TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 226, 1977

LINEAR OPERATORS FOR WHICH T^*T AND $T + T^*$ COMMUTE. II

BY

STEPHEN L. CAMPBELL(1) AND RALPH GELLAR

ABSTRACT. Let θ denote the set of bounded linear operators T, acting on a separable Hilbert space $\mathcal{H}$, such that T^*T and $T + T^*$ commute. It is shown that such operators are G_1. A complete structure theory is developed for the case when $\sigma(T)$ does not intersect the real axis. Using this structure theory, several nonhyponormal operators in θ with special properties are constructed.

1. Let θ denote the set of bounded linear operators T, acting on a separable Hilbert space $\mathcal{H}$, such that T^*T and $T + T^*$ commute. It is shown that such operators are G_1. A complete structure theory is developed for the case when $\sigma(T)$ does not intersect the real axis. Using this structure theory, nonhyponormal operators in θ are constructed. Some results on the structure of $\sigma(T)$ are also obtained.

2. **Introduction.** The class θ has been studied in [3], [4], [5], and considered in [8], [9]. Our notation and terminology will be that of [5]. We shall review it briefly. If $T \in \theta$, then $4T^*T - (T^* + T)^2 \geqslant 0$ [5]. Define

$$C = \frac{(T^* + T) + i\sqrt{4T^*T - (T^* + T)^2}}{2}. \tag{1}$$

Then C is normal, $\sigma(C)$ is contained in the closed upper half-plane, $C^*C = T^*T$, and $T + T^* = C + C^*$ [5]. In particular,

$$(\lambda - T^*)(\lambda - T) = (\lambda - C^*)(\lambda - C)$$

for all λ. If $T \in \theta$ and T is completely nonnormal, then $\sigma(T) = \sigma(T^*)$, $\sigma(C) \subsetneq \sigma(T)$, $\partial\sigma(T) \subsetneq \sigma(C) \cup \sigma(C^*)$, and $\sigma_p(T) = 0$ [4], [5]. The spectral measure for C is denoted by $F(\cdot)$. Any operator E such that $E^2 = E$ will be called a projection. The real numbers are denoted by $\mathfrak{R}$. UHP (LHP) is the open

Received by the editors November 19, 1975.

AMS (MOS) subject classifications (1970). Primary 47A15; Secondary 47B20, 47B99.

Key words and phrases. Operator such that T^*T and $T + T^*$ commute, spectrum, hyponormal operator, G_1 operator, subnormal operator, normal extension.

(1) Supported in part by a grant from the North Carolina Engineering Foundation.

upper (lower) half-plane, $\overline{\mathrm{UHP}}$ ($\overline{\mathrm{LHP}}$) are their closures. The restriction of an operator B to subspace $\mathfrak{M}$ is denoted $B|\mathfrak{M}$.

3. $T \in \theta$ with $\sigma(T) \cap \mathfrak{R} = \varnothing$. Our first result will be fundamental in the sequel.

THEOREM 1. *Suppose that C is a normal operator on $\mathcal{H}$ and E is a projection such that*
(i) $C^*(I - E) = (I - E)C^*(I - E)$, $\{EC^*(I - E) = 0\}$,
(ii) $CE = ECE$, $\{(I - E)CE = 0\}$,
(iii) $E^*(C - C^*)(I - E) = 0$.
Let

$$T = CE + C^*(I - E). \tag{2}$$

Then $T \in \theta$.

PROOF. Suppose that C, E satisfy conditions (i), (ii), (iii). Note that by (iii) and (i):

$$\begin{aligned} E^*C^{*2}(I - E) &= E^*C^*(I - E)C^*(I - E) = E^*C(I - E)C^*(I - E) \\ &= E^*CC^*(I - E). \end{aligned}$$

Let $T = CE + C^*(I - E)$. Then

$$\begin{aligned} T + T^* &= CE + C^*(I - E) + E^*C^* + (I - E^*)C \\ &= C^* + C + [CE - C^*E + E^*C^* - E^*C]. \end{aligned}$$

But,

$$\begin{aligned} CE - C^*E + E^*C^* - E^*C &= (C - C^*)E + E^*(C^* - C) \\ &= (C - C^*)E + E^*(C^* - C)E = (I - E^*)(C - C^*)E = 0. \end{aligned}$$

Thus $T + T^* = C + C^*$. Hence $T^* = C + C^* - T$, or

$$T^* = C^*E + C(I - E). \tag{3}$$

Using (2), (3) we get

$$T^*T = [C^*E + C(I - E)][ECE + (I - E)C^*(I - E)] = C^*C.$$

Thus $T \in \theta$. □

Our next result shows that if $\sigma(T) \cap \mathfrak{R} = \varnothing$, then T is in the form of Theorem 1.

THEOREM 2. *Suppose that $T \in \theta$ and $\sigma(T) \cap \mathfrak{R} = \varnothing$. Let E be the projection* *obtained by integrating* $(\lambda - T)^{-1}$ *around that portion of* $\sigma(T)$ *in the upper half-plane. Let* C *be as in* (1). *Then* C, E *satisfy* (i), (ii), (iii) *and* $T = CE + C^*(I - E)$.

PROOF. Since $(\lambda - T^*)(\lambda - T) = (\lambda - C^*)(\lambda - C)$ for all λ, we have for all $\lambda \notin \sigma(C) \cup \sigma(C^*)$

$$(C - C^*)(\lambda - T)^{-1} = [(\lambda - C)^{-1} - (\lambda - C^*)^{-1}](\lambda - T^*).$$

Integrating this first around the upper portion of $\sigma(T)$ and then the lower portion of $\sigma(T)$ gives

$$(C - C^*)E = C - T^* \quad \text{or} \quad E = (C - C^*)^{-1}(C - T^*),$$

and

$$(C - C^*)(I - E) = -(C^* - T^*) \quad \text{or} \quad I - E = (C - C^*)^{-1}(T^* - C^*).$$

By definition of E, we have $TE = ET$. Now

$$\begin{aligned} CE &= C(C - C^*)^{-1}(T - C^*) = (C - C^*)^{-1}(CT - C^*C) \\ &= (C - C^*)^{-1}(C - T^*)T = ET = TE. \end{aligned}$$

Thus (ii) holds. Similarly, $C^*(I - E) = (I - E)T = T(I - E)$. Thus $T = CE + C^*(I - E)$. There remains only to check (iii);

$$\begin{aligned} E^*(C - C^*)(I - E) &= (C^* - T)(C^* - C)^{-1}(C - C^*)(I - E) \\ &= -C^*(I - E) + T(I - E) = 0. \quad \square \end{aligned}$$

One might suppose that the existence of the C, E in Theorem 1 is restrictive. The next theorem shows it is not.

THEOREM 3. *Let* C *be any normal operator such that* $\sigma(C) \subseteq UHP$. *Let* $\mathfrak{M}_1$ *be any invariant subspace for* C. *Let* $\mathfrak{M}_2 = (C - C^*)^{-1}\mathfrak{M}_1^\perp$. *Let* E *be the projection onto* $\mathfrak{M}_1$ *along* $\mathfrak{M}_2$. *Then* $T = CE + C^*(I - E) \in \theta$ *and* C, E *satisfy* (i), (ii), (iii).

PROOF. Let C, $\mathfrak{M}_1$, $\mathfrak{M}_2$ be as in the statement of the theorem. Clearly $\mathfrak{M}_1$ is C invariant. Thus $\mathfrak{M}_2$ is C^* invariant since $\mathfrak{M}_1^\perp$ is C^* invariant. Let $(C - C^*)^{1/2}$ denote an analytic square root of $C - C^*$. Now $(C - C^*)^{1/2}\mathfrak{M}_1 \oplus (C - C^*)^{-1/2}\mathfrak{M}_1^\perp = \mathfrak{K}$. Multiplying by $(C - C^*)^{-1/2}$ we see that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{K}$, + denoting a direct sum. Thus E is bounded. Conditions (i), (ii) are now immediate. Condition (iii) is equivalent to $(C - C^*)\mathfrak{M}_2 \subseteq \mathfrak{M}_1^\perp$. But this follows from the definition of $\mathfrak{M}_2$. $\square$

COROLLARY 1. *If $T \in \theta$, and $\sigma(T) \cap \mathfrak{R} = \varnothing$, then T is similar to the orthogonal sum of two subnormal operators, T_1, T_2 and $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, $\sigma(T_1) \cap \sigma(T_2) = \varnothing$.*

Thus if $T \in \theta$, $\sigma(T) \cap \mathfrak{R} = \varnothing$, and T is completely nonnormal, any results about the spectra of subnormal operators may be applied to T.

For a compact set X, let $C(X)$ denote the continuous functions on X and $\mathfrak{R}(X)$ the functions on X which are uniformly approximable by rational functions with poles off X. Then from [6] and the results of this section we have:

PROPOSITION 1. *A compact set Σ such that $\Sigma \cap \mathfrak{R} = \varnothing$ is the spectrum of a completely nonnormal $T \in \theta$ if and only if Σ is symmetric with respect to the real axis and $\mathfrak{R}(\Sigma \cap \overline{D}) \neq C(\Sigma \cap \overline{D})$ for every open disc D such that $\Sigma \cap D \neq \varnothing$.*

The only part that needs to be proved is that if C is normal with an invariant subspace $\mathfrak{M}$, C is the minimal normal extension of $C|\mathfrak{M}$, and $C|\mathfrak{M}$ is completely nonnormal, then the T generated by C, $\mathfrak{M}$ is completely nonnormal. We now examine the relationship between the complete nonnormality of T and the complete nonnormality of $C|\mathfrak{M}$.

First we need the following well-known result whose proof we omit.

PROPOSITION 2. *Suppose T is hyponormal. If the subspace $\mathfrak{M}$ is invariant under T and $T|\mathfrak{M}$ is normal, then $\mathfrak{M}$ reduces T.*

PROPOSITION 3. *Let $\mathfrak{M}_1 \subseteq N(T-C)$, $(\mathfrak{M}_2 \subseteq N(T-C^*))$ be C, (C^*) invariant subspaces. If $C|\mathfrak{M}_1$ $(C^*|\mathfrak{M}_2)$ has a normal summand, then T has a normal summand.*

The proof follows from Proposition 2 and the fact that $T\phi = C\phi$, $T^*\phi = C^*\phi$ for $\phi \in \mathfrak{M}_1$ $(T\phi = C^*\phi, T^*\phi = C\phi$ for $\phi \in \mathfrak{M}_2)$.

THEOREM 4. *Suppose that $T \in \theta$, $\sigma(T) \cap \mathfrak{R} = \varnothing$, and C, E are as in Theorem 1. Let $\mathfrak{M}_1 = E\mathfrak{K}$ and $\mathfrak{M}_2 = (I-E)\mathfrak{K}$. Then T is completely nonnormal if and only if both $C|\mathfrak{M}_1$, and $C^*|\mathfrak{M}_2$ are completely nonnormal.*

PROOF. Proposition 3 takes care of the only if part. Suppose now that T has a normal summand so that $T = T_1 \oplus T_2$ where T_2 is normal. Since $(\lambda - T)^{-1} = (\lambda - T_1)^{-1} \oplus (\lambda - T_2)^{-1}$, one of E or $(I-E)$ has a normal summand and C has a corresponding normal summand. Hence either $C|\mathfrak{M}_1$ or $C^*|\mathfrak{M}_2$ has a normal summand. □

Theorem 4 has the following interesting consequence.

THEOREM 5. *Let T, C, E, $\mathfrak{M}_1$ be as in Theorem 4. Then T is completely nonnormal if and only if $C|\mathfrak{M}_1$ is completely nonnormal and C is the minimal normal extension of $C|\mathfrak{M}_1$.*

PROOF. C is not the minimal normal extension of $C|\mathfrak{M}_1$ if and only if there is a subspace $\tilde{\mathfrak{N}} \subseteq \mathfrak{M}_1^\perp$ which reduces C. But from Theorem 3, $\mathfrak{M}_2 = (C - C^*)^{-1}\mathfrak{M}_1^\perp$. Clearly $(C - C^*)^{-1}\tilde{\mathfrak{N}} = \tilde{\mathfrak{N}}$. Thus C is not the minimal normal extension of $C|\mathfrak{M}_1$ if and only if $C^*|\mathfrak{M}_2$ has a normal summand. Theorem 5 now follows from Theorem 4. □

Theorems 1, 2, and 3 completely characterize $T \in \theta$ with $\sigma(T) \cap \mathfrak{R} = \varnothing$. When considering some specific examples in §5 we will need the following results.

THEOREM 6. *Suppose that $T \in \theta$, there exists C, E satisfying* (i), (ii), (iii), *and $C - C^*$ is one-to-one. If T is also hyponormal, then T is normal.*

PROOF. Suppose that $T \in \theta$, C and E satisfy (i)–(iii), $C - C^*$ is one-to-one, and T is hyponormal. Then

$$\begin{aligned} T^*T - TT^* &= C^*C - [CE + C^*(I-E)][E^*C^* + (I-E)^*C] \\ &= C^*C - CEE^*C^* - E(I-E)^*C \\ &\quad - C^*(I-E)E^*C^* - C^*(I-E)(I-E)^*C \\ &= CEE^*(C - C^*) + C^*EE^*(C^* - C) \\ &\quad + (C^* - C)EC + C^*E^*(C - C^*) \\ &= (C^* - C)EE^*(C^* - C) \\ &\quad + (C^* - C)EC + C^*E^*(C - C^*). \end{aligned} \tag{4}$$

Thus $(I - E^*)[T^*T - TT^*](I - E) = 0$. But $[T^*T - TT^*] \geqslant 0$ so that $[T^*T - TT^*](I - E) = 0$. Thus by (4), we have $(C^* - C)EC(I - E) = 0$. But $C^* - C$ is one-to-one. Hence $EC(I - E) = 0$, or $EC = ECE = CE$. Since C is normal we also have $EC^* = C^*E$ by Fuglede's theorem [10]. Thus (iii) becomes $(C - C^*)E^*(I - E) = 0$ or $E^*(I - E) = 0$. But then $E^* = E^*E$. Hence E is hermitian and reduces T. But $\sigma(TE) \subseteq$ UHP, $TE \in \theta$, implies T is normal [5]. □

COROLLARY 2. *If $T \in \theta$, $\sigma(T) \cap \mathfrak{R} = \varnothing$, and T is not normal, then T is not seminormal.*

COROLLARY 3. *If $T \in \theta$ is hyponormal and completely nonnormal, then there does not exist an E satisfying* (i), (ii), (iii) *where* (2) *holds.*

4. **Operators in θ are G_1.** An operator is called G_1 if for all $\lambda \notin \sigma(T)$, $\|(\lambda - T)^{-1}\|$ is the reciprocal of the distance from λ to $\sigma(T)$. That is,

$$\|(\lambda - T)^{-1}\| = 1/\rho(\lambda, \sigma(T)).$$

Hyponormal operators are always G_1 [16].

Theorem 7. *If $T \in \theta$, then T is G_1.*

Proof. We may assume that $T \in \theta$ and T is completely nonnormal. Let C be as in (1). Let D_ε be the complement of $\Re \times [-i\varepsilon, i\varepsilon]$. Then $(\lambda - T) \cdot (\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)$ is analytic on $\overline{\mathrm{UHP}}$ and

$$(\lambda - T)(\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon) = (\lambda - T)(\lambda - C^*)^{-1} F(D_\varepsilon)$$

for $\lambda \notin \sigma(C^*) \subseteq \sigma(T)$. But for any vector $\phi \in \mathcal{H}$ and any real λ,

$$\begin{aligned}
&\|(\lambda - T)(\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\phi\|^2 \\
&\quad = \langle(\lambda - T)(\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\phi, (\lambda - T)(\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\phi\rangle \\
&\quad = \langle(\lambda - C^*)(\lambda - C)(\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\phi, (\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\phi\rangle \\
&\quad = \langle(\lambda - CF(D_\varepsilon))F(D_\varepsilon)\phi, (\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\phi\rangle \\
&\quad = \langle F(D_\varepsilon)\phi, F(D_\varepsilon)\phi\rangle = \|F(D_\varepsilon)\phi\|^2 \leqslant \|\phi\|^2.
\end{aligned}$$

Also $\lim_{|\lambda| \to \infty} \|(\lambda - T)(\lambda - C^* F(D_\varepsilon))^{-1}\| = 1$. Thus

$$\|(\lambda - T)(\lambda - C^* F(D_\varepsilon))^{-1} F(D_\varepsilon)\| \leqslant 1$$

for all $\lambda \in \overline{\mathrm{UHP}}$. Hence $\|(\lambda - T)(\lambda - C^*)^{-1}\| \leqslant 1$ for all $\lambda \in \overline{\mathrm{UHP}}$, $\lambda \notin \sigma(T)$ since $F(D_\varepsilon)$ converges strongly to I as $\varepsilon \to 0$ [5]. Similarly $\|(\lambda - T) \cdot (\lambda - C)^{-1}\| \leqslant 1$ for all $\lambda \in \overline{\mathrm{LHP}}$, $\lambda \notin \sigma(T)$. Now if $\lambda \in \overline{\mathrm{UHP}}$, $\lambda \notin \sigma(T)$, we have

$$\begin{aligned}
\|(\bar\lambda - T)^{-1}\| &= \|(\lambda - T^*)^{-1}\| = \|(\lambda - T)(\lambda - C^*)^{-1}(\lambda - C)^{-1}\| \\
&\leqslant \|(\lambda - T)(\lambda - C^*)^{-1}\| \, \|(\lambda - C)^{-1}\| \\
&\leqslant \|(\lambda - C)^{-1}\| = 1/\rho(\lambda, \sigma(C)) = 1/\rho(\bar\lambda, \sigma(C^*)) \\
&= 1/\rho(\bar\lambda, \sigma(T)).
\end{aligned}$$

Similarly, if $\lambda \in \overline{\mathrm{LHP}}$, $\lambda \notin \sigma(T)$,

$$\|(\bar\lambda - T)^{-1}\| \leqslant 1/\rho(\bar\lambda, \sigma(T)).$$

Hence T is G_1. □

From [17, Theorem 1] and Theorem 7 we have:

Proposition 4. *Suppose that $T \in \theta$ is completely nonnormal. Then for any $z_0 \in \sigma(T)$ and disc D centered at z_0, $D \cap \sigma(T)$ cannot lie on a Jordan arc.*

While Propositions 1 and 4 are similar, they are not equivalent.

Knowing that $T \in \theta$ is G_1 allows alternative proofs of some of our earlier results. For example, that isolated points of $\sigma(T)$ are reducing eigenvalues for G_1 operators is known [14]. It also tells us that the convex hull of $\sigma(T)$ is the closure of the numerical range of T, $\mathrm{Cl}\, W(T)$ [12]. That is, T is convexoid. It does not however, provide an alternative proof of the fact that all eigenvalues of T are reducing [4]. Note that there are nonnormal compact G_1 operators [16], though there are no nonnormal compact operators in θ [4].

If $T \in \theta$, then T restricted to any reducing subspace is also in θ. Thus $T \in \theta$ are not only G_1 but also reduction-G_1 [1].

5. Examples and extension of the model. Our first example is, in a certain sense, canonical for $T \in \theta$, T completely nonnormal, $\sigma(T) \cap \mathfrak{R} = \varnothing$. Theorem 3 will be the basis for most of our constructions.

Example 1. Let H^2 be the usual Hardy space of the circle. Let C be multiplication by $e^{i\theta} + 2i$ in L^2 of the circle. Let $\mathfrak{M}_1 = H^2$ and $\mathfrak{M}_2 = (2 + \sin\theta)^{-1} H^{2\perp}$. Let T be the operator generated by C, $\mathfrak{M}_1$, $\mathfrak{M}_2$. Then $T \in \theta$, T is completely nonnormal and $\sigma(T)$ is the union of two discs centered at $2i$, $-2i$ and of radius one. By Corollary 2, T is not hyponormal.

Example 1 shows that Conjecture (C) of [4] is false and the class of operators in θ is nontrivially larger then was conjectured there. It also shows that $\sigma(T)$ need not be connected as was suggested in [5].

The point spectrum of the adjoint of an operator is preserved by similarity. Hence $\sigma_p(T^*) = \{z \mid |z - 2i| < 1\} \cup \{z \mid |z + 2i| < 1\}$ for the T in Example 1 since $C|H^2$ is just $2i + S$, S a unilateral shift.

If α, β are real scalars and $T \in \theta$, then $\alpha T + \beta \in \theta$. By taking direct sums of these operators, T as in Example 1, it is possible to build a completely nonnormal nonhyponormal operator $T \in \theta$ whose spectrum is any closed set Σ whose interior is dense in Σ, and which is symmetric with respect to the real axis. Let Δ be a subset of the unit disc, equipped with a measure μ, so that $\mathfrak{R}(\Delta)$ is not dense in $L^2(\Delta, d\mu)$. Let $\mathfrak{R}^2(\Delta)$ be the L^2 closure of $\mathfrak{R}(\Delta)$. If Δ has no interior and we repeat the construction of Example 1 using $\mathfrak{R}^2(\Delta)$ instead of H^2, we get a $T \in \theta$, T completely nonnormal, T not hyponormal, and $\sigma(T)$ with no interior. For example, Δ could be chosen as a 'Swiss Cheese' space [14].

We shall now briefly consider two possible ways of extending the structure theory of Theorems 2 and 3 to operators with $\sigma(T) \cap \mathfrak{R} \neq \varnothing$. Note from the proof of Theorem 2, that if $\sigma(T) \cap \mathfrak{R} = \varnothing$, then $\mathfrak{M}_2 = N(C - T^*)$ while $\mathfrak{M}_1 = N(C - T)$. Conversely;

Proposition 5. *Suppose that $T \in \theta$ and C is* (1). *Let $\mathfrak{M}_1 = N(C - T)$, $\mathfrak{M}_2 = N(C^* - T)$. Then $\mathfrak{M}_1, \mathfrak{M}_2$ are T invariant, $T|\mathfrak{M}_1 = C|\mathfrak{M}_1$, and $T|\mathfrak{M}_2 = C^*|\mathfrak{M}_2$. Furthermore, if $C - C^*$ is one-to-one, then $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}$.*

Proof. Note that $C^* - T^* = T - C$, $T^* - C = C^* - T$, and $C^*C = T^*T$. Thus $C^*(T - C) = (T - C)T$ and $C(T - C^*) = (T - C^*)T$. □

There need not, however, exist a nontrivial null space for either $C - T$ or $C^* - T$.

Proposition 6. *Let S be a unilateral shift. Let C be as in* (1). *Then $N(C - S) = \{0\}$ and $N(C - S^*) = \{0\}$.*

Proof. Since $S^*S = I$, C is a unitary operator with spectrum on the upper half of the unit circle. Thus $C|\mathfrak{M}$ is normal for any invariant subspace $\mathfrak{M}$ of C. By Proposition 2, $N(C - S)$ and $N(C - S^*)$ reduce C. But $C - S = S^* - C^*$ and $C - S^* = S - C^*$. Thus $N(C - S)$, $N(C - S^*)$ reduce S. Since S is completely nonnormal, we have $N(C - S) = \{0\}$ and $N(C - S^*) = \{0\}$. □

Since operators in θ are G_1, another possible extension is to use the results of Stampfli [18] to generalize Theorem 2. In [18] a method is developed to integrate a scalar multiple of the resolvent around pieces of $\sigma(T)$. For example, if $\sigma(T) \subseteq D_{\varepsilon_1} \cup D_{\varepsilon_2}$ where D_{ε_i} are two discs, tangent say at 0, then [18] gives hyperinvariant subspaces $\mathfrak{M}_1, \mathfrak{M}_2$ for T such that $\sigma(T|\mathfrak{M}_1) \subseteq D_{\varepsilon_1}, \sigma(T|\mathfrak{M}_2) \subseteq D_{\varepsilon_2}$. If $\sigma(T) \cap \mathfrak{R} = \varnothing$, then this $\mathfrak{M}_1, \mathfrak{M}_2$ are complementary. In general, however, they need not be complementary. This difficulty is implicit in [18].

Example 2. Let $C_\varepsilon = e^{i\theta} + (1 + \varepsilon)i$ for $\varepsilon > 0$ on L^2 of the circle. Let $\mathfrak{M}_1(\varepsilon) = H^2$, $\mathfrak{M}_2(\varepsilon) = (\sin\theta + 1 + \varepsilon)^{-1}H^{2\perp}$, and E_ε be the projection onto $\mathfrak{M}_1(\varepsilon)$ along $\mathfrak{M}_2(\varepsilon)$. Assume for the moment that $\|E_\varepsilon\| \to \infty$ as $\varepsilon \to 0$. Define T_ε using C_ε, $\mathfrak{M}_1(\varepsilon)$, $\mathfrak{M}_2(\varepsilon)$. If T_ε, C_ε are multiplied by the same real scalar, then $T_\varepsilon = C_\varepsilon E_\varepsilon + C_\varepsilon^*(I - E_\varepsilon)$ still holds. Define

$$T = \sum_{i=1}^{\infty} \oplus T_{\varepsilon_i} / \|E_{\varepsilon_i}\| \quad \text{where } \varepsilon_i \to 0.$$

If $\varepsilon_i \to 0$ not too fast, we have $T \in \theta$, $\sigma(T)$ is connected, and $\sigma(T) \cap \mathfrak{R} = \{0\}$. Let $\mathfrak{M}_1, \mathfrak{M}_2$ be the subspaces generated by Stampfli's theorem. Using f_1, f_2 nonzero except at zero, we have $\mathfrak{M}_1, \mathfrak{M}_2$ are hyperinvariant for T, $\sigma(T|\mathfrak{M}_1) \subseteq \sigma(T) \cap \overline{\text{UHP}}$, $\sigma(T|\mathfrak{M}_2) \subseteq \sigma(T) \cap \overline{\text{LHP}}$, $\mathfrak{M}_1 + \mathfrak{M}_2$ is dense, and $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}$. The integrals used to define $\mathfrak{M}_1, \mathfrak{M}_2$ are the orthogonal sum of the corresponding integral on each L^2 space. Since f_1, f_2 were assumed nonzero away from zero, we have

$$\bigcup_n \left[\sum_{i=1}^{n} \oplus \mathfrak{M}_1(\varepsilon_i)\right] \subseteq \mathfrak{M}_1, \qquad \bigcup_n \left[\sum_{i=1}^{n} \oplus \mathfrak{M}_2(\varepsilon_i)\right] \subseteq \mathfrak{M}_2.$$

Thus to show that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not complementary it suffices to show $\|E_{\varepsilon_i}\| \to \infty$. To see that $\|E_\varepsilon\| \to \infty$ as $\varepsilon \to 0$, let α_ε, β_ε be the two roots of $z^2 + 2(1+\varepsilon)iz - 1$. One root has modulus greater then one, the other has modulus less then one. Assume $|\alpha_\varepsilon| < 1$, $1 < |\beta_\varepsilon|$. Note that $\alpha_\varepsilon, \beta_\varepsilon \to -i$ as $\varepsilon \to 0$. Let $f_\varepsilon = -\bar{\beta}_\varepsilon(z-\beta_\varepsilon)^{-1}$, $\tilde{f}_\varepsilon = (1-\alpha_\varepsilon \bar{z})\bar{z}$. Note that $f_\varepsilon \in H^2$ and $\tilde{f}_\varepsilon \in H^{2\perp}$. Now let

$$g_\varepsilon = f_\varepsilon + (z - \bar{z} + 2i(\varepsilon+1))^{-1}\tilde{f}_\varepsilon,$$

and observe that $f_\varepsilon \in \mathfrak{M}_1(\varepsilon)$, $(z-\bar{z}+2i(\varepsilon+1))^{-1}\tilde{f}_\varepsilon \in \mathfrak{M}_2(\varepsilon)$. Thus $E_\varepsilon g_\varepsilon = f_\varepsilon$ and $\|f_\varepsilon\| \to \infty$ as $\varepsilon \to 0$ since $\beta_\varepsilon \to -i$. But for $|z| = 1$,

$$\begin{aligned} g_\varepsilon &= -\bar{\beta}_\varepsilon(z-\beta_\varepsilon)^{-1} + z(z^2 + 2i(\varepsilon+1)z - 1)^{-1}(1-\alpha_\varepsilon\bar{z})\bar{z} \\ &= -\bar{\beta}_\varepsilon(z-\beta_\varepsilon)^{-1} + (z-\alpha_\varepsilon)^{-1}(z-\beta_\varepsilon)^{-1}(1-\alpha_\varepsilon\bar{z}) \\ &= (z-\beta_\varepsilon)^{-1}(-\bar{\beta}_\varepsilon + \bar{z}). \end{aligned}$$

Thus $\|g_\varepsilon\| = 1$, $\|E_\varepsilon g_\varepsilon\| \to \infty$, and hence $\|E_\varepsilon\| \to \infty$ as desired.

It would be of interest to know if for every completely nonnormal $T \in \theta$ such that $\sigma(T) \cap \mathfrak{R}$ is a single point, one has $\mathfrak{M}_1, \mathfrak{M}_2$ as in Example 2. Provided $\sigma(T) \cap \overline{\text{LHP}}$ and $\sigma(T) \cap \overline{\text{UHP}}$ are separated by the appropriate curves, Stampfli's result gives an $\mathfrak{M}_1, \mathfrak{M}_2$ hyperinvariant for T such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}$. The difficulty is in showing $\mathfrak{M}_1 + \mathfrak{M}_2$ is dense.

If one considers the special case in [18, Theorem 1] where $f_i(\lambda) = \lambda^m$, $i = 1, 2$, m an integer $\geqslant 1$, one can show that $\mathfrak{M}_1 + \mathfrak{M}_2$ is dense if $0 \notin \sigma_p(T^{*m})$, since

$$T^m = \int_{\partial D_1} \lambda^m(\lambda - T)^{-1}\,d\lambda + \int_{\partial D_2} \lambda^m(\lambda - T)^{-1}\,d\lambda.$$

Putnam has shown that if $0 \in \sigma_p(T^*)$, $0 \in \partial\sigma(T)$, and there exists $\lambda_n \to 0$ such that $|\lambda_n|\,\|(T^* - \lambda_n)^{-1}\| \to 1$ as $n \to \infty$, then 0 is a reducing eigenvalue [13]. Putnam's result is thus one way of getting $\overline{\mathfrak{M}_1 + \mathfrak{M}_2} = \mathcal{H}$ for completely nonnormal T. However, this result and its subsequent generalizations, force $\partial\sigma(T)$ to approach 0 almost vertically in order to apply them. Our next result does much better for operators in θ.

THEOREM 8. *Suppose that there exist lines $y^2 = ax^2$, $a > 0$ and fixed, such that all points in $\sigma(T)$ except zero lie either above both lines or below both lines. Suppose that $T \in \theta$ and T is completely nonnormal. Then $0 \notin \sigma_p(T^*)$.*

PROOF. Suppose that $T^*\phi = 0$, $\|\phi\| = 1$. Note that for real ε, $\varepsilon \neq 0$, $(\varepsilon - C^*)^{-1}(\varepsilon - T^*)$ is unitary. Thus $1 = \|\phi\| = \|(\varepsilon - C^*)^{-1}(\varepsilon - T^*)\phi\| = \|\varepsilon(\varepsilon - C^*)^{-1}\phi\|$. Now

$$\varepsilon(\varepsilon - C^*)^{-1}\phi = \int_{\sigma(C)} \frac{\varepsilon}{(\varepsilon - \bar{\lambda})} F(d\lambda)\phi.$$

But $|\varepsilon(\varepsilon - \bar{\lambda})^{-1}| \leqslant |\varepsilon|\,|\varepsilon - \bar{\lambda}_0|^{-1}$ where $\bar{\lambda}_0$ is on the two lines. Since the ratio between ε and the distance from ε to the nearest point on a line is a constant K, we have $|\varepsilon(\varepsilon - \bar{\lambda})^{-1}| \leqslant K$ all $\bar{\lambda} \in \sigma(C)$ and K is independent of ε. From [5] we have 0 is not a point mass of $F(\cdot)$. Hence there exists $\varepsilon_1 > 0$ such that $\|F(\{|z| < \varepsilon_1\})\phi\| < (2K)^{-1}$. Also there is an $\varepsilon_0 > 0$ such that $\{\lambda\colon |\varepsilon_0(\varepsilon_0 - \bar{\lambda})| \geqslant 1/2\} \subseteq \{z\colon |z| < \varepsilon_1\}$. Now

$$\int_{\sigma(C)} \frac{\varepsilon_0}{(\varepsilon_0 - \bar{\lambda})} F(d\lambda)\phi = \int_{|\lambda| < \varepsilon_1} \frac{\varepsilon_0}{\varepsilon_0 - \bar{\lambda}} F(d\lambda)\phi + \int_{|\lambda| \geqslant \varepsilon_1} \frac{\varepsilon_0}{\varepsilon_0 - \bar{\lambda}} F(d\lambda)\phi.$$

But

$$\left\| \int_{|\lambda| < \varepsilon_1} \frac{\varepsilon_0}{\varepsilon_0 - \bar{\lambda}} F(d\lambda)\phi \right\| < K(2K)^{-1} = 1/2$$

and

$$\left\| \int_{|\lambda| \geqslant \varepsilon_1} \varepsilon_0(\varepsilon_0 - \bar{\lambda})^{-1} F(d\lambda)\phi \right\| < \|\phi\|/2 = 1/2.$$

Thus $\|\varepsilon_0(\varepsilon_0 - C^*)^{-1}\phi\| < \|\phi\|$ which is a contradiction. □

One can weaken the assumptions of Theorem 9 to only $T \in \theta$, T completely nonnormal and there exists real ε_n, $\varepsilon_n \notin \sigma(T)$, $\varepsilon_n \to 0$, such that $\varepsilon_n \rho(\varepsilon_n, \sigma(T))^{-1}$ is bounded independently of n.

The example on pp. 280–281 of [13] shows that Theorem 8 is not true for T which are not in θ but are G_1.

Regardless of whether or not the subspaces generated by Stampfli's theorem have dense sum, their existence gives much information about $\sigma(T)$.

THEOREM 9. *Suppose that $T \in \theta$, $\sigma(T) \cap \mathfrak{R} = \{0\}$, and T is completely nonnormal. Suppose further that there exist functions f_1, f_2 and domains D_1, D_2 satisfying the assumptions of* [18, *Theorem* 1 *and Theorem* 1′]. *Let $\mathfrak{M}_1, \mathfrak{M}_2$ be the closure of the ranges of*

$$A = \int_{\partial D_1} f_1(\lambda)(\lambda - T)^{-1}\,d\lambda, \qquad B = \int_{\partial D_2} f_2(\lambda)(\lambda - T)^{-1}\,d\lambda$$

respectively. Let C be as in (1). *Then $T|\mathfrak{M}_1 = C|\mathfrak{M}_1$, and $T|\mathfrak{M}_2 = C^*|\mathfrak{M}_2$, $\sigma(T|\mathfrak{M}_1) \subseteq \overline{UHP}$, and $\sigma(T|\mathfrak{M}_2) \subseteq \overline{LHP}$.*

PROOF. The only part that needs proof is $T|\mathfrak{M}_1 = C|\mathfrak{M}_1$ and $T|\mathfrak{M}_2 = C^*|\mathfrak{M}_2$. The rest is done in [18]. First note that

$$\int_{\partial D_1}(C-\lambda)f_1(\lambda)(\lambda-T)^{-1}d\lambda$$
$$=\int_{\partial D_1}(C-\lambda)f_1(\lambda)(\lambda-C)^{-1}(\lambda-C^*)^{-1}(\lambda-T^*)d\lambda$$
$$=-\int_{\partial\Omega_1}f_1(\lambda)(\lambda-C^*)^{-1}(\lambda-T^*)d\lambda=0.$$

But then

$$0=\int_{\partial D_1}(C-\lambda)f_1(\lambda)(\lambda-T)^{-1}d\lambda$$
$$=C\int_{\partial D_1}f_1(\lambda)(\lambda-T)^{-1}d\lambda-\int_{\partial D_1}\lambda f_1(\lambda)(\lambda-T)^{-1}d\lambda$$
$$=CA-TA \quad \text{as desired.}$$

The proof that $(C^*-T)B=0$ is similar. □

6. **Comments and more examples.** While the results of [4], [5] and this paper have developed many basic properties of the class θ, numerous questions remain. For convenience, let (Q) denote the class of quasinormals [2] and (QA) denote operators of the form T_1+T_2 where $T_1\in(Q)$, $T_1T_2=T_2T_1$, and T_2 is selfadjoint. Then $(Q)\subset(QA)\subset\theta$ and all inclusions are proper. An obvious problem is to determine what types of restrictions on operators in θ force them to be in (Q) or (QA). In particular, are there $T\in\theta$ which are subnormal and not in (QA)?

It was shown in [4] that if T^*T-TT^* has a kernel, then operators in θ have a block decomposition much like the operators in (QA). If $T\in\theta$ and T^*T-TT^* has rank one, then $T\in(QA)$.

THEOREM 10. *Suppose that $T\in\theta$ and T^*T-TT^* has rank one. Then $T=[\lambda_1+\lambda_2S]\oplus N$ where λ_1 is real, $\lambda_2>0$, S is a unilateral shift of multiplicity one, and N is normal.*

PROOF. Suppose that $T\in\theta$, T^*T-TT^* has rank one, and T is completely nonnormal. Then by [4] T has the scalar matrix,

$$T=\begin{bmatrix} a_1 & 0 & 0 & \cdot \\ b_1 & a_2 & 0 & \cdot \\ 0 & b_2 & a_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

all b_i are nonzero, and the (1, 1) entry acts on the range of $[T^*T-TT^*]$. From (2) of [4] we have $\bar{b}_ia_{i+1}=a_i\bar{b}_i$, $|a_{i+1}|^2+|b_{i+1}|^2=|b_i|^2+|a_{i+1}|^2$. Also $\bar{a}_1|b_1|^2=|b_1|^2a_1$ since $T\in\theta$. Let $\lambda_1=a_1$. Then λ_1 is real and $a_i=\lambda_1$ for all i. Also $|b_i|^2$ is independent of i. Let $\lambda_2=|b_i|$ and recall that weighted shifts are unitarily equivalent if their weight sequences have the same moduli [10]. □

However, if $T^*T - TT^*$ has rank greater then one, the situation is different. We shall now construct a $T \in \theta$ such that T is hyponormal, $T \notin (QA)$, and $T^*T - TT^*$ has rank two.

EXAMPLE 3. Let T be given by

$$T = \begin{bmatrix} A_1 & 0 & 0 & 0 & \cdot \\ B_1 & A_2 & 0 & 0 & \cdot \\ 0 & B_2 & A_3 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{5}$$

on countably many copies of a two dimensional Hilbert space. Let

$$A_i = \begin{bmatrix} 0 & e_i \\ f_i & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} \delta_i & 0 \\ 0 & \gamma_i \end{bmatrix},$$

where $e_i, f_i, \delta_i, \gamma_i$ are real scalars. Then $T^*T - TT^*$ has matrix $\mathrm{Diag}\{D, 0, 0, \ldots\}$ if and only if

$$A_1^*A_1 + B_1^*B_1 - A_1A_1^* = D, \tag{6}$$

$$A_i^*A_i + B_i^*B_i = B_{i-1}B_{i-1}^* + A_iA_i^*, \qquad i \geqslant 2, \tag{7}$$

and

$$B_i^*A_{i+1} = A_iB_i^*, \qquad i \geqslant 1. \tag{8}$$

If (6), (7), (8) are satisfied, then $T \in \theta$ if A_1^*D is hermitian [4]. Take $0 < \alpha < 1$ and $c = (2 + 2\alpha)^{-1/2}$. Set $e_1 = c\alpha, f_1 = c$, and

$$\delta_1 = \gamma_1 = (1 + |c|^2(\alpha^2 - 1))^{1/2} = (\alpha + |c|^2(1 - \alpha^2))^{1/2}.$$

Equation (6) gives $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]$. Equation (7) becomes

$$e_{i+1} = e_i\gamma_i/\delta_i, \qquad f_{i+1} = f_i\delta_i/\gamma_i, \qquad i \geqslant 1, \tag{9}$$

while (8) is

$$\delta_{i+1}^2 = \delta_i^2 + e_{i+1}^2 - f_{i+1}^2, \qquad \gamma_{i+1}^2 = \gamma_i^2 + f_{i+1}^2 - e_{i+1}^2. \tag{10}$$

Note that given $e_i, f_i, \delta_i, \gamma_i$, then e_{i+1}, f_{i+1} are determined by (9). Then (10), if consistent, gives a unique positive $\delta_{i+1}, \gamma_{i+1}$. A straightforward computation yields that $e_1 = e_7, f_1 = f_7, \delta_1 = \delta_7, \gamma_1 = \gamma_7$. Thus the sequences A_i, B_i, defined by (9), (10), our initial conditions and the requirement $\delta_i, \gamma_i \geqslant 0$, are well defined and bounded. Furthermore, A_1^*D is hermitian so $T \in \theta$. But

$DA_1^* D$ is not hermitian so $T \notin (QA)$ [4]. Note also that T is hyponormal since $D > 0$.

For the convenience of the reader interested in studying this example more carefully we give the B_i, A_i, explicitly. As noted, $A_{i+6} = A_i$, $B_{i+6} = B_i$. The blocks are

$$B_1 = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} 0 & c\alpha \\ c & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & c\alpha \\ c & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & c\sqrt{\alpha} \\ c\sqrt{\alpha} & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & c \\ c\alpha & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & c \\ c\alpha & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & c\sqrt{\alpha} \\ c\sqrt{\alpha} & 0 \end{bmatrix}.$$

In Example 1, the two components of $\sigma(T)$ were not spectral sets since the projections obtained by integrating the resolvent were not hermitian. Hence $\sigma(T)$ was not a spectral set of T.

EXAMPLE 4. Let $\{T_i\}$ be a family of operators in θ constructed as in Example 1 such that $\bigcup_i \sigma(T_i)$ is dense in the unit disc. Let $T = \Sigma \oplus T_i$. Each T_i has norm no greater than one. So T is a contraction such that $\sigma(T)$ is the unit disc. Thus $\sigma(T)$ is a spectral set for T [15, p. 441]. Note that T is nonhyponormal and completely nonnormal.

However, if $T \in \theta$, $\sigma(T)$ is the unit disc and $\sigma(C)$ is contained in the unit circle, then T is an isometry since $T^* T = C^* C = I$. For a related result see [7].

If $T \in (Q)$, then $T^n \in (Q)$ for all positive integers n. Which other operators in θ have powers also in θ? As a partial answer we note that

PROPOSITION 7. *If $T \in (QA)$, then $T^2 \in \theta$ if and only if $T \in (Q)$.*

PROOF. Using the canonical form for (Q) given in [2] it is easy to reduce the problem to showing that $(\alpha + S)^2 \notin \theta$ for all real $\alpha \neq 0$ where S is a unilateral shift. It suffices to show that $T = 2\alpha S + S^2 \notin \theta$. But

$$T^* T = (S^* + 2\alpha)(S + 2\alpha) = 4\alpha^2 + I + 2\alpha(S + S^*),$$

$$T + T^* = 2\alpha(S + S^*) + S^2 + S^{*2}.$$

Thus $T \notin \theta$ if $S^* + S$ and $S^2 + S^{*2}$ do not commute. But

$$\begin{aligned}[(S^* + S)(S^2 + S^{*2}) - (S^2 + S^{*2})(S^* + S)]S \\ = (S^* + S)(S^3 + S^*) - (S^2 + S^{*2})(1 + S^2) \\ = S^2 + S^{*2} + S^4 + SS^* - S^2 - S^{*2} - S^4 - 1 = SS^* - 1 \neq 0.\end{aligned}$$

Thus $T \notin \theta$.

Note that if T is a weighted bilateral shift with positive weights whose smallest period is k, then $T^{nk} \in \theta$, $T^m \notin \theta$ for all $m \neq nk$, where $n \geqslant 0$.

The structure of the spectral measure of C and the structure of T are, of course, related. It was shown in [5] that if $T \in \theta$ is completely nonnormal, then $F(\mathfrak{R}) = 0$. Since eigenspaces of $T + T^*$ reduce T if T is hyponormal [11], we have

PROPOSITION 8. *If $T \in \theta$, T is hyponormal, and T is completely nonnormal, then $F(L) = 0$ for any verticle line L.*

EXAMPLE 5. Let Δ be the boundary of $\{x + iy: |x| \leqslant 1, |y| \leqslant 1\}$ equipped with linear Lebesgue measure. Let $\mathfrak{M}_1 = H^2(\Delta)$, C be the operator of multiplication by $z + 2i$ and define T as in Example 1. Then T is completely nonnormal, $T \in \theta$, $\sigma(C)$ is a square centered at $2i$, and $F(\{z: \operatorname{Re} z = 1\}) \neq 0$.

Consideration of the shift shows that one can have $T \in \theta$, $\sigma_p(T^*) \neq \varnothing$, and $\sigma_p(C) = \varnothing$. The converse is not possible.

PROPOSITION 9. *If $T \in \theta$ and $\lambda \in \sigma_p(C)$, then at least one of the following must hold*:

(a) λ *is a reducing eigenvalue of* T,

(b) $\bar{\lambda}$ *is a reducing eigenvalue of* T,

(c) $\lambda, \bar{\lambda}$ *are both eigenvalues of* T^*.

PROOF. Suppose that $T \in \theta$ and $C\phi = \lambda\phi$. Then

$$(\lambda - T^*)(\lambda - T)\phi = (\lambda - C^*)(\lambda - C)\phi = 0,$$

and

$$(\bar{\lambda} - T^*)(\bar{\lambda} - T)\phi = (\bar{\lambda} - C)(\bar{\lambda} - C^*)\phi = 0. \quad \square$$

The next example shows that (c) of Proposition 9 is actually possible. It is based on an operator first constructed by Sarason [10, Problem 156].

EXAMPLE 6. Let $\mathcal{K}_0$ be a one-dimensional Hilbert space, $g \in \mathcal{K}_0$ of norm one. Let $\mathcal{K}$ be the orthogonal sum of L^2 of the circle and $\mathcal{K}_0$. Let $\tilde{S} = \mathfrak{M}_z \oplus 0$, where $\mathfrak{M}_z$ is multiplication by z in L^2. Let $\mathfrak{M}_1$ be the $\tilde{S}$ invariant subspace generated by $1 \oplus g$ and zH^2. $\tilde{S}$ is the minimal normal dilation of $\tilde{S}|\mathfrak{M}_1$. Let $C = \tilde{S} + 2i$ and define T as in Theorem 3. Then $T \in \theta$. T is completely nonnormal by Theorem 5, so $\sigma_p(T) = \varnothing$. But $2i \in \sigma_p(C)$ since $0 \in \sigma_p(\tilde{S})$.

Note that in Example 6, $\partial\sigma(C) \not\subseteq \partial\sigma(T)$. Since $\tilde{S}|\mathfrak{M}_1$ and $\tilde{S}^*|\mathfrak{M}_2$ are both unitarily equivalent to a unilateral shift we have that the T of Example 1 is similar to the T of Example 6. However, the C of Example 1 has no point spectrum and hence is not similar to the C of Example 6.

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Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607