agree when (p_n) increases to p. We also show that when $\lambda_{(p_n)}$ is not equal to $\varnothing$ or to l_1, then $\lambda_{(p_n)}$ is not locally convex, and $l_{(p_n)}$ contains no locally bounded infinite-dimensional subspaces if and only if $\lim p_n = 0$. Finally, we show that any time $\lambda_{(p_n)}$ is isomorphic to $\lambda_{(q_n)}$, where (p_n) and (q_n) increase to one, and $l_{(q_n)} \neq^s l_1$, then $\lambda_{(p_n)}$ must equal $\lambda_{(q_n)}$.

2. **Notation.** In addition to the terminology used in §1, we find it convenient to use the following notation and conventions. The set R_p, $0 < p \leqslant 1$, is the collection of all sequences (x_n) of real numbers in $(0,p)$ which increase to p. The letter R denotes R_1. Unless otherwise stated, the letters p and q will always represent numbers in the interval (0, 1]. The symbol $\varnothing$ will represent the space of all finitely nonzero sequences of real numbers equipped with the strongest vector topology. The vector e_n is the vector $(0,\ldots,0,1,0,\ldots)$ where the nonzero entry is in the nth position. A block basic sequence $\{z_n\}$ is a sequence of nonzero vectors of the form $z_n = \sum_{i=k_{n-1}}^{k_n} a_i e_i$ where (n_k) is a strictly increasing sequence of nonnegative integers. The notation $[x_n]_\tau$ denotes the τ-closed linear span of the sequence $\{x_n\}$. The letter E represents the set of all real numbers, and the space $E \oplus l_{(p_n)}$ is the space of all sequences $(x_0, x_1, \ldots)$ such that $x_0 \in E$ and $(x_1, x_2, \ldots) \in l_{(p_n)}$. The equality $(x_n) = O(y_n)$ means that $|x_n| \leqslant M|y_n|$ for some M. For $0 < p \leqslant 1$, l_p has the usual meaning and $\|x\|_p$ is $\sum_1^\infty |x_n|^p$ when $x = (x_1, x_2, \ldots)$. Finally, $\| \ \|_\infty$ is the usual l_∞-norm of a bounded sequence.

3. **Main results.** Before proving our first theorem, we make some observations. For a fixed p, $0 < p \leqslant 1$, there is an uncountable number of distinct

spaces $l_{(p_n)}$ such that (p_n) is in R_p. Hence the $l_{(p_n)}$ spaces occur in great abundance. If $\inf p_n > 0$, then $l_{(p_n)}$ is locally bounded and a set is bounded in $l_{(p_n)}$ if and only if it is metrically bounded (cf. [10]). Also if (p_n) is an enumeration of the rational numbers in (0, 1), then $l_{(p_n)}$ contains a complemented isomorphic copy of each $l_{(q_n)}$. Hence $l_{(p_n)}$ is universal, in the terminology of [3], for the class $\{l_{(q_n)}: 0 < q_n \leqslant 1\}$. Our first theorem is a standard type result which is useful in the following.

PROPOSITION 1. *Let X be an infinite-dimensional closed subspace of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) where $0 < p_n \leqslant 1$ and $\inf p_n > 0$. Then X contains an infinite-dimensional subspace Y which is $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) isomorphic to a subspace Z of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) where the subspace Z is the closed linear span of a block basic sequence.*

PROOF. Let $\{x_j\}$ be a sequence of linearly independent elements of X such that $\|x_j\|_{(p_n)} = 1$. By taking linear combinations and normalizing if necessary, we can assume that $x_n = (0, \ldots, 0, x^n_{k_{n-1}+1}, x^n_{k_{n-1}+2}, \ldots)$ where (k_n) is a strictly increasing sequence of nonnegative integers such that

$$\|(0, \ldots, 0, x^n_{k_n+1}, x^n_{k_n+2}, \ldots)\|_{(p_n)} < \varepsilon_n.$$

Since $l_{(p_n)}$ is locally bounded when $\inf p_n > 0$, we can apply Theorem 1′ of [4]. This theorem implies that $[x_n]_{(p_n)}$ is isomorphic to $[y_n]_{(p_n)}$ if (ε_n) is chosen sufficiently small and $y_n = (0, \ldots, 0, x^n_{k_{n-1}}, \ldots, x^n_{k_n}, 0, \ldots)$; and the isomorphism is the natural mapping taking x_n to y_n for each n.

Since $\lambda_{(p_n)}$ is also locally bounded when $\inf p_n > 0$, the proof just given for $l_{(p_n)}$ also applies to $\lambda_{(p_n)}$. □

The following theorem is a generalization of a result in [10]. Since the proof is similar, it will be omitted.

THEOREM 2. *Suppose that $0 < p_n < q_n \leqslant 1$. Then $l_{(p_n)}$ is equal to $l_{(q_n)}$ if and only if there exists a number $K > 1$ such that $\sum (1/K^{p_n/(q_n-p_n)}) < \infty$ (equivalently $\sum (1/K^{q_n/(q_n-p_n)}) < \infty$).*

COROLLARY 3. *Suppose that $0 < p_n < q_n \leqslant 1$ and $\inf p_n > 0$. Then $l_{(p_n)}$ is equal to $l_{(q_n)}$ if and only if there exists a number $K > 1$ such that $\sum (1/K^{1/(q_n-p_n)})$ $< \infty$.*

COROLLARY 4. *Suppose (p_n) is in R_p, and suppose $(j(n))$ and $(k(n))$ are increasing sequences of positive integers such that $j(n) < k(n) \leqslant j(n+1)$. Then $l_{(p_{j(n)})} = l_{(p_{k(n)})}$ (and hence $\lambda_{(p_{j(n)})} = \lambda_{(p_{k(n)})}$).*

PROOF. Let b_n be defined by $p_{k(n)} = p_{j(n)} + 1/b_n$. Since $(p_n) \in R_p$, it follows that $\sum_{n=1}^{\infty} (1/b_n) < \infty$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{2^{1/(p_{k(n)}-p_{j(n)})}} = \sum \frac{1}{2^{b_n}} < \infty.$$

Hence Corollary 3 implies that $l_{(p_{j(n)})} = l_{(p_{k(n)})}$.

It is easy to see that $\lambda_{(r_n)}$ equals $\lambda_{(s_n)}$ when $l_{(r_n)}$ equals $l_{(s_n)}$ because $|x|_{(r_n)} = \infty$ implies $\sum |x_{\pi(n)}|^{r_n} = \infty$ for some π. □

If (p_n) is not required to be in R_p, the conclusion of Corollary 4 does not necessarily follow. For example, consider the following: Let $p_n = 1/2^n, j(n) = n$, and $k(n) = n + 1$. Then $l_{(p_n)} \neq l_{(p_{n+1})}$ follows from Theorem 2. Note also that for this choice of the sequence (p_n), $l_{(p_n)}$ is not equal to $E \oplus l_{(p_n)}$. (However, $l_{(p_n)}$ must be equal to $E \oplus l_{(p_n)}$ when $(p_n) \in R_p$, and $\lambda_{(p_n)}$ must be equal to $E \oplus \lambda_{(p_n)}$ always.)

We will show next that there are $l_{(p_n)}$ spaces where (p_n) is in R_p such that $l_{(p_n)} \neq l_{(p_{2n})}$. This is perhaps somewhat surprising in view of Corollary 4.

THEOREM 5. *There exists a sequence (p_n) in R such that $l_{(p_n)}$ is not equal to $l_{(p_{2n})}$.*

PROOF. Choose a sequence (b_n) of positive numbers such that $\sum_{n=0}^{\infty} (1/b_n) = B < 1$ and $\sum_{n=0}^{\infty} (2^n/K^{b_n})$ diverges for $K = 1, 2, 3, \ldots$. Let $p_1 = 1 - B$, let $p_{2^{k+1}} = p_{2^k} + 1/b_k$ for $k = 0, 1, 2, \ldots$, and let $p_n = p_{2^k}$ for $2^k \leqslant n < 2^{k+1}$, $k = 0, 1, 2, \ldots$. Then for $K > 1$, the series $\sum_{n=1}^{\infty} (1/K^{1/(p_{2n}-p_n)})$ equals the series $\sum_{n=0}^{\infty} (2^n/K^{b_n})$, and hence diverges. Corollary 3 implies that $l_{(p_n)}$ is not equal to $l_{(p_{2n})}$. □

THEOREM 6. *Suppose $0 < p_n < q_n \leqslant 1$ for $n = 1, 2, \ldots$. If $(n(1/n)^{p_n/q_n})$ is a bounded sequence, then $l_{(p_n)}$ is equal to $l_{(q_n)}$. If $\lim n(1/n)^{p_n/q_n} = \infty$, then $l_{(p_n)}$ is not equal to $l_{(q_n)}$.*

PROOF. Let $M_n = n(1/n)^{p_n/q_n}$. Then $p_n/q_n = 1 - (\ln M_n)/(\ln n)$. Hence

$$q_n/(q_n - p_n) = (\ln n)/(\ln M_n).$$

Thus $\sum 1/(K^{q_n/(q_n-p_n)}) = \sum 1/K^{(\ln n)/(\ln M_n)})$. If r is chosen such that $K = e^r$, this last series becomes $\sum 1/(n^{r/\ln M_n})$. The result now follows from Theorem 2.□

It is easy to see and will prove useful to note now that one can actually construct a sequence (p_n) in R_p such that $\lim n(1/n)^{p_n/p} = \infty$.

COROLLARY 7. *If $(p_n) \in R_p$, $0 < p \leqslant 1$, and if the sequence $\{x_n\}$ is $l_{(p_n)}$-bounded where*

$$x_n = (\underbrace{(1/n)^{1/p}, \ldots, (1/n)^{1/p}}_{n \text{ terms}}, 0, \ldots)$$

then $l_{(p_n)}$ is equal to l_p.

PROOF. The conditions imply that $\sup n(1/n)^{p_n/p} < \infty$, and hence the corollary follows from Theorem 6. □

The following example shows that the boundedness condition of Theorem 6 is not necessary.

EXAMPLE. There are sequences (p_n) and (q_n) such that $0 < p_n < q_n \leqslant 1$ for which the sequence $(n(1/n)^{p_n/q_n})$ is not bounded but $l_{(p_n)}$ is equal to $l_{(q_n)}$.

PROOF. One can select the sequences (p_n) and (q_n) and a strictly increasing sequence (n_k) of positive integers such that, in the notation of the proof of Theorem 6, $M_{n_k} \geqslant k$ while $\sum (1/n^{r/\ln M_n})$ converges for some $r > 0$. □

It is not possible to construct an example like the one above when $q_n = q$ for all n and (p_n) is in R_q. This is true because the inequality $n(1/n)^{p_n/q} \leqslant (1/n)^{p_1/q} + (1/n)^{p_2/q} + \cdots + (1/n)^{p_n/q}$ implies that the sequence $\{((1/n)^{1/q}, (1/n)^{1/q}, \ldots, (1/n)^{1/q}, 0, \ldots)\}$ with n nonzero terms would not be $l_{(p_n)}$-bounded while the members of this sequence are l_q-normalized.

The preceding discussion suggests the following question: Suppose (p_n) and (q_n) are in R_p and $p_n \leqslant q_n$ for all n. If there exists a number M such that $\|x\|_{(q_n)} = 1$ implies $\|x\|_{(p_n)} \leqslant M$ for all x of the form $x = (r, r, \ldots, r, 0, \ldots)$, must $\lambda_{(p_n)}$ equal $\lambda_{(q_n)}$?

THEOREM 8. *Suppose that* $0 < p_n < q_n \leqslant 1$. *Then* $l_{(p_n)}$ *equals* $l_{(q_n)}$ *if and only if there exists a permutation* π *of the positive integers such that* $(q_{\pi(n)} - p_{\pi(n)})/p_{\pi(n)} = O(1/\ln n)$.

PROOF. Suppose there exists a permutation π having the above property. We can assume that π is the identity. Then there exists an M such that $(q_n - p_n)/p_n \leqslant M/\ln n$. Thus

$$p_n/(q_n - p_n) \geqslant (\ln n)/M$$

and

$$\sum 1/(K^{p_n/(q_n - p_n)}) \leqslant \sum 1/(K^{(\ln n)/M}) \leqslant \sum 1/(n^{(\ln n)/M}).$$

Hence the series $\sum 1/(K^{p_n/(q_n - p_n)})$ converges for all large K. This implies, by Theorem 2, that $l_{(p_n)} = l_{(q_n)}$.

Conversely, if $l_{(p_n)}$ equals $l_{(q_n)}$, there exists a $K > 1$ such that the series $\sum 1/(K^{p_n/(q_n - p_n)})$ converges by Theorem 2. Choose a permutation π such that the terms of the series $\sum 1/(K^{p_{\pi(n)}/(q_{\pi(n)} - p_{\pi(n)})})$ are in decreasing order. Then

$$\lim n(1/(K^{p_{\pi(n)}/(q_{\pi(n)} - p_{\pi(n)})})) = 0.$$

Thus $\ln n - (p_{\pi(n)}/(q_{\pi(n)} - p_{\pi(n)}))(\ln K) < 0$ for all large values of n. This implies that $(q_{\pi(n)} - p_{\pi(n)})/p_{\pi(n)} \leqslant (\ln K)/(\ln n)$ for all large values of n. □

We omit the proof by Corollary 9 and Corollary 10 because these proofs are implicitly contained in [10].

COROLLARY 9. *If* $(p_n) \in R_p$, *then* $l_{(p_n)}$ *equals* l_p *if and only if* $(p - p_n) = O(1/\ln n)$.

COROLLARY 10. *If* $0 < p_n \leqslant q_n \leqslant 1$ *and if* $\sum (q_n/p_n - 1) < \infty$, *then* $l_{(p_n)}$ *equals* $l_{(q_n)}$.

PROPOSITION 11. *Suppose* $(p_n) \in R_p$ *and* $a_n = (1/n)(p_1 + p_2 + \cdots + p_n)$; *then* $l_{(p_n)} = l_p$ *if and only if the sequence* $(n(1/n)^{a_n/p})_{n=1}^{\infty}$ *is bounded.*

PROOF. It is a well-known fact that the geometric mean is less than or equal to the arithmetic mean, i.e., $(c_1 c_2 \cdots c_n)^{1/n} \leqslant (1/n)(c_1 + c_2 + \cdots c_n)$. If we let $c_k = (1/n)^{p_k/p}$, $k = 1, 2, \ldots, n$, in this last expression, we obtain the inequality

$$[(1/n)^{p_1/p}(1/n)^{p_2/p} \cdots (1/n)^{p_n/p}]^{1/n} \leqslant (1/n)[(1/n)^{p_1/p} + (1/n)^{p_2/p} + \cdots + (1/n)^{p_n/p}].$$

Hence $n(1/n)^{a_n/p} \leqslant (1/n)^{p_1/p} + \cdots + (1/n)^{p_n/p}$. The sequence $\{x_n\}$ where $x_n = ((1/n)^{1/p}, (1/n)^{1/p}, \ldots, (1/n)^{1/p}, 0, \ldots)$ is (topologically) bounded in l_p. Thus if $l_{(p_n)} = l_p$, $\{x_n\}$ is bounded in $l_{(p_n)}$, and this implies that $(n(1/n)^{a_n/p})$ is bounded by the last inequality. Conversely, suppose $(n(1/n)^{a_n/p})$ is bounded. Since $a_n \leqslant p_n$, $n(1/n)^{a_n/p} \geqslant n(1/n)^{p_n/p}$. Hence Theorem 6 implies that $l_{(p_n)} = l_p$. □

Combining Corollary 3, Corollary 9, and Proposition 11, we see the following four conditions are equivalent for any sequence (ε_n) converging monotonically to zero such that $\varepsilon_1 < 1$: (1) There exists a number $K > 1$ such that $\sum 1/K^{1/\varepsilon_n} < \infty$; (2) $\varepsilon_n = O(1/(\ln n))$; (3) $((\ln n)((1/n)(\sum_{j=1}^{n} \varepsilon_j)))$ is a bounded sequence; and (4) $l_{(1-\varepsilon_n)} = l_1$.

COROLLARY 12. *Suppose* $(p_n) \in R_p$ *and* $l_{(p_n)} = l_p$; *then* $l_{(a_n)} = l_p$ *where* $a_n = (1/n)(p_1 + p_2 + \cdots + p_n)$.

PROOF. If $l_{(p_n)}$ equals l_p, then Proposition 11 implies that $n(1/n)^{a_n/p} \leqslant M$, for some M. Hence, the proof of Theorem 6 implies that $l_{(a_n)}$ equals l_p. □

Unfortunately we do not know the answer to the following question: If (p_n) is in R_p and if $a_n = (1/n)(p_1 + \cdots + p_n)$, is $l_{(a_n)}$ equal to $l_{(p_n)}$?

THEOREM 13. *If* $(p_n) \in R_p$, *then* $\lambda_{(p_n)} = l_{(p_n)}$ *if and only if* $l_{(p_n)} = l_p$.

PROOF. Clearly $\lambda_{(p_n)} \subset l_{(p_n)} \subset l_p$. Suppose $l_{(p_n)} = l_p$ and $(x_n) \in l_{(p_n)}$. Then $(x_{\pi(n)}) \in l_{(p_n)}$ for any permutation π. If $\{y_n\} \in l_{(p_n)} \backslash \lambda_{(p_n)}$, then one can find a permuation π such that $\{y_{\pi(n)}\} \notin l_{(p_n)}$. Hence $\lambda_{(p_n)} = l_p$. Conversely, suppose that $\lambda_{(p_n)} = l_{(p_n)}$ and $l_{(p_n)} \neq l_p$. Choose an element $x = (x_1, x_2, \ldots)$ in $l_p \backslash l_{(p_n)}$. Clearly one can find an element of the form $\tilde{x} = (x_1, 0, \ldots, 0, x_2, 0, \ldots, 0, x_3, \ldots)$ which is in $l_{(p_n)}$. Since $l_{(p_n)} = \lambda_{(p_n)}$, $\tilde{x}$ is in $\lambda_{(p_n)}$. However, this implies that x is in $\lambda_{(p_n)}$ which is clearly impossible. □

We remarked after the proof of Theorem 6 that there are sequences (p_n) in R_p such that $\lim n(1/n)^{p_n/p} = \infty$. One can also construct a sequence (p_n) in R such that $l_{(p_n)} \neq l_1$ while $l_{(p_n)}$ has the following property: There exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ (where $x_n = (1/n, \ldots, 1/n, 0, \ldots)$ has n nonzero terms) such that $(\|x_{n_k}\|_{(p_n)})$ is bounded. These facts lead us to make the following definition.

Definition. *If (p_n) is in R_p, then $l_{(p_n)}$ is strongly not equal to l_p (written $l_{(p_n)} \neq^s l_p$) if and only if $\lim \|x_m\|_{(p_n)} = \infty$ where $x_m = ((1/m)^{1/p}, (1/m)^{1/p}, \ldots, (1/m)^{1/p}, 0, \ldots)$ has m nonzero entries.*

In order to utilize this definition, we need the following lemmas.

Lemma 14. *Suppose $0 < b < 1$, $(p_n) \in R$, $rb \geqslant 1$, and $p_n < 1$; then the minimum, $\min\{\|x\|_{(p_n)}: x = (x_1, \ldots, x_r, 0, \ldots), \|x\|_\infty = b, \|x\|_1 = 1$, and $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0\}$ is attained at a point of the form $x = (b, b, \ldots, b, c, c, \ldots, c, 0, \ldots)$ where $c \geqslant 0$.*

Proof. Assume that the minimum does not occur at a point of the form given above. Then the minimum must occur at a point x having at least three distinct entries. We will show that x cannot be of the form

$$x = (b, \ldots, b, \underset{\uparrow n_1}{y}, \cdots, \underset{\uparrow n_2}{y}, z, \cdots, \underset{\uparrow n_3}{z}, 0, \cdots)$$

where $b > y > z > 0$. It will be clear from the proof of this claim that x cannot have more than two distinct entries. We note if $0 < d < 1$, the function g_d given by $g_d(t) = t^p + (d-t)^q$, $0 \leqslant t \leqslant d$, $0 < p \leqslant q < 1$, is strictly increasing in $[0, t_0]$ and strictly decreasing in $[t_0, d]$ where t_0 is the solution to the equation $p/t^{1-p} = q/(d-t)^{1-q}$. We note further that if $0 < r \leqslant p$ and $q \leqslant s < 1$, then the solution to $r/t^{1-r} = s/(d-t)^{1-s}$ is not greater than t_0. Thus if one cannot lower the value of y in the n_2 entry and raise the value of z in the $n_2 + 1$ entry the same amount to decrease the value of $\|x\|_{(p_n)}$, then it must be true that y is greater than t_0. But if y is greater than t_0, then y is greater than the solution to the equation $p_{n_1}/t^{1-p_{n_1}} = p_{n_3}/t^{1-p_{n_3}}$. Hence we can increase y in the n_1 entry and decrease z in the n_3 entry a comparable amount to decrease the value of $\|x\|_{(p_n)}$. This contradicts the fact that x was chosen so that $\|x\|_{(p_n)}$ was a minimum. □

The following lemma generalizes a result contained in [8].

Lemma 15. *Given any positive number ε, there exists a positive number δ such that $0 < a < b < \delta$ implies that $a^p + b^q < a^q + b^p$ whenever $\varepsilon \leqslant p < q \leqslant 1$.*

Proof. Suppose that the lemma is not true. Then there exist convergent sequences (a_n), (b_n), (p_n), and (q_n) such that $\varepsilon \leqslant p_n < q_n \leqslant 1$, $0 < a_n < b_n$

$< 1/n$, and $a_n^{p_n} + b_n^{q_n} \geqslant a_n^{q_n} + b_n^{p_n}$. By the intermediate value theorem, there exists a number c_n in $(0, a_n)$ such that $c_n^{p_n} + b_n^{q_n} = c_n^{q_n} + b_n^{p_n}$. Let $f_n(x) = x^{p_n}$, and let $g_n(x) = x^{q_n}$. Applying the generalized mean value theorem to f_n and g_n on $[c_n, b_n]$, we obtain ξ_n in (c_n, b_n) such that

$$1 = \frac{b_n^{p_n} - c_n^{p_n}}{b_n^{q_n} - c_n^{q_n}} = \frac{p_n \xi_n^{(p_n - 1)}}{q_n \xi_n^{(q_n - 1)}}.$$

Thus $\xi_n = (q_n/p_n)^{1/(p_n - q_n)}$. Let $p = \lim p_n$ and $q = \lim q_n$. If $p \neq q$, then $\lim \xi_n = (q/p)^{1/(p-q)}$, and $\lim \xi_n \neq 0$. This contradicts the fact that $0 < \xi_n < 1/n$. If $p = q$, let $s_n = (q_n - p_n)/p_n$. Then $\xi_n = [(1 + s_n)^{1/s_n}]^{-1/p_n}$. Hence $\lim \xi_n = e^{-1/p}$, $p \geqslant \varepsilon > 0$. Again this contradicts the fact that $0 < \xi_n < 1/n$. □

DEFINITION. *Let $x = (x_1, x_2, \ldots)$ be any sequence of real numbers such that $\lim x_n = 0$. Then $\hat{x} = (|x_{j_1}|, |x_{j_2}|, \ldots)$ where $|x_{j_1}| = \max_j \{|x_j|\}$, and $|x_{j_n}| = \max_{j \neq j_1, \ldots, j_{n-1}} \{|x_j|\}$, for $n = 2, 3, \ldots$.*

THEOREM 16. *If $(p_n) \in R_p$, there exists a positive number ε such that $|x|_{(p_n)} = \|\hat{x}\|_{(p_n)}$ whenever $\|x\|_\infty < \varepsilon$.*

PROOF. This follows immediately from Lemma 15. □

We have already observed in the proof of Corollary 4 that $\lambda_{(p_n)}$ equals $\lambda_{(q_n)}$ when $l_{(p_n)}$ equals $l_{(q_n)}$. The converse of this statement is trivially false when we allow $\lim p_n$ to be zero, for in this case $\lambda_{(p_n)}$ equals $\varnothing$, but $l_{(p_n)}$ is not equal to $\varnothing$. We are now in a position to show that the converse is false even when the sequence (p_n) is bounded away from zero. This fact is shown in Theorem 17.

THEOREM 17. *There are sequences (p_n) and (q_n) in R such that $0 < p_n \leqslant q_n$ and $\lambda_{(p_n)} = \lambda_{(q_n)}$ while $l_{(p_n)} \neq l_{(q_n)}$.*

PROOF. Choose (p_n) in R as in the proof of Theorem 5. Then $l_{(p_n)}$ is not equal to $l_{(p_{2n})}$; but $l_{(p_{2n})}$ is equal to $l_{(p_{2n-1})}$ by Corollary 4. Let $q_n = p_{2n-1}$ for $n = 1, 2, \ldots$. Clearly $p_n \leqslant p_{2n-1} = q_n$, and this implies that $\lambda_{(q_n)}$ contains $\lambda_{(p_n)}$. Let $x = (x_1, x_2, \ldots)$ be an element of $\lambda_{(q_n)}$. Without loss of generality, we can assume that $x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots \geqslant 0$ and $x_1 < \varepsilon$ where $\varepsilon > 0$ is the number given in the statement of Theorem 16. Note that

$$\frac{\sum_{i=1}^n x_i^{p_i}}{2 \sum_{i=1}^n x^{q_i}} \leqslant \frac{\sum_{i=1}^n x^{p_i}}{(x_1^{p_1} + x_2^{p_3} + \cdots + x_n^{p_{2n-1}}) + (x_1^{p_2} + x_2^{p_4} + \cdots + x_n^{p_{2n}})}$$

$$\leqslant \frac{\sum_{i=1}^n x_i^{p_i}}{\sum_{i=1}^{2n} x_i^{p_i}} \leqslant 1.$$

Hence $|x|_{(p_n)} \leqslant 2|x|_{(q_n)}$, and this implies that $\lambda_{(p_n)} \supset \lambda_{(q_n)}$. □

We now turn our attention to some theorems involving the notion $l_{(p_n)} \neq^s l_p$.

THEOREM 18. *Suppose* $(p_n) \in R$, $x_k = (x_{k1}, x_{k2}, \ldots)$, $x_{k1} \geqslant x_{k2} \geqslant \cdots \geqslant 0$, $\|x_k\|_1 = 1$ *and* $\lim \|x_k\|_\infty = 0$; *then* $\lim \|x_k\|_{(p_n)} = \infty$ *if* $l_{(p_n)} \neq^s l_1$.

PROOF. Suppose $l_{(p_n)} \neq^s l_1$ and x_k satisfies the above conditions. By using Lemma 14, we can construct a sequence $\{y_k\}$ such that $\frac{1}{2}\|y_k\|_\infty = \|x_k\|_\infty = b_k$, $\|y_k\|_1 = 1$, $\|y_k\|_{(p_n)} \leqslant 2\|x_k\|_{(p_n)}$, and $y_k = (2b_k, \ldots, 2b_k, c_k, \ldots, c_k, 0, \ldots)$, $0 \leqslant c_k \leqslant 2b_k$. Since $\lim b_k = 0$ and $\|y_k\|_1 = 1$, the fact that $l_{(p_n)} \neq^s l_1$ implies that $\lim \|y_k\|_{(p_n)} = \infty$. This of course implies that $\lim \|x_k\|_{(p_n)} = \infty$. □

We are unable to determine if Theorem 18 generalizes to l_p in the following way: Suppose (p_n) is in R_p, $\|x_k\|_p = 1$, and $\lim \|x_k\|_\infty = 0$ where x_k is as above. Is $\lim \|x_k\|_{(p_n)} = \infty$ if $l_{(p_n)} \neq^s l_p$?

COROLLARY 19. *Suppose* $(p_n) \in R$, $x_k \in \lambda_{(p_n)}$, $\|x_k\|_1 = 1$, $l_{(p_n)} \neq^s l_1$, *and* $\lim_{k\to\infty} \|x_k\|_\infty = 0$; *then* $\lim |x_k|_{(p_n)} = \infty$.

PROOF. If $l_{(p_n)} \neq^s l_1$, then Theorem 18 implies the result because any rearrangement of a sequence in $\lambda_{(p_n)}$ has the same $\lambda_{(p_n)}$ norm. □

THEOREM 20. *Suppose* $(p_n) \in R$; *then* $l_{(p_n)} \neq^s l_1$ *if and only if there is no infinite-dimensional subspace of* $\lambda_{(p_n)}$ *on which the* $\lambda_{(p_n)}$*- and the* l_1*-topologies agree.*

PROOF. Suppose $l_{(p_n)} \neq^s l_1$ and the $\lambda_{(p_n)}$-topology and the l_1-topology agree on an infinite-dimensional subspace X. By checking the proof of Proposition 1, we observe that we may assume that X is closed and contains a block basic sequence $\{x_n\}$. By taking linear combinations of the x_n's, we can obtain a sequence $\{y_n\}$ in X such that $\|y_n\|_1 = 1$ and $\|y_n\|_\infty \to 0$. By Corollary 19, $|y_n|_{(p_n)} \to \infty$. This is a contradiction.

Conversely, suppose that it is not true that $l_{(p_n)} \neq^s l_1$. Then there exists a strictly increasing sequence (m_k) of positive integers such that

$$x_k = (\underbrace{x^k, x^k, \ldots, x^k}_{m_k}, 0, \cdots)$$

has the property that $\|x_k\|_1 = 1$ and $\|x_k\|_{(p_n)} \leqslant M$ for some M and all k. Choose a strictly increasing subsequence (n_k) of the sequence (m_k) such that $l_{(p_{n_k})}$ is equivalent to l_1. Then Theorem 13 implies that $\lambda_{(p_{n_k})}$ is equivalent to l_1. Let

$$y_1 = (\underbrace{y^1, \ldots, y^1}_{n_1}, 0, \cdots),$$

$$y_2 = (\underbrace{0, \ldots, 0}_{n_1}, \underbrace{y^2, \cdots, y^2}_{n_1+n_2}, 0, \cdots),$$

$$y_3 = (\underbrace{0, \ldots,}_{n_1+n_2+1} \underbrace{y^3, \cdots, y^3}_{n_1+n_2+n_3}, 0, \cdots),$$

etc., where y^j is chosen such that $\|y_j\|_1 = 1$. The conditions imply that $|y_j|_{(p_n)} \leqslant M$. Hence if $\sum a_j y_j$ is any element in $[y_j]_{\lambda_{(p_n)}}$, Theorem 16 implies that there is a permutation π of the natural numbers such that

$$\begin{aligned} |\textstyle\sum a_j y_j|_{(p_n)} &= (|a_{\pi(1)} y^{\pi(1)}|^{p_1} + |a_{\pi(1)} y^{\pi(1)}|^{p_2} + \cdots + |a_{\pi(1)} y^{\pi(1)}|^{p_{n_{\pi(1)}}}) \\ &\quad + (|a_{\pi(2)} y^{\pi(2)}|^{p_{n_{\pi(1)}+1}} + |a_{\pi(2)} y^{\pi(2)}|^{p_{(n_{\pi(1)}+2)}} + \cdots + |a_{\pi(2)} y^{\pi(2)}|^{p_{(n_{\pi(1)}+n_{\pi(2)})}}) + \cdots \\ &\leqslant M(|a_{\pi(1)}|^{p_1} + |a_{\pi(2)}|^{p_{n_{\pi(1)}}} + |a_{\pi(3)}|^{p_{(n_{\pi(1)}+n_{\pi(2)})}} + \cdots) \\ &\leqslant M(|a_{\pi(1)}|^{p_1} + |a_{\pi(2)}|^{p_{n_{\pi(1)}}} + |a_{\pi(3)}|^{p_{n_{\pi(2)}}} + \cdots) \\ &\leqslant M|(a_1, a_2, \ldots)|_{(p_{n_k})} \end{aligned}$$

when the $|a_j|, j = 1, 2, \ldots$, are sufficiently small. Since $|\ |_{(p_{n_k})}$ is equivalent to $\|\ \|_1$, the above implies that the l_1-topology on $[y_j]_{\lambda_{(p_n)}}$ is stronger than the $\lambda_{(p_n)}$-topology on $[y_j]_{\lambda_{(p_n)}}$. This implies that the two topologies agree on $[y_j]_{\lambda_{(p_n)}}$. □

If (q_n) increases "rapidly" to $q = 1$, then the unit vector basis in $l_{(q_n)}$ is equivalent to the unit vector basis in l_1, and the $l_{(q_n)}$-topology agrees with the l_1-topology on $l_{(q_n)}$. Hence we have the following.

COROLLARY 21. *Suppose $(p_n) \in R$, $l_{(p_n)} \neq^{s} l_1$, and (n_k) is an increasing sequence of positive integers chosen such that the $l_{(p_n)}$-basic sequence $\{e_{n_k}\}$ is equivalent to the unit vector basis in l_1. Then the $\lambda_{(p_n)}$-closed linear span of $\{e_{n_k}\}$ contains no infinite-dimensional subspace where the $l_{(p_n)}$-topology and the $\lambda_{(p_n)}$-topology agree.*

If $l_{(p_n)} \neq l_1$, then $l_{(p_n)}$ is not locally convex. This is easy to show, and is shown in [10]. Proposition 22 shows that the analogous result holds in $\lambda_{(p_n)}$.

PROPOSITION 22. *If $\lambda_{(p_n)}$ is not equal to $\varnothing$ or to l_1, then $\lambda_{(p_n)}$ is not locally convex.*

PROOF. If $\lambda_{(p_n)} \neq \varnothing$ there exists p, $0 < p \leqslant 1$, such that $p \leqslant p_n$ for all n. The set $\{x \in \lambda_{(p_n)}: |x|_{(p_n)} < \varepsilon\}$ contains the set $\{x \in \lambda_{(p_n)}: \|x\|_p < \varepsilon\}$. Since the convex hull of the last set contains an "l_1 ball", the convex hull of the first set contains an "l_1 ball". This implies that the strongest locally convex topology weaker than the $\lambda_{(p_n)}$-topology is the l_1-topology. Hence, if $\lambda_{(p_n)}$ is locally convex it must be equal to l_1. □

PROPOSITION 23. *Suppose (p_n) is in R_p and $\{x_n\}$ is a block basic sequence such that $|x_n|_{(p_n)} = 1$ and $0 < b \leqslant \|x_n\|_\infty \leqslant B$. Then the $\lambda_{(p_n)}$-closed linear span of $\{x_n\}$ is isomorphic to $\lambda_{(p_n)}$.*

PROOF. Let T be the linear mapping of $[x_n]$ into $\lambda_{(p_n)}$ defined by $Tx_n = e_n$. If $(c_1, c_2, \ldots, c_k, 0, \ldots)$ is any finitely nonzero sequence such that $|c_j| \leqslant 1$, then there exists a permutation π of the natural numbers such that $|(c_1, c_2, \ldots, c_k, 0, \ldots)|_{(p_n)} = |c_1|^{p_{\pi(1)}} + \cdots + |c_k|^{p_{\pi(k)}}$. Let $\|x_i\|_\infty = a_i$. Then $\varepsilon \leqslant a_i \leqslant 1$ implies that $a_i^{p_{\pi(i)}}/\varepsilon \geqslant 1$. Hence

$$\begin{aligned} |(c_1, c_2, \ldots, c_k, 0, \ldots)|_{(p_n)} &= c_1^{p_{\pi(1)}} + \cdots + c_k^{p_{\pi(k)}} \\ &\leqslant (1/\varepsilon)[(c_1 a_1)^{p_{\pi(1)}} + \cdots + (c_k a_k)^{p_{\pi(k)}}] \\ &\leqslant (1/\varepsilon)[|c_1 x_1 + \cdots + c_k x_k|_{(p_n)}]. \end{aligned}$$

This shows that T is continuous. Also

$$|c_1 x_1 + \cdots + c_k x_k|_{(p_n)} \leqslant |(c_1, \ldots, c_k, \quad 0, \ldots)|_{(p_n)}$$

implies that T^{-1} is continuous. Since T is one-to-one, the theorem follows. □

The $l_{(p_n)}$ part of the next theorem appears in [11], but its proof is also included here for convenience.

THEOREM 24. *If (p_n) is in R_p, then any infinite-dimensional closed subspace X of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) contains a subspace isomorphic to l_p.*

PROOF. We first prove the theorem for $l_{(p_n)}$. By Proposition 1, we can assume that X contains an $l_{(p_n)}$-normalized block basic sequence $\{x_n\}$, and by taking linear combinations if necessary, we can assume that

$$x_n = (0, \ldots, 0, x_{k_n}^n, \ldots, x_{k_{n+1}-1}^n, 0, \ldots)$$

where $\{k_n\}$ is an increasing sequence of positive integers such that $l_{(p_{k_n})} = l_p$.

Let T: $[x_n] \to l_p$ be the linear map satisfying $T(x_k) = e_k$. Since

$$\|c_1 x_1 + c_2 x_2 + \cdots\|_{(p_n)} \leqslant |c_1|^{p_{k_1}} + |c_2|^{p_{k_2}} + |c_3|^{p_{k_3}} + \cdots$$

when $|c_j| \leqslant 1$, and since $l_{(p_{k_n})} = l_p$, T must map onto all of l_p. Since T is clearly continuous, the open mapping theorem implies that T is an isomorphism, and this completes the proof for $l_{(p_n)}$.

Let X be a closed infinite-dimensional subspace of $\lambda_{(p_n)}$. Because of Proposition 1, we can assume that X contains a block basic sequence $\{x_n\}$. By taking linear combinations of the x_n's, we can obtain a block basic sequence $\{y_k\}$ such that $\|y_k\|_\infty = a_k$, $|y_k|_{(p_n)} = 1$, and each y_k is of the form

$$y_k = (0, \ldots, 0, *, \ldots, *, a_k, *, \ldots, *, a_k, *, \ldots, *, a_k, *, \ldots, *, 0, \ldots)$$

where y_k contains m_k a_k's and (m_k) is a strictly increasing sequence of positive integers chosen so that $l_{(p_{s_j})} = l_p$ where $s_j = \sum_{i=1}^{j} m_i$. Then Theorem 13 implies that $\lambda_{(p_{s_j})} = l_p$. Let T be the mapping of the $\lambda_{(p_n)}$-closed linear span of $\{y_k\}$ into l_p satisfying $T(y_k) = e_k$. We will show that T is a $\lambda_{(p_n)}$-to-(l_p)

isomorphism. Let (c_k) be an element of l_p. Then by Theorem 16, if $\|(c_k)\|_p$ is sufficiently small, we have

$$\begin{aligned}|\sum c_k y_k|_{(p_n)} &= \|(\sum c_k y_k)\hat{}\ \|_{(p_n)} \\ &= |c_1 a_1|^{p_{k_1}} + \cdots + |c_1 a_1|^{p_{k_1+m_1-1}} + (\text{other } c_1 y_1 \text{ terms}) \\ &\quad + |c_2 a_2|^{p_{k_2}} + \cdots + |c_2 a_2|^{p_{k_2+m_2-1}} + (\text{other } c_2 y_2 \text{ terms}) + \cdots \\ &\leqslant |c_1|^{p_{s_{\pi(1)}}} + |c_2|^{p_{s_{\pi(2)}}} + |c_3|^{p_{s_{\pi(3)}}} + \cdots\end{aligned}$$

for some permutation π of the natural numbers. This series converges since $\lambda_{(p_{s_i})}$ equals l_p, and thus T is onto. Clearly T is one-to-one, and the graph of T is closed because $\{y_k\}$ is a $\lambda_{(p_n)}$-Schauder basis and $\{e_k\}$ is an l_p-Schauder basis. Hence T is continuous by the Closed Graph Theorem. The Open Mapping Theorem then implies that T is an isomorphism. □

It is interesting to compare Theorem 20 and Theorem 24: Together these theorems show that there are cases where (p_n) is in R and $l_{(p_n)}$ has subspaces isomorphic to l_1 but these subspaces do not have the topology "inherited" from l_1. It is also interesting to note that there are choices of (p_n) in R_p such that the $\lambda_{(p_n)}$- and the l_p-topology do not agree on any infinite-dimensional subspaces (cf. Theorem 20) while Theorem 25 below shows that there is always an infinite-dimensional subspace where the $\lambda_{(p_n)}$- and the $l_{(p_n)}$-topology agree when (p_n) is in R.

THEOREM 25. *If (p_n) is in R_p, then $\lambda_{(p_n)}$ contains an infinite-dimensional subspace where the $\lambda_{(p_n)}$-topology and the $l_{(p_n)}$-topology agree.*

PROOF. Let

$$\begin{aligned} x_1 &= (x^1, \ldots, \underset{\longrightarrow m_1}{x^1}, 0, \cdots), \\ x_2 &= (0, \ldots, 0, \underset{\to m_1+1}{x^2}, \cdots, \underset{\longrightarrow m_1+m_2}{x^2}, 0, \cdots), \\ x_3 &= (0, \ldots, 0, \underset{\to m_1+m_2+1}{x^3}, \cdots, \underset{\longrightarrow m_1+m_2+m_3}{x^3}, 0, \cdots),\end{aligned}$$

etc; where (m_k) is a strictly increasing sequence of positive integers chosen so that $m_0 = 1$, $\|x_j\|_{(p_n)} = 1$, $|x_j|_{(p_n)} \leqslant 2$, and $l_{(p_{m_k})}$ equals l_p. Note that Theorem 13 implies that $\lambda_{(p_{m_k})}$ also equals l_p. Let $x = \sum a_j x_j$ where $a = (a_1, a_2, \ldots)$ is a finitely nonzero sequence. If $|a_j|, j = 1, 2, \ldots,$ is sufficiently small, Theorem 16 implies that there exists a permutation π of the positive integers such that

$$\begin{aligned}|x|_{(p_n)} &= |a_{\pi(1)} x^{\pi(1)}|^{p_1} + \cdots + |a_{\pi(1)} x^{\pi(1)}|^{p_{m_{\pi(1)}}} \\ &\quad + |a_{\pi(2)} x^{\pi(2)}|^{p_{(m_{\pi(1)}+1)}} + \cdots + |a_{\pi(2)} x^{\pi(2)}|^{p_{(m_{\pi(1)}+m_{\pi(2)})}} + \cdots \\ &\leqslant 2(|a_{\pi(1)}|^{p_1} + |a_{\pi(2)}|^{p_{m_{\pi(1)}}} + |a_{\pi(3)}|^{p_{m_{\pi(2)}}} + \cdots).\end{aligned}$$

The above implies that $|x|_{(p_n)} \leqslant 2|a|_{(p_{m_k})}$. Also when $|a_j| \leqslant 1$, we have

$$\begin{aligned}
&|a_1|^{p_{m_1}} + |a_2|^{p_{(m_1+m_2)}} + |a_3|^{p(m_1+m_2+m_3)} + \cdots \\
&\quad = |a_1|^{p_{m_1}}\|x_1\|_{(p_n)} + |a_2|^{p_{(m_1+m_2)}}\|x_2\|_{(p_n)} + |a_3|^{p_{(m_1+m_2+m_3)}}\|x_3\|_{(p_n)} + \cdots \\
&\quad \leqslant (|a_1x^1|^{p_1} + \cdots + |a_1x^1|^{p_{m_1}}) + (|a_2x^2|^{p_{(m_1+1)}} + \cdots + |a_2x^2|^{p_{(m_1+m_2)}}) + \cdots \\
&\quad = \|x\|_{(p_n)}.
\end{aligned}$$

If we let $s_j = \sum_1^j m_i$, then $\| \|_{(p_{s_j})}$ is equivalent to $\| \|_p$. Since $| |_{(p_{m_k})}$ is also equivalent to $\| \|_p$, the above inequalities imply that the $l_{(p_n)}$-topology is stronger than the $\lambda_{(p_n)}$-topology on $sp(x_n)$. This means that the two topologies agree on $sp(x_n)$ and hence on the closure of $sp(x_n)$. □

THEOREM 26. *Suppose (p_n) and (q_n) are in R, $l_{(q_n)} \neq^s l_1$, and $\lambda_{(p_n)}$ is isomorphic to $\lambda_{(q_n)}$; then $\lambda_{(p_n)}$ equals $\lambda_{(q_n)}$.*

PROOF. Let T: $\lambda_{(p_n)} \to \lambda_{(q_n)}$ be the given isomorphism, and let $f_k = T(e_k)$, $k = 1, 2, \ldots.$ Since $\{f_n\}$ is a $\lambda_{(q_n)}$-bounded sequence, the set of the kth coordinates of the sequence $\{f_n\}$ forms a bounded set. Thus we can apply the proof of Proposition 1 to construct two strictly increasing sequences (m_j) and (n_j) of positive integers such that $m_j < n_j < m_{j+1}$ and the sequence $\{f_{m_j} - f_{n_j}\}$ is a $\lambda_{(q_n)}$-basic sequence equivalent to the "truncated" block basic sequence $\{\tilde{f}_{m_j} - \tilde{f}_{n_j}\}$. Since T can be extended to an l_1-to-l_1 isomorphism, the sequence $\{\tilde{f}_{m_j} - \tilde{f}_{n_j}\}$ is l_1-bounded away from zero. Thus Corollary 19 implies that the sequence $(\|\tilde{f}_{m_j} - \tilde{f}_{n_j}\|_\infty)$ is bounded away from zero. Hence Proposition 23 implies that $\lambda_{(q_n)}$ is isomorphic to $[\tilde{f}_{m_j} - \tilde{f}_{n_j}]_{\lambda_{(q_n)}}$ and hence to $[f_{m_j} - f_{n_j}]_{\lambda_{(q_n)}}$. Since $\lambda_{(p_n)}$ is isomorphic to $[e_{m_j} - e_{n_j}]_{\lambda_{(p_n)}}$ and all of these isomorphisms are given by the natural mappings, $\lambda_{(p_n)}$ equals $\lambda_{(q_n)}$. □

We are unable to answer the following question: If (p_n) and (q_n) are in R_p and if $\lambda_{(p_n)}$ (resp., $l_{(p_n)}$) is isomorphic to $\lambda_{(q_n)}$ (resp., $l_{(q_n)}$), must $\lambda_{(p_n)}$ (resp., $l_{(p_n)}$) equal $\lambda_{(q_n)}$ (resp., $l_{(q_n)}$)?

We mentioned at the beginning of this section that if $p_n \geqslant p > 0$, then $l_{(p_n)}$ is a locally bounded space and a subset of this space is bounded if and only if the subset is metrically bounded. (It is easy to see that the same is also true for $\lambda_{(p_n)}$.) The following is an extension of this idea.

THEOREM 27. *Suppose $0 < p_n < 1$ for all n. Then $l_{(p_n)}$ contains no infinite-dimensional locally bounded subspace if and only if* $\lim p_n = 0$.

PROOF. If $\lim p_n \neq 0$, then (p_n) contains a subsequence (q_n) which is bounded away from 0. Since $l_{(q_n)}$ is locally bounded, then $l_{(p_n)}$ contains a locally bounded subspace. Conversely, let $\lim p_n = 0$ and let X be an infinite-dimensional subspace of $l_{(p_n)}$. Select a sequence $\{x_n\}$ in X such that x_n is of the form $x_n = (0, \ldots, 0, x^n_{k_n}, x^n_{k_n+1}, \ldots)$ where (k_n) is a strictly increasing sequence of positive integers. Suppose that X contains a bounded neighborhood N of 0 and ε is a positive number such that $\|x\|_{(p_n)} \leqslant \varepsilon$ implies that x in is N. Without loss of generality, assume that $\|x_n\|_{(p_n)} = \varepsilon$. Since $N_\varepsilon = \{x: \|x_n\|_{(p_n)} \leqslant \varepsilon\}$ is bounded, there exists a positive number α such that $N_{\varepsilon/2} \supset \alpha N_\varepsilon$. However, $\lim \|\alpha x_n\|_{(p_n)} = \lim \|x_n\|_{(p_n)} = \varepsilon$. □

Suppose $0 < p_n \leqslant 1$. Clearly $\lambda_{(p_n)}$ is the intersection $\bigcap_{\pi \in \Pi} l_{p_{\pi(n)}}$ where Π is the set of all permutations of the natural numbers. Let $\mathfrak{T}_\pi$ denote the "sup topology" obtained from the F-seminorms $\| \; \|_{(p_{\pi(n)})}$. With this notation, we have the following.

THEOREM 28. *If $(p_n) \in R_p$, then $\mathfrak{T}_\pi$ lies between the l_p-topology and the $\lambda_{(p_n)}$-topology. Furthermore, $\mathfrak{T}_\pi$ is metrizable if and only if $\lambda_{(p_n)} = l_p$.*

PROOF. It is clear that $\mathfrak{T}_\pi$ is stronger than the l_p-topology and weaker than the $\lambda_{(p_n)}$-topology. Since $\lambda_{(p_n)} = l_p$ implies that the $\lambda_{(p_n)}$-topology is equal to the l_p-topology, $\mathfrak{T}_\pi$ is metrizable when $\lambda_{(p_n)} = l_p$.

Conversely, suppose $\mathfrak{T}_\pi$ is metrizable and $\lambda_{(p_n)} \neq l_p$. Then there exists permutations $\{\pi_k\}_{k=1}^\infty$ and a decreasing sequence of positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ such that the sets $U_k = \{x: \|x\|_{(p_{\pi_k(n)})} < \varepsilon_k\}$, $k = 1, 2, \ldots$, form a neighborhood base at zero for $\mathfrak{T}_\pi$. By Theorem 6, there exists a positive integer n_1 such that

$$\Bigg\| \frac{\varepsilon_1}{2} \underbrace{\left(\left(\frac{1}{n_1}\right)^{1/p}, \left(\frac{1}{n_1}\right)^{1/p}, \ldots, \left(\frac{1}{n_1}\right)^{1/p}\right.}_{n_1 \text{ times}} \left., 0, \ldots \right) \Bigg\|_{(p_n)} > 1.$$

Since $\lim_{n\to\infty} p_{\pi_1(n)} = p$, there exists a point z_1 of the form

$$z_1 = \frac{\varepsilon_1}{2}\Big(0, \ldots, 0, \underset{\uparrow m_1}{\left(\frac{1}{n_1}\right)^{1/p}}, \ldots, \underset{\uparrow m_2+n_2-1}{\left(\frac{1}{n_1}\right)^{1/p}}, 0, \cdots \Big)$$

such that z_1 is in U_1. We will construct a permutation (q_n) of (p_n) in stages. The first $m_1 + n_1 - 1$ terms of (q_n) are

$$(p_{n_1+1}, \ldots, p_{k_1}, \underset{\uparrow m_1}{p_1}, p_2, \ldots, \underset{\uparrow m_1+n_1-1}{p_{n_1}})$$

where $k_1 = m_1 + n_1 - 1$. Since $l_{(p_n)} \neq l_p$, $l_{(p_{k_1+n})} \neq l_p$. Again, by Theorem 6, there exists a positive integer n_2 such that

$$\Bigg\| \frac{\varepsilon_2}{2} \underbrace{\left(\left(\frac{1}{n_2}\right)^{1/p}, \ldots, \left(\frac{1}{n_2}\right)^{1/p}\right.}_{n_2 \text{ times}} \left., 0, \ldots \right) \Bigg\|_{(p_{k_1+n})} > 1.$$

Since $\lim_{n\to\infty} p_{\pi_1(n)} = p$ and $\lim_{n\to\infty} p_{\pi_2(n)} = p$, there exists a point z_2 of the form

$$z_2 = \frac{\varepsilon_2}{2}\Big(0,\ldots,0,\underbrace{\Big(\frac{1}{n_2}\Big)^{1/p}}_{m_2},\ldots,\underbrace{\Big(\frac{1}{n_2}\Big)^{1/p}}_{m_2+n_2-1},0,\ \cdots\ \Big)$$

such that z_2 is in U_1 and U_2 and $m_2 > m_1 + n_1$. The first $m_2 + n_2 - 1$ terms of (q_n) are

$$(p_{n_1+1},\ldots,p_{k_1},p_1,p_2,\ldots,p_{n_1},p_{k_1+n_2+1},\ldots,p_{k_2},\underbrace{p_{k_1+1}}_{m_2},\ldots,\underbrace{p_{k_1+n_2}}_{m_2+n_2-1=k_2}\quad)$$

where $k_2 = m_2 + n_2 - 1$. Continue this process inductively to obtain (q_n) and $\{z_n\}$ such that z_n is in $\bigcap_{j=1}^n U_j$ and $\|z_k\|_{(q_n)} > 1$. Let U be defined by $U = \{x\colon \|x\|_{(q_n)} < 1\}$. Then clearly U is a neighborhood of 0 in $\mathfrak{T}_\pi$, but there does not exist any integer, n, such that $\bigcap_{k=1}^n U_k$ is contained in U. This is a contradiction. □

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