

INTERSECTION PROPERTIES OF BALLS AND SUBSPACES IN BANACH SPACES⁽¹⁾

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ABSTRACT. We study intersection properties of balls in Banach spaces using a new technique. With this technique we give new and simple proofs of some results of Lindenstrauss and others, characterizing Banach spaces with $L_1(\mu)$ dual spaces by intersection properties of balls, and we solve some open problems in the isometric theory of Banach spaces. We also give new proofs of some results of Alfsen and Effros characterizing M -ideals by intersection properties of balls, and we improve some of their results. In the last section we apply these results on function algebras, G -spaces and order unit spaces and we give new and simple proofs for some representation theorems for those Banach spaces with $L_1(\mu)$ dual spaces whose unit ball contains extreme points.

Introduction. A Banach space A is said to be a \mathfrak{P}_1 -space if for every Banach space X containing A , there is a linear projection P from X onto A with $\|P\| \leq 1$. It is easy to see [11, p. 123] that a Banach space A is a \mathfrak{P}_1 -space if and only if A has the following property: For every Banach space Y , for every linear subspace X of Y and for every bounded linear operator $T: X \rightarrow A$, there exists a linear extension $\tilde{T}: Y \rightarrow A$ of T with $\|T\| = \|\tilde{T}\|$. An example of a space with this property is $l_\infty(\Gamma)$ for some set $\Gamma \neq \emptyset$. This is shown by application of the Hahn-Banach theorem coordinatewise.

In 1950 Nachbin [39] and Goodner [18] characterized the real \mathfrak{P}_1 -spaces whose unit balls have extreme points: They are (up to isometry) the $C(K)$ -spaces for which the compact Hausdorff space K is extremally disconnected (i.e. the closure of every open set is open). In 1952 Kelley [30] showed that the assumption on the unit ball was superfluous. This was shown to hold also in the complex case by Hasumi [21] in 1958. In 1955 Grothendieck [19] showed that a real Banach space A is isometric to an $L_1(\mu)$ -space for some measure μ if and only if its dual A^* is a \mathfrak{P}_1 -space. This result was later proved for the complex case by Sakai [42].

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In the above mentioned paper [39] from 1950 Nachbin proved that a real Banach space is a \mathfrak{P}_1 -space if and only if its closed balls have the "binary intersection property" (i.e. if every family of closed balls, any two members of which intersect, has a nonvoid intersection). In 1973 Hustad [24], [25] observed that for real spaces a family of closed balls are mutually intersecting if and only if it has the "weak intersection property" (i.e. for every linear functional φ with $\|\varphi\| = 1$, the images of the balls by φ shall have a nonvoid intersection), and that it is the weak intersection property which is the relevant one in the complex case. Thus he proved that a complex Banach space is a \mathfrak{P}_1 -space if and only if every family of closed balls with the weak intersection property has a nonvoid intersection [25].

In 1964 Lindenstrauss [35] proved that for a real Banach space A the following properties are equivalent:

- (i) A^{**} is a \mathfrak{P}_1 -space.
- (ii) Every family of closed balls $\{B(a_i, r_i)\}_{i \in I}$ in A , any two members of which intersect, and with $\{a_i\}_{i \in I}$ relatively norm compact, has a nonempty intersection.
- (iii) Every family $\{B(a_i, r_i)\}_{i=1}^4$ of 4 balls such that $\|a_i - a_j\| \leq r_i + r_j$ all $i, j = 1, \dots, 4$, has a nonempty intersection.
- (iv) For every compact operator T from a Banach space X into A and for every Banach space $Y \supseteq X$ and for every $\varepsilon > 0$, the operator T admits a compact extension $\tilde{T}: Y \rightarrow A$ such that $\|\tilde{T}\| \leq (1 + \varepsilon)\|T\|$. (It suffices to have such extensions for spaces X and Y with $\dim Y/X = 1$.)

A Banach space A is said to be a Lindenstrauss-space if A^* is isometric to an $L_1(\mu)$ -space for some measure μ , which by the Grothendieck-Sakai result is equivalent to A^{**} being a \mathfrak{P}_1 -space.

In [35] Lindenstrauss showed that in the real l_1 every family of 3 mutually intersecting closed balls has a nonempty intersection.

In 1969 Lazar [34] characterized those real Banach spaces which have the following property: For every pair of Banach spaces $X \subseteq Y$ and for every compact operator $T: X \rightarrow A$, there exists a compact extension $\tilde{T}: Y \rightarrow A$ with $\|\tilde{T}\| = \|T\|$. They are precisely the polyhedral spaces with dual spaces isometric to $L_1(\mu)$ -space. ("Polyhedral" means that the unit ball of all finite dimensional subspaces are polytopes.)

Hustad [24] defined a (real or complex) Banach space A to be an "almost $E(n)$ -space" if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of n closed balls in A with the weak intersection property and for every $\varepsilon > 0$, the corresponding balls $B(a_i, r_i + \varepsilon)$ have nonempty intersection; the word "almost" is omitted if one can take $\varepsilon = 0$ as well. By our previous remarks, condition (iii) above is equivalent to A being an $E(4)$ -space. (Actually, Hustad gave another definition of almost $E(n)$ -spaces and the above definition is a theorem of his.) In the

same paper Hustad proved that for a complex Banach space A the following properties are equivalent:

- (i)' A^{**} is a \mathfrak{P}_1 -space.
- (ii)' For every family of closed balls $\{B(a_i, r_i)\}_{i \in I}$ in A with the weak intersection property and with $\{a_i\}_{i \in I}$ relatively norm compact and for every $\epsilon > 0$, the corresponding balls $\{B(a_i, r_i + \epsilon)\}_{i \in I}$ have nonempty intersection.
- (iii)' A is an $E(7)$ -space.
- (iv)' For an arbitrary compact operator T from a Banach space X to A and for every Banach space $Y \supseteq X$ and for every $\epsilon > 0$, the operator T admits an extension $\tilde{T}: Y \rightarrow A$ such that $\|\tilde{T}\| \leq (1 + \epsilon)\|T\|$. (It suffices to consider spaces X and Y with $\dim Y/X = 1$.)

In 1960 Cunningham [9] introduced the L -projections and the L -summands. A linear projection e in a Banach space A is said to be an L -projection if $\|x\| = \|e(x)\| + \|x - e(x)\|$ all $x \in A$. A subspace J of A is said to be an L -ideal or an L -summand if J is the range of an L -projection. Cunningham [9] showed that the $L_1(\mu)$ -spaces have "sufficiently many" L -projections and that this characterizes $L_1(\mu)$ -spaces. L -summands were studied further by Alfsen and Effros in [4]. Alfsen and Effros [4] defined a closed subspace J of A to be an " M -ideal" if J^0 is an L -summand. We say that a linear subspace J of A has the n.I.P. (n -ball property or n -intersection property) if given n closed balls $\{B(a_i, r_i)\}_{i=1}^n$ in A such that if $J \cap B(a_i, r_i) \neq \emptyset$ for all i and $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$, then $J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset$ all $\epsilon > 0$. One of the main theorems of Alfsen and Effros in [4] is Theorem 5.8, by which the following statements are equivalent:

- (a) J is an M -ideal.
- (b) J has the n.I.P. for all n .
- (c) J has the 3.I.P.

Alfsen and Effros proved this as a theorem on dominated extension of linear functionals.

In §1 we begin by giving necessary and sufficient conditions that a finite family of balls and a linear subspace (or a convex cone) have nonempty intersection. These are results that will be used several times in the work.

In §2 we consider the extreme point structure of the unit ball of the Banach space

$$H_r^n(A^*, (0)) = \left\{ (f_1, \dots, f_n) : \text{all } f_i \in A^* \text{ and } \sum_{i=1}^n f_i = 0 \right\}$$

with norm $\|(f_1, \dots, f_n)\| = \sum_{i=1}^n r_i \|f_i\|$ where $r = (r_1, \dots, r_n)$ is in the positive cone of \mathbf{R}^n . In Theorem 2.10 we show that when $n > k \geq 2$ then the following two statements are equivalent:

(i) For every family $\{B(a_i, r_i)\}_{i=1}^n$ of n balls in A such that any k of them intersect, then $\bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ all $\varepsilon > 0$.

(ii) The extreme points of the unit ball of $H_r^n(A^*, (0))$ have at most k nonzero components.

A similar characterization of almost $E(n)$ -spaces is given in Theorem 2.14. Here we show that A is an almost $E(n)$ -space if and only if the extreme points of the unit ball of $H_r^n(A^*, (0))$ are of the form $(z_1 g, \dots, z_n g)$ where z_i are scalars and $g \in A^*$.

We end the section by using the separation results from §1 and the "principle of local reflexivity" [36] to show that A and its bidual A^{**} have "almost" the same intersection properties.

§3 is devoted to the study of geometrical properties of the unit ball of real Banach spaces with 3.2.I.P. (i.e. $E(3)$ -spaces). We show in Theorem 3.2 that A has the 3.2.I.P. if and only if A has a reminiscence of the Riesz decomposition property, which we call the R_3 -property. It is then proved that A enjoys the R_3 -property if and only if A^* does. From this it follows that A has the 3.2.I.P. if and only if A^* has the 3.2.I.P. (Corollary 3.3). In Theorem 3.5 we show that if A has the 3.2.I.P. and F is a proper maximal face of the unit ball $B(0, 1)$ of A , then $B(0, 1) = \text{co}(F \cup -F)$. (Corollary 3.3 and Theorem 3.5 generalize results of Hanner [20] to infinite dimensional spaces.) These results are used to show that A has the 4.2.I.P. (i.e. A is an $E(4)$ -space) if and only if A^* is isometric to an $L_1(\mu)$ -space (Corollary 3.11) and that A is isometric to an $L_1(\mu)$ -space if and only if A^* has the 4.2.I.P. (Theorem 3.12). These are known results of Grothendieck [19] and Lindenstrauss [35], but the new proofs may be of some interest.

In §4 we study Banach spaces with the 3.2.I.P. but not the 4.2.I.P. We show in Theorem 4.3 that a real Banach space A with the 3.2.I.P. will fail to have the 4.2.I.P. if and only if there exists an isometry from l_∞^3 into A^* . This is used in the proof of Theorem 4.6 which states that A has the 4.2.I.P. if and only if

$$B(0, 1) \cap B(x, 1) \cap B(y, 1) \cap B((x+y)/3, 1/3) \neq \emptyset$$

for all $x, y \in A$ with $\|x\|, \|y\|, \|x-y\| \leq 2$. This generalizes a result of Lindenstrauss [35, Lemma 6.5]. In Theorem 4.8 we show that one can not replace $B((x+y)/3, 1/3)$ by $B((x+y)/3, 1)$ in Theorem 4.6.

In §5 we study L -projections and L -summands. These were introduced by Cunningham [9] and studied by Alfsen and Effros in [4]. We introduce the notion of a semi L -summand. This is a generalization of L -summand. A closed subspace J of A is said to be a semi L -summand if $\|x+y\| = \|x\| + \|y\|$, all $x \in J$ and all $y \in J'$. (J' is defined by Alfsen and Effros [4].) In Theorem 5.6 we show that a subspace J of A is a semi L -summand if and only if for all $x \in A$, there exists a unique $y \in J$ such that $\|x-y\| = d(x, J)$ and if this

minimizing element y enjoys the property $\|x\| = \|y\| + \|x - y\|$. In Proposition 5.9 we show that the set of semi L -summands are closed under the operation of taking intersections, but not under the operation of taking sums. In Theorem 5.8 we show that A is an almost $E(n)$ -space for all n if and only if A^* has "sufficiently many" L -projections. (This is made precise in Theorem 5.8(ii).) In Theorem 5.10 and Corollary 5.11 L -summands J are characterized by the geometry of the unit ball of $H_r^n(A, J)$, and in Theorem 5.12 and Corollary 5.13 semi L -summands J are characterized by the geometry of the unit ball of $H_r^2(A, J)$.

Following Alfsen and Effros [4] we say that a closed subspace J of A is an M -ideal if its annihilator J^0 is an L -summand in A^* . We also say that J is a semi M -ideal if J^0 is a semi L -summand. In Theorem 6.9 we give a new proof of the above quoted result of Alfsen and Effros (Theorem 5.8 in [4]), and in Theorem 6.10 we show that a closed subspace J of A is a semi M -ideal if and only if J has the 2.I.P. In both theorems we may assume that the balls have radius 1. In Theorem 6.14 we show that if J is a closed subspace of A , the polar J^0 will be a semi M -ideal if and only if J is a semi L -summand. We prove in Theorem 6.16 that J is an L -summand if and only if J^0 is an M -ideal. This solves Problems 1 and 2 of Alfsen and Effros [4] affirmatively. By means of the example preceding Lemma 6.1 we also show that their Problem 3 has a negative solution.

§7 contains applications. First we give a new proof of a representation theorem for Banach spaces whose dual spaces are isometric to $L_1(\mu)$ -space and whose unit balls have extreme points. The real version of this theorem was proved by Nachbin [39], Kadison [27] and Lindenstrauss [35], and the complex version was proved by Hirsberg and Lazar [23]. For the complex case our proof is considerably simpler than the original one. Next we consider semi M -ideals and semi L -summands in uniform algebras. In Theorem 7.6 we show that the semi M -ideals and the M -ideals will coincide in uniform algebras. In fact they are exactly the annihilator ideals $J = \{a \in A : a = 0 \text{ on } F\}$ of generalized peak sets $F \subseteq X$. (The connection between M -ideals and peak sets is due to Hirsberg [22].)

A subspace A of $C(X)$ is said to be a G -space if there exists a set $S = \{(x_\alpha, y_\alpha, \lambda_\alpha)\} \subseteq X \times X \times [-1, 1]$ such that $A = \{f \in C(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha), \text{ all } \alpha\}$. The G -spaces were introduced by Grothedieck [19] and characterized by Lindenstrauss and Wulbert [38] as those subspaces A of $C(X)$ such that for all $f, g \in A$

$$\max(f, g, 0) + \min(f, g, 0) \in A.$$

In Theorem 7.10 we present a new and simplified approach to this result and also to Effros' characterization of the G -spaces as those spaces A such that A^*

is isometric to an $L_1(\mu)$ -space and $\overline{\partial_e A_1^*} \subseteq [0, 1] \partial_e A_1^*$ (ω^* -closure) [13].

In Theorem 7.18 we show that in an Archimedean order unit space a semi M -ideal is a strongly Archimedean order ideal in the terminology of Alfsen [1]. By combining this with results of Alfsen and Effros [4] and of Størmer [44], we can prove that in the selfadjoint part of a C^* -algebra with unit, the semi M -ideals and the M -ideals coincide: in fact they will coincide with the selfadjoint parts of closed two-sided ideals of the C^* -algebra. (The connection between M -ideals and two-sided ideals is due to Alfsen and Effros [4].)

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NOTATION. A will denote a real or complex Banach space. The closed ball in A with center a and radius $r \geq 0$ is denoted by $B(a, r)$. The unit ball $B(0, 1)$ will be written A_1 . We denote the dual space of A by A^* . The following symbols are standard:

N: the set of integers $n \geq 1$.

R: the set of real numbers.

C: the set of complex numbers.

K will denote either **R** or **C**.

If $a \in A$ and $S \subseteq A$ with $S \neq \emptyset$, then we write the distance from a to S as follows

$$d(a, S) = \inf\{\|a - x\| : x \in S\}.$$

Let S be a subset of A . The closure of S is denoted \bar{S} , and the convex hull of S is denoted $\text{co}(S)$. If S is convex, then $\partial_e S$ is the set of extreme points of S . If S is a convex set and F is a subset of S , then we say that F is a *face* of S if F is convex and if $y, z \in F$ whenever $x = \lambda y + (1 - \lambda)z \in F$ with $\lambda \in (0, 1)$ and $y, z \in S$. A face F of S is *proper* if $F \neq S$ and $F \neq \emptyset$.

A convex cone S is said to have the *Riesz decomposition property* if for all $x_1, \dots, x_n; y_1, \dots, y_m \in S$ such that $\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$, there exists $z_{ij} \in S$ such that

$$x_i = \sum_{j=1}^m z_{ij}, \quad y_j = \sum_{i=1}^n z_{ij}, \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

For $n \in \mathbf{N}$, A^n will denote the direct sum $A \oplus A \oplus \dots \oplus A$ of n copies of A . For $r = (r_i)_{i=1}^n \in \mathbf{R}^n$ with all $r_i > 0$, we will organize A^n to a Banach space under each one of the two norms:

$$\|(a_1, \dots, a_n)\|_{1,r} = \sum_{i=1}^n r_i \|a_i\|$$

and

$$\|(a_1, \dots, a_n)\|_{\infty, r} = \max_{1 \leq i \leq n} (r_i^{-1} \|a_i\|).$$

It is easy to see that we may identify $(A^n, \|\cdot\|_{1,r})^*$ and $(A^{*n}, \|\cdot\|_{\infty, r})$ and also $(A^n, \|\cdot\|_{\infty, r})^*$ and $(A^{*n}, \|\cdot\|_{1,r})$.

For a linear subspace J of A , we will consider the following spaces:

$$\begin{aligned} J^n &= \{(a_1, \dots, a_n) \in A^n : \text{all } a_i \in J\}, \\ \Delta(J, n) &= \{(a_1, \dots, a_n) \in A^n : a_1 = \dots = a_n \in J\}, \\ H_r^n(A, J) &= \left\{ (a_1, \dots, a_n) \in (A^n, \|\cdot\|_{1,r}) : \sum_{i=1}^n a_i \in J \right\}. \end{aligned}$$

Note that we always consider $H_r^n(A, J)$ as a subspace of $(A^n, \|\cdot\|_{1,r})$. The space $H_r^n(A, J)$ will play a fundamental role in this work. This space generalizes the space $H_r^n(C, (0))$ used by Hustad [24] in his work on intersection properties of balls in complex Banach spaces.

1. Separating a finite family of convex sets. We shall now prove some results which will be fundamental in the sequel. The problem of separating a finite family of convex sets will be handled by embedding the sets into a product space and then separating the product set from the diagonal. (This will be made precise below.) Similar techniques have been used by various authors. (See e.g. [36].) The following result generalizes Corollary 1.3 in [24] by O. Hustad. In the following A shall be a real or complex Banach space.

THEOREM 1.1. *Let J be a linear subspace of A . Let $n \in \mathbb{N}$, let $a_1, \dots, a_n \in A$ and let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$. Then the following statements are equivalent:*

- (i) $J \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ all $\varepsilon > 0$,
- (ii) $|\sum_{i=1}^n f_i(a_i)| \leq \sum_{i=1}^n r_i \|f_i\|$ all $(f_1, \dots, f_n) \in H_r^n(A^*, J^0)$.

PROOF. (i) \Rightarrow (ii). Let $\varepsilon > 0$ and let $a \in J \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon)$. Then for $(f_1, \dots, f_n) \in H_r^n(A^*, J^0)$

$$\left| \sum_{i=1}^n f_i(a_i) \right| = \left| \sum_{i=1}^n f_i(a_i - a) \right| \leq \sum_{i=1}^n (r_i + \varepsilon) \|f_i\|.$$

Since $\varepsilon > 0$ is arbitrary, (ii) follows.

Suppose next that (i) is false. Then for some $\varepsilon > 0$

$$\Delta(J, n) \cap B((a_1, \dots, a_n), 1 + \varepsilon) = \emptyset$$

in $(A^n, \|\cdot\|_{\infty, r})$. Hence $\Delta(J, n)$ and $B = B((a_1, \dots, a_n), 1)$ can be strongly separated. Let $(f_1, \dots, f_n) \in (A^{*n}, \|\cdot\|_{1,r})$ be such that with $b = (b_1, \dots, b_n)$:

$$(1.1) \quad \sup_{b \in B} \operatorname{Re} \left(\sum_{i=1}^n f_i(b_i) \right) < \inf_{x \in J} \operatorname{Re} \left(\sum_{i=1}^n f_i(x) \right).$$

Since $\operatorname{Re}(\sum_{i=1}^n f_i) = 0$ on J , we also have $\sum_{i=1}^n f_i = 0$ on J , so (f_1, \dots, f_n) is in the polar $H_r^n(A^*, J^0)$ of $\Delta(J, n)$. Clearly the right-hand side of (1.1) is zero. On the left-hand side of (1.1) we write $b_i = a_i + y_i$ where $\|y_i\| \leq r_i$, and we get

$$(1.2) \quad \begin{aligned} \sup_{b \in B} \operatorname{Re} \left(\sum_{i=1}^n f_i(b_i) \right) &= \sup_{\|y_i\| \leq r_i} \operatorname{Re} \left[\sum_{i=1}^n f_i(a_i) + \sum_{i=1}^n f_i(y_i) \right] \\ &= \operatorname{Re} \left(\sum_{i=1}^n f_i(a_i) \right) + \sum_{i=1}^n r_i \|f_i\|. \end{aligned}$$

Now (1.1) and (1.2) yield

$$\sum_{i=1}^n r_i \|f_i\| < -\operatorname{Re} \left(\sum_{i=1}^n f_i(a_i) \right) \leq \left| \sum_{i=1}^n f_i(a_i) \right|.$$

This contradicts (ii), and the proof is complete.

If J is a closed subspace of A , then the polar of $H_r^n(A, J)$ is $\Delta(J^0, n) \subseteq (A^{*n}, \|\cdot\|_{\infty, r})$. An argument similar to the argument used to prove Theorem 1.1 now gives

THEOREM 1.2. *Let J be a closed subspace of A . Let $n \in \mathbb{N}$, let $f_1, \dots, f_n \in A^*$ and let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$. Then the following statements are equivalent:*

- (i) $J^0 \cap \bigcap_{i=1}^n B(f_i, r_i) \neq \emptyset$.
- (ii) $|\sum_{i=1}^n f_i(a_i)| \leq \sum_{i=1}^n r_i \|a_i\|$ all $(a_1, \dots, a_n) \in H_r^n(A, J)$.

COROLLARY 1.3. *Let J be a linear subspace of A . Let $n \in \mathbb{N}$, let $a_1, \dots, a_n \in A$ and let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$. Then the following statements are equivalent:*

- (i) $J \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ all $\varepsilon > 0$ in A .
- (ii) $J^{00} \cap \bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ in A^{**} .

REMARK. Corollary 1.3 generalizes Lemma 5.8 in [35]. See also Klee [31].

If we replace the subspace J by a convex cone in Theorem 1.1, then we get the following result:

THEOREM 1.4. *Let A be a real Banach space and let C be a convex cone in A . Let $n \in \mathbb{N}$, let $a_1, \dots, a_n \in A$ and let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$. Then the following statements are equivalent:*

- (i) $C \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ all $\varepsilon > 0$.
- (ii) $-\sum_{i=1}^n f_i(a_i) \leq \sum_{i=1}^n r_i \|f_i\|$ all $(f_1, \dots, f_n) \in A^{*n}$ with

$$\sum_{i=1}^n f_i \geq 0$$

on C .

2. Characterizations of almost $E(n)$ -spaces. In this section we will show how various intersection properties of a Banach space A are reflected in the structure of the set of the extreme points of the unit ball $H_r^n(A^*, J^0)_1$ of $H_r^n(A^*, J^0)$. First we need some definitions and some preliminary results. Unless otherwise is stated, A is allowed to be either a real or a complex Banach space, and the field of scalars will be denoted by \mathbf{K} .

DEFINITION. A family $\{B(a_i, r_i)\}_{i \in I}$ of balls in A is said to have the *weak intersection property* if for any $f \in A_1^*$,

$$\bigcap_{i \in I} B(f(a_i), r_i) \neq \emptyset \quad \text{in } \mathbf{C} \text{ or } \mathbf{R}.$$

Theorem 1.1 (with $A = \mathbf{C}$ (or \mathbf{R}) and $J = A$) gives immediately the following result of Hustad [24].

PROPOSITION 2.1. Let $\{B(a_i, r_i)\}_{i=1}^n$ be a finite family of balls in a Banach space A . Then the following statements are equivalent:

- (i) $\{B(a_i, r_i)\}_{i=1}^n$ has the weak intersection property.
- (ii) $|\sum_{i=1}^n z_i f(a_i)| \leq \sum_{i=1}^n |z_i| r_i$ for all $f \in A_1^*$ and all $z = (z_1, \dots, z_n) \in H_r^n(\mathbf{K}, (0))$.

Helly's theorem [47] implies the following results.

PROPOSITION 2.2. Let $\{B(a_i, r_i)\}_{i=1}^n$ be a finite family of balls in a real Banach space A . Then the following statements are equivalent:

- (i) $\{B(a_i, r_i)\}_{i=1}^n$ has the weak intersection property.
- (ii) The balls $\{B(a_i, r_i)\}_{i=1}^n$ are mutually intersecting.
- (iii) $\|a_i - a_j\| \leq r_i + r_j$ all $i, j = 1, 2, \dots, n$.

PROPOSITION 2.3. Let $\{B(a_i, r_i)\}_{i=1}^n$ be a finite family of balls in a complex Banach space A . Then the following statements are equivalent:

- (i) $\{B(a_i, r_i)\}_{i=1}^n$ has the weak intersection property.
- (ii) Every subfamily of 3 balls has the weak intersection property.

In [24] Hustad defined the notion of an almost $E(n)$ -space. He gave a characterization of almost $E(n)$ -spaces in terms of intersection properties of balls [24, Proposition 1.13]. Since we will be mainly concerned with intersection properties of balls, we will take this characterization as our definition.

DEFINITION. We say that A is an *almost $E(n)$ -space* if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of n balls in A with the weak intersection property we have

$$(2.1) \quad \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

We say that A is an $E(n)$ -space if we can take $\epsilon = 0$ in (2.1).

The following result was proved by Lindenstrauss [35] for the real case, and by Hustad [24] for the complex case.

LEMMA 2.4. *Let $n \geq 3$. Let $\epsilon > 0$, and assume $\{B(a_i, r_i)\}_{i=1}^n$ are n balls in A such that $\bigcap_{i=1}^n B(a_i, r_i + \epsilon) = \emptyset$. Then there exist $R > 0$ and $b_1, \dots, b_n \in A$ such that*

$$B(a_i, r_i) \subseteq B(b_i, R) \quad \text{and} \quad \bigcap_{i=1}^n B(b_i, R + \epsilon/2) = \emptyset.$$

DEFINITION. Let $n, k \in \mathbb{N}$ with $n \geq k \geq 2$. We say that A has the *almost $n.k.$ intersection property* (almost $n.k.$ I.P.) if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of n balls in A such that for any k of them

$$\bigcap_{j=1}^k B(a_{i_j}, r_{i_j}) \neq \emptyset,$$

we have

$$\bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

We say that A has the *restricted almost $n.k.$ intersection property* (almost $R.n.k.$ I.P.) if for every family $\{B(a_i, r)\}_{i=1}^n$ of n balls in A with common radius r such that for any k of them

$$\bigcap_{j=1}^k B(a_{i_j}, r) \neq \emptyset,$$

we have

$$\bigcap_{i=1}^n B(a_i, r + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

The word *almost* is omitted if we can take $\epsilon = 0$ as well.

COROLLARY 2.5. *The statements below are related as follows: (i) \Rightarrow (ii) \Leftrightarrow (iii).*

(i) A is an almost $E(n)$ -space.

(ii) A has the almost $n.3.$ I.P.

(iii) A has the almost $R.n.3.$ I.P.

COROLLARY 2.6. *A has the almost $n.k.$ I.P. if and only if A has the almost $R.n.k.$ I.P.*

REMARK. In case A is a real Banach space, then it follows from Proposition 2.2 that (almost) $E(n)$ is the same as (almost) n.2.I.P.

Lindenstrauss [35] has proved

THEOREM 2.7. *Let A be a real Banach space and let $n \geq 3$. Then the following statements are equivalent:*

- (i) A has the n.2.I.P.
- (ii) A has the almost n.2.I.P.
- (iii) A has the R.n.2.I.P.
- (iv) A has the almost R.n.2.I.P.

We now come to the new results of this section. First an important lemma.

LEMMA 2.8. *Let $n, k \in \mathbb{N}$ with $n > k \geq 2$, let J be a linear subspace of A , let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$, and let S be a compact (ω^*) circled subset of $H_r^n(A^*, J^0)_1$. Then the following statements are equivalent:*

- (i) $\partial_{\omega^*} H_r^n(A^*, J^0)_1 \subseteq S$.
- (ii) $\text{co}(S) = H_r^n(A^*, J^0)_1$ (ω^* -closure).
- (iii) *If $a_1, \dots, a_n \in A$ are such that $|\sum_{i=1}^n f_i(a_i)| \leq 1$ for all $(f_1, \dots, f_n) \in S$, then $J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset$ for all $\epsilon > 0$.*

PROOF. The equivalence of (i) and (ii) follows from Milmans' theorem [11].

(ii) \Rightarrow (iii) Let $a_1, \dots, a_n \in A$ be such that

$$\left| \sum_{i=1}^n f_i(a_i) \right| \leq 1 \quad \text{for all } (f_1, \dots, f_n) \in S.$$

Then, since $(f_1, \dots, f_n) \rightarrow \sum_{i=1}^n f_i(a_i)$ is a ω^* -continuous linear functional, we have $|\sum_{i=1}^n f_i(a_i)| \leq 1$ for all $(f_1, \dots, f_n) \in H_r^n(A^*, J^0)_1$, so

$$\left| \sum_{i=1}^n f_i(a_i) \right| \leq \sum_{i=1}^n r_i \|f_i\|$$

for all $(f_1, \dots, f_n) \in H_r^n(A^*, J^0)$, and (iii) follows from Theorem 1.1.

(iii) \Rightarrow (ii). Here assume that (ii) is false. Then there exists

$$(f_1, \dots, f_n) \in H_r^n(A^*, J^0)_1 \setminus \overline{\text{co}}(S),$$

and by Hahn-Banach (f_1, \dots, f_n) and $\overline{\text{co}}(S)$ can be strongly separated by a ω^* -continuous linear functional. Thus there exists $(a_1, \dots, a_n) \in A^n$ such that if $g = (g_1, \dots, g_n)$ then:

$$\text{Re} \left(\sum_{i=1}^n f_i(a_i) \right) > 1 \geq \sup_{g \in S} \text{Re} \left(\sum_{i=1}^n g_i(a_i) \right).$$

Hence

$$\left| \sum_{i=1}^n f_i(a_i) \right| > 1 \geq \left| \sum_{i=1}^n g_i(a_i) \right| \quad \text{all } (g_1, \dots, g_n) \in S.$$

By Theorem 1.1 it follows that

$$J \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) = \emptyset$$

for some $\varepsilon > 0$, and this concludes the proof of the lemma.

LEMMA 2.9. *Let $n \geq k \geq 2$, and let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$. Let S be the subset of $H_r^n(A^*, (0))_1$ consisting of those elements which have at most k coordinates different from zero. Then S is circled and ω^* -compact.*

PROOF. Let $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Then the subset of $H_r^n(A^*, (0))_1$ consisting of all $f = (f_1, \dots, f_n)$ such that $f_i = 0$ if $i \neq i_j, j = 1, \dots, k$, will be circled and ω^* -compact, and since S is a finite union of such sets, S is also circled and ω^* -compact. This completes the proof of the lemma.

THEOREM 2.10. *Let $n, k \in \mathbb{N}$ with $n > k \geq 2$. Then the following statements are equivalent:*

- (i) *A has the almost $n.k.I.P.$*
- (ii) *A has the almost $R.n.k.I.P.$*
- (iii) *If $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$ and if $(f_1, \dots, f_n) \in \partial_e H_r^n(A^*, (0))_1$, then at most k coordinates of (f_1, \dots, f_n) are different from zero.*
- (iv) *If $r = (1, 1, \dots, 1) \in \mathbb{R}^n$ and if $(f_1, \dots, f_n) \in \partial_e H_r^n(A^*, (0))_1$, then at most k coordinates of (f_1, \dots, f_n) are different from zero.*

PROOF. (iii) \Rightarrow (iv) is trivial, and (i) \Leftrightarrow (ii) is just Corollary 2.6.

(iv) \Rightarrow (ii). Let $\{B(a_i, 1)\}_{i=1}^n$ be n balls in A such that any k of them have a nonempty intersection. Then by Lemma 2.9, Lemma 2.8 and Theorem 1.1

$$\bigcap_{i=1}^n B(a_i, 1 + \varepsilon) \neq \emptyset \quad \text{all } \varepsilon > 0,$$

and this shows that (ii) is a consequence of (iv).

(i) \Rightarrow (iii). Let $r = (r_i)_{i=1}^n \in \mathbb{R}^n$ with all $r_i > 0$, and let S be as in Lemma 2.9. Let $a_1, \dots, a_n \in A$ be such that $|\sum_{i=1}^n f_i(a_i)| \leq 1$ all $(f_1, \dots, f_n) \in S$. By Theorem 1.1 for $1 \leq i_1 < i_2 < \dots < i_k \leq n$:

$$\bigcap_{j=1}^k B(a_{i_j}, r_{i_j} + \varepsilon) \neq \emptyset \quad \text{all } \varepsilon > 0.$$

But then by (i)

$$\bigcap_{i=1}^n B(a_i, r_i + 2\varepsilon) \neq \emptyset \quad \text{all } \varepsilon > 0,$$

and by Lemma 2 the desired result follows. The proof is complete.

For $1 \leq j < k \leq n$ we define the following sets ($r = (1, \dots, 1) \in \mathbb{R}^n$ and $f = (f_1, \dots, f_n) \in A^{*n}$):

$$(2.2) \quad S_{jk} = \{f \in H_r^n(A^*, (0)) : f_i = 0 \text{ if } i \neq j \text{ and } i \neq k\} \\ \text{and } \|f_j\| + \|f_k\| \leq 1\}.$$

With this notation we get

COROLLARY 2.11. *Let $n \geq 3$. Then the following statements are equivalent for a real Banach space A :*

- (i) *A has the n.2.I.P.*
- (ii) *For $r = (1, \dots, 1) \in \mathbb{R}^n$ we have*

$$H_r^n(A^*, (0))_1 = \text{co} \left(\bigcup_{1 \leq j < k \leq n} S_{jk} \right).$$

- (iii) *For $r = (1, \dots, 1) \in \mathbb{R}^n$ we have*

$$\partial_e H_r^n(A^*, (0))_1 \subseteq \bigcup_{1 \leq j < k \leq n} S_{jk}.$$

PROOF. This result follows from Theorem 2.7, Theorem 2.10 and Lemma 2.8. We should only remark that all S_{jk} are convex and ω^* -compact, so we do not need to take the closure in (ii).

Using Corollary 2.11, one can easily prove the following theorem of Lindenstrauss [35] and Hanner [20].

THEOREM 2.12. *If a real Banach space has the 4.2.I.P. then it has the n.2.I.P. for all natural numbers n .*

PROOF. Let $n \geq 4$ and let $r = (1, \dots, 1)$ denote $(1, \dots, 1)$ in \mathbb{R}^n and also the corresponding point $(1, \dots, 1)$ in \mathbb{R}^{n+1} . By Corollary 2.11 it is enough to prove that

$$H_r^n(A^*, (0))_1 = \text{co} \left(\bigcup_{1 \leq i < j \leq n} S_{ij} \right)$$

implies that $\partial_e H_r^{n+1}(A^*, (0))_1 \subseteq \bigcup_{1 \leq i < j \leq n+1} S_{ij}$. Assume

$$(f_1, \dots, f_{n+1}) \in \partial_e H_r^{n+1}(A^*, (0))_1 \setminus \bigcup_{1 \leq i < j \leq n+1} S_{ij}.$$

Let $\alpha = \|\sum_{i=1}^{n-1} f_i\| + \sum_{i=1}^{n-1} \|f_i\|$. Since $(f_1, \dots, f_{n+1}) \notin S_{n(n+1)}$, we have $\alpha > 0$, so

$$\frac{1}{\alpha} \left(f_1, \dots, f_{n-1}, -\sum_{i=1}^{n-1} f_i \right) \in H_r^n(A^*, (0))_1 = \text{co} \left(\bigcup_{1 \leq i < j \leq n} S_{ij} \right).$$

Let $1 \leq i < j < n$. Then

$$\begin{aligned} \frac{1}{\alpha} \left(f_1, \dots, f_{n-1}, -\sum_{i=1}^{n-1} f_i \right) \\ = \lambda(0, \dots, g, \dots, -g, \dots, 0) + (1 - \lambda)(h_1, \dots, h_n), \end{aligned}$$

where $\lambda \in [0, 1]$, $(0, \dots, g, \dots, -g, \dots, 0) \in S_{ij}$ and

$$(h_1, \dots, h_n) \in \text{co} \left(\bigcup_{k \neq i, m \neq j} S_{km} \right).$$

From this it follows that

$$\alpha^{-1} \|f_i\| = \|\lambda g\| + \|(1 - \lambda)h_i\|, \quad \alpha^{-1} \|f_j\| = \|\lambda g\| + \|(1 - \lambda)h_j\|.$$

Now we may write

$$\begin{aligned} (f_1, \dots, f_{n+1}) &= (0, \dots, \alpha\lambda g, \dots, -\alpha\lambda g, \dots, 0) \\ &\quad + (f_1, \dots, f_i - \alpha\lambda g, \dots, f_j + \alpha\lambda g, \dots, f_{n+1}). \end{aligned}$$

This gives a convex combination in $H_r^{n+1}(A^*, (0))_1$. Since (f_1, \dots, f_{n+1}) is an extreme point of $H_r^{n+1}(A^*, (0))_1$ we must have $\lambda g = 0$. Hence

$$\frac{1}{\alpha} \left(f_1, \dots, f_{n-1}, -\sum_{i=1}^{n-1} f_i \right) \in \text{co} \left(\bigcup_{i=1}^{n-1} S_{in} \right).$$

Now we have

$$\frac{1}{\alpha} \left(f_1, \dots, f_{n-1}, -\sum_{i=1}^{n-1} f_i \right) = \sum_{i=1}^{n-1} \lambda_i (0, \dots, g_i, \dots, -g_i, \dots, 0)$$

where $2\|g_i\| \leq 1$, $\lambda_i \geq 0$, $\sum_{i=1}^{n-1} \lambda_i = 1$, and where $\lambda_i > 0$ if $f_i \neq 0$. In particular $\alpha\lambda_i g_i = f_i$, and we have

$$\begin{aligned} 1 &= \sum_{i=1}^{n-1} \left\| \frac{1}{\alpha} f_i \right\| + \left\| \sum_{i=1}^{n-1} \frac{1}{\alpha} f_i \right\| \\ &= \sum_{i=1}^{n-1} \|\lambda_i g_i\| + \left\| \sum_{i=1}^{n-1} \lambda_i g_i \right\| \leq 2 \sum_{i=1}^{n-1} \lambda_i \|g_i\| = 1. \end{aligned}$$

Hence $\|\sum_{i=1}^{n-1} f_i\| = \sum_{i=1}^{n-1} \|f_i\|$. But then

$$\begin{aligned}
2(\|f_n\| + \|f_{n+1}\|) &\geq \|f_n + f_{n+1}\| + \|f_n\| + \|f_{n+1}\| \\
&= \left\| \sum_{i=1}^{n-1} f_i \right\| + \|f_n\| + \|f_{n+1}\| = \sum_{i=1}^{n+1} \|f_i\| = 1.
\end{aligned}$$

Similarly we get

$$2(\|f_i\| + \|f_j\|) \geq 1 \quad \text{all } i, j; i \neq j.$$

This is impossible if $n \geq 4$, so we must have

$$\partial_e H_r^{n+1}(A^*, (0))_1 \subseteq \bigcup_{1 \leq i < j \leq n+1} S_{ij}$$

and the proof is complete.

Our next lemma is valid for complex as well as real spaces.

LEMMA 2.13. Let $n \geq 2$, let $r = (r_i)_{i=1}^n \in \mathbf{R}^n$ with all $r_i > 0$.

Then the set $S_r^n = \{(f_1, \dots, f_n) \in H_r^n(A^*, (0))_1 : f_i = z_i g, g \in A^*, (z_1, \dots, z_n) \in H_r^n(\mathbf{K}, (0))\}$ is ω^* -compact.

PROOF. Observe that we may take $\|g\| = 1$. Use compactness of A_1^* and of $H_r^n(\mathbf{K}, (0))_1$, and the result will follow.

THEOREM 2.14. Let $n \geq 2$. Then the following statements are equivalent:

- (i) A is an almost $E(n)$ -space.
- (ii) If $r = (r_i)_{i=1}^n \in \mathbf{R}^n$ with all $r_i > 0$, and if $(f_1, \dots, f_n) \in \partial_e H_r^n(A^*, (0))_1$, then $f_i = z_i g$ for some $g \in \partial_e A_1^*$ and some $(z_1, \dots, z_n) \in \partial_e H_r^n(\mathbf{K}, (0))_1$.
- (iii) If $r = (1, \dots, 1) \in \mathbf{R}^n$, and if $(f_1, \dots, f_n) \in \partial_e H_r^n(A_1^*(0))_1$, then $f_i = z_i g$ for some $g \in \partial_e A_1^*$ and some $(z_1, \dots, z_n) \in \partial_e H_r^n(\mathbf{K}, (0))_1$.

PROOF. Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A . By Proposition 2.1 these balls have the weak intersection property if and only if $|\sum_{i=1}^n f_i(a_i)| \leq 1$ for all $(f_1, \dots, f_n) \in S_r^n$, where S_r^n is defined as in Lemma 2.13. The theorem now follows from Lemmas 2.8, 2.13 and 2.4. The proof is complete.

REMARK. From Helly's theorem and Theorem 2.10 it follows that if $(z_1, \dots, z_n) \in \partial_e H_r^n(\mathbf{R}, (0))_1$, then $z_i \neq 0$ for exactly two indices and if $(z_1, \dots, z_n) \in \partial_e H_r^n(\mathbf{C}, (0))_1$, then $z_i \neq 0$ for at most three indices. In [24, Theorem 3.6] Hustad has characterized the extreme points of the unit ball of $H_r^n(\mathbf{C}, (0))$.

EXAMPLE. Let $A = \mathbf{C}^2$ with $\|(z, w)\| = |z| + |w|$. Let $x = (3^{-1}, 3^{-1})$, $y = (3^{-1}e^{2\pi i/3}, 3^{-1}e^{-2\pi i/3})$ and $z = (3^{-1}e^{-2\pi i/3}, 3^{-1}e^{2\pi i/3})$. Using Theorem 3.6 in [24], we can show that $(x, y, z) \in \partial_e H_r^3(A^*, (0))_1$ where $r = (1, 1, 1)$. By Theorem 2.14 A is not an $E(3)$ -space.

THEOREM 2.15. *A has the almost n.k.I.P. if and only if A^{**} has the n.k.I.P.*

PROOF. Assume A^{**} has the n.k.I.P. Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A such that any k of them have nonempty intersection. Then $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ in A^{**} . By Corollary 1.3

$$\bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset \quad \text{in } A, \text{ all } \varepsilon > 0,$$

so A has the almost n.k.I.P.

Assume next that A has the almost n.k.I.P. and assume that A^{**} does not have the n.k.I.P. Then there exist balls $\{B(a_i, r_i)\}_{i=1}^n$ in A^{**} such that any k of them have nonempty intersection and $\bigcap_{i=1}^n B(a_i, r_i) = \emptyset$. Choose $b_1, \dots, b_m \in A^{**}$ such that if $1 \leq i_1 < \dots < i_k \leq n$, then $b_p \in \bigcap_{j=1}^k B(a_{i_j}, r_{i_j})$ for some b_p . Then by Theorem 1.2 there exist $f_1, \dots, f_n \in A^*$ such that $\sum_{i=1}^n f_i = 0$ and $|\sum_{i=1}^n f_i(a_i)| > \sum_{i=1}^n r_i \|f_i\|$. Choose $\theta > 0$ such that

$$\left| \sum_{i=1}^n f_i(a_i) \right| > (1 + \theta) \sum_{i=1}^n r_i \|f_i\|.$$

Let $U = \text{span}(a_1, \dots, a_n, b_1, \dots, b_m)$ and let $F = \text{span}(f_1, \dots, f_n)$. By the “principle of local reflexivity” (see [36] or [12]) there exists a linear operator $T: U \rightarrow A$ such that:

- (i) $T(a) = a$ if $a \in U \cap A$,
- (ii) $f(T(a)) = a(f)$ for $a \in U$ and $f \in F$,
- (iii) $(1 - \theta)\|a\| \leq \|T(a)\| \leq (1 + \theta)\|a\|$ all $a \in U$.

Now the balls $\{B(T(a_i), (1 + \theta)r_i)\}_{i=1}^n$ have the property that any k of them have a nonempty intersection. In fact, if $1 \leq i_1 < \dots < i_k \leq n$ and $b_p \in \bigcap_{j=1}^k B(a_{i_j}, r_{i_j})$ then

$$T(b_p) \in \bigcap_{j=1}^k B(T(a_{i_j}), (1 + \theta)r_{i_j}).$$

Since A has the almost n.k.I.P.

$$\bigcap_{i=1}^n B(T(a_i), (1 + \theta)r_i + \varepsilon) \neq \emptyset \quad \text{all } \varepsilon > 0.$$

By Theorem 1.1 and by the choice of θ we have

$$\begin{aligned} (1 + \theta) \sum_{i=1}^n r_i \|f_i\| &< \left| \sum_{i=1}^n f_i(a_i) \right| \\ &= \left| \sum_{i=1}^n f_i(T(a_i)) \right| \leq (1 + \theta) \sum_{i=1}^n r_i \|f_i\|. \end{aligned}$$

This contradiction completes the proof.

A similar proof gives

THEOREM 2.16. *A is an almost $E(n)$ -space if and only if A^{**} is an $E(n)$ -space.*

REMARK. The argument used to prove Theorem 2.15 can also be used to prove a conjecture of Lindenstrauss (see [35, p. 56]). Later on we shall give another proof of this conjecture (Corollary 4.7).

By the compactness of the balls in A^{**} it follows from Theorems 2.14 and 2.6 that A is an almost $E(n)$ -space for all n if and only if every family of balls in A^{**} with the weak intersection property has a nonempty intersection. By the characterizations of Nachbin [39] and Hustad [24], we conclude that A is an almost $E(n)$ -space for all n if and only if A^{**} is a \mathfrak{P}_1 -space.

Theorem 2.17 summarizes some of the most important results on spaces having \mathfrak{P}_1 -biduals.

THEOREM 2.17. *Let A be a real or complex Banach space. The following statements are equivalent:*

- (i) *A is an $E(n)$ -space for all n .*
- (ii) *A^* is isometric to an $L_1(\mu)$ -space for some measure μ .*
- (iii) *A^{**} is a \mathfrak{P}_1 -space.*
- (iv) *For every Banach space B such that $A \subseteq B$, there exists a projection P in B^* such that $P(B^*) = A^0$ and $\|I - P\| \leq 1$.*

PROOF. (i) \Rightarrow (iii) follows from Theorem 2.16 and from results of Nachbin [39] and Hustad [24].

(iii) \Rightarrow (ii) is proved by Grothendieck [19] and Sakai [42]. (See also Corollary 3.11 and Theorem 3.12.)

(ii) \Rightarrow (iv) is proved in Lacey [32, Lemma 22.3]. That this lemma is valid in the complex case follows from results of Sakai [42]. See also [24], [8] and [14].

(iv) \Rightarrow (i). For some set Γ we may imbed A isometrically into $B = l_\infty(\Gamma)$. Let P be a projection as in (iv). Let $\{B(a_i, r_i)\}_{i=1}^n$ be a family of n balls in A with the weak intersection property. Then since B is a \mathfrak{P}_1 -space: $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$. Let $f_1, \dots, f_n \in B^*$ be such that $\sum_{i=1}^n f_i \in A^0$. Let $a \in \bigcap_{i=1}^n B(a_i, r_i)$. Since $\sum_{i=1}^n (I - P)(f_i) = 0$ we get

$$\begin{aligned} \left| \sum_{i=1}^n f_i(a_i) \right| &\leq \left| \sum_{i=1}^n P(f_i)(a_i) \right| + \left| \sum_{i=1}^n (I - P)(f_i)(a_i) \right| \\ &= \left| \sum_{i=1}^n (I - P)(f_i)(a_i - a) \right| \\ &\leq \sum_{i=1}^n \|(I - P)(f_i)\| \|a_i - a\| \leq \sum_{i=1}^n \|f_i\| r_i. \end{aligned}$$

By Theorem 1.1 $\bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ in A , all $\varepsilon > 0$, so A is an almost $E(n)$ -space. An induction argument similar to the argument in the proof of

Proposition 6.5 now gives that A is an $E(n-1)$ -space. This completes the proof.

REMARK. The equivalence of the statements in Theorem 2.17 is well known, but the proofs of some of the implications seem to be new. Among these is the proof of the implication (iv) \Rightarrow (i), which gives a nice application of Theorem 1.1.

3. Banach spaces with the 3.2.I.P. In this section A will be a real Banach space. We shall study Banach spaces A with the 3.2.I.P., and the main tool will be the following notion.

DEFINITION. A real Banach space A is said to have the R_3 -property if for all $x, y \in A$, there exists $z, u, v \in A$ such that:

- (i) $x = z + u$ and $\|x\| = \|z\| + \|u\|$,
- (ii) $y = z + v$ and $\|y\| = \|z\| + \|v\|$,
- (iii) $\|x - y\| = \|u - v\| = \|u\| + \|v\|$.

THEOREM 3.1. Let $r = (1, 1, 1) \in \mathbb{R}^3$. Then the following statements are equivalent:

- (i) A has the 3.2.I.P.
- (ii) $H_r^3(A^*, (0))_1 = \text{co}(S_{12} \cup S_{13} \cup S_{23})$.
- (iii) A^* has the R_3 -property.

PROOF. (i) \Leftrightarrow (ii) is just Corollary 2.11.

(ii) \Rightarrow (iii). Let $f_1, f_2 \in A^*$. Then $(f_1, -f_2, f_2 - f_1) \in H_r^3(A^*, (0))$ and we may assume $\|f_1\| + \|f_2\| + \|f_1 - f_2\| = 1$. By (ii) there exist $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $g_1, g_2, g_3 \in A^*$ such that $\|g_i\| \leq 2^{-1}$, $\sum_{i=1}^3 \lambda_i = 1$ and

$$(f_1, -f_2, f_2 - f_1) = \lambda_1(g_1, -g_1, 0) + \lambda_2(g_2, 0, -g_2) + \lambda_3(0, -g_3, g_3).$$

From this it follows that

$$f_1 = \lambda_1 g_1 + \lambda_2 g_2, \quad \|f_1\| = \|\lambda_1 g_1\| + \|\lambda_2 g_2\|,$$

$$f_2 = \lambda_1 g_1 + \lambda_3 g_3, \quad \|f_2\| = \|\lambda_1 g_1\| + \|\lambda_3 g_3\|,$$

$$\|f_1 - f_2\| = \|\lambda_2 g_2 - \lambda_3 g_3\| = \|\lambda_2 g_2\| + \|\lambda_3 g_3\|,$$

so A^* has the R_3 -property.

(iii) \Rightarrow (ii). Let $(f_1, f_2, f_3) \in H_r^3(A^*, (0))_1$. By the R_3 -property, we can find $g, h, k \in A^*$ such that

$$f_1 = g + h, \quad \|f_1\| = \|g\| + \|h\|,$$

$$-f_2 = g + k, \quad \|f_2\| = \|g\| + \|k\|,$$

$$\|f_3\| = \|f_1 + f_2\| = \|h - k\| = \|h\| + \|k\|.$$

Now

$$(f_1, f_2, f_3) = (g, -g, 0) + (h, 0, -h) + (0, -k, k) \in \text{co}(S_{12} \cup S_{13} \cup S_{23}).$$

This completes the proof.

THEOREM 3.2. *A real Banach space A has the 3.2.I.P. if and only if it has the R_3 -property.*

PROOF. Assume A has the 3.2.I.P. and let $x, y \in A$. Define $r_0, r_1, r_2 \geq 0$ by

$$2r_0 = \|x\| + \|y\| - \|x - y\|,$$

$$2r_1 = \|x\| - \|y\| + \|x - y\|,$$

$$2r_2 = -\|x\| + \|y\| + \|x - y\|.$$

Then $\|x\| = r_0 + r_1$, $\|y\| = r_0 + r_2$ and $\|x - y\| = r_1 + r_2$, so $B(0, r_0)$, $B(x, r_1)$ and $B(y, r_2)$ are mutually intersecting. Let

$$z \in B(0, r_0) \cap B(x, r_1) \cap B(y, r_2)$$

and let $u = x - z$ and $v = y - z$. Then

$$x = z + u, \quad \|x\| \leq \|z\| + \|u\| \leq r_0 + r_1 = \|x\|,$$

$$y = z + v, \quad \|y\| \leq \|z\| + \|v\| \leq r_0 + r_2 = \|y\|,$$

$$\|x - y\| = \|u - v\| \leq \|u\| + \|v\| \leq r_1 + r_2 = \|x - y\|,$$

so A has the R_3 -property.

Suppose next that A has the R_3 -property, and assume that three mutually intersecting balls in A are given. By translating and considering balls with smaller radii we may (as in [35, Theorem 4.6]) assume that these balls are $B(0, r_0)$, $B(x, r_1)$ and $B(y, r_2)$, where $\|x\| = r_0 + r_1$, $\|y\| = r_0 + r_2$ and $\|x - y\| \leq r_1 + r_2$. Let $z, u, v \in A$ be as in the definition of the R_3 -property. Then we have $0 \leq r_1 - \|u\| = \|z\| - r_0 = r_2 - \|v\|$. Define $w = \|z\|^{-1} r_0 z$ ($w = 0$ if $z = 0$). Then

$$\|0 - w\| = r_0,$$

$$\|x - w\| = \|u + (\|z\| - r_0)z/\|z\|\| \leq \|u\| + (\|z\| - r_0) = r_1,$$

$$\|y - w\| = \|v + (\|z\| - r_0)z/\|z\|\| \leq \|v\| + (\|z\| - r_0) = r_2,$$

so

$$w \in B(0, r_0) \cap B(x, r_1) \cap B(y, r_2),$$

and the proof is complete.

COROLLARY 3.3. *A has the 3.2.I.P. if and only if A^* has the 3.2.I.P.*

REMARK. Hanner [20] proved Corollary 3.3 for finite-dimensional spaces. The general case was an open problem.

It is easy to see that a real $L_1(\mu)$ -space has the R_3 -property, so Theorem 3.2 gives a new proof of Lindenstrauss' result [35] that every real $L_1(\mu)$ -space has the 3.2.I.P.

The following notion was introduced by Fullerton [17].

A Banach space A is called a *CL-space* if for every maximal proper face F of A_1 , $A_1 = \text{co}(F \cup -F)$.

It is easy to see that the maximal proper faces of A_1 coincide with the maximal convex subsets of the boundary of A_1 . Note also that every maximal proper face of A_1 is normclosed. (See [4] or [17].)

We now want to show that every Banach space with the 3.2.I.P. is a *CL-space*. Before we can do so, we need some more terminology.

A nonempty subset $C \subseteq A$ is a *cone* if $\lambda C \subseteq C$ all $\lambda \geq 0$. A cone C is *proper* if $C \cap -C = (0)$ and *convex* if $C + C \subseteq C$. If S is a subset of A , then cone $S = \bigcup_{\lambda \geq 0} \lambda S$ is the smallest cone containing S . We say that a cone C is a *facial cone* if $C = (0)$ or $C = \text{cone } F$ for some proper face F of A_1 . If $x \in A$ and $x \neq 0$, then

$$C(x) = \text{cone}(\text{face}(x/\|x\|))$$

is the smallest facial cone containing x . We note that $\text{face}(x/\|x\|)$, the smallest face of A_1 containing $x/\|x\|$, is well defined for every $x \neq 0$, and that $y \in \text{face}(x/\|x\|)$ if and only if $x/\|x\| = \lambda y + (1 - \lambda)z$ for some $\lambda \in (0, 1]$ and some $z \in A_1$. (See [2].) We define $C(0) = (0)$.

Our terminology here coincides with that of Alfsen and Effros [4]. In [4] they also defined the *complementary cone* C' of a given cone C in A , to be the set

$$C' = \{x \in A : C \cap C(x) = (0)\}.$$

We say that a cone C is *hereditary* if $x \in C$ and $\|x\| = \|x - y\| + \|y\|$ implies that $y \in C$. Alfsen and Effros showed [4, Lemma 2.7] that a cone C is hereditary if and only if C is a union of facial cones. Further, Alfsen and Effros [4] showed that the norm is additive on a facial cone, and they proved the following result [4, Theorem 2.9] which we will use several times:

THEOREM 3.4. *Suppose C is a norm-closed convex cone in a Banach space A . Then every $p \in A$ admits a decomposition $p = q + r$, $\|p\| = \|q\| + \|r\|$, where $q \in C$ and $r \in C'$.*

We can now state

THEOREM 3.5. *Every Banach space with the R_3 -property is a CL-space.*

PROOF. Let F be a maximal proper face of A_1 , and let $C = \bigcup_{\lambda \leq 0} \lambda F$. Then C is a norm-closed convex cone in A . Let $x \in A$. By Theorem 3.4 there exists $y \in C$ and $z \in C'$ such that $x = y + z$, $\|x\| = \|y\| + \|z\|$. Now it clearly suffices to show that $C' = -C$. Since F is a proper face, we have $-C \subseteq C'$. Now let $w \in C$ and $t \in C'$. By the R_3 -property, there exists $s, u, v \in A$ such that

$$\begin{aligned} t &= s + u, & \|t\| &= \|s\| + \|u\|, \\ w &= s + v, & \|w\| &= \|s\| + \|v\|, \\ \|t - w\| &= \|u - v\| = \|u\| + \|v\|. \end{aligned}$$

Since both C and C' are unions of facial cones, they are hereditary, so we have $s \in C \cap C' = (0)$. Hence $s = 0$. (See [4, Lemma 2.7].) But then we have $\|t - w\| = \|t\| + \|w\|$. Assume $t \neq 0$, and define $F_t = \text{co}(F \cup \{-t/\|t\|\})$. Then we just showed that F_t is a convex subset of the boundary of A_1 , and that $F \subseteq F_t$. Since F is maximal, we have $F = F_t$, so $-t/\|t\| \in F$, i.e. $t \in -C$. Hence $C' \subseteq -C$. This completes the proof.

COROLLARY 3.6. *Let A be a Banach space with the 3.2.I.P. Then both A and A^* are CL-spaces.*

REMARK. Corollary 3.6 improves a result of Lindenstrauss [35, Theorem 4.8]. For finite dimensional spaces the corollary was proved in the works of Fullerton [17], Hanner [20] and Lindenstrauss [35].

PROPOSITION 3.7. *Let A be a real Banach space with the 3.2.I.P. Let F be a maximal proper face of A_1 and let $C = \bigcup_{\lambda \geq 0} \lambda F$. We order A by writing $x \geq y$ if $x - y \in C$, and we write $[y, x] = \{z \in A : y \leq z \leq x\}$. If $x, y \in C$ and $w \in [0, x] \cap [0, y]$, then there exists an element $z \in [0, x] \cap [0, y]$ such that $w \leq z$ and*

$$2\|z\| = \|x + y\| - \|x - y\|.$$

PROOF. Assume $w \in [0, x] \cap [0, y]$. Define $u = x - w$ and $v = y - w$. Then $u, v \in C$. By Theorem 3.2 we can find $a, b, c \in A$ such that

$$\begin{aligned} u &= a + b, & \|u\| &= \|a\| + \|b\|, \\ v &= a + c, & \|v\| &= \|a\| + \|c\|, \\ \|u - v\| &= \|b - c\| = \|b\| + \|c\|. \end{aligned}$$

Define $z = a + w$. Since C is a facial cone, we have $a, b, c \in C$, so $z \in C$. It is clear that $0 \leq w \leq z \leq x, y$, and we have

$$\begin{aligned}
2\|z\| &= 2(\|a\| + \|w\|) \\
&= \|u\| + \|w\| + \|v\| + \|w\| - (\|b\| + \|c\|) \\
&= \|x\| + \|y\| - \|u - v\| = \|x + y\| - \|x - y\|.
\end{aligned}$$

The proof is complete.

A real Banach space A with a closed convex cone C such that

$$\|x + y\| = \|x\| + \|y\| \quad \text{for all } x, y \in C$$

and such that for all $z \in A$, there exist $x, y \in C$ with

$$(3.1) \quad z = x - y, \quad \|z\| = \|x\| + \|y\|$$

is called an A -space [4]. We say that an A -space A is *uniquely generated* if the elements x and y in the decomposition (3.1) are unique.

COROLLARY 3.8. *Let A be a real Banach space. Then A is isometric to an $L_1(\mu)$ -space for some measure μ if and only if A is a uniquely generated A -space with the 3.2.I.P.*

PROOF. It is well known that an $L_1(\mu)$ -space is a uniquely generated A -space, and Lindenstrauss has shown that an $L_1(\mu)$ -space has the 3.2.I.P. (See also the remark after Corollary 3.3.)

Assume that A is a uniquely generated A -space with the 3.2.I.P. Let C be the positive cone in A . Then C is a facial cone generated by a maximal proper face of A_+ . Let $x, y \in C$. Let $0 \leq z, w \leq x, y$ be such that

$$2\|z\| = 2\|w\| = \|x + y\| - \|x - y\|.$$

Then we have

$$\begin{aligned}
\|x - y\| &= \|x + y\| - 2\|z\| = (\|x\| - \|z\|) + (\|y\| - \|z\|) \\
&= \|x - z\| + \|y - z\| = \|x - w\| + \|y - w\|.
\end{aligned}$$

Since A is uniquely generated, $x - z = x - w$ and $z = w$. Hence $[0, x] \cap [0, y] = [0, z]$, and we have shown that C generates a lattice-ordering in A . By a theorem of Kakutani [28] A is isometric to an $L_1(\mu)$ -space, and the proof is complete.

In analogy with the R_3 -property we define

DEFINITION. A real Banach space A is said to have the R_4 -property if for all $x_1, x_2, x_3 \in A$, there exist $u_{ij} \in A$, $1 \leq i < j \leq 4$, such that

- (i) $x_1 = u_{12} + u_{13} + u_{14}$ and $\|x_1\| = \|u_{12}\| + \|u_{13}\| + \|u_{14}\|$,
- (ii) $x_2 = -u_{12} + u_{23} + u_{24}$ and $\|x_2\| = \|u_{12}\| + \|u_{23}\| + \|u_{24}\|$,
- (iii) $x_3 = -u_{13} - u_{23} + u_{34}$ and $\|x_3\| = \|u_{13}\| + \|u_{23}\| + \|u_{34}\|$,
- (iv) $\|x_1 + x_2 + x_3\| = \|u_{14} + u_{24} + u_{34}\| = \|u_{14}\| + \|u_{24}\| + \|u_{34}\|$.

A proof similar to the proof of Theorem 3.1 gives

THEOREM 3.9. *Let $r = (1, 1, 1, 1) \in \mathbb{R}^4$. The following statements are equivalent:*

- (i) A has the 4.2.I.P.
- (ii) $H_r^4(A^*, (0))_1 = \text{co}(\cup_{1 \leq i < j \leq 4} S_{ij})$.
- (iii) A^* has the R_4 -property.

THEOREM 3.10. *A real Banach space A is isometric to an $L_1(\mu)$ -space if and only if A has the R_4 -property.*

PROOF. That every $L_1(\mu)$ -space has the R_4 -property follows easily by taking three functions and their sum, decomposing them into positive and negative parts, and then using the Riesz decomposition property.

Assume that A has the R_4 -property. Let F be a maximal proper face of A_1 and let $C = \cup_{\lambda > 0} \lambda F$. Let $x, y \in C$, and let z be such that $0 \leq z \leq x + y$. Since C is a facial cone, the R_4 -property implies that there exist $u_2, u_3, u_4, u_5 \in C$ such that $x = u_2 + u_3, y = u_4 + u_5$ and $z = u_2 + u_4$ (use R_4 on x, y and $-z$). But then C has the Riesz decomposition property, so if $0 \leq z, w \leq x, y$ and $2\|z\| = 2\|w\| = \|x + y\| - \|x - y\|$, then $z, w \leq u \leq x, y$ for some $u \in C$. But then $z = u = w$, so $[0, x] \cap [0, y] = [0, z]$. Hence C defines a lattice-ordering in A . By a theorem of Kakutani [28] A is isometric to an $L_1(\mu)$ -space, and the proof is complete.

COROLLARY 3.11. *A real Banach space A has the 4.2.I.P. if and only if A^* is isometric to an $L_1(\mu)$ -space.*

REMARK. Corollary 3.11 is a well-known result of Lindenstrauss [35].

THEOREM 3.12. *A real Banach space A is isometric to an $L_1(\mu)$ -space if and only if A^* has the 4.2.I.P.*

PROOF. Assume A is isometric to an $L_1(\mu)$ -space. Then A has the R_4 -property, and it easily follows from Theorem 1.2 that A^* has the 4.2.I.P.

Assume next that A^* has the 4.2.I.P. Let F be a maximal proper face of A_1 and let $C = \cup_{\lambda > 0} \lambda F$. By Theorem 3.2 and Corollary 3.3 A has the R_3 -property, and A is a CL -space by Theorem 3.5. As in the proof of Theorem 3.10 it is enough to show that C has the Riesz decomposition property. Let $e \in \partial_e A_1^*$ be such that $e(x) = 1$ all $x \in F$. Then the dual cone of C is

$$\begin{aligned} C^* &= \{f \in A^*: f(x) \geq 0 \text{ all } x \in C\} \\ &= \{f \in A^*: f = \lambda(e + g), \lambda \geq 0 \text{ and } g \in A_1^*\}. \end{aligned}$$

By Theorem 6.1 in Lindenstrauss [35], C^* has the Riesz decomposition property. By a result of Andô [5] C also has the Riesz decomposition property. This completes the proof.

REMARK. Theorem 3.12 is a well-known result of Grothendieck [19] and Lindenstrauss [35].

We remark here, that the above mentioned result of Andô, as well as its converse, can be proved by the separation technique used in the proof of Theorem 1.1.

4. Intersection properties and extreme points of $H_r^n(A^*, (0))_1$. In this section as in the preceding section we will study real Banach spaces with the 3.2.I.P. The R_3 -property will now be used to obtain information on the set $\partial_e H_r^n(A^*, (0))_1$ of extreme points of $H_r^n(A^*, (0))_1$, and we will then translate this into information about intersection properties of A .

LEMMA 4.1. *Let A be a real Banach space with the 3.2.I.P. and let $n \geq 4$. Let $r = (1, \dots, 1) \in \mathbb{R}^n$ and assume*

$$(x_1, \dots, x_n) \in \partial_e H_r^n(A^*, (0))_1 \setminus \bigcup_{1 \leq j < k \leq n} S_{jk}.$$

Then $\|x_i \pm x_j\| = \|x_i\| + \|x_j\|$ for all $i, j; i \neq j$.

PROOF. Let $1 \leq i < j \leq n$. By Theorem 3.1 A^* has the R_3 -property. Hence we can write

$$\begin{aligned} x_i &= z + u, & \|x_i\| &= \|z\| + \|u\|, \\ x_j &= z + v, & \|x_j\| &= \|z\| + \|v\|, \\ \|x_i - x_j\| &= \|u - v\| = \|u\| + \|v\|. \end{aligned}$$

Then we have

$$\begin{aligned} \|x_i + z\| + \|x_j - z\| &= \|2z + u\| + \|v\| = 2\|z\| + \|u\| + \|v\| \\ &= \|z + u\| + \|z + v\| = \|x_i\| + \|x_j\|, \end{aligned}$$

and

$$\|x_i - z\| + \|x_j + z\| = \|x_i\| + \|x_j\|.$$

We can now write

$$\begin{aligned} (x_1, \dots, x_n) &= \frac{1}{2}(x_1, \dots, x_i + z, \dots, x_j - z, \dots, x_n) \\ &\quad + \frac{1}{2}(x_1, \dots, x_i - z, \dots, x_j + z, \dots, x_n). \end{aligned}$$

This is a convex combination of elements in $H_r^n(A^*, (0))_1$. Since (x_1, \dots, x_n) is an extreme point, we must have $z = 0$, so

$$\|x_i - x_j\| = \|u - v\| = \|u\| + \|v\| = \|x_i\| + \|x_j\|.$$

Applying the R_3 -property once more, we have

$$\begin{aligned} x_i &= z + u, & \|x_i\| &= \|z\| + \|u\|, \\ -x_j &= z + v, & \|x_j\| &= \|z\| + \|v\|, \\ \|x_i + x_j\| &= \|u - v\| = \|u\| + \|v\|. \end{aligned}$$

We can now write

$$(x_1, \dots, x_n) = (0, \dots, z, \dots, -z, \dots, 0) + (x_1, \dots, u, \dots, -v, \dots, x_n),$$

which gives a convex combination of elements in $H_r^n(A^*, (0))_1$. Since (x_1, \dots, x_n) is an extreme point of $H_r^n(A^*, (0))_1$ and $(x_1, \dots, x_n) \notin S_{ij}$, we must have $z = 0$, so $\|x_i + x_j\| = \|x_i\| + \|x_j\|$. This completes the proof of the lemma.

LEMMA 4.2. *Assume A is a real Banach space with the 3.2.I.P. and let $r = (1, \dots, 1) \in \mathbf{R}^4$. Assume*

$$(x_1, \dots, x_4) \in \partial_e H_r^4(A^*, (0))_1 \setminus \bigcup_{1 \leq i < j \leq 4} S_{ij}.$$

Then $\|x_i\| = 4^{-1}$ for all i , and if $\varepsilon_i = \pm 1$ ($i = 1, \dots, 4$) with $\sum_{i=1}^4 \varepsilon_i = 0$, then $\|\varepsilon_1 x_1 + \dots + \varepsilon_4 x_4\| = 1$.

PROOF. By Lemma 4.1 we have $\|x_i \pm x_j\| = \|x_i\| + \|x_j\|$ for all i, j ; $i \neq j$. Since $\sum_{i=1}^4 x_i = 0$, we get

$$\begin{aligned} \|x_1\| + \|x_2\| &= \|x_1 + x_2\| = \|x_3 + x_4\| = \|x_3\| + \|x_4\|, \\ \|x_1\| + \|x_3\| &= \|x_1 + x_3\| = \|x_2 + x_4\| = \|x_2\| + \|x_4\|, \\ \|x_1\| + \|x_4\| &= \|x_1 + x_4\| = \|x_2 + x_3\| = \|x_2\| + \|x_3\|. \end{aligned}$$

Hence

$$\|x_1\| - \|x_4\| = \|x_3\| - \|x_2\| = \|x_2\| - \|x_3\|$$

and

$$\|x_1\| - \|x_2\| = \|x_4\| - \|x_3\| = \|x_3\| - \|x_4\|.$$

This gives $\|x_i\| = 4^{-1}$ for all i since $\sum_{i=1}^4 \|x_i\| = 1$. Let $\varepsilon_i = \pm 1$ with $\sum_{i=1}^4 \varepsilon_i = 0$. We may assume $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = \varepsilon_4 = -1$. Then $\|\varepsilon_1 x_1 + \dots + \varepsilon_4 x_4\| = \|2x_1 + 2x_2\| = 1$. The proof is complete.

l_∞^n will denote \mathbf{R}^n with maximum norm, and l_1^n will denote \mathbf{R}^n with the norm $\|(x_1, \dots, x_n)\| = \sum_{i=1}^n |x_i|$.

THEOREM 4.3. *Assume A is a real Banach space with the 3.2.I.P. Then the following statements are equivalent:*

- (i) *A does not have the 4.2.I.P.*
- (ii) *There exists an isometry $T: l_\infty^3 \rightarrow A^*$.*

PROOF. If A does not have the 4.2.I.P., then by Corollary 2.11 there exists an element

$$(x_1, \dots, x_4) \in \partial_e H_r^4(A^*, (0)) \setminus \bigcup_{1 \leq i < j \leq 4} S_{ij}$$

where $r = (1, \dots, 1) \in \mathbf{R}^4$.

Define a linear operator $T: l_\infty^3 \rightarrow A^*$ by

$$T(1, 1, 1) = 4x_1, \quad T(-1, 1, -1) = 4x_2, \quad T(1, -1, -1) = 4x_3.$$

Then

$$\begin{aligned} T(-1, -1, 1) &= -T((1, 1, 1) + (-1, 1, -1) + (1, -1, -1)) \\ &= -4(x_1 + x_2 + x_3) = 4x_4. \end{aligned}$$

Let $x \in l_\infty^3$ with $\|x\| = 1$. Then there exist $a, b, c, d \in \mathbf{R}$ with $|a| + |b| + |c| + |d| = 1$ such that

$$x = a(1, 1, 1) + b(-1, 1, -1) + c(1, -1, -1) + d(-1, -1, 1)$$

and such that two of the elements a, b, c, d are ≥ 0 and two of them are ≤ 0 . By Lemma 4.2 we have

$$\begin{aligned} \|T(x)\| &= \|4ax_1 + 4bx_2 + 4cx_3 + 4dx_4\| \\ &= 4|a|\|x_1\| + 4|b|\|x_2\| + 4|c|\|x_3\| + 4|d|\|x_4\| \\ &= |a| + |b| + |c| + |d| = 1. \end{aligned}$$

Hence T is an isometry.

Conversely assume $T: l_\infty^3 \rightarrow A^*$ is an isometry. Define $x_1, \dots, x_4 \in A^*$ by

$$\begin{aligned} 4x_1 &= T(1, 1, 1), & 4x_2 &= T(-1, 1, -1), \\ 4x_3 &= T(1, -1, -1), & 4x_4 &= T(-1, -1, 1). \end{aligned}$$

Then $\sum_{i=1}^4 x_i = 0$ and $\sum_{i=1}^4 \|x_i\| = 1$. We also have $\|x_i \pm x_j\| = \|x_i\| + \|x_j\|$ for all $i, j; i \neq j$. Let $\alpha \in \langle 0, 1 \rangle$ and let $(y_1, \dots, y_4), (z_1, \dots, z_4) \in H_r^4(A^*(0))_1$ be such that

$$(x_1, \dots, x_4) = \alpha(y_1, \dots, y_4) + (1 - \alpha)(z_1, \dots, z_4).$$

Then

$$\begin{aligned} 1 &= \sum_{i=1}^4 \|x_i\| = \sum_{i=1}^4 \|\alpha y_i + (1 - \alpha)z_i\| \\ &\leq \alpha \sum_{i=1}^4 \|y_i\| + (1 - \alpha) \sum_{i=1}^4 \|z_i\| \leq 1 \end{aligned}$$

so

$$\|x_i\| = \alpha\|y_i\| + (1 - \alpha)\|z_i\| \quad \text{all } i.$$

Hence for $i \neq j$

$$\begin{aligned} \|x_i + x_j\| &\leq \alpha\|y_i + y_j\| + (1 - \alpha)\|z_i + z_j\| \\ &\leq \alpha\|y_i\| + \alpha\|y_j\| + (1 - \alpha)\|z_i\| + (1 - \alpha)\|z_j\| \\ &= \|x_i\| + \|x_j\|, \end{aligned}$$

so

$$\|y_i + y_j\| = \|y_i\| + \|y_j\|, \quad \|z_i + z_j\| = \|z_i\| + \|z_j\|.$$

From this it follows that

$$(x_1, \dots, x_4) \notin \text{co}\left(\bigcup_{1 \leq i < j \leq 4} S_{ij}\right).$$

Now it follows by Corollary 2.11 that A does not have the 4.2.I.P.

COROLLARY 4.4. *A space A with the 3.2.I.P. will fail to have the 4.2.I.P. if and only if it admits a quotient space A/J isometric to l_1^3 .*

COROLLARY 4.5. *Assume that A has the 3.2.I.P. but not the 4.2.I.P. Let $\epsilon > 0$. Then there exists a linear operator $S: l_1^3 \rightarrow A$ such that*

$$\|x\| \leq \|S(x)\| \leq (1 + \epsilon)\|x\|$$

for all $x \in l_1^3$ and there exists a projection P in A such that $P(A) = S(l_1^3)$ and $\|P\| \leq 1 + \epsilon$.

PROOF. By Corollary 4.4 there exist a subspace J of A and an isometry $B: l_1^3 \rightarrow A/J$. Let $Q: A \rightarrow A/J$ be the quotient map. A lifting gives the existence of S (see [32, Theorem 18.9]), and P is given by $P = S \circ B^{-1} \circ Q$. This concludes the proof.

REMARK. We can show that (ii) \Leftrightarrow (iii) in Theorem 3.9 holds if we replace A^* by A . Then using Theorem 3.10 and modifying the arguments in Lemmas 4.1 and 4.2 (see the proof of Theorem 5.14), we find that a real Banach space A with the 3.2.I.P. is nonisometric to an $L_1(\mu)$ -space if and only if there exists an isometry $T: l_\infty^3 \rightarrow A$.

Alfsen and Effros [4] observed that in l_1^3 we have

$$B((0,0,0), 1) \cap B((1,-1,0), 1) \cap B((1,0,-1), 1) = \{(1,0,0)\}.$$

THEOREM 4.6. *A has the 4.2.I.P. if and only if for all $x, y \in A$ with $\|x\|, \|y\|, \|x - y\| \leq 2$, we have*

$$B(0, 1) \cap B(x, 1) \cap B(y, 1) \cap B((x + y)/3, 1/3) \neq \emptyset.$$

PROOF. To prove the nontrivial part of the theorem we assume that

$$B(0, 1) \cap B(x, 1) \cap B(y, 1) \cap B((x + y)/3, 1/3) = \emptyset,$$

for all $x, y \in A$ with $\|x\|, \|y\|, \|x - y\| \leq 2$. Then clearly A has the 3.2.I.P. If A does not have the 4.2.I.P., let $1 > \varepsilon > 0$ and let S and P be as in Corollary 4.5. Let $x = S(1, -1, 0)$ and $y = S(1, 0, -1)$, and let

$$z \in B(0, 1 + \varepsilon) \cap B(x, 1 + \varepsilon) \cap B(y, 1 + \varepsilon) \cap B((x + y)/3, 1/3 + \varepsilon).$$

Then we have

$$\begin{aligned} P(z) &\in B(0, 1 + 3\varepsilon) \cap B(x, 1 + 3\varepsilon) \cap B(y, 1 + 3\varepsilon) \\ &\cap B((x + y)/3, 1/3 + 3\varepsilon). \end{aligned}$$

Since the balls in l_1^3 are compact and $\varepsilon > 0$ is arbitrary, we have

$$\begin{aligned} \emptyset &\neq B((0,0,0), 1) \cap B((1,-1,0), 1) \\ &\cap B((1,0,-1), 1) \cap B((2/3, -1/3, -1/3), 1/3). \end{aligned}$$

This contradiction completes the proof.

REMARK. The proof of Theorem 4.6 shows that in Theorem 4.6 it suffices to require that

$$B(0, 1 + \varepsilon) \cap B(x, 1 + \varepsilon) \cap B(y, 1 + \varepsilon) \cap B((x + y)/3, r + \varepsilon) \neq \emptyset$$

for all $\varepsilon > 0$ and some $r \in [1/3, 1]$.

COROLLARY 4.7. *A real Banach space A has the 4.2.I.P. if and only if for every family of four mutually intersecting balls $\{B(a_i, r_i)\}_{i=1}^4$ in A with $\dim \text{span}(a_1, \dots, a_4) \leq 2$ we have*

$$\bigcap_{i=1}^4 B(a_i, r_i + \varepsilon) \neq \emptyset \quad \text{all } \varepsilon > 0.$$

REMARK. Corollary 4.7 was proved by Lindenstrauss [35, Lemma 6.5] for the case when the unit ball of A contains extreme points. The general case was an open problem. (See also the remark after Theorem 2.16.) Independently, J. Stern [49] has given a proof of Corollary 4.7 using ultraproducts, "the principle of local reflexivity" and Lindenstrauss' result that Corollary 4.7 holds if the unit ball of A contains an extreme point.

At this point the following result may seem a little surprising.

THEOREM 4.8. *Let A be a real Banach space. Then the following statements are equivalent:*

- (i) A has the 3.2.I.P.
- (ii) If $\{B(a_i, 1)\}_{i=1}^4$ are four mutually intersecting balls in A with $\dim \text{span}(a_1, \dots, a_4) \leq 2$, then

$$\bigcap_{i=1}^4 B(a_i, 1 + \varepsilon) \neq \emptyset \quad \text{for all } \varepsilon > 0.$$

PROOF. One implication is trivial. So assume $\{B(a_i, r_i)\}_{i=1}^4$ are four mutually intersecting balls in A with $\dim \text{span}(a_1, \dots, a_4) \leq 2$.

Case 1. $\text{co}(a_1, \dots, a_4)$ has less than 4 extreme points. In this case we may assume (by renumbering if necessary) that $a_4 = \alpha a_1 + \beta a_2 + \gamma a_3$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. Let $x \in \bigcap_{i=1}^3 B(a_i, 1)$. Then clearly also $x \in B(a_4, 1)$.

Case 2. $\text{co}(a_1, \dots, a_4)$ has 4 extreme points. Then (by renumbering if necessary) the line between a_1 and a_2 and the line between a_3 and a_4 intersect in a point x . By translating we may assume $x = 0$. Now choose $a, b \in A$ with $\|a\| = \|b\| = 1$ and $p, q, s, t \geq 0$ such that $a_1 = sa, a_2 = -ta, a_3 = pb, a_4 = -qb$. Let $(x_1, \dots, x_4) \in \partial_e H_r^4(A^*, (0))_1$ where $r = (1, 1, 1, 1)$. If $(x_1, \dots, x_4) \in S_{ij}$ for some i, j , then since $\|a_i - a_j\| \leq 2$,

$$\left| \sum_{i=1}^4 x_i(a_i) \right| \leq 1.$$

If $(x_1, \dots, x_4) \notin \bigcup_{1 \leq i < j \leq 4} S_{ij}$ then by Lemma 4.2, $\|x_i\| = 4^{-1}$ for all i . Hence

$$\left| \sum_{i=1}^4 x_i(a_i) \right| \leq \frac{1}{4} \sum_{i=1}^4 \|a_i\| = \frac{1}{4}(s + t) + \frac{1}{4}(p + q) \leq 1.$$

Now Theorem 1.1 gives

$$\bigcap_{i=1}^4 B(a_i, 1 + \varepsilon) \neq \emptyset$$

for all $\varepsilon > 0$. The proof is complete.

REMARK. Let $A = l_1^3$, and let $r = (1, 1, 1, 1)$ and

$$(x_1, \dots, x_4) \in \partial_e H_r^4(A^*, (0))_1 \setminus \bigcup_{1 \leq i < j \leq 4} S_{ij}.$$

From Lemmas 4.1 and 4.2 it follows that if $x_i = [x_{i1}, x_{i2}, x_{i3}]$, then for each j , $|x_{ij}| = 1/4$ for at least three indices i . But then since $\sum_{i=1}^4 x_i = 0$, we have $|x_{ij}| = 1/4$ for all i, j , so $4x_i \in \partial_e(l_\infty^3)_1$. Now the unit ball of l_∞^3 has 8 extreme points, and we can write down the different extreme points of $H_r^4(A^*, (0))_1$ with more than two nonzero components. A typical element is

$$([1, 1, 1], [1, -1, -1], [-1, 1, -1], [-1, -1, 1])/4.$$

More generally, if $n \geq 3$ and $r = (1, \dots, 1) \in \mathbb{R}^{n+1}$ then $(x_1, \dots, x_{n+1}) \in \partial_e H_r^{n+1}(l_\infty^n, (0))_1$ where

$$x_k = (-1, \dots, -1, 1, -1, \dots, -1)/(2n-2), \quad k = 1, \dots, n,$$

and

$$x_{n+1} = (n-2)(1, \dots, 1)/(2n-2).$$

This shows that l_1^n does not have the $(n+1).n$.I.P.

In [26] Hustad gave a characterization of the extreme points of $H_r^n(l_\infty^m, (0))_1$. He then showed that for $3 < n \leq m+1$, l_1^m does not have the $n.(n-1)$.I.P. by proving the existence of suitable extreme points.

The following lemma is an easy consequence of Theorem 4.6.

LEMMA 4.9. *Let A be a real Banach space, let $r = (1, 1, 1, 1/3) \in \mathbb{R}^4$, let $Y = H_r^4(\mathbb{R}, (0))$ and let $Z = (\mathbb{R}^4, \|\cdot\|_r)$. If every linear operator $T: Y \rightarrow A$ has, for every $\varepsilon > 0$, an extension $\tilde{T}: Z \rightarrow A$ with $\|\tilde{T}\| \leq (1 + \varepsilon)\|T\|$, then A has the 4.2.I.P.*

PROOF. Let $x, y \in A$ with $\|x\|, \|y\|, \|x - y\| \leq 2$. Define $T: Y \rightarrow A$ by

$$T(z_1, z_2, z_3, z_4) = z_2x + z_3y + z_4((x + y)/3).$$

Since the balls $B(0, 1)$, $B(x, 1)$, $B(y, 1)$, $B((x + y)/3, 1/3)$ have the weak intersection property, it follows that $\|T\| \leq 1$. (See [24, Corollary 1.4].) Hence, by Lemma 1.1 of Hustad [24],

$$\begin{aligned} \tilde{T}(1, 0, 0, 0) &\in B(0, 1 + \varepsilon) \cap B(x, 1 + \varepsilon) \cap B(y, 1 + \varepsilon) \\ &\cap B((x + y)/3, (1 + \varepsilon)/3). \end{aligned}$$

By Theorem 4.6 (see also the remark following Theorem 4.6), we get that A has the 4.2.I.P. The proof is complete.

A real Banach space is called *polyhedral* if the unit balls of all its finite dimensional subspaces are polytopes.

The following theorem improves Theorem 3 of Lazar [34]. Its proof is an obvious modification of the proof of Theorem 3 in [34] using Lemma 4.9.

THEOREM 4.10. *Let A be a real Banach space. Then the following statements are equivalent.*

- (i) A is polyhedral and A^* is isometric to an $L_1(\mu)$ -space.
- (ii) For every pair of Banach spaces $Y \subseteq Z$ and every operator $T: Y \rightarrow A$ with $\dim T(Y) \leq 2$ there is a compact extension $\tilde{T}: Z \rightarrow A$ such that $\|\tilde{T}\| = \|T\|$.

A modification of the proof of Theorem 5.4 of Lindenstrauss [35] using Theorem 4.6 gives the following result.

THEOREM 4.11. *Let A be a real Banach space. Then the following statements are equivalent:*

- (i) A has the 4.2.I.P.
- (ii) For every pair of Banach spaces $Y \subseteq Z$ with $\dim Y \leq 2$ and $\dim Z/Y = 1$ and every operator $T: Y \rightarrow A$, there is, for every $\epsilon > 0$, an extension $\tilde{T}: Z \rightarrow A$ such that $\|\tilde{T}\| \leq (1 + \epsilon)\|T\|$.

5. Semi L -summands and L -summands. In this section A will be a real or a complex Banach space and J will be a closed linear subspace of A .

We say that a linear projection P in A is an *L -projection* if

$$\|x\| = \|Px\| + \|x - Px\| \quad \text{all } x \in A.$$

A linear subspace J of A is called an *L -summand* if J is the range of an L -projection.

The “model” of L -projections is given by the following example. Let χ be the characteristic function of a measurable set in a measure space (X, μ) . Then $f \rightarrow f\chi$ is an L -projection in $L_1(X, \mu)$.

The L -projections were first studied by Cunningham [9] and later by Alfsen and Effros ([4] or [9]).

The first two results below are due to Cunningham. The proofs can be found in [4].

THEOREM 5.1. *Let P and Q be two L -projections in A . Then P and Q commute, and PQ is an L -projection.*

COROLLARY 5.2. *Let J be an L -summand in A . Then J is the range of a unique L -projection, and J is closed.*

In §3 the *complementary cone* J' of J was defined to be

$$J' = \{x \in A: x = 0 \text{ or } J \cap \text{face}(x/\|x\|) = \emptyset\}.$$

This cone is also well defined in complex spaces, and we remark that Theorem 3.4 holds also in the complex case.

The next result was proved in the real case by Alfsen and Effros [4, Proposition 3.1], and extended to the complex case by Hirsberg [22, Theorem 1.2].

THEOREM 5.3. *Let A be a real or complex Banach space and let J be a closed subspace. Then the following statements are equivalent:*

- (i) J is an L -summand.
- (ii) J' is a convex cone.

Moreover, if J is an L -summand and P is the unique L -projection onto J , then $J' = (I - P)(A)$.

We will generalize the notion of an L -summand in the following way. We say that a closed subspace J of A is a *semi L -summand* if

$$\|x + y\| = \|x\| + \|y\| \quad \text{all } x \in J, y \in J'.$$

It is easy to see that every L -summand is a semi L -summand. The converse is not true, as will be shown later on. However in $L_1(\mu)$ -space the L -summands and the semi L -summands will coincide.

LEMMA 5.4. *Let J be a closed subspace of A . Then we have*

$$\{y \in A: \|x + y\| = \|x\| + \|y\| \text{ all } x \in J\} \subseteq J',$$

and

$$\{y \in A: d(y, J) = \|y\|\} \subseteq J'.$$

PROOF. Suppose $y \in A$ and that $\|y\| = d(y, J)$. By Theorem 3.4 we can write

$$y = y_1 + y_2, \quad \|y\| = \|y_1\| + \|y_2\|$$

where $y_1 \in J, y_2 \in J'$. But then

$$d(y, J) = \|y\| = \|y_1\| + \|y - y_1\| \geq \|y_1\| + d(y, J)$$

so $y = y_2 \in J'$, and the proof is complete.

THEOREM 5.5. *Let A be a real or complex $L_1(\mu)$ -space and let J be a semi L -summand in A . Then J is an L -summand.*

PROOF. By Theorem 5.3 it is enough to show that J' is convex. Let $g, h \in J'$ and let $f \in J$. Then $\|f \pm g\| = \|f\| + \|g\|$ and $\|f \pm h\| = \|f\|$

+ $\|h\|$. From this a straightforward calculation gives $\|f + g + h\| = \|f\| + \|g + h\|$, so by Lemma 5.4, $g + h \in J'$. This completes the proof.

REMARKS. (1) We have seen that every L -summand J is a semi L -summand.

(2) Every semi L -summand J is a hereditary subspace.

In fact, if $x \in J$ and $y \in A$ are such that $\|x\| = \|y\| + \|x - y\|$, then by Theorem 3.4 we can write $y = y_1 + y_2$ where $y_1 \in J$ and $y_2 \in J'$. So

$$\|x\| = \|y_1\| + \|y_2\| + \|x - y_1\| + \|y_2\| \geq \|x\| + 2\|y_2\|,$$

and this shows that $y = y_1 \in J$.

(3) If A has the R_3 -property and J is a closed hereditary subspace of A , then J is a semi L -summand. In fact, if $x \in J$ and $y \in J'$ then we can write

$$x = z + u, \quad \|x\| = \|z\| + \|u\|,$$

$$-y = z + v, \quad \|y\| = \|z\| + \|v\|,$$

$$\|x + y\| = \|u - v\| = \|u\| + \|v\|.$$

Since both J and J' are hereditary, we have $z \in J \cap J' = (0)$, so $\|x + y\| = \|x\| + \|y\|$.

(4) If A has the R_4 -property and J is a closed hereditary subspace of A , then J is an L -summand. This follows from Theorems 5.5 and 3.9, or by a direct argument using Theorem 3.4 and the R_4 -property to prove that J' is convex.

We have the following characterization of semi L -summands.

THEOREM 5.6. *Let J be a closed subspace of a real or complex Banach space A . Then the following statements are equivalent:*

- (i) J is a semi L -summand.
- (ii) Every element in A has a unique decomposition w.r. to J and J' .
- (iii) For every $x \in A$, there exists a unique $y \in J$ such that $\|x - y\| = d(x, J)$, and moreover this unique y satisfies $\|x\| = \|y\| + \|x - y\|$.

PROOF. (i) \Rightarrow (ii). Let $x \in A$ and write $x = y_i + z_i$ ($i = 1, 2$) where $y_i \in J$ and $z_i \in J'$. We may assume $\|z_1\| \geq \|z_2\|$. Now $(y_1 - y_2) + z_1 = z_2$, so $\|z_1\| \geq \|z_2\| = \|y_1 - y_2\| + \|z_1\|$. Hence $y_1 = y_2$ and $z_1 = z_2$.

(ii) \Rightarrow (i) is an easy consequence of Theorem 3.4.

(ii) \Rightarrow (iii). Let $x \in A$. By Theorem 3.4 there exists a $y \in J$ such that $x - y \in J'$ and $\|x\| = \|y\| + \|x - y\|$. Now if $z \in J$ and $\|x - z\| = d(x, J)$, then by Theorem 3.4 we can write for some $u \in J$ and $v \in J'$: $x - z = u + v$, $\|x - z\| = \|u\| + \|v\|$. So

$$\begin{aligned} d(x, J) &= \|x - z\| = \|u\| + \|v\| \\ &= \|u\| + \|x - z - u\| \geq \|u\| + d(x, J). \end{aligned}$$

Hence $x - z = v \in J'$.

By (ii) it now follows that $y = z$.

(iii) \Rightarrow (i). Let $y \in J$ and $z \in J'$ and write $x = y + z$. Now let $u \in J$ be such that $\|x - u\| = d(x, J)$ and

$$\|x\| = \|u\| + \|x - u\|.$$

Using (iii) on z gives $\|z\| = d(z, J)$, so

$$\begin{aligned} \|x - u\| &= \|y + z - u\| \geq d(z, J) = \|z\| = \|x - y\| \\ &\geq d(x, J) = \|x - u\|. \end{aligned}$$

By the uniqueness of u , we have $u = y$. Hence $\|y + z\| = \|y\| + \|z\|$, and the proof is complete.

We will now characterize dual $L_1(\mu)$ -spaces using L -projections.

LEMMA 5.7. *Let $n \geq 2$, let $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with all $r_i > 0$, let $(x_1, \dots, x_n) \in \partial_e H_r^n(A^*, (0))_1$, and let P be an L -projection in A^* . Then we have $P(x_i) = x_i$ for all i or $P(x_i) = 0$ for all i .*

PROOF. Since $\sum_{i=1}^n x_i = 0$, we get $P(\sum_{i=1}^n x_i) = 0$. Thus

$$\begin{aligned} 1 &= \|(x_1, \dots, x_n)\| = \sum_{i=1}^n r_i \|x_i\| \\ &= \sum_{i=1}^n r_i \|P(x_i)\| + \sum_{i=1}^n r_i \|(I - P)(x_i)\| \\ &= \|(Px_1, \dots, Px_n)\| + \|(x_1 - Px_1, \dots, x_n - Px_n)\|, \end{aligned}$$

so if (Px_1, \dots, Px_n) and $(x_1 - Px_1, \dots, x_n - Px_n)$ both are different from 0, then we may write (x_1, \dots, x_n) as a proper convex combination of elements in $H_r^n(A^*, (0))_1$, and we obtain a contradiction. Hence all $Px_i = x_i$ or all $Px_i = 0$.

For each L -projection P in A^* we define

$$N_P = \{x \in A_1^* : P(x) = x \text{ or } P(x) = 0\},$$

and generally we define

$$N = \bigcap \{N_P : P \text{ is an } L\text{-projection in } A^*\}.$$

Since $x \in J \cup J'$ for every $x \in \partial_e A_1$ and for every hereditary subspace J of A it follows that $[0, 1]\partial_e A_1^* \subseteq N$. We also see that if $A^* = \mathbb{R}^3$ with the norm

$$\|(x, y, z)\| = |x| + \sqrt{y^2 + z^2},$$

then $[0, 1]\partial_e A_1^* = N$.

THEOREM 5.8. *Let A be a real or complex Banach space. Then the following statements are equivalent:*

- (i) A^* is isometric to an $L_1(\mu)$ -space.
- (ii) $[0, 1] \partial_e A_1^* = N$ and $\text{span}(x)$ is an L -summand for every $x \in \partial_e A_1^*$.
- (iii) If $n \geq 2$, if $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with all $r_i > 0$, and if $(x_1, \dots, x_n) \in \partial_e H_r^n(A^*, (0))_1$, then there exists $(z_1, \dots, z_n) \in \partial_e H_r^n(\mathbb{K}, (0))_1$ and $y \in \partial_e A_1^*$ such that $(x_1, \dots, x_n) = (z_1 y, \dots, z_n y)$.
- (iv) A is an almost $E(n)$ -space for all n .
- (v) Every family of closed balls in A^{**} with the weak intersection property has a nonempty intersection.

PROOF. (i) \Rightarrow (ii). $[0, 1] \partial_e A_1^* \subseteq N$ is always true. Let A^* be isometric to an $L_1(X, \mu)$ -space. Let $\chi \in L_1(X, \mu)$ be the characteristic function of a measurable set. Then $f \rightarrow f\chi$ is an L -projection in $L_1(X, \mu)$. Now (ii) follows since the extreme points of the unit ball of $L_1(X, \mu)$ are exactly the functions $z\mu(B)^{-1}\chi_B$ where $z \in \mathbb{K}$ with $|z| = 1$ and B is an atom.

(ii) \Rightarrow (iii). Let $n \geq 2$, let $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with all $r_i > 0$, and let $(x_1, \dots, x_n) \in \partial_e H_r^n(A^*, (0))_1$. By Lemma 5.7 we have $P(x_i) = x_i$ or $P(x_i) = 0$ for all i and every L -projection P in A^* . By (ii) $x_1, \dots, x_n \in [0, \infty) \partial_e A_1^*$. Let $y_i \in \partial_e A_1^*$ and let $z_i \in \mathbb{K}$ be such that $x_i = z_i y_i$. If P_i is the L -projection onto $\text{span}(y_i)$, then by Lemma 5.7, $z_i = 0$, or $z_i \neq 0$ and $P_i(y_j) = y_j \in \text{span}(y_i)$ for all j . Hence we may assume $y_1 = \dots = y_n = y$. Clearly $(z_1, \dots, z_n) \in \partial_e H_r^n(\mathbb{K}, (0))_1$.

(iii) \Rightarrow (iv) is part of Theorem 2.14.

(iv) \Rightarrow (v) follows from Theorem 2.16 and from the fact that the closed balls in A^{**} are compact in the ω^* -topology.

(v) \Rightarrow (i). In the real case Theorem 3.12 or Corollary 2.12 implies that A^* is isometric to an $L_1(\mu)$ -space. In the complex case A^{**} is isometric to a $C(K)$ -space for some compact Hausdorff space K , by the theorem in [25] and Theorem 7.20 in [8]. Hence by Proposition 1.18.1 and Corollary 1.13.3 in [42], A^* is isometric to an $L_1(\mu)$ -space.

REMARK. In the real case (i) \Leftrightarrow (v) was first proved by Nachbin [39], Goodner [18] and Grothendieck [19] while (i) \Leftrightarrow (iv) was first proved by Lindenstrauss [35]. In the complex case (i) \Leftrightarrow (iv) \Leftrightarrow (v) was first proved by Hustad [24].

In [4] and [9] it is proved that the collection of L -summands is closed under the operations of taking sums and intersections. For semi L -summands we have

PROPOSITION 5.9. *Let J_1 and J_2 be two semi L -summands in a Banach space A . Then*

- (i) $J_1 \cap J_2$ is a semi L -summand with complementary cone $(J_1 \cap J_2)' = (J_1 \cap J_2') + J_1'$.

(ii) $J_1 + J_2$ need not be a semi L -summand.

(iii) If J_1 is an L -summand, then $J_1 + J_2$ is a semi L -summand with complementary cone $(J_1 + J_2)' = J_1' \cap J_2'$.

PROOF. (i) Let $x \in A$. By Theorem 3.4 we may write $x = x_1 + x_2$ where $x_1 \in J_1$ and $x_2 \in F_1'$. Again Theorem 3.4 gives $x_1 = y_1 + y_2$ where $y_1 \in J_2$ and $y_2 \in J_2'$. Since semi L -summands are hereditary, we get

$$x = y_1 + y_2 + x_2 \in (J_1 \cap J_2) + (J_1 \cap J_2') + J_1'.$$

Hence

$$\begin{aligned} \|x\| &\leq \|y_1\| + \|y_2 + x_2\| \leq \|y_1\| + \|y_2\| + \|x_2\| \\ &= \|x_1\| + \|x_2\| = \|x\|, \end{aligned}$$

so $\|x\| = \|y_1\| + \|y_2 + x_2\|$.

Assuming that $x \in (J_1 \cap J_2)'$, we get $y_1 = 0$. Hence

$$(J_1 \cap J_2)' \subseteq (J_1 \cap J_2') + J_1'.$$

Assuming that $x \in J_1 \cap J_2$ and $z = z_1 + z_2 \in (J_1 \cap J_2') + J_1'$, we get

$$\begin{aligned} \|x + z\| &= \|x + z_1\| + \|z_2\| \\ &= \|x\| + \|z_1\| + \|z_2\| = \|x\| + \|z\|, \end{aligned}$$

since J_1 and J_2 are semi L -summands. Hence by Lemma 5.4

$$(J_1 \cap J_2') + J_1' \subseteq (J_1 \cap J_2)'$$

and (i) is proved.

(ii) follows from Theorem 7.8.

(iii) From Theorem 3.4 we get $J_1 + J_2 = J_1 + (J_1' \cap J_2)$. To show that $J_1 + J_2$ is closed, we argue as follows. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $J_1 + J_2$. Then we can write

$$x_n = y_n + z_n \in J_1 + (J_1' \cap J_2).$$

Using that J_1 is an L -summand, we get

$$\|x_n - x_m\| = \|y_n - y_m\| + \|z_n - z_m\|,$$

so both $(y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ are Cauchy sequences. Since both J_1 and J_2 are closed, it follows that the sequence $(x_n)_{n=1}^\infty$ converges to an element in $J_1 + J_2$.

Let $x \in A$. Using Theorem 3.4 we can split x w.r. to J_1 and J_1' , and then split the J_1' component w.r. to J_2 and J_2' . Thus we get

$$A = J_1 + (J'_1 \cap J_2) + (J'_1 \cap J'_2) = (J_1 + J_2) + (J'_1 \cap J'_2).$$

Let $x \in J_1 + J_2$, write $x = y + u \in J_1 + (J'_1 \cap J_2)$, and let $z \in J'_1 \cap J'_2$. Using that J_1 and J_2 are semi L -summands and that J'_1 is convex, we get

$$\begin{aligned} \|x + z\| &= \|y + u + z\| = \|y\| + \|u + z\| \\ &= \|y\| + \|u\| + \|z\| = \|y + u\| + \|z\| = \|x\| + \|z\|. \end{aligned}$$

By Lemma 5.4 $J'_1 \cap J'_2 \subseteq (J_1 + J_2)'$. Let $v \in (J_1 + J_2)'$ and write $v = x + z \in (J_1 + J_2) + (J'_1 \cap J'_2)$. Then we have $x = 0$, since $(J_1 + J_2)'$ is hereditary. Hence $(J_1 + J_2)' \subseteq J'_1 \cap J'_2$. The proof is complete.

We shall close this section by giving some new characterizations of L -summands and semi L -summands.

THEOREM 5.10. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is an L -summand.
- (ii) For all $n \geq 2$ and all $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with $r_i > 0$, the unit ball of $H_r^n(A, J)$ is the convex hull of the unit ball of $H_r^n(A, (0))$ and the unit ball of $(J^n, \|\cdot\|_{1,r})$.
- (iii) For $r = (1, 1, 1) \in \mathbb{R}^3$, the unit ball $H_r^3(A, J)_1$ is the convex hull of the unit balls $H_r^3(A, (0))_1$ and $(J^3, \|\cdot\|_{1,r})_1$.

PROOF. (i) \Rightarrow (ii). Let P be the L -projection onto J . Let $n \geq 2$ and let $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with all $r_i > 0$. Let $(x_1, \dots, x_n) \in H_r^n(A, J)$. Then

$$(x_1, \dots, x_n) = (Px_1, \dots, Px_n) + (x_1 - Px_1, \dots, x_n - Px_n)$$

gives us the desired convex combination.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). By Theorem 5.3 it suffices to show that J' is convex. Let $x_1, x_2 \in J'$. Using Theorem 3.4 we may write $x_1 + x_2 = y - x_3$, where $y \in J$ and $x_3 \in J'$. We may assume $\|x_1\| + \|x_2\| + \|x_3\| = 1$. Since $(x_1, x_2, x_3) \in H_r^3(A, J)_1$, we may write

$$(x_1, x_2, x_3) = \alpha(y_1, y_2, y_3) + (1 - \alpha)(z_1, z_2, z_3)$$

where $\alpha \in [0, 1]$, $(y_1, y_2, y_3) \in (J^3, \|\cdot\|_{1,r})_1$ and $(z_1, z_2, z_3) \in H_r^3(A, (0))_1$.

Then we get

$$x_i = \alpha y_i + (1 - \alpha)z_i, \quad \|x_i\| = \alpha\|y_i\| + (1 - \alpha)\|z_i\|, \quad i = 1, 2, 3.$$

Since $x_i \in J'$ and J' is hereditary, we get $\alpha y_i = 0$ all i , so $(x_1, x_2, x_3) \in H_r^3(A, (0))_1$. Hence $y = \sum_{i=1}^3 x_i = 0$, and the proof is complete.

COROLLARY 5.11. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J^0 is an L -summand.
- (ii) If $n \geq 2$ and if $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with all $r_i > 0$, then

$$\partial_e H_r^n(A^*, J^0)_1 \subseteq \partial_e (J^{0n}, \|\cdot\|_{1,r})_1 \cup \partial_e H_r^n(A^*, (0))_1.$$

- (iii) If $r = (1, 1, 1) \in \mathbb{R}^3$ then

$$\partial_e H_r^3(A^*, J^0)_1 \subseteq \partial_e (J^{03}, \|\cdot\|_{1,r})_1 \cup \partial_e H_r^3(A^*, (0))_1.$$

THEOREM 5.12. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is a semi L -summand.
- (ii) For $r = (r_1, r_2)$ with $r_i > 0$ we have

$$H_r^2(A, J)_1 = \text{co}(H_r^2(A, (0))_1 \cup (J^2, \|\cdot\|_{1,r})_1).$$

- (iii) For $r = (1, 1)$ we have

$$H_r^2(A, J)_1 = \text{co}(H_r^2(A, (0))_1 \cup (J^2, \|\cdot\|_{1,r})_1).$$

PROOF. (i) \Rightarrow (ii). Let $r = (r_1, r_2)$ with $r_i > 0$ and let $x_1, x_2 \in A$ with $x_1 + x_2 \in J$. Using Theorem 3.4 we may write $x_i = y_i + z_i$ where $y_i \in J$ and $z_i \in J'$. Then $x_1 = y_1 + z_1 = (x_1 + x_2) - x_2 = (x_1 + x_2 - y_2) - z_2 \in J + J'$. By Theorem 5.6 (ii) we get $z_1 = -z_2$. Hence the equation $(x_1, x_2) = (y_1, y_2) + (z_1, -z_1)$ gives us the desired convex combination.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). Let $x_1 \in J$ and $x_2 \in J'$. We may assume $\|x_2\| + \|x_1 - x_2\| = 1$, so $(x_2, x_1 - x_2) \in H_r^2(A, J)_1$. Hence

$$(x_2, x_1 - x_2) = \alpha(y_1, y_2) + (1 - \alpha)(z_1, z_2)$$

where $\alpha \in [0, 1]$, $(y_1, y_2) \in (J^2, \|\cdot\|_{1,r})_1$ and $(z_1, z_2) \in H_r^2(A, (0))_1$. Then

$$x_2 = \alpha y_1 + (1 - \alpha) z_1, \quad \|x_2\| = \alpha \|y_1\| + (1 - \alpha) \|z_1\|,$$

$$x_1 - x_2 = \alpha y_2 + (1 - \alpha) z_2, \quad \|x_1 - x_2\| = \alpha \|y_2\| + (1 - \alpha) \|z_2\|.$$

Since J' is hereditary, we get $\alpha y_1 \in J \cap J' = (0)$. Hence $x_2 = (1 - \alpha) z_1 = -(1 - \alpha) z_2$, so $x_1 = \alpha y_2$. This implies $\|x_1 - x_2\| = \|x_1\| + \|x_2\|$. The proof is complete.

COROLLARY 5.13. *Let J be a closed subspace of A . The following statements are equivalent:*

- (i) J^0 is a semi L -summand.
- (ii) If $r = (r_1, r_2)$ with $r_i > 0$, then

$$\partial_e H_r^2(A^*, J^0)_1 \subseteq \partial_e(J^{02}, \|\cdot\|_{1,r})_1 \cup \partial_e H_r^2(A^*, (0))_1.$$

- (iii) If $r = (1, 1)$, then

$$\partial_e H^2(A^*, J^0)_1 \subseteq \partial_e(J^{02}, \|\cdot\|_{1,r})_1 \cup \partial_e H_r^2(A^*, (0))_1.$$

THEOREM 5.14. *Let A be a real Banach space with the 3.2.I.P. A semi L -summand $J \subseteq A$ will fail to be an L -summand if and only if there exists an isometry $T: l_\infty^3 \rightarrow A$ such that $T(1, 1, 1) \in J$ and $T(-1, 1, 1), T(1, -1, 1), T(1, 1, -1) \in J'$.*

PROOF. Assume first that there exists an isometry with the properties stated in the theorem. Then it is easily verified that J' is not convex, so by Theorem 5.3 J is not an L -summand.

Next assume that A is not an L -summand. Let $r = (1, 1, 1)$. By Theorem 5.10 there exists an element

$$(x_1, x_2, x_3) \in H_r^3(A, J)_1 \setminus \text{co}(H_r^3(A, 0))_1 \cup (J^3, \|\cdot\|_{1,r})_1.$$

By Theorem 3.4 we can write each $x_i = y_i + z_i$ where $y_i \in J$ and $z_i \in J'$: But then $(x_1, x_2, x_3) = (y_1, y_2, y_3) + (z_1, z_2, z_3)$ gives rise to a convex combination in $H_r^3(A, J)_1$. Hence we may assume $x_i = z_i \in J'$ for all i .

Since A has the 3.2.I.P., Theorem 3.2 gives

$$\begin{aligned} x_1 &= z + u, & \|x_1\| &= \|z\| + \|u\|, \\ -x_2 &= z + v, & \|x_2\| &= \|z\| + \|v\|, \\ \|x_1 + x_2\| &= \|u - v\| = \|u\| + \|v\|. \end{aligned}$$

Hence the decomposition

$$(x_1, x_2, x_3) = (z, -z, 0) + (u, -v, x_3)$$

gives rise to a convex combination in $H_r^3(A, J)_1$. Thus we may assume $z = 0$ and $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$.

In the same manner, we get that we may assume

$$\|x_1 + x_3\| = \|x_1\| + \|x_3\| \quad \text{and} \quad \|x_2 + x_3\| = \|x_2\| + \|x_3\|.$$

Let $x = \sum_{i=1}^3 x_i$. By assumption $x \neq 0$ and $x \in J$. We may also assume $\|x_1\| \geq \|x_2\| \geq \|x_3\|$. Then using that J is a semi L -ideal, we get

$$\|x_1\| + \|x_2\| = \|x_1 + x_2\| = \|x - x_3\| = \|x\| + \|x_3\| \leq \|x\| + \|x_2\|$$

and

$$\|x_2\| + \|x_3\| = \|x_2 + x_3\| = \|x - x_1\| = \|x\| + \|x_1\| \geq \|x\| + \|x_2\|,$$

so

$$\|x\| \geq \|x_1\| \geq \|x_2\| \geq \|x_3\| \geq \|x\|.$$

Multiplying by a scalar, we may assume

$$\|x\| = \|x_1\| = \|x_2\| = \|x_3\| = 1.$$

Now the map $T: l_\infty^3 \rightarrow A$ defined by $T(1, 1, 1) = x$, $T(-1, 1, 1) = x_1$, $T(1, -1, 1) = x_2$ and $T(1, 1, -1) = x_3$, is an isometry. The proof is complete.

6. Semi M -ideals and M -ideals. Following Alfsen and Effros [4], we say that a closed linear subspace J of A is an M -ideal if J^0 is an L -summand.

We also say that a closed linear subspace J of A is a *semi M -ideal* if J^0 is a semi L -summand.

DEFINITION. We say that a closed subspace J of A has the *n -intersection property (n.I.P.)* if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of balls in A with the properties

(i) $J \cap B(a_i, r_i) \neq \emptyset$, for $i = 1, \dots, n$,

(ii) $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$,

we have

(iii) $J \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ all $\varepsilon > 0$.

We say that J has the *restricted n -intersection property (R.n.I.P.)* if (i) and (ii) imply (iii) for every family $\{B(a_i, r_i)\}_{i=1}^n$ where all $r_i = 1$.

If we can take $\varepsilon = 0$ in (iii), then we say that J has the *strong n.I.P.*

REMARK. It is clear that the strong n.I.P. implies the n.I.P. The converse is false. In fact, there exist a Banach space A with a closed subspace J such that J has the n.I.P. for all n but not the strong 2.I.P. The example is as follows. Let H be a separable Hilbert space, let A be the ordered Banach space of all selfadjoint operators on A and let J be the compact operators in A . Then J has the n.I.P. for all n (see [4, p. 126]). In [43] Stefánsson constructed an element k in J such that $-I \leq k \leq I$, and a projection p on H such that $k + p$ is a noncompact projection on H , and $k + p$ is the least upper bound for 0 and k . Then

$$k \in B(k + \frac{1}{2}I, \frac{1}{2}) \cap J, \quad 0 \in B(\frac{1}{2}I, \frac{1}{2}) \cap J,$$

and

$$\frac{1}{2}(k + I) \in B(k + \frac{1}{2}I, \frac{1}{2}) \cap B(\frac{1}{2}I, \frac{1}{2}).$$

Assume $v \in B(k + \frac{1}{2}I, \frac{1}{2}) \cap B(\frac{1}{2}I, \frac{1}{2}) \cap J$. Then

$$-\frac{1}{2}I \leq k + \frac{1}{2}I - v \leq \frac{1}{2}I \quad \text{and} \quad -\frac{1}{2}I \leq \frac{1}{2}I - v \leq \frac{1}{2}I,$$

and this implies $0, k \leq v$. Hence $0, k \leq k + p \leq v$. Since v is compact and $k + p$ is noncompact, we get a contradiction by using that J^+ is a facial cone.

This example solves Problem 3 of Alfsen and Effros [4] to the negative.

LEMMA 6.1. *Let J be a closed subspace of A with the R.2.I.P. and let $\epsilon > 0$. If $x \in J$ with $\|x\| = 1$ and $a \in A_1$, then there exists $z \in J$ such that*

$$\|x + a - z\| \leq 1 + \epsilon, \quad \|x - a + z\| \leq 1 + \epsilon.$$

PROOF. We have

$$a \in B(a + x, 1) \cap B(a - x, 1) \quad \text{and} \quad \pm x \in J \cap B(a \pm x, 1).$$

Now every element

$$z \in J \cap B(a + x, 1 + \epsilon) \cap B(a - x, 1 + \epsilon)$$

fulfills the requirements.

COROLLARY 6.2. *Let J be a closed subspace of A with the strong 2.I.P. Let $x \in J$ with $\|x\| = 1$ and let $a \in A_1$. Then there exists $z \in J$ such that*

$$\|x + a - z\| \leq 1, \quad \|x - a + z\| \leq 1.$$

COROLLARY 6.3. *Let J be a proper closed subspace of A with the strong 2.I.P. If $x \in J$, then $x \notin \partial_e A_1$.*

PROOF. Let $x \in J$ with $\|x\| = 1$. Choose $a \in A_1$ such that $a \notin J$. Let z be as in Corollary 6.2. Then the element

$$x = \frac{1}{2}(x + a - z) + \frac{1}{2}(x - a + z)$$

is a proper convex combination in A_1 .

The next lemma was first proved by Hustad [24]. We will give a simplified proof below.

LEMMA 6.4. *Let A be a complex Banach space and let $\epsilon > 0$. Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A with the weak intersection property. If $a \in A$ satisfies*

$$\|a - a_i\|^2 \leq r_i^2 + \epsilon^2, \quad i = 1, \dots, n,$$

then $\{B(a_i, r_i)\}_{i=1}^n \cup \{B(a, \epsilon)\}$ has the weak intersection property.

PROOF. Using Proposition 2.3 we see that it suffices to show that if $a_1, a_2, a \in \mathbb{C}$ are such that $\|a_1 - a_2\| \leq r_1 + r_2$, $\|a_i - a\|^2 \leq r_i^2 + \varepsilon^2$ ($i = 1, 2$) then

$$B(a_1, r_1) \cap B(a_2, r_2) \cap B(a, \varepsilon) \neq \emptyset \text{ in } \mathbb{C}.$$

If $B(a_1, r_1) \subseteq B(a_2, r_2)$ (or vice versa), then there is nothing to prove since $r_i^2 + \varepsilon^2 \leq (r_i + \varepsilon)^2$. Hence we may assume that the boundaries of $B(a_1, r_1)$ and $B(a_2, r_2)$ intersect in two different points e and f . Let S_1 be the cone of \mathbb{C} containing a_1 and determined by the lines from a_2 through e and f . Let S_2 be the cone of \mathbb{C} containing a_2 and determined by the lines from a_1 through e and f . If $a \in S_1 \cup S_2$, then

$$B(a_1, r_1) \cap B(a_2, r_2) \cap B(a, \varepsilon) \neq \emptyset,$$

since $r_i^2 + \varepsilon^2 \leq (r_i + \varepsilon)^2$, $i = 1, 2$. The rest of \mathbb{C} consists of two sectors T_1 and T_2 . Let T_1 be the sector with vertex e , and T_2 the sector with vertex f . Suppose $a \in T_1$. Then an inspection of the triangles $a_1 ea$ and $a_2 ea$ shows that for at least one of these triangles, the angle at e is between $\pi/2$ and π . Hence the distance from e to a is less than ε , so

$$e \in B(a_1, r_1) \cap B(a_2, r_2) \cap B(a, \varepsilon).$$

The case $a \in T_2$ is treated similarly, and this completes the proof.

PROPOSITION 6.5. Let $n \geq 1$ and let J be a closed subspace of a real or complex Banach space A with the $(n+1)$.I.P. If A is an almost $E(n+1)$ -space, then J has the strong n .I.P.

PROOF. Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A such that

$$J \cap B(a_i, r_i) \neq \emptyset \text{ all } i$$

and

$$\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset.$$

Let $\varepsilon > 0$, let $\varepsilon_m = \varepsilon \cdot 2^{-m}$ and let

$$0 < \theta_m \leq \min\{\sqrt{r_i^2 + \varepsilon_m^2} - r_i : i = 1, \dots, n\}, \quad m = 1, 2, \dots$$

Now if $a \in A$ with $\|a - a_i\| \leq r_i + \theta_m$ all i , then $\{B(a_i, r_i)\}_{i=1}^n \cup \{B(a, \varepsilon_m)\}$ has the weak intersection property.

The conclusion will now follow from an induction argument similar to the one used by Aronszajn and Panitchpakdi [6, p. 418].

Suppose we have found $(x_k)_{k=1}^p$ in A such that for $k = 1, \dots, p-1$

$$x_{k+1} \in J \cap B(x_k, \varepsilon_k + \theta_{k+1}) \cap \bigcap_{i=1}^n B(a_i, r_i + \theta_{k+1}).$$

Then $\{B(a_i, r_i)\}_{i=1}^n \cup \{B(x_p, \varepsilon_p)\}$ has the weak intersection property. Since A is an almost $E(n+1)$ -space, we have

$$\bigcap_{i=1}^n B(a_i, r_i + \tfrac{1}{2}\theta_{p+1}) \cap B(x_p, \varepsilon_p + \tfrac{1}{2}\theta_{p+1}) \neq \emptyset.$$

Using that J has the $(n+1)$.I.P., we find

$$x_{p+1} \in J \cap B(x_p, \varepsilon_p + \theta_p) \cap \bigcap_{i=1}^n B(a_i, r_i + \theta_p).$$

Now $(x_k)_{k=1}^\infty$ is a Cauchy sequence converging to some element

$$x \in J \cap \bigcap_{i=1}^n B(a_i, r_i).$$

The proof is complete.

COROLLARY 6.6. *Let J be a closed subspace of A with the 2.I.P. If $a \in A$, then there exists $x \in J$ with $\|a - x\| = d(a, J)$.*

PROOF. Every Banach space is an $E(2)$ -space.

LEMMA 6.7. *Let J be a closed subspace of A with the strong 2.I.P. Let F be a face of A_1 with $F \cap J = \emptyset$ and let $a \in F$. Then $d(a, J) = 1$.*

PROOF. Let $r = d(a, J) > 0$. Suppose $r < 1$. Then

$$(1-r)a \in B(a, r) \cap B(0, 1-r), \quad 0 \in J \cap B(0, 1-r),$$

and

$$J \cap B(a, r) \neq \emptyset$$

by Corollary 6.6. Let $x \in J \cap B(a, r) \cap B(0, 1-r)$. Then

$$1 = \|a\| \leq \|a - x\| + \|x\| \leq r + (1-r) = 1,$$

so $\|x\| = 1-r$. But then the element

$$a = \|x\|(x/\|x\|) + (1-\|x\|)((a-x)/\|a-x\|)$$

is a proper convex combination, so $x/\|x\| \in F \cap J$. This contradiction shows that $r = 1$.

COROLLARY 6.8. *Let J be a proper closed subspace of A with the strong 2.I.P. If $a \in \partial_e A_1$, then $d(a, J) = 1$.*

In the next theorem (i) \Leftrightarrow (ii) \Leftrightarrow (iii) was proved by Alfsen and Effros [4] by different methods. (iv) and (v) are new characterizations of M -ideals.

THEOREM 6.9. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is an M -ideal.
- (ii) J has the $n.I.P.$ for all n .
- (iii) J has the $3.I.P.$
- (iv) J has the $R.n.I.P.$ for all n .
- (v) J has the $R.3.I.P.$

PROOF. (ii) \Rightarrow (iii) \Rightarrow (v) and (ii) \Rightarrow (iv) \Rightarrow (v) are trivial.

(i) \Rightarrow (ii). Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A such that $J \cap B(a_i, r_i) \neq \emptyset$ all i , and $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$. Let $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ and let

$$S = H_r^n(A^*, (0))_1 \cup (J^{0n}, \|\cdot\|_{1,r})_1.$$

Since

$$J \cap B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } i, \text{ all } \epsilon > 0$$

if and only if

$$\left| \sum_{i=1}^n f_i(a_i) \right| \leq \sum_{i=1}^n r_i \|f_i\| \quad \text{all } f_1, \dots, f_n \in J^0,$$

and

$$\bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0$$

if and only if

$$\left| \sum_{i=1}^n f_i(a_i) \right| \leq \sum_{i=1}^n r_i \|f_i\| \quad \text{all } (f_1, \dots, f_n) \in H_r^n(A^*, (0)),$$

Theorem 5.10 and Lemma 2.8 will imply

$$J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

(v) \Rightarrow (i) also follows from Theorem 5.10 and Lemma 2.8. The proof is complete.

Similarly we get from Theorem 5.12 and Lemma 2.8:

THEOREM 6.10. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is a semi M -ideal.
- (ii) J has the 2.I.P.
- (iii) J has the R.2.I.P.

Combining Theorems 5.10 and 5.12 with Theorem 1.2 we get:

THEOREM 6.11. *Let J be a closed subspace of A . If J is an L -summand, then J^0 has the n .I.P. for all n , and if J is a semi L -summand then J^0 has the 2.I.P.*

We will now show that the word if can be replaced by if and only if in Theorem 6.11. This will solve Problems 1 and 2 of Alfsen and Effros [4] to the affirmative.

LEMMA 6.12. *Let J be a closed subspace of A and assume that for all $x \in J^0$ with $\|x\| = 1$ and all $y \in A_1^*$, there exists $z \in J^0$ such that*

$$\|x + y - z\| \leq 1, \quad \|x - y + z\| \leq 1.$$

Let $a \in A$ and let $\epsilon > 0$. Suppose $b, c \in J$ and

$$\|a - b\| < d(a, J) + \epsilon, \quad \|a - c\| < d(a, J) + \epsilon.$$

Then

$$\|b - c\| \leq 4\epsilon + 2\epsilon(d(a, J) + \epsilon).$$

PROOF. Since J^0 is isometric to the dual space of A/J , we can find $x \in J^0$ with $\|x\| = 1$ such that $d(a, J) \geq x(a) = x(a - b) > \|a - b\| - \epsilon$. We can also find $y \in A^*$ with $\|y\| = 1$ and $y(b - c) > \|b - c\| - \epsilon$. Now choose $z \in J^0$ such that $\|x + y - z\| \leq 1 + \epsilon$, $\|x - y + z\| \leq 1 + \epsilon$. We have

$$x(a - c) = x(a - b) + x(b - c) = x(a - b) > \|a - b\| - \epsilon,$$

$$(x + y - z)(b - c) = y(b - c) > \|b - c\| - \epsilon,$$

$$\|a - b\| \geq d(a, J) > \|a - c\| - \epsilon,$$

$$\operatorname{Re}(x + y - z)(a - c) \leq \|x + y - z\| \|a - c\| \leq (1 + \epsilon) \|a - c\|$$

and

$$\begin{aligned} \|a - b\| - \epsilon &< x(a - b) \\ &= \frac{1}{2}(x + y - z)(a - b) + \frac{1}{2}(x - y + z)(a - b) \\ &= \frac{1}{2}\operatorname{Re}(x + y - z)(a - b) + \frac{1}{2}\operatorname{Re}(x - y + z)(a - b) \\ &\leq \frac{1}{2}\operatorname{Re}(x + y - z)(a - b) + \frac{1}{2}(1 + \epsilon)\|a - b\|. \end{aligned}$$

Hence

$$\operatorname{Re}(x + y - z)(a - b) > (1 - \epsilon)\|a - b\| - 2\epsilon.$$

Hence also

$$\begin{aligned} & (1 - \epsilon)\|a - b\| - 2\epsilon - (1 + \epsilon)\|a - c\| \\ & < \operatorname{Re}(x + y - z)(a - b) - \operatorname{Re}(x + y - z)(a - c) \\ & = \operatorname{Re}(x + y - z)(c - b) \\ & = (x + y - z)(c - b) = y(c - b) < \epsilon - \|b - c\|, \end{aligned}$$

so

$$\begin{aligned} \|b - c\| & < 3\epsilon + (1 + \epsilon)\|a - c\| - (1 - \epsilon)\|a - b\| \\ & < 3\epsilon + \|a - c\| + \epsilon\|a - c\| + (1 - \epsilon)(\epsilon - \|a - c\|) \\ & = 3\epsilon + 2\epsilon\|a - c\| + \epsilon(1 - \epsilon) \\ & < 4\epsilon + 2\epsilon(d(a, J) + \epsilon), \end{aligned}$$

and the lemma is proved.

LEMMA 6.13. *Let J be a closed subspace of A and assume that for all $x \in J^0$ with $\|x\| = 1$ and all $y \in A_1^*$, there exists $z \in J^0$ such that $\|x + y - z\| \leq 1$, $\|x - y + z\| \leq 1$. Then A is a semi L -summand.*

PROOF. Let $a \in A$. Choose a sequence $(b_n)_{n=1}^\infty$ in J such that $\|a - b_n\| < d(a, J) + 2^{-n}$. From Lemma 6.12 it follows that $(b_n)_{n=1}^\infty$ is a Cauchy sequence. Hence $(b_n)_{n=1}^\infty$ converges to an element $b \in J$ and

$$\|a - b\| = d(a, J).$$

It follows from Lemma 6.12 that b is unique with these properties. By Theorem 5.6 it is enough to prove that $\|a\| = \|b\| + \|a - b\|$. Put $c = a - b$. Then

$$\|c\| = \|a - b\| = d(a, J) = d(a - b, J) = d(c, J).$$

We can now choose sequences $(x_n)_{n=1}^\infty \subseteq J^0$ and $(y_n)_{n=1}^\infty \subseteq A^*$ such that $\|x_n\| = 1 = \|y_n\|$ all n and

$$\|c\| \geq x_n(c) > \|c\| - 1/n, \quad \|b\| \geq y_n(b) > \|b\| - 1/n.$$

Then there exists a sequence $(z_n)_{n=1}^\infty \subseteq J^0$ such that $\|x_n + y_n - z_n\|, \|x_n - y_n + z_n\| \leq 1 + 1/n$. Clearly

$$|(x_n + y_n - z_n)(c)| \leq (1 + 1/n)\|c\|, \quad |(x_n - y_n + z_n)(c)| \leq (1 + 1/n)\|c\|.$$

Going to a subsequence if necessary, we may assume

$$(x_n + y_n - z_n)(c) \rightarrow u \in \mathbf{C} \quad \text{as } n \rightarrow \infty,$$

$$(x_n - y_n + z_n)(c) \rightarrow v \in \mathbf{C} \quad \text{as } n \rightarrow \infty,$$

and $|u|, |v| \leq \|c\|$. Furthermore we have

$$\begin{aligned} \|c\| &= \lim x_n(c) \\ &= \lim \left(\frac{1}{2}(x_n + y_n - z_n)(c) + \frac{1}{2}(x_n - y_n + z_n)(c) \right) \\ &= \frac{1}{2}(u + v) = \frac{1}{2}|u + v| \leq \frac{1}{2}|u| + \frac{1}{2}|v| \leq \|c\|. \end{aligned}$$

Hence $u = v = \|c\|$. But then

$$\begin{aligned} \|b\| + \|c\| &= |\lim(x_n + y_n - z_n)(c) + \lim y_n(b)| \\ &= |\lim(x_n + y_n - z_n)(c) + \lim(x_n + y_n - z_n)(b)| \\ &= \lim |(x_n + y_n - z_n)(a)| \\ &\leq \lim(1 + 1/n)\|a\| = \|a\| \leq \|b\| + \|c\|. \end{aligned}$$

This completes the proof.

We now have:

THEOREM 6.14. *Let J be a closed subspace of A . Then the following properties are equivalent:*

- (i) J is a semi L -summand.
- (ii) J^0 is a semi M -ideal.
- (iii) For all $x \in J^0$ with $\|x\| \leq 1$ and all $y \in A_1^*$ we have $J^0 \cap B(y + x, 1) \cap B(y - x, 1) \neq \emptyset$.
- (iv) For all $x \in J^0$ with $\|x\| = 1$ and all $y \in A_1^*$, there exists $z \in J^0$ such that

$$\|x + y - z\| = 1 = \|x - y + z\|.$$

PROOF. (i) \Rightarrow (ii) \Rightarrow (iii) follows from Theorems 6.10 and 6.11.

(iii) \Rightarrow (iv) is Corollary 6.2.

(iv) \Rightarrow (i) is Lemma 6.13.

The arguments in Lemmas 6.12 and 6.13 can be dualized, and we get

THEOREM 6.15. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is a semi M -ideal.
- (ii) For all $\varepsilon > 0$, all $x \in J_1$ and all $y \in A_1$, we have $J \cap B(y + x, 1 + \varepsilon) \cap B(y - x, 1 + \varepsilon) \neq \emptyset$.
- (iii) For all $\varepsilon > 0$, all $x \in J$ with $\|x\| = 1$ and all $y \in A_1$, there exists

$z \in J$ such that $\|x + y - z\|, \|x - y + z\| \leq 1 + \varepsilon$.

For M -ideals we have

THEOREM 6.16. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is an L -summand.
- (ii) J^0 is an M -ideal.
- (iii) $J^0 \cap \bigcap_{i=1}^3 B(y + x_i, 1) \neq \emptyset$ whenever $x_i \in J^0$ with $\|x_i\| \leq 1$ ($i = 1, 2, 3$) and $y \in A_1^*$.

PROOF. (i) \Rightarrow (ii) \Rightarrow (iii) follows from Theorems 6.11 and 6.9.

(iii) \Rightarrow (i). By Theorem 5.3 it is enough to show that J' is convex. Let $a_1, a_2 \in J'$. By Theorem 3.4 we have $a_1 + a_2 = b - a_3$, where $b \in J$ and $a_3 \in J'$. By Theorem 6.14 we have $\|a_i\| = d(a_i, J)$. Let $\varepsilon > 0$ and let $x_i \in J^0$ with $\|x_i\| = 1$ and $x_i(a_i) > \|a_i\| - \varepsilon$ for $i = 1, 2, 3$. Let $y \in A_1^*$ be such that $y(b) > \|b\| - \varepsilon$. Now we can find an element

$$z \in J^0 \cap \bigcap_{i=1}^3 B(y + x_i, 1 + \varepsilon).$$

Hence

$$\begin{aligned} \|b\| - \varepsilon + \sum_{i=1}^3 (\|a_i\| - \varepsilon) &< y(b) + \sum_{i=1}^3 x_i(a_i) \\ &= y\left(\sum_{i=1}^3 a_i\right) + \sum_{i=1}^3 x_i(a_i) = \sum_{i=1}^3 (y + x_i)(a_i) \\ &= \sum_{i=1}^3 (y + x_i)(a_i) - z(b) = \sum_{i=1}^3 (y + x_i - z)(a_i) \\ &\leq \sum_{i=1}^3 (1 + \varepsilon)\|a_i\|. \end{aligned}$$

Hence

$$\|b\| < 4\varepsilon + \varepsilon \sum_{i=1}^3 \|a_i\|.$$

Since $\varepsilon > 0$ is arbitrary, we get $b = 0$, so $a_1 + a_2 = -a_3 \in J'$, and the theorem is proved.

Dually we get

THEOREM 6.17. *Let J be a closed subspace of A . Then the following statements are equivalent:*

- (i) J is an M -ideal.
- (ii) For all $\varepsilon > 0$, all $x_i \in J$ with $\|x_i\| \leq 1$ ($i = 1, 2, 3$) and all $y \in A_1$, we have $J \cap \bigcap_{i=1}^3 B(y + x_i, 1 + \varepsilon) \neq \emptyset$.

Let P be a linear projection in A . We say that P is an M -projection if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \quad \text{all } x \in A.$$

In [4] Alfsen and Effros proved that a linear projection P in A is an M -projection if and only if its adjoint projection P^* in A^* is an L -projection, and that P is an L -projection if and only if P^* is an M -projection.

We say that a closed subspace J of A is an M -summand if J is the range of an M -projection, and that J is an L -ideal if J^0 is an M -summand.

It follows that every M -summand is an M -ideal. From the results in [8] it follows that c_0 is an M -ideal in l_∞ , but not an M -summand. Clearly every L -summand is an L -ideal. In [48] Perdrizet showed that in certain spaces all L -ideals are L -summands and later Cunningham, Effros and Roy [10] proved that in general the L -ideals and the L -summands coincide. In Theorem 6.16 we showed that a closed subspace J of A is an L -summand if and only if J^0 is an M -ideal. Hence every ω^* -closed M -ideal in A^* is an M -summand. Evans [46] has characterized the M -summands by intersection properties of balls and he has proved that every ω^* -closed M -ideal in A^* is an M -summand.

LEMMA 6.18. *Let J_1 and J_2 be two semi M -ideals in A . Then $J_1^0 + J_2^0$ is ω^* -closed in A^* .*

PROOF. By definition J_1^0 and J_2^0 are semi L -summands. Let $x \in J_1^0 + J_2^0$. Using Theorem 3.4 we find $x = x_1 + x_2 \in J_1^0 + (J_1^0 \cap J_2^0)$ so $\|x\| = \|x_1\| + \|x_2\|$. By ω^* -compactness of A_1^* this immediately gives that $A_1^* \cap (J_1^0 + J_2^0)$ is ω^* -closed. By the Krein-Smulian theorem [11], it follows that $J_1^0 + J_2^0$ is ω^* -closed.

In [4] Alfsen and Effros proved that finite sums of M -ideals are M -ideals.

COROLLARY 6.19. *Let J_1 and J_2 be two semi M -ideals in A . Then $J_1 + J_2$ is a semi M -ideal. If J_1 and J_2 are M -ideals, then $J_1 + J_2$ is an M -ideal.*

PROOF. By Lemma 6.18 and a theorem of Reiter [41], it follows that $J_1 + J_2$ is closed and that $(J_1 + J_2)^0 = J_1^0 \cap J_2^0$. Now Proposition 5.9 gives that $J_1 + J_2$ is a semi M -ideal.

PROPOSITION 6.20. *Let (J_α) be a family of closed subspaces of A . Let $J = (\overline{\sum_\alpha J_\alpha})$ (norm-closure).*

- (i) *If every J_α is a semi M -ideal, then J is a semi M -ideal.*
- (ii) *If every J_α is an M -ideal, then J is an M -ideal.*

PROOF. (i) Let $a_1, a_2 \in A$ and let $r_1, r_2 > 0$ be such that $\|a_1 - a_2\| \leq r_1 + r_2$ and $d(a_i, J) \leq r_i$ ($i = 1, 2$). Let $\varepsilon > 0$. Then there exist $J_1, J_2 \in (J_\alpha)$ such that $d(a_i, J_i) \leq r_i + \varepsilon$ ($i = 1, 2$). By Corollary 6.19

$$(J_1 + J_2) \cap \bigcap_{i=1}^2 B(a_i, r_i + 2\epsilon) \neq \emptyset,$$

so also

$$J \cap \bigcap_{i=1}^2 B(a_i, r_i + 2\epsilon) \neq \emptyset.$$

By Theorem 6.10 J is a semi M -ideal.

The proof of (ii) is similar.

REMARKS. From Proposition 5.9 it follows that the intersection of two semi M -ideals need not be a semi M -ideal, whereas the intersection of an M -ideal and a semi M -ideal will be a semi M -ideal.

Statement (ii) in Proposition 6.20 is proved by Alfsen and Effros in [4]. Here they also proved that finite intersections of M -ideals are M -ideals.

For $p \in \langle 1, \infty \rangle$ there is a natural definition of L_p -projection, L_p -summand and L_p -ideal. As is the case with $L = L_1$ -ideals, one can verify that the L_p -ideals and the L_p -summands coincide. This was first proved by H. Fakhoury [50].

7. Applications.

A. The Hirsberg-Lazar theorem.

THEOREM 7.1. *Let A be a (real or complex) almost $E(3)$ -space and suppose A_1 contains an extreme point e . Suppose that $\text{span}(p)$ is an L -summand for all $p \in \partial_e A_1^*$. Define $S = \{f \in A^*: \|f\| = 1 = f(e)\}$ and define $\Phi: A \rightarrow C(S)$ by $\Phi(x)(f) = f(x)$. Then Φ is an isometry into $C(S)$ and $\Phi(e) = 1$.*

PROOF. Clearly $\Phi(e) = 1$ and $\|\Phi(x)\| \leq \|x\|$ for all $x \in A$. Let $x \in A$. We only have to prove that $\|x\| \leq \|\Phi(x)\|$. Let $p \in \partial_e A_1^*$ be such that $\|x\| = p(x)$. Define $J_p = \{y \in A: p(y) = 0\}$. Then $J_p^0 = \text{span}(p)$, so J_p is an M -ideal. By Theorem 6.9, Proposition 6.5 and Corollary 6.8 it follows that $d(e, J_p) = 1$. Since the dual space of A/J_p is isometric to $J_p = \text{span}(p)$, we get that $|p(e)| = 1$. Hence for some z with $|z| = 1$, we have $zp \in S$, so $\|x\| = |zp(x)| \leq \|\Phi(x)\|$, and the proof is complete.

COROLLARY 7.2. *Let A be a (real or complex) Banach space whose dual space is isometric to an $L_1(\mu)$ -space. Assume that the unit ball A_1 contains an extreme point e . Let S and Φ be as in Theorem 7.1. Then Φ is a linear isometry of A onto the ω^* -continuous affine (real or complex valued) functions on S and $\Phi(e) = 1$.*

PROOF. It follows from Theorems 5.8 and 7.1 that Φ is a linear isometry, and the surjectivity follows by an argument from [1, p. 74] or [40].

REMARKS. For the complex case Corollary 7.2 was first proved by Hirsberg and Lazar in [23]. For the real case, Theorem 7.1 can be sharpened so we get

the following result of Nachbin [39], Kadison [27] and Lindenstrauss [35] (see Theorem 4.7 in [35]):

THEOREM 7.3. *Let A be a real Banach space and suppose A_1 contains an extreme point e . Let S and Φ be defined as in Theorem 7.1. Suppose that for every family $\{B(a_i, r_i)\}_{i=1}^3$ of three mutually intersecting balls in A such that $B(a_1, r_1) \cap B(a_2, r_2) = \{e\}$, we have $e \in B(a_3, r_3)$. Then $\text{span}(e)$ is a semi L -summand, $\varphi(e) = 1$, and Φ is an isometry of A onto the ω^* -continuous affine real-valued functions on S .*

PROOF. First we will show that $\text{span}(e)$ is a semi L -summand. Let $x \in A \setminus \text{span}(e)$ and let $r = d(x, \text{span}(e)) > 0$. Suppose $a, b \in \text{span}(e) \cap B(x, r)$ and $2s = \|a - b\| > 0$. Choose ε such that $s \geq \varepsilon > 0$ and $r > \varepsilon > 0$. Then the balls $\{B(x, r - \varepsilon), B(a, s), B(b, s)\}$ can be translated by an element y in $\text{span}(e)$ such that $B(a - y, s) \cap B(b - y, s) = \{e\}$ and the balls $B(x - y, r - \varepsilon), B(a - y, s), B(b - y, s)$ are mutually intersecting. In fact, $y = a - e$ will do. But then by the assumption, $e \in B(x - y, r - \varepsilon)$, so $d(x, \text{span}(e)) \leq r - \varepsilon$. This is a contradiction, so $\text{span}(e) \cap B(x, r)$ must consist of one point z . Then $\|x\| \leq \|z\| + \|x - z\|$.

Suppose $\|x\| < \|z\| + \|x - z\|$. Then $\|x - z\| < \|x\|$, since $\|x\| = \|x - z\| = r$ implies $z = 0$ and $\|x\| = \|z\| + \|x - z\|$. From the assumption it follows that

$$\begin{aligned} \text{span}(e) \cap B(0, \|x\| \|x - z\|) \\ \cap B(z, \|z\| + \|x - z\| - \|x\|) \cap B(x, \|x - z\|) \neq \emptyset. \end{aligned}$$

Since $\text{span}(e) \cap B(x, \|x - z\|) = \{z\}$ we have $z \in B(0, \|x\| - \|x - z\|)$, so $\|z\| + \|x - z\| \leq \|x\|$. This contradiction shows that $\|x\| = \|z\| + \|x - z\|$. By Theorem 5.6 $\text{span}(e)$ is a semi L -summand.

Let $J = (\text{span}(e))^0$. Then J is a ω^* -closed semi M -ideal in A^* and for all $p \in \partial_e A_1^*$, $d(p, J) = 1$ by virtue of Theorem 6.10 and Corollary 6.8. Since $\text{span}(e)^*$ is isometric to A^*/J , we get $|p(e)| = 1$, so $p \in S$ or $-p \in S$. But then Φ is an isometry of A into the ω^* -continuous affine real-valued functions on S and $\Phi(e) = 1$. The surjectivity argument can be found in [1, Corollary I.1.5]. The proof is complete.

COROLLARY 7.4. *Let A be a real Banach space and let $e \in \partial_e A_1$. Then the following statements are equivalent:*

- (i) *For every family $\{B(a_i, r_i)\}_{i=1}^3$ of three mutually intersecting balls in A such that $B(a_1, r_1) \cap B(a_2, r_2) = \{e\}$, we have $e \in B(a_3, r_3)$.*
- (ii) *There is a linear isometry Φ of A onto the space of continuous affine functions on a compact convex set, with $\Phi(e) = 1$.*
- (iii) *$\text{span}(e)$ is a semi L -summand.*

PROOF. (i) \Rightarrow (iii) \Rightarrow (ii) follows from Theorem 7.3.

(ii) \Rightarrow (i) is proved in [35, Theorem 4.7].

B. *M-ideals and L-summands in spaces of continuous functions and uniform algebras.* Let X be a compact Hausdorff space and let A be a Banach space of continuous real- or complex-valued functions on X containing the constants. We say that a subset F of X is a *peak set* for A if there exists an element $f \in A$ such that $f(x) = 1$ for all $x \in F$ and $|f(x)| < 1$ for all $x \in X \setminus F$. We say that F is a *generalized peak set* for A if F is an intersection of peak sets. It is easily verified that the notions of peak set and generalized peak set will coincide for G_δ -sets.

THEOREM 7.5. *Let A be as above and let J be a semi M -ideal in A . Then there exists a generalized peak set $F \subseteq X$ for A such that $J = \{f \in A: f(x) = 0 \text{ for all } x \in F\}$.*

PROOF. Define $F = \{x \in X: f(x) = 0 \text{ for all } f \in J\}$. Let q be an extreme point of the unit ball of J^0 in A^* . Since J^0 is a semi L -ideal, $q \in \partial_e A_1^*$. Let $\Phi: X \rightarrow A_1^*$ be the natural map. Then there exist $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $x \in X$ such that $q = \lambda\Phi(x)$. Clearly $x \in F$ and $F(x) = 0$ for all $f \in J$. Assume $g \in A$ and $g(y) = 0$ for all $y \in F$. Then $q(g) = 0$, so by the Krein-Milman theorem [11] $p(g) = 0$ for all $p \in J^0$. Hence $g \in J$ and $J = \{f \in A: f(x) = 0 \text{ for all } x \in F\}$.

Let $x \in X \setminus F$, and let $f \in J$ with $\|f\| = 1$ and $f(x) = 2\varepsilon > 0$. Then $1 \in B(1 + f, 1) \cap B(1 - f, 1)$ and $\pm f \in J \cap B(1 \pm f, 1)$. From the proof of Proposition 6.5 it follows that there exists an element $g \in J \cap B(1 - f, 1) \cap B(1 + f, 1 + \varepsilon)$. But then $\|1 - f - g\| = 1$ and

$$(1 - f - g)(y) = 1 \quad \text{for all } y \in F.$$

We have

$$1 + \varepsilon \geq \|1 + f - g\| \geq \operatorname{Re}(1 + f(x) - g(x)) = 1 + 2\varepsilon - \operatorname{Re} g(x),$$

so $\operatorname{Re} g(x) \geq \varepsilon$. Hence

$$\operatorname{Re}(1 - f(x) - g(x)) \leq 1 - 2\varepsilon - \varepsilon = 1 - 3\varepsilon < 1.$$

Let $F_x = \{y \in X: 1 - f(y) - g(y) = 1\}$. Then F_x is a peak set, $F \subseteq F_x$ and $x \notin F_x$. Hence F is the intersection of a family of peak sets. The proof is complete.

THEOREM 7.6. *Assume A is a Banach algebra of continuous complex- (or real-) valued functions on X containing the constants and separating points. Let J be a*

closed subspace of A . Then the following statements are equivalent:

- (i) J is an M -ideal.
- (ii) J is a semi M -ideal.
- (iii) There exists a generalized peak set F for A such that $J = \{f \in A: f(x) = 0 \text{ for all } x \in F\}$.

PROOF. (i) \Rightarrow (ii) is trivial and (ii) \Rightarrow (iii) is proved in Theorem 7.5.

(iii) \Rightarrow (i). Let $f_1, f_2, f_3 \in J_1$, let $g \in A_1$ and let $\varepsilon > 0$. Let $G = \{x \in X: \sum_{i=1}^3 |f_i(x)| \geq \varepsilon\}$. Then G is a compact set disjoint from F . Hence there exists an $h \in A_1$ such that $h(x) = 1$ for all $x \in F$ and $|h(x)| < 1$ for all $x \in G$. For some m , we have $|h^m(x)| < \varepsilon$ for all $x \in G$. Then we get

$$g(1 - h^m) \in J \cap \bigcap_{i=1}^3 B(g + f_i, 1 + \varepsilon).$$

By Theorem 6.17 it follows that J is an M -ideal. The proof is complete.

COROLLARY 7.7. Assume $A = C(X)$ (= the Banach space of all real- or complex-valued continuous functions on X). Then a closed subspace J of A is an M -ideal if and only if $J = \{f \in A: f(x) = 0 \text{ all } x \in F\}$ for some closed subset F of X .

REMARK. The equivalence of (i) and (iii) in Theorem 7.6 was proved by Hirsberg in [22].

A Banach space A always contains the two trivial L -summands (0) and A . The next result shows that there need not be any other L -summands in A .

THEOREM 7.8. Let $A = C(X)$ (= the Banach space of all real-valued continuous functions on a compact Hausdorff space X). Let J be a closed subspace of A . Then J is a semi L -summand in A if and only if $J = A$, $J = (0)$ or $J = \text{span}(f)$ for some $f \in \partial_e A_1$.

It follows from Corollary 7.4 that $\text{span}(f)$ is a semi L -summand for all $f \in \partial_e A_1$. That there is no other nontrivial semi L -summand in A , follows from the following observation: Assume J is a closed hereditary subspace of $C(X)$ and that $f \in J$ is such that $\|f\| \neq |f(x)|$ for some $x \in X$, then every $g \in C(X)$ with $g(y) = 0$ for $|f(y)| = \|f\|$ will belong to J .

The complementary cones J' of the semi L -ideals $J = \text{span}(f)$ in Theorem 7.8 are clearly not convex. Hence we get:

COROLLARY 7.9. The only L -summands in $C(X)$ is (0) and $C(X)$.

REMARK. Since real $C(X)$ -spaces have the 4.2.I.P. their dual spaces are isometric to $L_1(\mu)$ -spaces. The projections of norm 1 in $L_1(\mu)$ -spaces ($1 \leq p < \infty$) are completely described by several authors (see [37]). It follows that

$L_1(\mu)$ -spaces do not contain any nontrivial M -projections. Hence $C(X)$ do not contain any nontrivial L -projections.

C. G -spaces. A real Banach space A is said to be a G -space if there exists a compact Hausdorff space X and a set $S = \{(x_\alpha, y_\alpha, \lambda_\alpha)\} \subseteq X \times X \times [-1, 1]$ such that A is isometric to $Y = \{f \in C(X): f(x_\alpha) = \lambda_\alpha f(y_\alpha) \text{ all } (x_\alpha, y_\alpha, \lambda_\alpha) \in S\}$. The G -spaces were introduced by Grothendieck [19] and they have been studied by various authors, notably by Lindenstrauss [35], Lindenstrauss and Wulbert [38] and Effros [13].

We will here give new proofs of some results of Lindenstrauss, Wulbert and Effros.

THEOREM 7.10. *Let A be a real Banach space. Then the following statements are equivalent:*

- (i) A is a G -space.
- (ii) *There exist a compact Hausdorff space X and a subspace Y of $C(X)$ isometric to A such that $\max(f, g, 0) + \min(f, g, 0) \in Y$ for all $f, g \in Y$.*
- (iii) *If $x, y \in A$, then there exists $z \in A$ such that*

$$p(z) = \max(p(x), p(y), 0) + \min(p(x), p(y), 0)$$

for all $p \in \partial_e A_1^*$.

- (iv) A^* is isometric to an $L_1(\mu)$ -space and $\overline{\partial_e A_1^*} \subseteq [0, 1]\partial_e A_1^*$ (ω^* -closure).

PROOF. (i) \Rightarrow (ii) is a verification. (See [35, Lemma 6.7].)

(ii) \Rightarrow (iii) is straightforward.

(iv) \Rightarrow (i) is proved in Effros [13, Theorem 6.3].

(iii) \Rightarrow (iv). Let $x, y \in A$ be such that $\|x\|, \|y\|, \|x - y\| \leq 2$. Let $z \in A$ be such that

$$2p(z) = \max(p(x), p(y), 0) + \min(p(x), p(y), 0)$$

for all $p \in \partial_e A_1^*$. An easy verification (see [35, Theorem 6.9]) shows that $z \in B(0, 1) \cap B(x, 1) \cap B(y, 1)$. Let $p \in \partial_e A_1^*$. If $0 \leq p(y) \leq p(x)$, then

$$\begin{aligned} 6p(z) - 2p(x) - 2p(y) \\ = 3p(x) + 0 - 2p(x) - 2p(y) = p(x) - 2p(y), \end{aligned}$$

so $-p(y) \leq p(6z - 2x - 2y) \leq p(x)$. Hence

$$-\max(\|x\|, \|y\|) \leq p(6z - 2x - 2y) \leq \max(\|x\|, \|y\|).$$

Similarly we treat the cases $p(x) \geq 0 \geq p(y)$, $0 \geq p(x) \geq p(y)$, $0 \geq p(y) \geq p(x)$, $p(y) \geq 0 \geq p(x)$, and $p(y) \geq p(x) \geq 0$. Then we get

$$\|z - (x + y)/3\| \leq \max(\|x\|, \|y\|)/6 \leq 1/3.$$

By Theorem 4.6 and Corollary 3.11 we get that A^* is isometric to an $L_1(\mu)$ -space.

Suppose now that $p \in \overline{\partial_e A_1^*}$ (ω^* -closure). Then clearly p satisfies the equality in (iii). We may assume $\|p\| = 1$. In order to show that $p \in \partial_e A_1^*$, it suffices to show that if $q \in A^*$ and $\|p\| = \|q\| + \|p - q\|$, then $q \in \text{span}(p)$. Define $J = \{x \in A : p(x) = 0\}$. Then $J^0 = \text{span}(p)$. Let $\varepsilon > 0$. Let $x \in J$ with $\|x\| = 1$ and choose $y \in A$ with $\|y\| = 1$ such that $p(y) > \|p\| - \varepsilon = 1 - \varepsilon$. Let $z \in A$ be such that

$$f(z) = \max(f(y+x), f(y-x), 0) + \min(f(y+x), f(y-x), 0)$$

for all $f \in \partial_e A_1^*$. Then

$$\begin{aligned} p(z) &= \max(p(y+x), p(y-x), 0) + \min(p(y+x), p(y-x), 0) \\ &= \max(p(y), 0) + \min(p(y), 0), \end{aligned}$$

so $p(z) = p(y)$. Hence $z - y \in J$. Define $u = z - y$. If $f \in \partial_e A_1^*$, then

$$f(z) = 2f(y) + \max(f(x), -f(x), -f(y)) + \min(f(x), -f(x), -f(y)).$$

Verification gives

$$f(y-u) = 2f(y) - f(z) = \begin{cases} f(y) - f(x) & \text{if } f(y) \geq f(x) \geq 0, \\ f(y) + f(x) & \text{if } f(y) \geq -f(x) \geq 0, \\ 0 & \text{if } |f(y)| \leq |f(x)|, \\ f(y) - f(x) & \text{if } f(y) \leq f(x) \leq 0, \\ f(y) + f(x) & \text{if } f(y) \leq -f(x) \leq 0. \end{cases}$$

Hence we get

$$\|x + y - u\| \leq \max(\|x\|, \|y\|) = 1$$

and

$$\|y - x - u\| \leq \max(\|x\|, \|y\|) = 1,$$

so

$$u \in J \cap B(y+x, 1) \cap B(y-x, 1).$$

Now

$$x = \frac{1}{2}(x+y-u) + \frac{1}{2}(x-y+u).$$

Since $p(x+y-u) > 1 - \varepsilon$, $p(x-y+u) < -1 + \varepsilon$ and $1 = \|p\| = \|q\| + \|p - q\|$, we get $q(x+y-u) > \|q\| - \varepsilon$ and $q(x-y+u) < -\|q\| + \varepsilon$, so

$|q(x)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $q(x) = 0$. Hence $q \in J^0 = \text{span}(p)$, the proof is complete.

REMARKS. (1) In [35] Lindenstrauss proved that the dual space of a G -space is isometric to an $L_1(\mu)$ -space. In [38] Lindenstrauss and Wulbert gave a direct proof of the implication (ii) \Rightarrow (i) in Theorem 7.11. Their proof is rather technical.

(2) In [13] Effros proved (iv) \Rightarrow (i) and for separable spaces also (i) \Rightarrow (iv) of Theorem 7.11. The assumption of separability was later removed by Fakhoury [16] and Taylor [45]. Effros [13] also proved that A is a G -space if and only if A^* is isometric to an $L_1(\mu)$ -space and has Hausdorff structure topology on $\partial_e A_1^*$. (See Effros' paper [13] for the definition of the structure topology.) Uttersrud [51] has proved (using Theorem 5.8) that if the structure topology on $\partial_e A_1^*$ is Hausdorff, the A^* is isometric to an $L_1(\mu)$ -space and hence A is isometric to a G -space.

(3) A real Banach space A is said to be an M -space if A is a G -space such that all $0 \leq \lambda_\alpha \leq 1$ for all α (λ_α as in the definition of G -space). Arguments similar to those used in the proof of Theorem 7.11 can be used to prove the equivalence of the following statements:

- (i) A is an M -space.
- (ii) A is isometric to a sublattice of a $C(X)$ -space.
- (iii) A^* is isometric to an $L_1(\mu)$ -space and there exists a maximal proper face F of A_1^* such that $\overline{\partial_e F} \subseteq [0, 1]\partial_e F$ (ω^* -closure). The equivalence of (i) and (ii) is a result of Kakutani [29].

(4) By the method of proof used in Theorem 7.11, we can obtain characterizations similar to (iv) in Theorem 7.11 and (3)(iii) above for $C(K)$ -, $C_0(LK)$ -, C_Σ - and C_0 -spaces. (See Lindenstrauss and Wulbert [38], Fakhoury [16], Ka Sing Lau [33] and Olsen [40].)

(5) In the proof of Theorem 7.11 we proved that J is a semi M -ideal. By Theorem 5.5 we then get that $\text{span}(p)$ is an L -summand for all $p \in \overline{\partial_e A_1^*}$ (ω^* -closure).

D. *Semi M -ideals in order unit spaces.* Let $A = A(K)$ be the Banach space of all continuous affine real-valued functions on a convex compact subset K of a locally convex Hausdorff space. As in Theorem 7.1 we define $S = \{f \in A^*: \|f\| = f(1) = 1\}$. It is proved in Alfsen [1] that

$$A_1^* = \text{co}(S \cup -S)$$

and that K is homeomorphic to S under the map $x \rightarrow \hat{x}$ where $\hat{x}(a) = a(x)$. Hence we can and shall identify K and S . A and A^* are partially ordered with positive cones $A^+ = \{f \in A: f \geq 0\}$ and $A^{*+} = \bigcup_{\lambda > 0} \lambda K$ respectively.

PROPOSITION 7.11. *Let J be a semi M -ideal in A . Then J^0 is positively generated.*

PROOF. Let $x \in J^0$. Then since $A_1^* = \text{co}(K \cup -K)$, we can write $x = y - z$ where $y, z \in A^{**}$ and $\|x\| = \|y\| + \|z\|$. Since J^0 is a semi L -summand, J^0 is hereditary (see the remarks after Theorem 5.5). Hence $y, z \in J^0 \cap A^{**}$.

Let J be a subspace of A , and let $\varphi: A \rightarrow A/J$ be the canonical map. $\varphi(A^+)$ is a (possibly improper) convex cone in A/J . We say that J is an *order ideal* in A if $a, b \in J, c \in A, a \leq c \leq b$, implies $c \in J$. In [1, Proposition II.1.1] it is proved that J is an order ideal if and only if $\varphi(A^+)$ is a proper convex cone, i.e. $\varphi(A^+) \cap -\varphi(A^+) = (0)$. Clearly, if J is a semi M -ideal in A , then by Proposition 7.11 J is an order ideal in A .

PROPOSITION 7.12. *Let J be a semi M -ideal in A . Then $\varphi(A^+)$ is closed in the quotient topology.*

PROOF. Let $F = J^0 \cap K$. Then by Proposition 7.11, $J_1^0 = \text{co}(F \cup -F)$.

Let $x \in A$ be such that $\varphi(x) \in \overline{\varphi(A^+)}$. By Hahn-Banach $\varphi(x) \in \varphi(A^+)$ if and only if $f(x) \geq 0$ for all $f \in F$. Without loss of generality, we may assume $0 \leq f(x) \leq 2$ for all $f \in F$. But then $-1 \leq f(x - 1) \leq 1$ for all

$$f \in \text{co}(F \cup -F).$$

Since the dual space of A/J is isometric to J^0 , we get $\|\varphi(x - 1)\| \leq 1$, so $d(x - 1, J) \leq 1$.

By Corollary 6.6 there exists an element $y \in J \cap B(x - 1, 1)$. But then $-1 \leq x - 1 - y \leq 1$, so $0 \leq x - y \leq 2$. Hence $\varphi(x) = \varphi(x - y) \in \varphi(A^+)$. This completes the proof.

Propositions 7.11 and 7.12 imply: (i) if $\varphi(x) \in A/J$, then $-\eta\varphi(1) \leq \varphi(x) \leq \eta\varphi(1)$ for some η , and (ii) if $\eta\varphi(y) \leq \varphi(1)$ for all $\eta = 1, 2, \dots$ then $\varphi(y) \leq 0$. Hence if J is a semi M -ideal, the A/J with positive cone $\varphi(A^+)$ will be an *Archimedean ordered vector space* (with terminology of [1]).

We now want to prove that if J is a semi M -ideal in A , then J is positively generated. We write $J^+ = J \cap A^+$.

LEMMA 7.13. *Let J be a semi M -ideal in A . Then*

$$A_1^* \cap (J^0 + A^{**}) \subseteq A^{**} + (J^0 \cap A_1^*).$$

PROOF. Let $x \in A_1^* \cap (J^0 + A^{**})$. Then we can write $x = y + z$ where $y \in J^0$ and $z \in A^{**}$. By Theorem 3.4 we may write $z = z_1 + z_2$, $\|z\| = \|z_1\| + \|z_2\|$ where $z_1 \in J^0$ and $z_2 \in J^{0'}$. Since $z \in A^{**}$ we get

$$\|z_1\| + \|z_2\| = \|z\| = z(1) = z_1(1) + z_2(1) \leq \|z_1\| + \|z_2\|.$$

Hence $z_2(1) = \|z_2\|$, so $z_2 \in A^{**}$. But then

$$x = (y + z_1) + z_2 \in J^0 + (A^{*+} \cap J^{0'}),$$

and since J^0 is a semi L -summand

$$1 \geq \|x\| = \|y + z_1\| + \|z_2\| \geq \|y + z_1\|.$$

Hence $x \in A^{*+} + (J^0 \cap A_1^*)$. The proof is complete.

Taking polars in the formula in Lemma 7.13 (see [7, Proposition 1.1]), we get

COROLLARY 7.14. *If J is a semi M -ideal in A , then*

$$A^+ \cap (J + A_1) \subseteq \overline{J^+ + A_1}.$$

LEMMA 7.15. *Let J be a semi M -ideal in A . Then $J^0 + A^{*+}$ is ω^* -closed and $-(J \cap A^+)^0 = J^0 + A^{*+}$.*

PROOF. Let (x_α) be a net in $A_1^* \cap (J^0 + A^{*+})$. In the proof of Lemma 7.13 we showed that we could write $x_\alpha = y_\alpha + z_\alpha$, $\|x_\alpha\| = \|y_\alpha\| + \|z_\alpha\| \leq 1$, where $y_\alpha \in J^0$ and $z_\alpha \in A^{*+}$. Since $A_1^* \cap J^0$ and $A^{*+} \cap A_1^*$ both are ω^* -compact, we may assume, passing to subnets if necessary, that $y_\alpha \rightarrow y \in J^0$ and $z_\alpha \rightarrow z \in A^{*+}$ (ω^* -convergence). Using that the norm is ω^* -lower semicontinuous, we get

$$\begin{aligned} \|y + z\| &\leq \|y\| + \|z\| \leq \liminf \|y_\alpha\| + \liminf \|z_\alpha\| \\ &\leq \liminf (\|y_\alpha\| + \|z_\alpha\|) \leq \liminf \|x_\alpha\| \leq 1. \end{aligned}$$

Hence a subnet of (x_α) converges to $y + z \in A_1^* \cap (J^0 + A^{*+})$. By the Krein-Šmulian theorem, $J^0 + A^{*+}$ is ω^* -closed.

The formula $-(J \cap A^+)^0 = J^0 + A^{*+}$ now follows from Asimow [7, Proposition 1.1]. The proof is complete.

COROLLARY 7.16. *Let J be a semi M -ideal in A and let $a \in A^+$. Then $d(a, J) = d(a, J^+)$.*

PROOF. Suppose $r = d(a, J) > 0$. Let $f \in A^*$ be such that $f \geq 0$ on $J \cap A^+$. Then $f \in -(J \cap A^+)^0 = J^0 + A^{*+}$. Hence we may write $f = g + h$, $\|f\| = \|g\| + \|h\|$, where $g \in J^0$ and $h \in A^{*+}$. Let $x \in J \cap B(a, r)$. Then

$$-f(a) = -g(a) - h(a) \leq -g(a) = -g(a - x) \leq r\|g\| \leq r\|f\|.$$

By Theorem 1.4 we get $d(a, J \cap A^+) \leq r$. The proof is complete.

PROPOSITION 7.17. *Let J be a semi M -ideal in A . For all $\delta \in [0, 1)$ we have $J \cap B(0, \delta) \subseteq (J^+ \cap A_1) - (J^+ \cap A_1)$.*

PROOF. Let $x \in J$ with $\|x\| = 1$ and let $\varepsilon > 0$. By Theorem 6.10 and the proof of Proposition 6.5 there exists an element

$$y \in J \cap B(1 - x, 1) \cap B(1 + x, 1 + \varepsilon).$$

Since $y \in B(1 - x, 1)$ we have $0 \leq x + y \leq 2$. Since $y \in B(1 + x, 1 + \varepsilon)$ we have $-(2 + \varepsilon) \leq x - y \leq \varepsilon$. Hence by Corollary 7.14

$$\begin{aligned} y - x + \varepsilon &\in A^+ \cap (J + B(0, \varepsilon)) \subseteq \overline{(J \cap A^+) + B(0, \varepsilon)} \\ &\subseteq (J \cap A^+) + B(0, 2\varepsilon). \end{aligned}$$

Now choose $z \in J \cap A^+$ and $u \in B(0, 2\varepsilon)$ such that $y - x + \varepsilon = z + u$. Then $\|z\| \leq 2 + 3\varepsilon$ and $y - x = z + (u - \varepsilon)$. But then

$$\begin{aligned} x &= 2^{-1}(x + y) - 2^{-1}(y - x) = 2^{-1}(x + y) - 2^{-1}z - 2^{-1}(u - \varepsilon) \\ &= 2^{-1}(x + y) - (2 + 3\varepsilon)^{-1}z - 2^{-1}(u - \varepsilon) + ((2 + 3\varepsilon)^{-1} - 2^{-1})z. \end{aligned}$$

We have

$$\begin{aligned} &\|2^{-1}(u - \varepsilon) + (2^{-1} - (2 + 3\varepsilon)^{-1})z\| \\ &\leq 2^{-1}3\varepsilon + (2 + 3\varepsilon)(2^{-1} - (2 + 3\varepsilon)^{-1}) = 3\varepsilon. \end{aligned}$$

Hence $x \in (J^+ \cap A_1) - (J^+ \cap A_1) + B(0, 3\varepsilon)$, so

$$J \cap A_1 \subseteq \overline{(J^+ \cap A_1) - (J^+ \cap A_1)}.$$

The Tukey-Klee-Ellis theorem (see [15, Lemma 7]) now gives that for $\delta \in [0, 1]$, we have

$$J \cap B(0, \delta) \subseteq (J^+ \cap A_1) - (J^+ \cap A_1).$$

The proof is complete.

Following [1] we say that a closed subspace J of A is a *strongly Archimedean order ideal* if

- (i) A/J is an Archimedean ordered vector space.
- (ii) J is positively generated.
- (iii) J^0 is positively generated.

Propositions 7.11, 7.12 and 7.17 show that:

THEOREM 7.18. *If J is a semi M -ideal in A , then J is a strongly Archimedean order ideal.*

By Theorem 7.18, [44, Theorem 5.2] and [4, Proposition 6.18, part II] we get:

COROLLARY 7.19. *If J is a closed subspace of the selfadjoint part of a C^* -algebra with unit, then the following statements are equivalent:*

- (i) J is a semi M -ideal.
- (ii) J is an M -ideal.
- (iii) J is the selfadjoint part of a two-sided ideal of the algebra.

PROOF. (ii) \Leftrightarrow (iii) is due to Alfsen and Effros [4].

(ii) \Rightarrow (i) is trivial.

(i) \Rightarrow (iii) follows from Theorems 7.18 and 5.2 in Størmer [44].

REMARK. Let $A = l_1^3$ and let $e = (1, 0, 0)$. From Corollary 7.4 and Theorem 7.8 it follows that A is isometric to an $A(K)$ -space, and that $J = \{(x, y, z): x + y + z = 0\}$ is a semi M -ideal in A which is not an M -ideal.

This example was first used by Alfsen and Effros in the proof of Theorem 5.8 in [4]. There they proved that J has the 2.I.P. but not the 3.I.P. by an elementary but somewhat technical geometric argument.

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