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A GENERAL EXTREMAL PROBLEM FOR THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS(1)

BY

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ABSTRACT. For $\beta \geqslant 0$, K_β denotes the set of functions $f(z) = z + a_2 z^2 + \cdots$ defined on the unit disc U with the representation $f'(z) = ap^\beta(z)s(z)/z$, where $a \in C$, p is an analytic function with positive real part in U, and s is a normalized starlike function. If $0 \leqslant \beta \leqslant 1$, and $\zeta \in U$, let $F(u,v)$ be analytic in a neighborhood of $\{(f(\zeta),\zeta): f \in K_\beta\}$. Then $\max\{\operatorname{Re} F(f(\zeta),\zeta): f \in K_\beta\}$ occurs for a function of the form

$$f(z) = (\beta+1)^{-1}(x-y)^{-1}[(1+xz)^{\beta+1}(1+yz)^{-\beta-1} - 1],$$

where $|x| = |y| = 1$ and $x \neq y$. If $0 < \beta < 1$ these are the only extremal functions. A consequence of this result is the determination of the value region $\{f(\zeta)/\zeta: f \in K_\beta\}$ as $\{(\beta+1)^{-1}(s-t)^{-1}[(1+s)^{\beta+1}(1+t)^{-\beta-1} - 1]: |s|, |t| \leqslant |\zeta|\}$.

0. Introduction. Let U denote the unit disc $\{z: |z| < 1\}$ and S^* the family of univalent functions $s(z) = z + a_2 z^2 + \cdots$ that map U onto a starlike domain. An analytic function $f(z) = z + a_2 z^2 + \cdots$ defined on U is said to be close-to-convex of order β, $\beta \geqslant 0$, if there exists a complex number a, a function $s \in S^*$, and a function p analytic and with positive real part in U such that

$$f'(z) = ap^\beta(z)s(z)/z. \tag{0.1}$$

Since $f'(0) = 1$, we may always assume that $|a| = |p(0)| = 1$. The family of all functions which are close-to-convex of order β will be denoted K_β. The class K_1 was introduced by Kaplan [6] and the general class K_β was introduced later by Pommerenke [11]. Geometrically the class K_β, $0 \leqslant \beta \leqslant 1$, is characterized by the property that the tangent vector to any image curve $f(|z| = r)$ never turns back on itself by an angle greater than $\beta\pi$, i.e.,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\{1 + zf''(z)/f'(z)\}\,d\theta > -\beta\pi$$

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for $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi + \theta_1$ and $z = re^{i\theta}$. We consider a general extremal problem for the class K_β, $0 < \beta \leqslant 1$, which we solve using variational methods. We suppose that a function $F(u, v)$ is given which is analytic in a neighborhood of $\{(f(\zeta), \zeta): f \in K_\beta\}$ where ζ is a fixed point in U. We show that $\max_{f \in K_\beta} \operatorname{Re} F(f(\zeta), \zeta)$ is attained for a function of the form

$$f(z) = \frac{1}{\beta + 1} \frac{1}{x - y}\left[\left(\frac{1 + xz}{1 + yz}\right)^{\beta+1} - 1\right]$$

where $|x| = |y| = 1$ and $x \neq y$. Such functions map U onto the complement of a wedge of angular opening $(1 - \beta)\pi$.

Kirwan [8] solved the corresponding extremal problem for the class V_k, $2 \leqslant k \leqslant 4$, of functions with boundary rotation at most $k\pi$. The class V_k, $k \geqslant 2$, consists of all locally univalent, analytic functions $f(z) = z + a_2 z^2 + \cdots$ defined in U which satisfy

$$\int_0^{2\pi} \left|\operatorname{Re}\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta \leqslant k\pi.$$

This integral measures the total variation of the argument of the tangent vector to $f(|z| = r)$. These functions were introduced by Paatero [10]. The classes V_k and K_β are related by the set inclusion $V_{2\beta+2} \subset K_\beta$ [3].

Kirwan based his proof on a variational formula for functions of bounded boundary rotation given by Schiffer and Tammi [13] and on the Julia variational formula [5]. The proof of the stated result for the class K_β, $0 < \beta \leqslant 1$, depends on the variational formula for starlike functions given by Hummel [4], the variational formula for functions with positive real part given by Robertson [12], and on the Julia variational formula as applied by Kirwan.

We apply this general theorem to investigate the value region of $f(\zeta)/\zeta$, where $\zeta \in U$ is fixed, $|\zeta| = r$. The case $\beta = 0$ was studied by Strohhäcker [14] and the case $\beta = 1$ by Biernacki [2]. Denoting this region by $D_{r,\beta}$ we show that

$$\partial D_{r,\beta} \subset \left\{\frac{1}{\beta + 1} \frac{1}{s - t}\left[\left(\frac{1 + s}{1 + t}\right)^{\beta+1} - 1\right]: |s| = |t| = r\right\}$$

and that

$$D_{r,\beta} = \left\{\frac{1}{\beta + 1} \frac{1}{s - t}\left[\left(\frac{1 + s}{1 + t}\right)^{\beta+1} - 1\right]: |s|, |t| \leqslant r\right\}.$$

From this result we may infer information about the radius of starlikeness of K_β, which is defined to be the radius of the largest disc centered at 0 such that every function in K_β is starlike in this disc. Denoting this number by $r_s(\beta)$ we show that

$$r_S(\beta) = \min\left\{r\colon \operatorname{Re}\frac{1}{\beta+1}\frac{1}{s-t}\left[\left(\frac{1+s}{1+t}\right)^{\beta+1} - 1\right] = 0 \text{ on } |s| = |t| = r\right\}.$$

Lewandowski [9] showed explicitly that $r_S(1) = 4\sqrt{2} - 5$.

1. The variational formulae and outline of proof. Let $0 < \beta \leqslant 1$ and suppose that $F(u, v)$ is analytic in a neighborhood of $\{(f(\zeta), \zeta)\colon f \in K_\beta\}$, where ζ is a fixed point in U. We wish to determine the functions in K_β which yield

$$\max_{f \in K_\beta} \operatorname{Re} F(f(\zeta), \zeta). \tag{1.1}$$

The method of proof involves the use of a number of variational formulae which we now describe.

Since $f \in K_\beta$, we have

$$f'(z) = ap^\beta(z)s(z)/z, \tag{1.2}$$

where $s \in S^*$, $\operatorname{Re} p(z) > 0$, $p(0) = e^{i\gamma}$ and $a = e^{-i\beta\gamma}$. The first step in the proof will be to vary the function $s \in S^*$ using the Hummel variation for starlike functions [4]. Hummel showed that given an $s \in S^*$, there exists a function $s^* \in S^*$ of the form

$$\begin{aligned} s^*(z) = s(z) + \varepsilon(1 - |z_0|^2)\Bigg\{ &\frac{e^{i\alpha}}{z_0}\left[\frac{s(z)}{2}\frac{z+z_0}{z-z_0} + \frac{s(z)}{2} - A(z_0)\left(\frac{zs'(z)}{2}\frac{z+z_0}{z-z_0} - \frac{zs'(z)}{2} + s(z)\right)\right] \\ &+ \frac{e^{-i\alpha}}{\bar{z}_0}\left[\frac{s(z)}{2}\frac{1+z\bar{z}_0}{1-z\bar{z}_0} - \frac{s(z)}{2} + \overline{A}(z_0)\left(\frac{zs'(z)}{2}\frac{1+z\bar{z}_0}{1-z\bar{z}_0} - \frac{zs'(z)}{2}\right)\right]\Bigg\} \\ &+ o(\varepsilon) \end{aligned} \tag{1.3}$$

where $A(z_0) = s(z_0)/z_0 s'(z_0)$, ε is a positive parameter tending to 0, α is an arbitrary real number, and z_0 is an arbitrary point in the unit disc. Also the estimate for $o(\varepsilon)$ is uniform on compact subsets of U. The varied function maps U onto a domain obtained from $s(U)$ by making a small radial displacement of the boundary of $s(U)$.

An application of this variation shows that the function f cannot be extremal for (1.1) unless s maps U onto the complement of a finite set of disjoint rays, i.e., $\operatorname{Re}\{zs'(z)/s(z)\} = 0$ on $|z| = 1$ except for a finite number of points.

The next part of the proof involves varying the function p by means of a formula established by Robertson [12] for functions which are subordinate to a given function.

Suppose that p is subordinate to a univalent function p_0 in U, i.e., $p(z) = p_0(\omega(z))$ where ω: $U \to U$ and $\omega(0) = 0$. Then

$$\omega(z) = (Q(z) - 1)/(Q(z) + 1),$$

where $\operatorname{Re} Q(z) > 0$ for $z \in U$. Robertson gave the following variational formula for $p(z)$ which yields a function p^* which is also subordinate to p_0:

$$(1.4) \qquad p^*(z) = p(z) - \rho^2(1 - |z_0|^2)\frac{(1 - \phi(p(z)))^2}{2\phi'(p(z))}B(z) + o(\rho^2)$$

where

$$B(z) = \frac{2\phi'(p(z))p'(z)}{(1 - \phi(p(z)))^2}\left\{\frac{e^{i\theta}z}{Q(z_0)(z_0 - z)} + \frac{z^2e^{-i\theta}}{\overline{Q(z_0)}(1 - \bar{z}_0 z)}\right\}$$
$$+Q(z)\left\{\frac{e^{i\theta}z}{Q(z_0)(z_0 - z)^2} + \frac{e^{-i\theta}z}{\overline{Q(z_0)}(1 - \bar{z}_0 z)^2}\right\}$$
$$+\left\{\frac{e^{-i\theta}z}{(1 - \bar{z}_0 z)^2} - \frac{e^{i\theta}z^2}{z_0(z_0 - z)^2} - \frac{e^{i\theta}z}{z_0(z_0 - z)}\right\},$$

$\phi = p_0^{-1}$, θ is an arbitrary real number, z_0 is an arbitrary point of U, ρ is a parameter tending to 0 and the estimate for $o(\rho^2)$ is uniform on compact subsets of U.

Using this variation we show that f cannot be extremal for (1.1) unless $\operatorname{Re} p(z) = 0$ on $|z| = 1$ except for a finite number of points. A calculation then yields that an extremal f must map U onto a polygonal domain. To finish the proof we employ the Julia variational formula as done in [8] in order to show that the extremal domain is the complement of an infinite wedge of opening $(1 - \beta)\pi$.

The form of the Julia variation required for our application is that

$$(1.5) \quad f^*(z) = f(z) + \frac{\varepsilon z f'(z)}{2\pi i}\int_\Gamma \frac{\xi + z}{\xi - z}\operatorname{Im}\left\{-\psi(w)\frac{|dw|}{dw}\right\}\frac{n(w)}{\xi^2 f'^2(\xi)}dw + o(\varepsilon).$$

Here $f(z) = z + a_2 z + \cdots$ maps U conformally onto the domain with boundary $\Gamma = \{w = f(\xi)\colon |\xi| = 1\}$, $\psi(w)$ is continuous and piecewise differentiable on Γ, $n(w)$ is the unit exterior normal to Γ at w and $f^*(z) = a_1^* z + a_2^* z + \cdots$ maps U conformally onto the domain D^* with boundary $\Gamma^* = \{w^* = w + \varepsilon\psi(w)\}$ where $\varepsilon > 0$.

We now outline the method of application of the Julia variation which we will use. This method was originated by Biernacki [2] and has been refined by others (e.g., [1], [8]). Let ζ be fixed, $|\zeta| < 1$. Let $f \in K_\beta$ and let f^* be defined by (1.5). We expand $F(f^*(\zeta), \zeta)$ in powers of ε, obtaining

$$F(f^*(\zeta),\zeta) = F(f(\zeta),\zeta)$$

$$(1.6) \qquad +\varepsilon\zeta f'(\zeta)\frac{F_1(f(\zeta),\zeta)}{2\pi i}\int_\Gamma \frac{z+\zeta}{z-\zeta}\operatorname{Im}\left\{-\psi(w)\frac{|dw|}{dw}\right\}\frac{n(w)}{zf'^2(z)}dw + o(\varepsilon)$$

where we have set $F_1(u,v) = \partial F(u,v)/\partial u$.

Let us assume that the boundary of $f(U)$ contains three disjoint analytic arcs, which then correspond to three arcs l_1, l_2, and l_3 on $|z| = 1$. For fixed ζ, $\zeta f'(\zeta)F_1(f(\zeta),\zeta)(z+\zeta)/(z-\zeta)$ traces out a circle as z varies on $|z| = 1$. Hence, among the three arcs l_1, l_2, and l_3 there are two, which we denote by γ_1 and γ_2, satisfying

$$(1.7) \qquad \begin{aligned} &\max_{z\in\gamma_1}\operatorname{Re}\{\zeta f'(\zeta)F_1(f(\zeta),\zeta)(z+\zeta)/(z-\zeta)\} \\ &\qquad < \min_{z\in\gamma_2}\operatorname{Re}\{\zeta f'(\zeta)F_1(f(\zeta),\zeta)(z+\zeta)/(z-\zeta)\}. \end{aligned}$$

Let us choose $\psi(w)$ so that

$$(1.8) \qquad \operatorname{Im}\left\{-\psi(w)\frac{|dw|}{dw}\right\} \begin{cases} < 0 & \text{if } w \in f(\gamma_1), \\ > 0 & \text{if } w \in f(\gamma_2), \\ = 0 & \text{if } w \in \Gamma\backslash(f(\gamma_1)\cup f(\gamma_2)); \end{cases}$$

and

$$(1.9) \qquad \int_\Gamma \operatorname{Im}\left\{-\psi(w)\frac{|dw|}{dw}\right\}\frac{n(w)}{z^2f'^2(z)}dw = 0.$$

An application of the variational formula (1.6) to f with this choice of $\psi(w)$ yields $f^{*\prime}(0) = 1$ by (1.9) and

$$\operatorname{Re}\left\{\zeta f'(\zeta)\frac{F_1(f(\zeta),\zeta)}{2\pi i}\int_\Gamma\frac{z+\zeta}{z-\zeta}\operatorname{Im}\left\{-\psi(w)\frac{|dw|}{dw}\right\}\frac{n(w)}{zf'^2(z)}dw\right\} > 0$$

by (1.7), (1.8) and (1.9). Hence, from (1.6) we conclude that

$$\operatorname{Re}F(f^*(\zeta),\zeta) > \operatorname{Re}F(f(\zeta),\zeta)$$

and, consequently, if $f^* \in K_\beta$, then f is not extremal for problem (1.1). The following lemma summarizes the above discussion.

LEMMA 1.1. *Let $f \in K_\beta$, $0 < \beta < 1$, and suppose that the boundary of $f(U)$ contains three disjoint analytic arcs. Moreover let $F(u,v)$ be analytic in $\bigcup_{f\in K_\beta}(f(\zeta),\zeta)$ where ζ is fixed, $|\zeta| < 1$. Then*

(a) *there exist two arcs γ_1 and γ_2 on ∂U that satisfy* (1.7);

(b) *in addition, if $\psi(w)$ is defined on $\partial f(U)$ to satisfy* (1.8) *and* (1.9) *and if f^**

defined by (1.5) *belongs to* K_β, *then* f *is not an extremal function for problem* (1.1).

2. Statement and proof of the main theorem. We are now ready to state the main theorem.

THEOREM 2.1. *Let* ζ *be a fixed point of* U *and suppose that* $F(u,v)$ *is analytic in a neighborhood of* $\{(f(\zeta),\zeta): f \in K_\beta\}$ *where* $0 < \beta \leqslant 1$. *Then* $\max_{f\in K_\beta} \operatorname{Re} F(f(\zeta),\zeta)$ *is assumed for a function of the form*

$$f(z) = \frac{1}{\beta+1}\frac{1}{x-y}\left[\left(\frac{1+xz}{1+yz}\right)^{\beta+1} - 1\right] \tag{2.1}$$

where $|x| = |y| = 1$ *and* $x \neq y$. *If* $0 < \beta < 1$ *and if* F *is nonconstant, functions of the form* (2.1) *are the only functions for which the maximum is attained.*

We remark that a function of the form (2.1) maps U onto the complement of a wedge with angular opening $(1-\beta)\pi$. We also note that the case $\beta = 0$ was established by Kirwan [8].

Before beginning the proof of the theorem, it is necessary to prove the following lemma.

LEMMA 2.2. *Let* $f \in K_\beta$ *be an extremal function for* (1.1) *and suppose that* $f'(z) = ap(z)^\beta s(z)/z$ *where* $|a| = 1$, $a \cdot p(0) = 1$, $\operatorname{Re} p(z) > 0$ *for* $|z| < 1$ *and* $s \in S^*$. *Then*

$$\operatorname{Im}\left\{\frac{\partial F}{\partial u}(f(\zeta),\zeta)\left[f(\zeta) - \int_0^\zeta ap^\beta(z)s'(z)\,dz\right]\right\} = 0 \tag{2.2)(a}$$

and

$$\operatorname{Im}\left\{\frac{\partial F}{\partial u}(f(\zeta),\zeta)\cdot\int_0^\zeta ap^{\beta-1}(z)s(z)p'(z)\,dz\right\} = 0. \tag{2.3)(b}$$

PROOF. We have

$$f(\xi) = \int_0^\xi ap^\beta(z)\frac{s(z)}{z}\,dz.$$

If $\tau \in \mathbf{R}$ the function $e^{i\tau}s(e^{-i\tau}z)$ is also in S^* and we define the function $f_\tau \in K_\beta$ by

$$f_\tau(\xi) = \int_0^\xi ap^\beta(z)e^{i\tau}\frac{s(e^{-i\tau}z)}{z}\,dz.$$

But

$$s(e^{-i\tau}z) = s(z - i\tau z + o(\tau)) = s(z) - s'(z)i\tau z + o(\tau)$$

and $e^{i\tau} = 1 + i\tau + o(\tau)$. Therefore

$$f_\tau(\xi) = \int_0^\xi ap^\beta(z)\left\{\frac{s(z) + i\tau[s(z) - zs'(z)]}{z} + o(\tau)\right\}dz$$
$$= f(\xi) + i\tau\left[f(\xi) - \int_0^\xi ap^\beta(z)s'(z)\,dz\right] + o(\tau).$$

In these formulae, the $o(\tau)$-estimate is uniform on compact subsets of U. Expanding $F(f_\tau(\zeta),\zeta)$ in powers of τ and recalling that f is an extremal function, we obtain

$$\operatorname{Re} F(f_\tau(\zeta),\zeta)$$
$$= \operatorname{Re} F(f(\zeta),\zeta) + \operatorname{Re}\left\{i\tau F_1 \cdot \left[f(\zeta) - \int_0^\zeta ap^\beta(z)s'(z)\,dz\right]\right\} + o(\tau)$$
$$\leqslant \operatorname{Re} F(f(\zeta),\zeta),$$

where we have set $F_1 = \partial F(f(\zeta),\zeta)/\partial u$. Hence

$$\operatorname{Re}\left\{i\tau F_1 \cdot \left[f(\zeta) - \int_0^\zeta ap^\beta(z)s'(z)\,dz\right]\right\} + o(\tau) \leqslant 0.$$

Since τ can take on either positive or negative values, we conclude that

$$0 = \operatorname{Re}\left\{iF_1 \cdot \left[f(\zeta) - \int_0^\zeta ap^\beta(z)s'(z)\,dz\right]\right\}$$
$$= -\operatorname{Im}\left\{F_1 \cdot \left[f(\zeta) - \int_0^\zeta ap^\beta(z)\,dz\right]\right\}$$

which is (2.2).

To obtain the second relation, we set

$$f_\tau(\xi) = \int_0^\xi ap^\beta(e^{-i\tau}z)\frac{s(z)}{z}\,dz,$$

where $\tau \in \mathbf{R}$. Then $f_\tau \in K_\beta$. Moreover,

$$p(e^{-i\tau}z) = p(z - i\tau z + o(\tau)) = p(z) - i\tau zp'(z) + o(\tau)$$

and, hence,

$$p^\beta(e^{-i\tau}z) = p^\beta(z) - \beta i\tau zp'(z)p^{\beta-1}(z) + o(\tau).$$

Here again the estimate $o(\tau)$ is uniform on compact subsets of U. Then

$$f_\tau(\xi) = \int_0^\xi a\left[p^\beta(z) - \beta i\tau z p'(z)p^{\beta-1}(z)\right]\frac{s(z)}{z}\,dz + o(\tau)$$
$$= f(\xi) - \beta i\tau \int_0^\xi ap'(z)p^{\beta-1}(z)s(z)\,dz + o(\tau).$$

We again expand $F(f_\tau(\zeta),\zeta)$ in powers of τ and obtain

$$\begin{aligned}\operatorname{Re} F(f_\tau(\zeta),\zeta) \\ &= \operatorname{Re} F(f(\zeta),\zeta) + \operatorname{Re} F_1 \cdot \left[-\beta i\tau \int_0^\zeta ap^{\beta-1}(z)p'(z)s(z)\,dz\right] + o(\tau) \\ &\leqslant \operatorname{Re} F(f(\zeta),\zeta).\end{aligned}$$

Since f is assumed to be an extremal function and since τ may assume both positive and negative values, it follows that

$$\operatorname{Re} F_1 \cdot i\int_0^\zeta ap^{\beta-1}(z)p'(z)s(z)\,dz = 0$$

i.e.,

$$\operatorname{Im} F_1 \cdot \int_0^\zeta ap^{\beta-1}(z)p'(z)s(z)\,dz = 0$$

which is the second relation.

PROOF OF THEOREM. Let $f \in K_\beta$ be an extremal function for (1.1). Then $f'(z) = ap^\beta(z)s(z)/z$ where $|a| = |p(0)| = 1$, $\operatorname{Re} p(z) > 0$ and $s \in S^*$. Our first aim is to show that $s(z)$ has an analytic extension to $|z| = 1$ with a finite number of points deleted and that, except for these exceptional points, $\operatorname{Re}\{zs'(z)/s(z)\} = 0$ on $|z| = 1$.

We apply the Hummel variation to s obtaining a function $s^* \in S^*$ and we set $f^{*\prime}(z) = ap^\beta(z)s^*(z)/z$. Then

$$f^*(\zeta) = \int_0^\zeta ap^\beta(z)\frac{s^*(z)}{z}\,dz$$

and expanding $F(f^*(\zeta),\zeta)$ in powers of ε with the aid of (1.3), we obtain

$$F(f^*(\zeta),\zeta) = F(f(\zeta),\zeta) + \partial F(f(\zeta),\zeta)/\partial u \cdot h(z_0) + o(\varepsilon) \tag{2.4}$$

where

$$\begin{aligned} h(z_0) = \varepsilon(1-|z_0|^2)\int_0^\zeta \frac{ap^\beta(z)}{z}\Bigg\{ &\frac{e^{i\alpha}}{z_0}\Bigg[\frac{s(z)}{2}\frac{z+z_0}{z-z_0} + \frac{s(z)}{2} \\ &- A(z_0)\left(\frac{zs'(z)}{2}\frac{z+z_0}{z-z_0} - \frac{zs'(z)}{2} + s(z)\right)\Bigg] \\ &+ \frac{e^{-i\alpha}}{\bar z_0}\Bigg[\frac{s(z)}{2}\frac{1+\bar z_0 z}{1-\bar z_0 z} - \frac{s(z)}{2} + \bar A(z_0) \\ &\cdot\left(\frac{zs'(z)}{2}\frac{1+\bar z_0 z}{1-\bar z_0 z} - \frac{zs'(z)}{2}\right)\Bigg]\Bigg\}\,dz. \end{aligned}$$

Since f is an extremal function for (1.1),

$$\operatorname{Re} F(f^*(\zeta),\zeta) \leqslant \operatorname{Re} F(f(\zeta),\zeta).$$

From (2.4) we obtain

$$\operatorname{Re} F_1 \cdot h(z_0) + o(\varepsilon) \leqslant 0,$$

where we have set $F_1 = \partial F(f(\zeta),\zeta)/\partial u$. Dividing by $\varepsilon(1-|z_0|^2)$ and using the fact that $\operatorname{Re}\{aM + \bar{a}N\} = \operatorname{Re}\{a(M+\bar{N})\}$, we have

$$0 \geqslant \operatorname{Re} e^{i\alpha}\Bigg\{F_1 \cdot \int_0^\zeta a\frac{p^\beta(z)}{z}\frac{1}{z_0}\Bigg[\frac{s(z)}{2}\frac{z+z_0}{z-z_0}+\frac{s(z)}{2} -A(z_0)\left(\frac{zs'(z)}{2}\frac{z+z_0}{z-z_0}-\frac{zs'(z)}{2}+s(z)\right)\Bigg]dz$$
$$+\overline{F_1\int_0^\zeta ap^\beta(z)\frac{1}{z_0}\left[\frac{s(z)}{2}\frac{1+\bar{z}_0 z}{1-\bar{z}_0 z}-\frac{s(z)}{2}+\bar{A}(z_0)\left(\frac{zs'(z)}{2}\frac{1+z\bar{z}_0}{1-\bar{z}_0 z}-\frac{zs'(z)}{2}\right)\right]dz}\Bigg\}.$$

But α is an arbitrary real number so we conclude that

$$0 = F_1\int_0^\zeta \frac{ap^\beta(z)}{z}\frac{1}{z_0}\Bigg(\frac{s(z)}{2}\frac{z+z_0}{z-z_0}+\frac{s(z)}{2} -A(z_0)\left[\frac{zs'(z)}{2}\frac{z+z_0}{z-z_0}-\frac{zs'(z)}{2}+s(z)\right]\Bigg)dz$$
$$+\overline{F_1\int_0^\zeta \frac{ap^\beta(z)}{z}\frac{1}{z_0}\left(\frac{s(z)}{2}\frac{1+\bar{z}_0 z}{1-\bar{z}_0 z}-\frac{s(z)}{2}+\bar{A}(z_0)\left[\frac{zs'(z)}{2}\frac{1+\bar{z}_0 z}{1-\bar{z}_0 z}-\frac{zs'(z)}{2}\right]\right)dz}.$$

The solution of the above equation for $A(z_0) = s(z_0)/z_0 s'(z_0)$ yields

$$z_0 s'(z_0)/s(z_0) = Q(z_0)/R(z_0)$$

where

$$Q(z_0) = F_1\int_0^\zeta \frac{ap^\beta(z)}{z}\left(\frac{zs'(z)}{2}\frac{z+z_0}{z-z_0}-\frac{zs'(z)}{2}+s(z)\right)dz$$
$$-\overline{F_1\int_0^\zeta \left(\frac{zs'(z)}{2}\frac{1+\bar{z}_0 z}{1-\bar{z}_0 z}-\frac{zs'(z)}{2}\right)\frac{ap^\beta(z)}{z}dz}$$

and

$$R(z_0) = F_1\int_0^\zeta \frac{ap^\beta(z)}{z}\left(\frac{s(z)}{2}\frac{z+z_0}{z-z_0}+\frac{s(z)}{2}\right)dz$$
$$+\overline{F_1\int_0^\zeta \frac{ap^\beta(z)}{z}\left(\frac{s(z)}{2}\frac{1+\bar{z}_0 z}{1-\bar{z}_0 z}-\frac{s(z)}{2}\right)dz}.$$

From their definitions it is clear that $Q(z_0)$ and $R(z_0)$ are not identically zero unless $F_1 = 0$. But by a result in [7], this can only occur if F is identically constant, in which case the theorem is trivial. The equation $z_0 s'(z_0)/s(z_0) = Q(z_0)/R(z_0)$ holds for each $z_0 \in U$. Let us replace z_0 by t. We see that $Q(t)$ and $R(t)$ are defined in the annulus $|\zeta| < |t| < 1/|\zeta|$ and, hence, s has an analytic extension to $|t| = 1$, except possibly at the finite number of zeros of R on $|t| = 1$. A simple calculation shows that if $|t| = 1$ then $R(t) = \overline{-R(1/\bar{t})}$ and

$$\begin{aligned}\frac{Q(t)}{R(t)} + \overline{\frac{Q(1/\bar{t})}{R(1/\bar{t})}} &= \frac{1}{R(t)}\left\{F_1 \int_0^\zeta ap^\beta(z)\left[\frac{s(z)}{z} - s'(z)\right]dz\right. \\ &\qquad \left. - \bar{F}_1 \overline{\int_0^\zeta ap^\beta(z)\left[\frac{s(z)}{z} - s'(z)\right]dz}\right\} \\ &= 2i \operatorname{Im} F_1\left\{f(\zeta) - \int_0^\zeta ap^\beta(z)s'(z)\,dz\right\}\Big/R(t) \\ &= 0\end{aligned}$$

by (2.2). Therefore

$$\operatorname{Re}\{ts'(t)/s(t)\} = 0 \quad \text{on } |t| = 1 \tag{2.5}$$

except for a finite number of points. This implies that $\arg s(e^{i\theta})$ is constant on a finite number of intervals of $|t| = 1$ whose closure is $|t| = 1$.

We now assume that the extremal $f \in K_\beta$ satisfies $f'(z) = ap(z)s(z)/z$, where $s \in S^*$ satisfies (2.5). Our aim is to show that $p(z)$ has an analytic extension to $|z| = 1$ with a finite number of points deleted and that except for these exceptional points, $\operatorname{Re} p(z) = 0$ on $|z| = 1$. We let $p(0) = e^{i\gamma}$ where $-\pi/2 < \gamma < \pi/2$. Then p is subordinate to $p_\gamma(z) = (e^{i\gamma} + z)/(1 - e^{i\gamma}z)$ and we will now vary p using (1.4). Then

$$p^{*\beta}(z) = p^\beta(z) + p^{\beta-1}(z) \cdot \beta(p^*(z) - p(z)) + o(\rho^2)$$

and

$$f^*(\zeta) = f(\zeta) + \int_0^\zeta \frac{k(z)}{z}(p^*(z) - p(z))\,dz + o(\rho^2)$$

where $k(z) = a\beta p^{\beta-1}(z)s(z)$. Expanding $F(f^*(\zeta),\zeta)$ in powers of ρ^2 and again setting $F_1 = \partial F(f(\zeta),\zeta)/\partial u$, we obtain

$$\operatorname{Re} F(f^*(\zeta),\zeta) = \operatorname{Re} F(f(\zeta),\zeta) + \operatorname{Re} F_1 \cdot \int_0^\zeta \frac{k(z)}{z}(p^*(z) - p(z))\,dz + o(\rho^2),$$

and since f is extremal,

$$\operatorname{Re} F_1 \cdot \int_0^\zeta \frac{k(z)}{z}(p^*(z) - p(z))\,dz + o(\rho^2) \leqslant 0.$$

If we set $l(z) = (1 - \phi(p(z)))^2/2\phi'(p(z))$, divide both sides of the above inequality by $-\rho^2(1 - |z_0|^2)$, and let ρ tend to 0, we get

$$0 \leqslant \operatorname{Re}\Bigg[F_1 e^{i\theta} \int_0^\zeta k(z)\bigg\{\frac{p'(z)}{Q(z_0)(z_0 - z)} + \frac{Q(z)l(z)}{Q(z_0)(z_0 - z)^2} - \bigg[\frac{zl(z)}{z_0(z_0 - z)^2} + \frac{l(z)}{z_0(z_0 - z)}\bigg]\bigg\}dz$$
$$+ F_1 e^{-i\theta} \int_0^\zeta k(z)\bigg\{\frac{p'(z)z}{\overline{Q(z_0)}(1 - \bar{z}_0 z)} + \frac{Q(z)l(z)}{\overline{Q(z_0)}(1 - \bar{z}_0 z)^2} + \frac{l(z)}{(1 - \bar{z}_0 z)^2}\bigg\}dz\Bigg].$$

Since $\operatorname{Re}\{aM + \bar{a}N\} = \operatorname{Re}\{a(M + \overline{N})\}$ and since θ is arbitrary, we find that

$$Q(z_0) = A(z_0)/B(z_0)$$

where

$$A(z_0) = F_1 \int_0^\zeta \left(\frac{k(z)p'(z)}{(z_0 - z)} + \frac{k(z)Q(z)l(z)}{(z_0 - z)^2}\right)dz + \overline{F_1 \int_0^\zeta \left(\frac{zk(z)p'(z)}{1 - \bar{z}_0 z} + \frac{k(z)l(z)Q(z)}{(1 - \bar{z}_0 z)^2}\right)dz}$$

and

$$B(z_0) = F_1 \int_0^\zeta \frac{k(z)l(z)}{(z_0 - z)^2}dz - \overline{F_1 \int_0^\zeta \frac{k(z)l(z)}{(1 - \bar{z}_0 z)^2}dz}.$$

The relation $Q(z_0) = A(z_0)/B(z_0)$ holds for every $z_0 \in U$. As before, we replace z_0 by t. We see that $A(t)$ and $B(t)$ are not identically zero since we can assume $F_1 \neq 0$ and also that A and B are defined in the annulus $|\zeta| < |t| < 1/|\zeta|$. Calculating $\overline{Q(1/\bar{t})}$, we obtain

$$\overline{Q(1/\bar{t})} = \left(F_1 \int_0^\zeta \left(\frac{zk(z)p'(z)t}{t - z} + \frac{k(z)l(z)Q(z)z_0^2}{(t - z)^2}\right)dz + \overline{F_1 \int_0^\zeta \left(\frac{k(z)p'(z)\bar{z}_0}{1 - \bar{t}z} + \frac{k(z)Q(z)l(z)\bar{t}^2}{(1 - \bar{t}z)^2}\right)dz}\right) \Big/ - t^2 B(t)$$

and hence

$$Q(t) + \overline{Q(1/\bar{t})} = \left(F_1 \cdot \int_0^\zeta k(z)p'(z)\,dz - \bar{F}_1 \overline{\int_0^\zeta k(z)p'(z)\,dz}\right) \Big/ tB(t)$$
$$= \left(2i \operatorname{Im} F_1 \int_0^\zeta k(z)p'(z)\,dz\right) \Big/ tB(t) = 0$$

by (2.3). If $|t| = 1$, then $\operatorname{Re} Q(t) = 0$ except for the finite number of zeros of B on $|t| = 1$. Since Q is defined on $|t| \leqslant 1$ and $Q(t) = -\overline{Q(1/\bar{t})}$ on $|t| = 1$, Q is meromorphic on $\mathbf{C}$. Then $\omega(t) = (Q(t) - 1)/(Q(t) + 1)$ is also meromorphic on $\mathbf{C}$ and

$$\omega(1/\bar{t}) = \frac{Q(1/\bar{t}) - 1}{Q(1/\bar{t}) + 1} = \frac{\overline{Q(t)} + 1}{\overline{Q(t)} - 1} = \frac{1}{\overline{\omega(t)}}.$$

This implies that $|\omega(t)| = 1$ if $|t| = 1$. Since $p(t) = p_\gamma(\omega(t))$, p has a meromorphic extension to $\mathbf{C}$ and

$$p(1/\bar{t}) = p_\gamma(\omega(1/\bar{t})) = p_\gamma(1/\overline{\omega(t)}) = \overline{-p_\gamma(\omega(t))} = \overline{-p(t)}.$$

Thus except for a finite number of points on $|t| = 1$, $\operatorname{Re} p(t) = 0$. We observe that for $z = re^{i\theta}$,

$$\operatorname{Re}\{zp'(z)/p(z)\} = (\partial/\partial\theta)\arg p(re^{i\theta}),$$

and, hence, except for a finite number of points on $|t| = 1$, $\operatorname{Re}\{tp'(t)/p(t)\} = 0$.

We see that the extremal function f of (1.1) has an analytic extension to $|z| = 1$ with a finite number of points deleted. We now show that f maps U onto a polygonal domain. Writing $f'(z) = ap(z)^\beta s(z)/z$ and using the above observation together with (2.5), we compute

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \operatorname{Re}\left\{\beta z \frac{p'(z)}{p(z)} + \frac{zs'(z)}{s(z)}\right\} = 0$$

for all but finitely many z with $|z| = 1$. But $\operatorname{Re}\{1 + zf''(z)/f'(z)\}$ for $|z| = r$ measures the rate of turn of the tangent vector to the curve $f(\{|z| = r\})$. Since f is piecewise analytic on $\{|z| = 1\}$ we see that the boundary of $f(U)$ is composed of a finite number of line segments, rays, or lines. Hence f maps U onto a polygonal domain.

We now show that the extremal polygon has only one finite vertex and that the exterior angular opening is $(1 - \beta)\pi$. For this we need the following geometric characterization due to Pommerenke [11]. Let $f(z) = z + a_2 z^2 + \cdots$ be a conformal map of U. Then $f \in K_\beta$, $0 \leqslant \beta \leqslant 1$, if and only if the following condition holds:

(C) The complement, E, of $f(U)$ is the union of rays which are disjoint (except that perhaps the origin of one ray may lie on another ray) and which have the property that for each ray in E the sector of opening $(1-\beta)\pi$ whose bisector is the given ray lies in E.

Suppose f is a polygonal mapping and $f(U)$ is not a wedge of exterior angular opening $(1-\beta)\pi$. Kirwan [8] has constructed a variation for polygonal domains which preserves the angle at each vertex. One easily checks, using condition (C), that if a polygonal domain is the image of a function f of class K_β, then the polygon obtained by this method of variation is also the image of a function in K_β. Hence we may define a function ψ on $\partial f(U)$ so that the hypotheses of Lemma 1.1 are satisfied and we conclude that f is not extremal for problem (1.1). Consequently f maps U onto a wedge of exterior angular opening $(1-\beta)\pi$ and therefore f has the form (2.1) where $|x| = |y| = 1$ and $x \neq y$. This completes the proof of the theorem in the case $0 < \beta < 1$.

Let $\beta = 1$ and let $f \in K_1$ be extremal for (1.1). For $0 < \gamma < 1$, we claim $f_\gamma(z) = f(\gamma z)/\gamma \in K_{\beta(\gamma)}$ with $\beta(\gamma) < 1$. To see this, write $f'(z) = ap(z)^\beta s(z)/z$ as in (1.2). Then $f'_\gamma(z) = ap(\gamma z)^\beta s(\gamma z)/\gamma z$. Now $s_\gamma(z) = s(\gamma z)/\gamma \in S^*$ and $|\arg p(\gamma z)| \leqslant A_\gamma \pi/2 < \pi/2$. If we set $p_\gamma(z) = p(\gamma z)^{1/A_\gamma}$, then p_γ has positive real part in U and $f'_\gamma(z) = ap_\gamma(z)^{\beta A_\gamma} s_\gamma(z)/z$, showing that $f_\gamma \in K_{\beta A_\gamma}$ where $\beta A_\gamma < 1$. Applying the theorem, in the case $\beta = \beta(\gamma)$, there is a function g_γ of the form (2.1) such that $\operatorname{Re} F(f_\gamma(\zeta),\zeta) \leqslant \operatorname{Re} F(g_\gamma(\zeta),\zeta)$. For some increasing sequence $\{\gamma_n\}$ tending to 1, $\{g_{\gamma_n}\}$ converges to a limit function g in K_1, which is again of the form (2.1), with $\beta \leqslant 1$. Hence $\operatorname{Re} F(f(\zeta),\zeta) \leqslant \operatorname{Re} F(g(\zeta),\zeta)$ and since f was assumed to be extremal, the inequality is in fact an equality. It remains only to show that g is of the form (2.1) with $\beta = 1$. If not, $g \in K_\beta$, $\beta < 1$, and hence could not be extremal for (1.1) in $K_{\beta'}$, $\beta < \beta' < 1$. This completes the proof of Theorem 2.1.

3. **Applications.** Let $0 < \beta < 1$ and let ζ be fixed, $|\zeta| = r$. We wish to investigate the value region of $f(\zeta)/\zeta$ as f varies through K_β. The case $\beta = 1$ was studied by Biernacki [2] and the case $\beta = 0$ by Strohhäcker [14]. We define

$$D_{r,\beta} = \{f(\zeta)/\zeta : f \in K_\beta\},$$

and let $\partial D_{r,\beta}$ denote the boundary of $D_{r,\beta}$.

THEOREM 3.1. *If* $|\zeta| = r < 1$, *then*

(a) $$\partial D_{r,\beta} \subset \left\{\frac{1}{\beta+1}\frac{1}{s-t}\left[\left(\frac{1+s}{1+t}\right)^{\beta+1} - 1\right] : |s| = |t| = r\right\};$$

(b) $$D_{r,\beta} = \left\{\frac{1}{\beta+1}\frac{1}{s-t}\left[\left(\frac{1+s}{1+t}\right)^{\beta+1} - 1\right] : |s|, |t| \leqslant r\right\}.$$

PROOF. (a) We first show that there exists a set $E \subset D_{r,\beta}$ which is everywhere dense in $\partial D_{r,\beta}$ and has the property that if $u_1 \in E$ there is a point $v \in \mathcal{C}D_{r,\beta}$ such that $\{u\colon |u - v| < |u_1 - v|\} \subset \mathcal{C}D_{r,\beta}$. To see this, let w be an arbitrary point of $\partial D_{r,\beta}$ and let $\varepsilon > 0$. Choose $v \in \mathcal{C}D_{r,\beta}$ such that $|w - v| < \varepsilon/2$. The family K_β is compact so $D_{r,\beta}$ is closed and hence there is $\delta > 0$ such that $\{u\colon |u - v| < \delta\} \subset \mathcal{C}D_{r,\beta}$ and the boundary of this disc meets $D_{r,\beta}$ in a point u_1. By the definition of v, we see that $\delta \leqslant \varepsilon/2$, which implies $|u_1 - w| < \varepsilon$. But $u_1 \in E$ and, hence E is everywhere dense in $\partial D_{r,\beta}$.

Choose an arbitrary $u_1 \in E$ and let v be the corresponding point in $\mathcal{C}D_{r,\beta}$. Then $f(\zeta)/\zeta \neq v$ for all $f \in K_\beta$ and $\log(f(\zeta)/\zeta - v)$ is analytic on $\{(f(\zeta), \zeta)\colon f \in K_\beta\}$. Applying Theorem 2.1 in the case $0 < \beta < 1$ we find that

$$m = \min_{f \in K_\beta} \log\left|\frac{f(z)}{z} - v\right| = \min_{f \in K_\beta} \operatorname{Re}\log\left(\frac{f(z)}{z} - v\right)$$

is attained only for a function of the form

$$F_{x,y}(z) = \frac{1}{\beta + 1}\frac{1}{x - y}\left[\left(\frac{1 + xz}{1 + yz}\right)^{\beta+1} - 1\right], \qquad x \neq y,\ |x| = |y| = 1.$$

By construction, $m = \log|u_1 - v|$ so that every point of E corresponds to some $F_{x,y}$. If $u \in \partial D_{r,\beta}$ is arbitrary, there is a sequence $\{u_n\} \subset E$ with $\lim_{n\to\infty} u_n = u$. For some $x_n \neq y_n$ of modulus 1, $u_n = F_{x_n,y_n}(\zeta)/\zeta$, and a subsequence of $\{F_{x_n,y_n}\}$ converges to a function again of the form $F_{x,y}$, where here it is possible that $x = y$. Hence

$$u = \frac{F_{x,y}(\zeta)}{\zeta} = \frac{1}{\beta + 1}\frac{1}{(x - y)\zeta}\left[\left(\frac{1 + x\zeta}{1 + y\zeta}\right)^{\beta+1} - 1\right].$$

If we set $x\zeta = s$ and $y\zeta = t$, we obtain the desired result.

(b) Set

$$f_\beta(s,t) = \frac{1}{\beta + 1}\frac{1}{s - t}\left[\left(\frac{1 + s}{1 + t}\right)^{\beta+1} - 1\right].$$

We first show

$$D_{r,\beta} \supset \{f_\beta(s,t)\colon |t| \leqslant |s| = r \text{ or } |s| \leqslant |t| = r\}. \tag{3.1}$$

To do this we fix t, $|t| = r$, and consider the curve Γ_β defined by

$$\Gamma_\beta(s) = f_\beta(s,t) = \frac{(1 + s)^{\beta+1} - (1 + t)^{\beta+1}}{(\beta + 1)(s - t)(1 + t)^{\beta+1}}, \qquad |s| = r.$$

Then $\Gamma_\beta(t) = (1 + t)^{-1}$ and $\Gamma_0(s) = (1 + t)^{-1}$, i.e., Γ_0 reduces to a single point.

Suppose $z \in \operatorname{int}\Gamma_\beta$ (interior of Γ_β), i.e., the index of z with respect to Γ_β, $n(\Gamma_\beta, z)$, is nonzero. We claim $z \in \Gamma_\alpha$ for some α, $0 < \alpha < \beta$. Otherwise, consider the function $H: \{|s| = r\} \times [0, \beta] \to \mathbf{C} - \{z\}$ defined by $H(s,\beta) = \Gamma_\beta(s)$. H is a homotopy of Γ_0 to Γ_β relative to $\{(1+t)^{-1}\}$ in $\mathbf{C}\backslash\{z\}$. This would imply

$$n(\Gamma_\beta, z) = \frac{1}{2\pi i}\int_{\Gamma_\beta}\frac{dw}{w-z} = \frac{1}{2\pi i}\int_{\Gamma_0}\frac{dw}{w-z} = n(\Gamma_0, z) = 0,$$

contradicting the assumption made on z. Consequently every point $z \in \operatorname{int}\Gamma_\beta$ corresponds to an extremal function in K_α for some α, $0 < \alpha < \beta$. Since $K_\alpha \subset K_\beta$, we conclude that $\operatorname{int}\Gamma_\beta \subset D_{r,\beta}$ and by the argument principle we obtain $\{f_\beta(s,t): |s| \leqslant |t| = r\} \subset D_{r,\beta}$. Interchanging the roles of s and t, the above argument shows that $\{f_\beta(s,t): |t| \leqslant |s| = r\} \subset D_{r,\beta}$, which gives us (3.1). Suppose $0 < r_1 < r < 1$ and $u = f(\xi)/\xi \in D_{r_1,\beta}$, $|\xi| = r_1$. If we define $g(z) = f(\tau z)/\tau$ where $\tau = r_1/r$, then $g \in K_\beta$. Set $\zeta = \xi/\tau$, so that $|\zeta| = r$ and, consequently,

$$u = f(\xi)/\xi = g(\zeta)/\zeta \in D_{r,\beta},$$

which shows that $D_{r_1,\beta} \subset D_{r,\beta}$. We note that $D_{0,\beta} = \{1\}$. Hence

$$\begin{aligned} D_{r,\beta} &= \bigcup_{0 \leqslant r_1 < r} D_{r_1,\beta} \\ &\supset \bigcup_{0 \leqslant r_1 \leqslant r} \{f_\beta(s,t): |s| \leqslant |t| = r_1 \text{ or } |t| \leqslant |s| = r_1\} \\ &= \{f_\beta(s,t): |s|, |t| \leqslant r\}. \end{aligned}$$

Let $C_{r,\beta}$ denote $\{f_\beta(s,t): |s|, |t| \leqslant r\}$. For $w \neq 1$ and $w \in D_{r,\beta}$, set $r_0 = \inf\{x: w \in D_{x,\beta}\}$. Clearly $w \in D_{r_0,\beta}$. For $f \in K_\beta$, let $\hat{f}$ denote the analytic function defined by $\hat{f}(z) = f(z)/z$. The set $\bigcup_{x<r_0} D_{x,\beta}$ is open since it equals

$$\bigcup_{x<r_0}\{\hat{f}(\zeta): |\zeta| = x, f \in K_\beta\} = \bigcup_{f \in K_\beta}\{\hat{f}(\zeta): |\zeta| < r_0\}.$$

We then have

$$\begin{aligned} D_{r_0,\beta} &= \bigcup_{x \leqslant r_0} D_{x,\beta} = \bigcup_{f \in K_\beta} \hat{f}(|z| \leqslant r_0) = \bigcup_{f \in K_\beta} \overline{\hat{f}(|z| < r_0)} \\ &\subset \overline{\bigcup_{f \in K_\beta} \hat{f}(|z| < r_0)} = \overline{\bigcup_{x<r_0} D_{x,\beta}} \subset \overline{D_{r_0,\beta}} = D_{r_0,\beta}, \end{aligned}$$

showing that $D_{r_0,\beta}$ is the closure of the open set $\bigcup_{x<r_0} D_{x,\beta}$. If V is an arbitrary disc about w, V meets $D_{x,\beta}$ for some x sufficiently near and less than r_0. Since $w \in \mathcal{C}D_{x,\beta}$, V contains a point of $\partial D_{x,\beta}$. Hence there exists a sequence

$w_n \in \partial D_{x_n,\beta}$ whereby $x_n \to r_0$ and $w_n \to w$. But by part (a) of the theorem, $w_n = f_\beta(s_n, t_n)$ where $|s_n| = |t_n| = x_n$ implying that $w = f_\beta(s, t)$ where $|s| = |t| = r_0$. Therefore $w \in C_{r_0,\beta} \subset C_{r,\beta}$, i.e., $D_{r,\beta} \subset C_{r,\beta}$, which completes the proof.

As a second application of Theorem 1.2 we consider the problem of determining the radius of starlikeness, $r_s(\beta)$, for the class K_β. The case $\beta = 1$ was solved by Lewandowski [9], who showed that $r_s(1) = 4\sqrt{2} - 5$.

Corollary 3.2. *If* $0 < \beta \leqslant 1$,

$$r_s(\beta) = \min\{r\colon \operatorname{Re} f_\beta(s,t) = 0 \text{ on } |s| = |t| = r\}.$$

Proof. The function

$$g(z) = \frac{f((z + \zeta)/(1 + \bar{\zeta} z)) - f(\zeta)}{f'(\zeta)(1 - |\zeta|^2)} \in K_\beta$$

if $f \in K_\beta$ and $|\zeta| < 1$. Setting $z = -\zeta$, we obtain

$$\frac{g(-\zeta)}{-\zeta} = \frac{1}{1 - |\zeta|^2} \frac{f(\zeta)}{\zeta f'(\zeta)}. \tag{3.2}$$

The function f is starlike in $|\zeta| < r$ if and only if $\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\} > 0$ for $|\zeta| < r$. Hence $r_s(\beta)$ is the largest value of $r, 0 < r < 1$, such that $\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\} \geqslant 0$ for every $f \in K_\beta$. By (3.2), $r_s(\beta)$ is also the largest value of r satisfying $\operatorname{Re}\{f(\zeta)/\zeta\} \geqslant 0$ and this is the smallest value of r such that $D_{r,\beta}$ cuts the imaginary axis, which must then occur at some point of $\partial D_{r,\beta}$. The proof is completed by applying Theorem 3.1(a).

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