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COMPLETELY UNSTABLE FLOWS ON 2-MANIFOLDS

BY

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ABSTRACT. Completely unstable flows on 2-manifolds are classified under both topological and C^r-equivalence ($1 < r < \infty$), in terms of the corresponding orbit spaces.

1. **Statement of results.** Let ϕ: $M \times \mathbf{R}^1 \to M$ be a continuous flow on the differentiable manifold M. A point $x \in M$ is *nonwandering* if $x \in J^+(x)$; here $J^+(x)$ denotes the set of limits of sequences $\{\phi(x_n, t_n)\}$, where $\{x_n\}$ converges to x and $\{t_n\}$ tends to infinity. We say that ϕ is *completely unstable* if there are *no* nonwandering points of ϕ.

If ψ: $N \times \mathbf{R}^1 \to N$ is another continuous flow, we say that ϕ and ψ are (*topologically*) *equivalent* if there is a homeomorphism h of M onto N that takes orbits of ϕ onto orbits of ψ, preserving sense. If ϕ and ψ are C^r ($1 \leq r \leq \infty$), they are *C^r-equivalent* if there is such an h that is a C^r-diffeomorphism.

We are concerned with flows (M, ϕ) in which M is an arbitrary 2-manifold (separable metric and without boundary), with given C^∞ structure, and ϕ is completely unstable. Our results constitute a classification of such flows, under both topological and C^r-equivalence, in terms of the associated orbit spaces. Let M/ϕ denote the space of orbits of ϕ with the quotient topology (the finest topology in which the projection π: $M \to M/\phi$ is continuous). It is known that M/ϕ has a countable basis of open sets homeomorphic with $\mathbf{R}^1$ (see [5]); but M/ϕ need not be Hausdorff. We refer to a space with these properties of M/ϕ as a *nonseparated* 1-*manifold*. In general, there are many topological types (M, ϕ) with a given orbit space; to obtain a classification we must impose additional structure. In §3 we define an "order" relation on certain points and subsets of M/ϕ—essentially: $p < q$ in M/ϕ iff $\pi^{-1}(q) \subseteq J^+(\pi^{-1}(p))$ in M. The resulting ordered orbit space, still denoted M/ϕ, completely determines both M and ϕ:

THEOREM 1. *If ϕ, ϕ' are completely unstable continuous flows on 2-manifolds M, M' respectively, then ϕ and ϕ' are topologically equivalent if and only if M/ϕ and M'/ϕ are order isomorphic.*

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Essentially this result, for the case M, $M' = \mathbf{R}^2$, is stated in [2]. Note that any noncritical flow on the plane is completely unstable. An earlier classification of noncritical flows in the plane appears in [4]. Theorem 1 is proved in §3 below, using a preliminary result derived in §2.

If M has a given C^r-structure, and ϕ is C^r, then there is a (unique) C^r-structure on M/ϕ with respect to which π is C^r (see [5]). M/ϕ, with this C^r-structure and the order indicated above, completely determines the C^r-equivalence class of (M, ϕ):

THEOREM 2. *Suppose ϕ, ϕ' are completely unstable C^r-flows on 2-manifolds M, M' respectively. Then ϕ and ϕ' are C^r-equivalent if and only if there is an order preserving C^r-diffeomorphism of M/ϕ onto M'/ϕ'.*

Comparing Theorems 1 and 2, we may obtain some information on the relation between topological and C^r-equivalence classes for completely unstable flows. We remark first that, if ϕ is any completely unstable continuous flow on a 2-manifold M, then ϕ is topologically equivalent to a C^∞-flow on M (this is proved in [7]); i.e., any topological equivalence class contains nonempty C^r-equivalence classes, for $1 \leqslant r \leqslant \infty$. We show further that any C^r-structure on M/ϕ corresponds to a C^r-flow on M, which is topologically equivalent to ϕ. Then Theorem 2 may be given the following form:

THEOREM 3. *Suppose ϕ is a completely unstable continuous flow on the 2-manifold M and fix $r \in \{1, \ldots, \infty\}$. Then the distinct C^r-equivalence classes contained in the topological equivalence class of ϕ are in 1-1 correspondence with the distinct C^r-structures on M/ϕ.*

On nonseparated 1-manifolds in a fairly large class, it is easy to construct a continuum of pairwise distinct C^r-structures. Thus we may apply Theorem 3 to prove, for example,

COROLLARY. *Suppose ϕ is a completely unstable continuous flow on the 2-manifold M and that ϕ has at most countably many separatrices but is not parallel. Then the topological equivalence class of (M, ϕ) contains a continuum of distinct C^r-equivalence classes, for any $r \in \{1, 2, \ldots, \infty\}$.*

Here *separatrix* may be defined as follows (cf. [5]): for $p \in M/\phi$, the orbit $\pi^{-1}(p)$ is a *separatrix* if there is a point $q \in M/\phi$ such that p and q cannot be separated with disjoint open sets in M/ϕ. A completely unstable flow is *parallel* if it has no separatrices.

The proofs of Theorems 2 and 3 and the corollary are given in §4.

2. Subdivision of noncritical flows on 2-manifolds. Let ϕ be a continuous flow without critical points on the 2-manifold M.

DEFINITIONS. A *parallel neighborhood* (or *p-neighborhood*) of a point $m \in M$

is a closed 2-cell $N \subseteq M$, that is a neighborhood of m and that is homeomorphic with the rectangle

$$R = \{(x, t) \in \mathbf{R}^2 |\ |x|, |t| \leqslant 1\}$$

under a map that takes each orbit segment in N onto a vertical segment in R. The boundary ∂N of N then consists of two orbit segments of ϕ–the *edges* of N, and two arcs that are local sections of ϕ–the *ends* of N. It is known that every point of M has a p-neighborhood (cf. [1, Chapter IV, §2] and [3, Theorem 1]).

By a *subdivision* of (M, ϕ) we mean a cover of M by a locally finite collection $\{N_i\}$ of p-neighborhoods which intersect *properly*: viz., for each $i \neq j$, $N_i \cap N_j$ is either empty, a subarc of an edge of each or a subarc of an end of each.

LEMMA 1. *Any noncritical flow* (M, ϕ) *admits a subdivision.*

PROOF. There is a locally finite covering $\{N_i\}$ of M by p-neighborhoods. Note that we may assume that each N_i is contained in a slightly "longer" p-neighborhood $\tilde{N}_i$, corresponding to say $\tilde{R} = \{(x, t) \in \mathbf{R}^2 |\ |x| \leqslant 1, |t| \leqslant 1 + \varepsilon\}$ (that is, $(\tilde{N}_i, N_i)$ is fiberwise homeomorphic with $(\tilde{R}, R)$).

Let εN_i denote the union of the two ends of N_i. We first show that we may adjust the N_i so that no three of the εN_i intersect in a single point and so that $\varepsilon N_i \cap \varepsilon N_j$ is at most finite for each $i \neq j$. For, suppose that we have adjusted $N_1, \ldots, N_{n-1}$ so that this is true for $i < j \leqslant n - 1$. We may assume that no point of $\varepsilon N_i \cap \varepsilon N_j$ $(i < j \leqslant n - 1)$ lies on εN_n. Cover the ends of N_n with new p-neighborhoods $T_1, \ldots, T_k$ which also miss $\varepsilon N_i \cap \varepsilon N_j$ $(i < j \leqslant n - 1)$. The interior of each εN_i $(i \leqslant n - 1)$ meets $\mathring{T}_1$ in at most countably many open arcs, only finitely many of which meet a "shorter" p-neighborhood $S_1 \subseteq T_1$ (see Figure 1). These finitely many, for each $i \leqslant n - 1$, may be

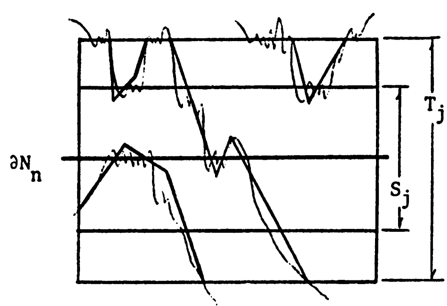

FIGURE 1

replaced by new arcs with the same endpoints, that are again local sections of ϕ, but that now meet $\varepsilon N_n \cap T_1$ in at most finitely many points. We may do this, for example, with an isotopy of M that is the identity off T_1, and that takes these arcs onto piecewise linear ones (in the piecewise linear structure induced by some parametrization of T_1) in general position with respect to $\varepsilon N_n \cap T_1$. Note that then no new intersections $\varepsilon N_i \cap \varepsilon N_j$ ($i < j \leqslant n - 1$) are introduced. Repeat with the adjusted arcs in $T_2, \ldots, T_k$ in succession. Our assertion now follows by induction; note that, because of the local finiteness, we can insure that no εN_i is adjusted more than finitely many times.

We next observe that by a slight modification of this argument we may assume that we have chosen $\tilde{N}_i \supseteq N_i$ (as above) so that no three of $\varepsilon \tilde{N}_i$, εN_j intersect and so that, for $i \neq j$, the various intersections $\varepsilon N_i \cap \varepsilon N_j$, $\varepsilon \tilde{N}_i \cap \varepsilon \tilde{N}_j$, $\varepsilon N_i \cap \varepsilon \tilde{N}_j$ are finite.

Finally, we obtain the desired subdivision by partitioning the N_i. Suppose we have determined p-neighborhoods $K_1, \ldots, K_{m(n-1)}$ satisfying:

(a) $K_1, \ldots, K_{m(n-1)}$ intersect properly;

(b) $N_1 \cup \cdots \cup N_{n-1} \subseteq K_1 \cup \cdots \cup K_{m(n-1)}$;

(c) $\varepsilon K_i \cap \varepsilon N_j$ and $\varepsilon K_i \cap \varepsilon \tilde{N}_j$ are finite ($i \leqslant m(n-1), j \geqslant n$).

Let A denote the set of orbit segments in $\tilde{N}_n$ which pass through a point of $\varepsilon K_i \cap \varepsilon \tilde{N}_n$ or $\varepsilon K_i \cap \varepsilon N_n$ for some $i \leqslant m(n-1)$, or meet an edge of some K_j ($j \leqslant m(n-1)$) that lies in $\tilde{N}_n$ (cf. Figure 2). The arcs of A partition $\tilde{N}_n$ into

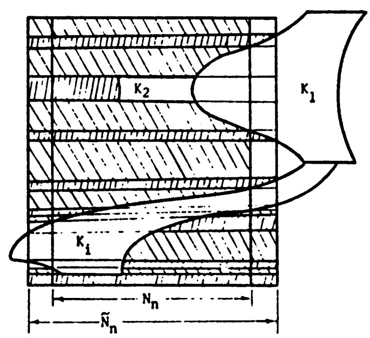

FIGURE 2

p-neighborhoods, which are further divided by the sections $\varepsilon K_1, \ldots, \varepsilon K_{m(n-1)}$, $\varepsilon \tilde{N}_n$ (or εN_n) into the required p-neighborhoods $K_{m(n-1)+1}, \ldots, K_{m(n)}$. It is not difficult to check that this may be done so that (a)–(c) are preserved. Hence our assertion follows by induction.

3. **Topological equivalence of completely unstable flows.** Let ϕ be a completely unstable flow on the 2-manifold M with orbit space M/ϕ.

DEFINITION. We define an order relation on certain points and subsets of M/ϕ as follows:

(1) Let $p, q \in M/\phi$, $p \neq q$. Define $p < q$ iff, for $x \in \pi^{-1}(p)$ and $y \in \pi^{-1}(q)$, we have $y \in J^+(x)$.

(2) Suppose $p, q \in M/\phi$ with both $p < q$ and $q < p$, and that there are

half-open intervals $I, J \subseteq M/\phi$ terminating at p, q respectively, with $I - \{p\} = J - \{q\}$. Choose disjoint sections $S, T \subseteq M$, lying over I, J respectively. Define $I < J$ iff orbits of ϕ meet S before T.

Throughout this section M/ϕ will denote the orbit space endowed with this order relation.

REMARKS. Note that two points of M/ϕ are related in this ordering exactly when they cannot be separated with disjoint open sets in M/ϕ. However there is very little restriction on the size or complexity of this set of *nonseparated* points. An example of Ważewski [8] shows that, even with $M = \mathbf{R}^2$ and $\phi \in C^\infty$, it may be all of M/ϕ.

In many cases, the orbit space with the order defined in (1) above is sufficient to determine the topological type of (M, ϕ). It can be proved, for example, that two noncritical flows on the plane are topologically equivalent iff the corresponding orbit spaces are homeomorphic by a homeomorphism which preserves the order defined by (1). The following example shows that this is not true in the generality of Theorem 1.

EXAMPLE. Let $M = \mathbf{R}^2 - \{0\}$. Let $A, B \subseteq (0, \infty)$ be closed sets with $A \cup B = (0, \infty)$, and define $A' = \{(0, y) | y \in A\} \subseteq M$ and $B' = \{(0, y) | -y \in B\}$. Let $f: M \to [0, 1]$ be a C^∞ function that is 0 exactly on $A' \cup B'$, and let ϕ be the flow on M defined (in polar coordinates) by $\dot{r} = 0, \dot{\theta} = f(r, \theta)$. Now, let ϕ_1 be the flow obtained applying this construction with

$$A_1 = [1, \infty) \cup (0, \tfrac{1}{2}] \cup \bigcup_{n=1}^{\infty} \left[\frac{2n}{2n+1}, \frac{2n+1}{2n+2} \right]$$

and

$$B_1 = \{1\} \cup \bigcup_{n=1}^{\infty} \left[\frac{2n-1}{2n}, \frac{2n}{2n+1} \right]$$

(see Figure 3). Let ϕ_2 be the flow defined choosing

$$A_2 = \{1\} \cup (0, \tfrac{1}{2}] \cup \bigcup_{n=1}^{\infty} \left[\frac{2n}{2n+1}, \frac{2n+1}{2n+2} \right]$$

and

$$B_2 = [1, \infty) \cup \bigcup_{n=1}^{\infty} \left[\frac{2n-1}{2n}, \frac{2n}{2n+1} \right].$$

In either case the orbit space may be described as follows: it consists of two copies of $(0, \infty)$ (corresponding to the sections S_i, T_i of Figure 3) in which each pair of points corresponding to the same coordinate $x \in (0, \infty)$ is identified, except when $x = n/(n+1)$ $(n \in \mathbf{Z}^+)$ or $x = 1$. Also, each pair

$\{p, q\}$ of nonseparated points (in either quotient space), according to (1) of the definition, satisfies both $p < q$ and $q < p$. However ϕ_1 and ϕ_2 are not topologically equivalent; in fact, it is easy to check that M/ϕ_1 and M/ϕ_2 are distinguished by the ordering defined in (2).

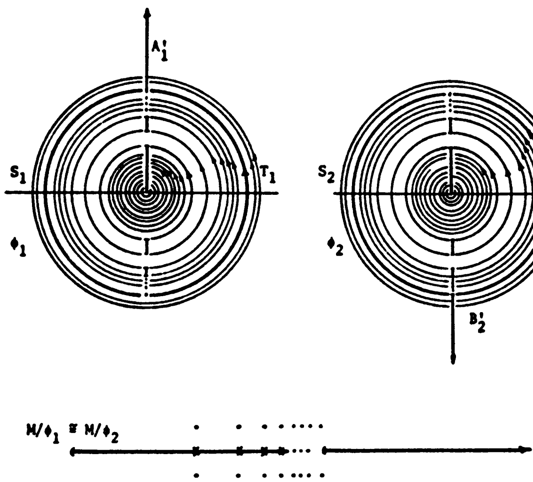

FIGURE 3

The main object of the present section is to prove that the ordered orbit space M/ϕ completely determines both M and ϕ. Thus let (M', ϕ') be another completely unstable flow and suppose that we are given an isomorphism (order preserving homeomorphism)

$$h: M'/\phi' \to M/\phi.$$

We will construct a topological equivalence k of ϕ' with ϕ that satisfies

$$h\pi' = \pi k.$$

Let $\{N_i'\}$ be a subdivision of (M', ϕ'). We will construct an isomorphic subdivision $\{N_i\}$ of (M, ϕ), and use it to define k.

Let S_i' denote the "initial" end of N_i' and let

$$T_i' = \pi'(S_i').$$

Note that we may assume no orbit of ϕ' crosses any S_i' more than once, so we may define $\alpha_i' = T_i' \to S_i'$ by

$$\pi' \circ \alpha_i' = 1_{T_i'}.$$

For $i \neq j$ let $A_{ij}' = T_i' \cap T_j'$ and define f_{ij}': $A_{ij}' \to \mathbf{R}^1$ by

$$f_{ij}'(x) = t \quad \text{iff} \quad \alpha_i'(x) \cdot t = \alpha_j'(x);$$

thus f_{ij}' is the time along the orbit $\pi'^{-1}(x)$ from its intersection with S_i' to its intersection with S_j'. Each f_{ij}' is continuous on its domain and can be zero only at a common endpoint of T_i' and T_j' (i.e., exactly when S_i' and S_j' have a common endpoint). However, as the examples above show, $\operatorname{sgn}(f'_{ij})$ need not be constant on A'_{ij}.

In order to construct $\{N_i\}$ we wish to construct sections S_i of ϕ over $T_i = h(T'_i)$ which are related by ϕ in the same way the S'_i are related by ϕ'; viz., for $i > j$ and $x \in A'_{ij}$, we want $\pi^{-1}(hx)$ to meet S_i before S_j if and only if $\pi'^{-1}(x)$ meets S'_i before S'_j. Equivalently, we require

$$\operatorname{sgn} f_{ij}(hx) = \operatorname{sgn} f'_{ij}(x) \qquad (x \in A'_{ij}).$$

Suppose we have obtained such sections $\alpha_j\colon T_j \twoheadrightarrow S_j \subseteq M$ for $j \leqslant n-1$, so that

$$\text{(a)} \qquad \operatorname{sgn} f_{ij}(x) = \operatorname{sgn} f'_{ij}(h^{-1}x) \qquad (x \in A_{ij}, j < i \leqslant n-1).$$

Let $\alpha_n\colon t_n \to M$ denote any continuous section with $\pi\alpha_n = 1$ and let $S_n = \alpha_n(T_n)$. We do not assume that S_n is disjoint from $S_1, \ldots, S_{n-1}$. We do assume in the following argument that S'_n does not have an endpoint in common with an S'_j ($j \leqslant n-1$). In the case there is such an endpoint, we may assume that S_n and the corresponding S_j have a common endpoint; the argument given is then easily adapted.

We will need to alter S_n a number of times; to simplify notation, each time an adjustment is made we agree to carry over the old notation to the adjusted section and time maps.

For $j \leqslant n-1$, let f_{nj} denote the time from S_n to S_j. We need to show that we can adjust S_n so that the resulting time maps f_{nj} satisfy

$$\operatorname{sgn} f_{nj}(hx) = \operatorname{sgn} f'_{nj}(x) \qquad (x \in A'_{nj}; j \leqslant n-1).$$

LEMMA 2. *Suppose e is an endpoint of a component interval I of A_{nj} with $e \notin A_{nj}$, $e = he'$ and $I = h(I')$. Then*

$$\lim_{\substack{x\to e\\ x\in I}} f_{nj}(x) = \lim_{\substack{x\to e'\\ x\in I'}} f'_{nj}(x) = \pm\infty.$$

PROOF. First note that, since $e \notin A_{nj}$, we must have

$$(*) \qquad \lim_{\substack{x\to e\\ x\in I}} f_{nj}(x) = +\infty \quad (\text{or } -\infty)$$

(similarly for f'_{nj}). We assume the limit in $(*)$ is $+\infty$; the argument in the remaining case is similar. Now $(*)$ holds if and only if

$$p = \lim_{\substack{x\to e\\ x\in I}} (\alpha_n(x)\cdot f_{nj}(x))$$

is in $J^+(\alpha_n(e))$. Let $d = \pi p$ and $d' = h^{-1}d$; then $e < d$. If we do not also have $d < e$, then, since h preserves order, we must have $e' < d'$, but not $d' < e'$, and it follows that $\lim_{x\to e'; x\in I'} f'_{nj}(x) = +\infty$ as asserted. If we do also have $d < e$, then by definition, $I \cup \{e\} < I \cup \{d\}$, so $I' \cup \{e'\} < I' \cup \{d'\}$ and we have the same result.

(Note that we may identify $\pi^{-1}(T_n)$ with $T_n \times \mathbf{R}^1$, since the map $\phi(\alpha_n(x), t) \leftrightarrow (x, t)$ $(x \in T_n, t \in \mathbf{R}^1)$ is a homeomorphism. We may then think of the intersections of the S_j $(j \leq n-1)$ with $\pi^{-1}(T_n)$ as the graphs of the functions f_{nj}.)

Now define functions u_n, l_n: $T_n \to [-\infty, \infty]$ as follows:

$$u_n(x) = \begin{cases} \min f_{nj}(x) & \text{all } j \leq n-1 \text{ with } \operatorname{sgn} f'_{nj}h^{-1}(x) > 0, \\ \infty & \text{if no such } j \text{ exist;} \end{cases}$$

$$l_n(x) = \begin{cases} \max f_{nj}(x) & \text{all } j \leq n-1 \text{ with } \operatorname{sgn} f'_{nj}h^{-1}(x) < 0, \\ -\infty & \text{if no such } j \text{ exist.} \end{cases}$$

Then we have

Lemma 3. *u_n is continuous, except possibly at an endpoint e of T_j with $e \in A_{nj}$; at such a point e both one-sided limits exist (one may be $+\infty$), and*

$$\lim_{\substack{x \to e \\ x \in A_{nj}}} u_n(x) = u_n(e).$$

Proof. First note that if $x \in T_n$ is in no A_{ni} $(i \leq n-1)$, so that $u_n(x) = \infty$, and $\{x_k\}$ is a sequence of points of T_n converging to x, then

$$\lim_{k \to \infty} u_n(x_k) = \infty.$$

For suppose not. We may assume that $u_n(x_k) = f_{nj}(x_k)$ for some fixed $j \leq n-1$. By Lemma 2 we may assume that the x_k are in distinct component intervals of A_{nj}. Suppose that $f_{nj}(x_k) \to s$, s finite or $-\infty$. Then we can pick y_k in the component of A_{nj} containing x_k, with

$$f_{nj}(y_k) = \begin{cases} s+1, & s \text{ finite,} \\ 0, & s = -\infty. \end{cases}$$

But then $\alpha_n(y_k) \cdot f_{nj}(y_k) \in S_j$ and, hence,

$$\lim_{k \to \infty} (\alpha_n(y_k) \cdot f_{nj}(y_k)) = \begin{cases} \alpha_n(x) \cdot (s+1), & s \text{ finite,} \\ \alpha_n(x), & s = -\infty, \end{cases}$$

is in S_j, contrary to our assumption that $x \notin A_{nj}$.

Next suppose that $x \in A_{nj_1} \cap \ldots \cap A_{nj_l}$ (and no other A_{ni}), but that x is not an endpoint of any T_{j_i} $(i = 1, \ldots, l)$. If x is adherent to A_{ni}, but $x \notin A_{ni}$, then, as above, $f_{ni}(y) \to \infty$ as $y \to x$ (with $y \in A_{ni}$, $\operatorname{sgn} f'_{ni}h^{-1}(y) = 1$). Thus there is a neighborhood N of x on which $u_n = \min(f_{nj_1}, \ldots, f_{nj_l})$, and this function is continuous at x.

Finally, if $x \in A_{nj}$ is an endpoint of T_j, the assertions of the lemma are clear.

Similarly, we may prove

Lemma 4. *l_n is continuous except possibly at an endpoint $e \in A_{nj}$ of T_j; in this case, both one-sided limits at e exist in $[-\infty, \infty)$, and*

$$\lim_{\substack{x \to e \\ x \in A_{nj}}} l_n(x) = l_n(e).$$

LEMMA 5. *$l_n(x) < u_n(x)$ for all $x \in T_n$.*

PROOF. Suppose x is a point at which both l_n and u_n are finite. Then $l_n(x) = f_{ni}(x)$ and $u_n(x) = f_{nj}(x)$ for some $i, j \leqslant n - 1$. Note that $f'_{ij}h^{-1}(x) > 0$. Hence, also

$$u_n(x) - l_n(x) = f_{nj}(x) - f_{ni}(x) = f_{ij}(x) > 0.$$

Now let U denote the subset of $\pi^{-1}(T_n)$ that corresponds to $\{(x, t) \in T_n \times \mathbf{R}^1 | l_n(x) < t < u_n(x)\}$. It follows from the preceding lemmas that U is fiber-wise homeomorphic with $T_n \times \mathbf{R}^1$. Thus we may assume that the section S_n is chosen to lie in U and is therefore related in the desired way to the S_j ($j \leqslant n - 1$).

By induction, we can then obtain a collection $\{S_n\}$ of sections related appropriately by ϕ. However we need, in addition, to insure that the resulting collection is locally finite at points of M. To do this we strengthen the induction as follows: Let a_k, b_k denote the endpoints of T_k and a'_k, b'_k the endpoints of T'_k. For $k \leqslant n - 1$, let $u_k, l_k: T_k \to [-\infty, \infty]$ and $u'_k, l'_k: T_k \to [-\infty, \infty]$ denote the analogues of u_n, l_n above. Assume by induction that, in addition to (a), the sections $S_1, \ldots, S_{n-1}$ satisfy

(b) for each $k \leqslant n - 1$, either

(b1) there exist $i, j \leqslant n - 1$ such that $a_k \in \mathring{T}_i \cap \mathring{T}_j$ and $\alpha_k(a_k)$ separates $\alpha_i(a_k)$ from $\alpha_j(a_k)$, or

(b2) $|f_{kj}(a_k)| > |f'_{kj}(a'_k)|$ for all $j \leqslant n - 1$;

(and similarly for b_k, $k \leqslant n - 1$).

(c) for each $k \leqslant n - 1$,

(c1) if $x \in T_k$ and $l_k(x) = -\infty$, then $u_k(x) > \min(u'_k(h^{-1}x), 1)$, and

(c2) if $x \in T_k$ and $u_k(x) = \infty$, then $l_k(x) < \max(l'_k(h^{-1}x), -1)$.

(These conditions say essentially that as each new section is added to M, it is added, whenever possible, at least as far from the preceding sections as the distance between the corresponding sections in M'.)

We have seen that we can add an nth section S_n so that (a) holds with n in place of $n - 1$; this we assume done. It remains to check that S_n may be adjusted so that then (b) and (c) hold for $k \leqslant n$.

LEMMA 6. *S_n may be adjusted so that* (a) *and* (b) *hold with n in place of $n - 1$.*

PROOF. *Case* 1. Fix $x \in \mathring{T}_n$. Let $p_1, \ldots, p_r$ denote the liftings of $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$ that lie on $\pi^{-1}(x)$ and do not separate liftings of interior points of $T_1, \ldots, T_{n-1}$; suppose $p_i = \alpha_{k_i}(a_{k_i})$ (or b_{k_i}), $i = 1, \ldots, r$. We may assume $r \geqslant 1$. First consider the case in which at least some of the p_i, say

$p_i, \ldots, p_s$ ($s \geq 1$), lie above $\alpha_n(x)$. If no lifting of x lies below $\alpha_n(x)$, then we can clearly adjust S_n so that (b2) holds at x, for $k \leq n-1$ and the resulting section. On the other hand, if $\alpha_j(x)$ is *any* lifting of x below $\alpha_n(x)$ (possibly one of the p_i) then, by induction, each $p_i \cdot f'_{k_i n}(h^{-1}x)$ ($i = 1, \ldots, s$) lies above $\alpha_j(x) \cdot f'_{jn}(h^{-1}x)$. It follows that there is an open interval of possible positions of $\alpha_n(x)$ for which (b2) holds In the contrary case (all of $p_1, \ldots, p_r$ lie below $\alpha_n(x)$) an analogous argument shows that (b2) can be recovered.

Case 2. Now suppose that x is an endpoint of T_n. If there are no liftings of x above $\alpha_n(x)$ (or none below) then we can clearly place $\alpha_n(x)$ so that (b2) is satisfied. Thus we assume there are liftings of x both above and below $\alpha_n(x)$. If there are liftings of interior points both above and below, then the argument of Case 1 applies. Hence we may assume that all liftings of interior points lie above (say). Now suppose $\alpha_j(x)$ is any lifting of x above $\alpha_n(x)$, and $\alpha_k(x)$ any below. Then x is an endpoint of T_k and (b2) is satisfied by induction. Thus $\alpha_k(x) \cdot f'_{kn}(H^{-1}x)$ is below $\alpha_j(x) \cdot f'_{jn}(h^{-1}x)$. Since this is true for each such pair, there is an open interval of satisfactory positions for the adjusted $\alpha_n(x)$.

Thus we now assume that S_n has been chosen so that both (a) and (b) hold with n in place of $n-1$.

LEMMA 7. *S_n may be adjusted so that* (a)–(c) *are satisfied for $S_1, \ldots, S_{n-1}, S_n$.*

PROOF. Define $I_- = \{x \in T_n | l_n(x) = -\infty\}$, $I_+ = \{x \in T_n | u_n(x) = \infty\}$. Let E_- (E_+) denote the set of those endpoints of the T_k ($k \leq n-1$) which are adherent to I_- (I_+), but not in I_- (I_+). By Lemmas 3 and 4, $I_- \cup E_-$ and $I_+ \cup E_+$ are closed.

If $e \in E_-$ then for some $k \leq n-1$ we have $e = a_k$ (or b_k) and $l_n(e) = f_{nk}(e)$ (see Figure 4). Since (b) holds we have $f_{kj}(e) > f'_{kj}(h^{-1}e)$ for all $j \leq n$ with $e \in T_j$. Hence we may adjust S_n so that on an open interval U_e containing e we have

$$(*) \qquad \begin{cases} u_n(x) > u'_n(h^{-1}x) & (\text{or } u_n(x) = u'_n(h^{-1}x) = \infty), \\ l_n(x) < l'_n(h^{-1}x) & (\text{or } l_n(x) = l'_n(h^{-1}x) = -\infty) \end{cases}$$

and so that (a), (b) continue to hold. Let $U = \bigcup_{e \in E_-} U_e$. Then $I_- - U$ is compact. Hence by Lemma 4 we may choose finitely many open intervals W_i ($i = 1, \ldots, s$), that cover $I_- - U$ and such that $l_n(x) < -2$ ($x \in W = \bigcup_{i=1}^s W_i$). Choose open intervals V_i with $\overline{V_i} \subseteq W_i$ and $I_- - U \subseteq V = \bigcup_{i=1}^s V_i$. Let $\delta\colon T_n \to [-1, 0]$ be a continuous function that is 0 off W and -1 on each $\overline{V_i}$. Replace S_n with the section defined by $\alpha_n(x) \cdot \delta(x)$ ($x \in T_n$). Note that we may assume that W contains no a_k (or b_k) with $\alpha_k(a_k)$ (or $\alpha_k(b_k)$) below S_n, so that (b) is not affected by this adjustment. The adjusted section satisfies (cl) for all $x \in T_n$, and also (c2), at least for $x \in O = V \cup W$.

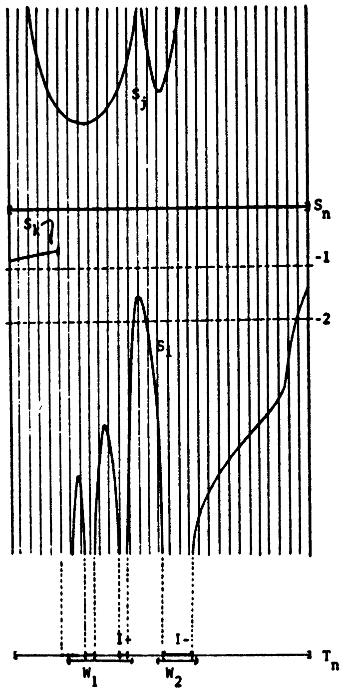

FIGURE 4

Now let $e \in E_+$, but $e \notin O$. Note that we may assume that O is chosen so $e \notin \bar{O}$. As above, we may adjust S_n on a neighborhood of e so that, afterward, (*) holds on an open interval B_e about e, and so that (a), (b) are not affected. Let $B = \bigcup_{e \in E_+ - O} B_e$. Then $I_+ - (B \cup O)$ is compact; the argument of the preceding paragraph allows us to adjust S_n on a sufficiently small neighborhood of this set, so that the resulting section satisfies (c).

Hence, by induction, we may construct a sequence $\{S_j\}$ of sections of ϕ satisfying (a)–(c) for all n. It is easy to check that $\{S_j\}$ is locally finite.

We can now obtain an equivalence $k: M' \to M$ of ϕ' with ϕ as follows. For each N_i' in the given subdivision of (M', ϕ'), let N_i denote the p-neighborhood in M bounded by the S_j that correspond to the S_j' in the ends of N_i', and the orbit segments of ϕ determined by the endpoints of the initial end. Define k by taking each N_i' homeomorphically fiberwise onto N_i, and so that $\pi k = h\pi'$ holds on each N_i'. Then k is an embedding since both $\{N_i\}$ and $\{N_i'\}$ are locally finite. Finally, since condition (c) holds, it is easily checked that k must be onto. Thus we have proved a slight strengthening of Theorem 1:

THEOREM 1$'$. *Suppose ϕ, ϕ' are completely unstable continuous flows on*

2-manifolds M, M' respectively, and that h: $M/\phi \to M'/\phi'$ is an isomorphism of the corresponding orbit spaces. Then there is an equivalence k of ϕ with ϕ' that satisfies $\pi' k = k\pi$.

REMARK. We may determine which ordered spaces can appear as the orbit space of a completely unstable flow. To do this we must first define order on a nonseparated 1-manifold, in a way which depends only on the 1-manifold (and not on its representation as the quotient space of a flow).

DEFINITION. Let K be a nonseparated 1-manifold. An *order* on K consists of a (countable) atlas $\{T_n\}$ of open subsets of K, each homeomorphic with (0, 1), and a collection $\{\sigma_{ij}\}$ of continuous functions σ_{ij}: $T_i \cap T_j \to \{-1, 1\}$ satisfying

(1) $\sigma_{ji} = -\sigma_{ij}$,

(2) if $\sigma_{ij}(x) = +1\ (-1)$ and $\sigma_{jk}(x) = +1\ (-1)$ then $\sigma_{ik}(x) = +1\ (-1)$,

(3) if $e \in T_i$ is an endpoint of a component interval I of $T_i \cap T_j$ and $e \in T_i \cap T_k$, then, for $x \in I \cap T_k$, $\sigma_{ij}(x)\sigma_{jk}(x) = -1$.

If ϕ is a completely unstable flow on the 2-manifold M, then M/ϕ may be ordered in this sense: choose disjoint sections $S_n \subseteq M$ of ϕ, each homeomorphic with an open interval, and so that $\{T_n = \pi(S_n)\}$ covers M/ϕ; define f_{ij}: $T_i \cap T_j \to \mathbf{R}^1$ as in the proof of Theorem 1 and set $\sigma_{ij} = \operatorname{sgn} f_{ij}$. It follows from the proof of Theorem 1 that this ordered space completely determines (M, ϕ). One may also prove the following:

Any nonseparated 1-manifold ordered in this sense can be realized as the ordered orbit space of a completely unstable flow (M, ϕ), where M is a Hausdorff 2-manifold.

4. **C^r-equivalence of completely unstable flows.** We can now give the proof of Theorem 2. We will need the following lemma; the proof is straightforward and so we omit it. As above, S_ε denotes $\{(x, y) \in \mathbf{R}^2 |\ |x|, |y| < 1 + \varepsilon\}$. A homeomorphism $f = (f_1, f_2)$: $S_\varepsilon \to \mathbf{R}^2$ (into) is said to be *fiberwise* if $f_1(x, y)$ is independent of y.

LEMMA 8. *Suppose $f = (f_1, f_2)$: $S_\varepsilon \to \mathbf{R}^2$ is a fiberwise homeomorphism, that is a C^r-diffeomorphism on an open set $U \subseteq S_\varepsilon$, that f_1 is C^r on all of S_ε and that f_{1x} is nonzero on S_ε. Fix $\eta > 0$. Then there is a fiberwise homeomorphism g: $S_\varepsilon \to \mathbf{R}^2$ satisfying*

(a) *$g_1(x, y) = f_1(x, y)$ for all $(x, y) \in S_\varepsilon$,*

(b) *$|f(x, y) - g(x, y)| < \eta$ for all $(x, y) \in S_\varepsilon$,*

(c) *g is a C^r-diffeomorphism on $U \cup \mathring{S}_{\varepsilon/4}$, and*

(d) *$g(x, y) = f(x, y)$ for all $(x, y) \in S_\varepsilon - S_{3\varepsilon/4}$.*

PROOF OF THEOREM 2. Let ϕ (ϕ') denote a completely unstable C^r-flow on the 2-manifold M (M'). Let M/ϕ (M'/ϕ') denote the corresponding orbit space, endowed with the order relation defined in §3 and the (unique)

C^r-structure with respect to which the natural projection $\pi: M \to M/\phi$ ($\pi': M' \to M'/\phi'$) is C^r (cf. §1). It is easy to check that a C^r-equivalence of ϕ with ϕ' induces a C^r-diffeomorphism of M/ϕ onto M'/ϕ' that preserves order. We prove the converse. Thus suppose we are given an order preserving C^r-diffeomorphism h of M/ϕ onto M'/ϕ'. By Theorem 1, there is a topological equivalence k of ϕ with ϕ' that makes the accompanying diagram commute. We wish to smooth k to a C^r-diffeomorphism, maintaining the commutativity. The preceding lemma will allow us to do this locally.

$$\begin{array}{ccc} M & \xrightarrow{\;k\;} & M' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M/\phi & \xrightarrow[\;h\;]{} & M'/\phi' \end{array}$$

Let $\{U_n\}$ $(n \in \mathbf{Z}^+)$ be a cover of M by open sets of the form $S_n \cdot \mathbf{R}^1$, where $S_n \cong (0, 1)$ is a C^r-section of ϕ. For each $n \in \mathbf{Z}^+$, define $W_n = U_1 \cup \ldots \cup U_n$. Suppose by induction that we have obtained an equivalence k_n, that covers h, and that is a C^r-diffeomorphism on W_n.

In W_{n+1}, $U_{n+1} - W_n$ and $\partial U_{n+1} \cap W_n$ are disjoint closed sets; hence, let G_{n+1} and H_{n+1} be disjoint open sets in W_{n+1}, with $U_{n+1} - W_n \subseteq G_{n+1}$, $\partial U_{n+1} \cap W_n \subseteq H_{n+1}$. We may also assume that

$$G_{n+1} \subseteq \{x \in M | \rho(x, U_{n+1} - W_n) < 1/(n+1)\},$$

where ρ is the given metric on M. We will replace k_n with an equivalence k_{n+1}, that differs from k_n only on G_{n+1}, and that is C^r on W_{n+1}.

Let $\{(N_i, O_i)\}$ $(i \in \mathbf{Z}^+)$ be a collection of p-neighborhood pairs in G_{n+1}, with $(N_i, O_i) \cong (S_\varepsilon, S_0)$ (some $\varepsilon > 0$), such that $\{\mathring{O}_i\}$ is a cover of $U_{n+1} - W_n$ and $\{N_i\}$ is locally finite at points of U_{n+1}. We now apply Lemma 8 to each N_i in succession. Assume by induction that we have adjusted $k_n = k_{n,0}$ to $k_{n,j}$ so that:

(i) $k_{n,j}$ is a C^r-diffeomorphism on $W_n \cup O_1 \cup \ldots \cup O_j$,

(ii) for $i \leqslant j$, $\rho'(k_{n,i-1}, k_{n,i}) < 2^{-i}$,

(iii) for $i \leqslant j$, $k_i = k_{i-1}$ off N_i.

Apply Lemma 8 to $k_{n,j}$ and N_{j+1}, with $U = W_n \cup O_1 \cup \ldots \cup O_j$ and $\eta = 2^{-(j+1)}$, to obtain $k_{n,j+1}$. By induction we obtain a sequence $\{k_{n,i}\}$ of equivalences, satisfying (i)–(iii) for all $j \in \mathbf{Z}^+$. Since $\{N_i\}$ is locally finite at points of U_{n+1}, the map l defined by

$$l(x) = \lim_{i \to \infty} k_{n,i}(x) \quad \text{for } x \in U_{n+1},$$

is a fiberwise diffeomorphism of U_{n+1} onto $k_n(U_{n+1})$ that covers $h|_{\pi(U_{n+1})}$. We must check that the extension of this map by k_n on $M - U_{n+1}$ remains an equivalence. Thus fix $x \in \partial U_{n+1}$. If $x \in W_n$, then k_n has not been adjusted on a neighborhood (H_{n+1}) of x. Hence, we may assume $x \in \partial U_{n+1} - W_n$. Fix $\delta > 0$. Pick j_0 so that $2^{-j_0+1} < \delta$. There is a neighborhood O of x that misses $N_1 \cup \ldots \cup N_{j_0}$, so if $y \in O \cap U_{n+1}$ we must have $\rho'(k_n(x), l(x)) < \delta$. Since δ was arbitrary, it follows that, if $\{y_i\} \subseteq U_{n+1}$ and $y_i \to x$ $(i \to \infty)$, then $l(y_i) \to k_n(x)$ $(i \to \infty)$. Thus k_n extends l continuously as desired.

Now denote by k_{n+1} the equivalence defined by

$$k_{n+1}(x) = \begin{cases} l(x), & x \in U_{n+1}, \\ k_n(x), & x \in M - U_{n+1}. \end{cases}$$

We have shown that k_{n+1} is an equivalence of ϕ with ϕ' that is a C^r-diffeomorphism on W_{n+1} and still covers h. By induction we obtain a sequence $\{k_n\}$ $(n \in \mathbf{Z}^+)$ of such equivalences, where, for each n, k_{n+1} differs from k_n at most on $G_{n+1} \subseteq \{x \in M | \rho(x, U_{n+1} - W_n) < 1/(n+1)\}$.

Finally, we show that, defining

$$k'(x) = \lim_{n \to \infty} k_n(x) \qquad (x \in M),$$

we obtain an equivalence covering h that is a C^r-diffeomorphism of M onto M'. It suffices to prove that, for any $x \in M$, there is a neighborhood O of x and an index n_0, such that $k_n(y) = k_{n_0}(y)$ for any $y \in O$ and all $n \geqslant n_0$. Fix $x \in M$. Let m be the least index with $x \in U_m$ and let $d = \rho(x, \partial U_m)$. Pick n_0 so that $1/n_0 < d/2$ and let O be the disc of radius $d/2$ centered at x. Then for any $n > n_0$ we see that $O \cap G_n = \varnothing$ and hence that k_n agrees with k_{n_0} on O, as asserted.

Proof of Theorem 3. Fix $r \in \{1, 2, \ldots, \infty\}$ and let ϕ denote a completely unstable (not necessarily C^r) flow on the 2-manifold M. It is proved in [7] that M/ϕ admits at least one C^r-structure and that any C^r-structure on M/ϕ lifts to a C^r-structure on M, with respect to which ϕ is C^r. By Munkres' theorem [6], the C^r-structure on a 2-manifold is unique; hence, there is a C^r-diffeomorphism h, of M with the C^r-structure lifted from M/ϕ, onto M with its given (original) C^r-structure. If we let ψ denote the flow induced by ϕ under h (viz., $\psi(x, t) = h\phi(h^{-1}x, t)$), then ψ is C^r in the given structure on M, and is topologically equivalent (under h^{-1}) to ϕ. Thus to any C^r-structure on M/ϕ we may assign a C^r-flow on M that is in the topological equivalence class of ϕ. It then follows from Theorem 2 that this assignment defines a 1-1 correspondence between the C^r-equivalence classes contained in the topological class of ϕ and the C^r-structures on M/ϕ.

Proof of Corollary. Fix $r \in \{1, 2, \ldots, \infty\}$. By Theorem 3 it suffices to construct a continuum of distinct C^r-structures on M/ϕ. Since there is at least one such structure, we may assume that we are given a C^r-atlas $\mathfrak{A} = \{(T_i, \alpha_i)\}$ $(i \in \mathbf{Z}^+)$ on M/ϕ. Here $\{T_i\}$ is an open cover of M/ϕ and each α_i is a homeomorphism of $(-1, 1)$, say, onto T_i. Fix $n, m \in \mathbf{Z}^+$ with

$T_n \cap T_m \neq \varnothing$. Let e_n be an endpoint of the component interval I of $T_n \cap T_m$, and let e_m be the corresponding point of T_m (see Figure 5; of course $T_n \cap T_m$ may be much more complicated than is indicated here). We may assume that e_n is in no T_i except T_n and that both α_n and α_m map $(0, \varepsilon)$ onto I, with $e_m = \alpha_m(0)$ and $e_n = \alpha_n(0)$.

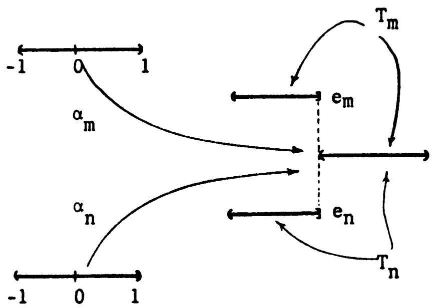

FIGURE 5

Now suppose that f is an orientation preserving homeomorphism of $(-1, 1)$ onto itself that fixes 0 and is a C^r-diffeomorphism in the complement of $\{0\}$. Note that if we alter the atlas $\mathfrak{A}$ only by replacing α_n with $\alpha_n \circ f$, then we obtain a new C^r-atlas on M/ϕ. For each real number $p > 1$, we construct in this manner an atlas $\mathfrak{A}_p$, using the homeomorphism f_p defined by

$$f_p(x) = \begin{cases} l(x^p), & x \in (0, 1), \\ x, & x \in (-1, 0], \end{cases}$$

where l is a fixed C^r-diffeomorphism of $(0, 1)$ onto itself that agrees with $\alpha_n^{-1}\alpha_m$ on some interval $(0, \delta)$ $(0 < \delta < \varepsilon)$. It is not difficult to check that no pair of the C^r (nonseparated) manifolds $\{(M, \mathfrak{A}_p)\}$ are even C^1-diffeomorphic, by a diffeomorphism which fixes e_n and e_m. For if d: $(M, \mathfrak{A}_p) \to (M, \mathfrak{A}_q)$ is such a diffeomorphism, let h, k: $(-1, 1) \to (-1, 1)$ denote the diffeomorphisms defined by $\alpha_m^{-1} d\alpha_m$, $f_q^{-1}\alpha_n^{-1} d\alpha_n f_p$ respectively. Then both

$$h'(0) = \lim_{x \to 0^+} \frac{h(x)}{x}$$

and

$$k'(0) = \lim_{x \to 0^+} \frac{k(x)}{x} = \lim_{x \to 0^+} \frac{(h(x^p))^{1/q}}{x} = \lim_{x \to 0^+} \left(\frac{h(x^p)}{x^p} \right)^{1/q} \cdot x^{(p-q)/q}$$

are finite and nonzero, an impossibility if $p \neq q$. But then, since the images of e_n and e_m under any diffeomorphism of M/ϕ must lie in the set of nonseparated points, which is countable by assumption, it follows that there are uncountably many distinct C^r-structures in the collection $\{\mathfrak{A}_p\}$.

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