

CHERN-SIMONS INVARIANTS OF REDUCTIVE HOMOGENEOUS SPACES

BY

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ABSTRACT. The geometric characteristic classes of Chern-Simons are computed for certain connections on the canonical bundle and tangent bundle over a reductive homogeneous space. This includes globally symmetric spaces with the Levi-Civita connection of any bi-invariant metric.

Introduction. In recent papers [9], [11], [18], Chern and Simons have developed a theory of secondary characteristic classes on manifolds. These classes were calculated by Heitsch and Lawson on compact Lie groups with bi-invariant metric. One result of this paper is to extend these calculations to Riemannian symmetric spaces.

§1 begins by quickly reviewing the main results of Chern and Simons. Later in this section we prove some general lemmas concerning extension of connections and Chern-Simons theory. In §2 we state the properties of the transgression map. In §3 some well-known results concerning invariant connections on reductive homogeneous spaces are recorded. §4 gives the main result, Theorem 4.5, relating the Chern-Simons invariants of the bundle of bases over a reductive homogeneous space with the canonical affine connection of the second kind to those of those of the defining bundle with its canonical connection. §5 specializes to symmetric spaces and Theorem 5.1 shows that the secondary classes are well-defined in some cases. In §6 we treat Lie groups as symmetric spaces and in particular recover the main result of [12]. §7 involves the calculation of secondary classes over Riemannian symmetric spaces fibered by the classical compact Lie groups. In §8 we calculate the Chern-Simons invariants restricting conformal immersions for the spaces of §7 and obtain the nontrivial invariants for $SU(2k+1)$ and $SU(2k+1)/SO(2k+1)$.

The main idea of the calculation is that the Chern-Simons invariants for the bundle of bases over G/H with canonical affine connection of the second kind are determined by the Chern-Simons invariants of the natural connection over the defining bundle. This last bundle may be treated using techniques from

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algebraic topology and the usual theory of characteristic classes.

1. Chern-Simons theory. In recent papers S. S. Chern and J. Simons have studied secondary characteristic classes on manifolds. These classes are defined when the invariant polynomials $P(\Omega)$ of the Weil homomorphism vanish and the secondary classes have interesting geometrical properties.

In their joint work Chern and Simons associate to triples (B, M, ϕ) consisting of a principal bundle B with connection ϕ over a manifold M forms $TP(\phi)$ on the bundle B for each invariant polynomial $P \in I^k(G)$ of degree k where G is the group of the bundle. If we set $\Omega_t = t\Omega + \frac{1}{2}(t^2 - t)[\phi, \phi]$ then by definition $TP(\phi) = k \int_0^1 P(\phi, \Omega_t) dt$. A calculation shows $dTP(\phi) = P(\Omega)$. Thus if $P(\Omega) = 0$, $TP(\phi)$ determines a real cohomology class in B . When B is the bundle of bases over M the forms $TP(\phi)$ are invariant under conformal change of metric where ϕ is chosen as the Levi-Civita connection associated to a metric. The main result of [9] is the following:

THEOREM 1.1. *Let M^n be an n -dimensional Riemannian manifold. Let $d(M) = \{E(M), M, \theta\}$ denote the $Gl(n, R)$ basis bundle over M equipped with the Riemannian connection θ . A necessary condition that M^n admit a conformal immersion in R^{n+k} is that $P_i^\perp(\Omega) = 0$ and $\{\frac{1}{2}TP_i^\perp(\theta)\} \in H^{4i-1}(E(M), Z)$ for $i > [k/2]$.*

In a later paper [18], Simons defined characters associated to triples (B, M, ϕ) . The idea was to define some geometric invariants in M corresponding to the $TP(\phi)$ in B . Since the homology structure of M is in general simpler than that of B it is reasonable to expect that such invariants would be easier to apply.

For each pair (P, u) where $P \in I^1(G)$ is an invariant polynomial of degree 1 and $u \in H^{2l}(B_G, Z)$ such that $W(P) = r(u)$ with

$$W: I^1(G) \rightarrow H^{2l}(B_G, R)$$

the Weil homomorphism and $r: H^{2l}(B_G, Z) \rightarrow H^{2l}(B_G, R)$ the coefficient homomorphism we have a character $S_{P,u}(\phi): Z_{2l-1}(M) \rightarrow R/Z$. $S_{P,u}(\phi)$ have a natural definition on torsion cycles $x \subset M$. If $x \in Z_{2l-1}^{\text{tor}}(M)$ and $\gamma \in Z^{2l-1}(M)$ represents the characteristic class u let $y \in Z_{2l}(M)$ be such that $\partial y = nx$. Then by definition $S_{P,u}(\phi)(x) = (\overline{P(\Omega)y - \gamma y})/n$ where the bar denotes reduction mod Z . Simons proves by elementary arguments that this definition on torsion cycles is independent of the choices y, γ .

To define $S_{P,u}(\phi)$ for general cycles we must assume that G , the group of B , has a finite number of connected components. Then there is a classifying map of (B, M, ϕ) into a sufficient approximation of the classifying space B_G with a canonical connection ω . Now it is well known that $H^{\text{odd}}(B_G, R) = 0$ since G

has a finite number of connected components. Then the image of any odd dimensional cycle in M is a torsion cycle in B_G . Simons defines $S_{P,u}(\phi)$ by the given definition on torsion cycles and the requirement of naturality under connection preserving bundle maps. He proves that the definition is independent of the choice of classifying map.

The characters $S_{P,u}(\phi)$ determine the forms $P(\Omega)$ and the characteristic classes u . These characters determine R/Z cohomology classes if and only if $P(\Omega) = 0$.

The characters $S_{P,u}(\phi)$ are closely related to the forms $TP(\phi)$. In fact we have the following [18]:

THEOREM 1.2. *If $\pi: B \rightarrow M$ is the projection then*

$$\pi^* S_{P,u}(\phi) = \overline{TP(\phi)}|_{Z_{2l-1}(E)}$$

where the bar denotes reduction mod Z .

COROLLARY 1.3. *If $S_{P,u}(\phi) = 0$ then $TP(\phi)$ is an integral class.*

The analogue of Theorem 1.1 is given by Simons [18]:

THEOREM 1.4. *Let M be an n -dimensional Riemannian manifold. A necessary condition that it admit a conformal immersion in R^{n+k} is that $Sp_i^+(\phi) = 0$ for $i > [k/2]$.*

For our later work we need some preliminary lemmas in Chern-Simons theory. They are established below.

LEMMA 1.5. *Let P be a principal G_1 bundle with connection ϕ and $f: G_1 \rightarrow G_2$ a homomorphism. Define $\hat{P} = P \times G_2 / \sim$ the associated principal bundle with connection $\hat{\phi}$. The bar denotes the usual equivalence relation $(p, g) = (ph^{-1}, f(h)g)$ and denote e the identity of G_2 . Let $i_f: P \rightarrow \hat{P}$ be the map $p \rightarrow \overline{(p, e)}$. Then $i_f^* \hat{\phi} = f_* \phi$ where f_* is the Lie algebra homomorphism induced by f .*

PROOF. For $p \in P$ and $Z \in \mathfrak{G}_1$ (resp. \mathfrak{G}_2) denote by \bar{Z} the vector field induced on the fiber containing p (resp. $i_f p$).

(a) For $X \in \mathfrak{G}_1$ if

$$\begin{aligned} (pe^{tX}) &= \overline{(pe^{tX}, e)} = \overline{(p, f(e^{tX})e)} = \overline{(p, ef(e^{tX}))} = \overline{(p, e)}f(e^{tX}) \\ &= \overline{(p, e)}e^{(f_* X)t} \end{aligned}$$

this shows $(i_f)_* X = \bar{f}_* \bar{X}$.

(b) The horizontal subspace at $i_f p$ is by definition the image of the horizontal space at p . Now let \bar{X} be an arbitrary vertical vector $i_f^* \hat{\phi}(\bar{X}) = \hat{\phi}((i_f)_* \bar{X}) = \hat{\phi}(\bar{f}_* \bar{X}) = f_* X = f_* \phi(\bar{X})$. Let Y be an arbitrary horizontal

vector then

$$(i_f^* \hat{\phi})(Y) = \hat{\phi}((i_f)_* Y) = 0 = f_* \phi(Y).$$

Since any tangent vector to P is the sum of a horizontal and a vertical vector this completes the proof.

COROLLARY 1.6. *With the assumptions as in Lemma 1.5 $i_f^* \hat{\Omega} = f_* \Omega$ where $\hat{\Omega}$ (resp. Ω) is the curvature form of $\hat{\phi}$ (resp. ϕ).*

Let $f: G_1 \rightarrow G_2$ be as in Lemma 1.5. Then f induces maps $f^*: I^l(G_2) \rightarrow I^l(G_1)$, $f^*: H^q(B_{G_2}) \rightarrow H^q(B_{G_1})$.

LEMMA 1.7. *For any $Q \in I^l(G)$ we have $(f^* Q)(\Omega) = Q(\hat{\Omega})$ considered as forms in the base manifold M .*

PROOF. $i_f^*(Q(\hat{\Omega})) = Q(i_f^* \hat{\Omega}) = Q(f_* \Omega) = (f^* Q)(\Omega)$. Now if p, \hat{p} are the projections in P, \hat{P} we have $\hat{p}i_f = p$. So for a form ω in the base M , $i_f^* \hat{p}^* \omega = p^* \omega$. The assertion follows.

We wish to relate the Simons characters of P, \hat{P} . For this we need the preliminary:

LEMMA 1.8. *If $W(Q) = r(u)$ for $Q \in I^l(G_2)$, $u \in H^{2l}(B_{G_2}, \mathbb{Z})$ then if $f: G_1 \rightarrow G_2$ is a homomorphism $W(f^* Q) = r(f^* u)$.*

PROOF. Let E_{G_1} be sufficiently n -classifying for G_1 and use f to extend it to a bundle \hat{E} with group G_2 . If θ is the canonical connection on E_G let $\hat{\theta}$ be the induced connection on \hat{E} . Let $\gamma: \hat{E} \rightarrow E_{G_2}$ be a connecting preserving for classifying map for \hat{E} where E_{G_2} is s -classifying for G_2 with $s \geq n$. Then by definition $f^* u = \gamma^* u$. $W(f^* Q) = (f^* Q)(\Omega) = i_f^* Q(\hat{\Omega})$ in the bundles E, \hat{E} where the third step in the proof follows from the proof of Lemma 1.7. Then in the space B_{G_1} , $W(f^* Q) = Q(\hat{\Omega}) = r(\gamma^* u) = r(f^* u)$ where the third step follows by naturality of the connection preserving bundle map γ and the relation $W(Q) = r(u)$.

LEMMA 1.9. *With the notation as in Lemma 1.5*

$$S_{Q,u}(\hat{\phi}) = S_{f^* Q, f^* u}(\phi).$$

PROOF. By Lemma 1.8 $S_{f^* Q, f^* u}(\phi)$ is well defined. Consider the diagram below:

$$\begin{array}{ccccc}
 G_2 & \longrightarrow & \hat{P} & \cdots \cdots \cdots & \hat{E} \\
 \uparrow & & \uparrow & & \uparrow \\
 G_1 & \longrightarrow & P & \xrightarrow{\quad G_1 \quad} & E_{G_1} \\
 & & \downarrow & \searrow & \downarrow \\
 & & M & & B_{G_2}
 \end{array}$$

Notation is as in the proof of Lemma 1.8. The maps are obtained by choosing a classifying map $P \rightarrow E_{G_1}$ and extending it to a map $P \rightarrow \hat{E}$. By naturality of the Simons character it is enough to verify the relation on the system E_{G_1}, \hat{E} . The real cohomology of B_{G_1} vanishes in the relevant odd dimension. Thus all cycles are torsion cycles. Let $\partial y = nx$,

$$\begin{aligned} S_{f^*Q, f^*u}(\phi)(x) &= \overline{(f^*Q(\Omega) - f^*u)(y)/n} \\ &= \overline{(Q(\hat{\Omega}) - u)(y)/n} = S_{Q,u}(\hat{\phi})(x) \end{aligned}$$

where the third step follows from $u(\hat{E}) = f^*u(E_{G_1})$ as in Lemma 1.8 by the definition of f^* .

LEMMA 1.10. *With notation as in Lemma 1.5*

$$i_f^* TQ(\hat{\phi}) = T(f^*Q)(\phi).$$

PROOF. This follows easily from Lemma 1.5, Corollary 1.6, and the definition of TP .

2. The transgression. There is a well-known map τ called the transgression which will be important in this paper. Let G be a compact Lie group and $\pi: EG \rightarrow BG$ its classifying object. For an arbitrary coefficient ring Λ , $\tau: H^i(BG, \Lambda) \rightarrow H^{i-1}(G, \Lambda)$ with $i \geq 1$. τ is defined as follows: Let γ_1 represent a class α_1 in $H^i(BG, \Lambda)$. $\pi^*\gamma_1$ is closed and therefore by acyclicity of EG there exists γ_2 a cochain in $Z^{i-1}(EG, \Lambda)$ s.t. $\delta\gamma_2 = \pi^*\gamma_1$. If $i: G \rightarrow EG$ is the inclusion of the fiber in the bundle then $\tau\alpha_1$ is the class $i^*\gamma_2$. It is straightforward to check that this definition of τ is independent of the choices made.

The following well-known lemma will be needed in the calculations made in the latter part of this paper:

LEMMA 2.1. *τ maps products to zero.*

PROOF. Let $\alpha = \alpha_1 \alpha_2 \in H^i(BG, \Lambda)$. Then if γ_1 represents α_1 , γ_2 represents α_2 , we have $\gamma_1 \gamma_2$ representing $\alpha_1 \alpha_2$. If $\delta\beta = \pi^*\gamma_1$ then $\delta(\beta\pi^*\gamma_2) = (\pi^*\gamma_1) \cdot (\pi^*\gamma_2) = \pi^*(\gamma_1 \gamma_2)$. But $i^*(\beta\pi^*\gamma_2) = (i^*\beta)(i^*\pi^*\gamma_2) = 0$, since $i^*\pi^*\gamma_2 = 0$.

If the coefficient ring is the real numbers then the transgression map is closely related to the forms $TP(\phi)$. In fact let $\pi: E \rightarrow B$ be a sufficiently close approximation to a classifying object. Then identifying the cohomology groups of B in lower dimensions with those of B_G we may define essentially the same transgression in $\pi: E \rightarrow B$. If ϕ is a connection on $\pi: E \rightarrow B$ let $P(\Omega)$ for $\deg P$ sufficiently small be the Weil forms in the base. Then

it is a direct consequence of the definitions that $\tau[P(\Omega)] = [i^* TP(\phi)]$. A straightforward calculation gives for $P \in I^k(G)$:

$$\tau_p = \left(-\frac{1}{2}\right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} P(\omega_G, [\omega_G, \omega_G])$$

where $p = [P(\Omega)]$ and ω_G is the Maurer-Cartan form on G .

3. Connections on reductive homogeneous spaces. The material in this section is well known and more details may be found in [14] and [17].

A homogeneous space is called reductive if there is a decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ of the Lie algebra of G , the total group, such that $\text{ad}(h)\mathfrak{M} \subset \mathfrak{M}$ all $h \in H$, the fiber group. If $xH \in G/H$ then $xH \rightarrow gxH$ defines an action of G on the homogeneous space G/H . Since for all $h \in H$, h maps eH to eH , the action of h induces a representation called the isotropy representation of H on the tangent space to G/H at eH . If \mathfrak{M} is identified with this tangent space then the isotropy representation is just the action of H on \mathfrak{M} .

Let p_0 denote the identity coset eH . There exists a neighborhood N^* of p_0 such that for $g \in G$ and $X \in \mathfrak{M}$ there is a vector field X^* in N^* defined by $(X^*)_{gp_0} = g_* X$ with g an element of a local section and where on the right hand side X is identified with an element of the tangent space at p_0 . Then the fundamental existence theorem of [17] concerning invariant affine connections on G/H is the following:

THEOREM 3.1. *Let G/H be a reductive homogeneous space with fixed decomposition of the Lie algebra $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$, $\text{ad}(H)\mathfrak{M} \subset \mathfrak{M}$. There exists a one to one correspondence between the set of G invariant affine connections on G/H and the set of all bilinear functions α on $\mathfrak{M} \times \mathfrak{M}$ with values in \mathfrak{M} which are invariant by $\text{ad}(H)$, that is, $\text{ad } h(\alpha(X, Y)) = \alpha(\text{ad } hX, \text{ad } hY)$ for $X, Y \in \mathfrak{M}$ and $h \in H$. The correspondence is given by*

$$\alpha(X, Y) = (\nabla_{Y^*} X^*)_{p_0}.$$

We shall be interested in the connection given by $\alpha \equiv 0$. This is called the canonical affine connection of the second kind.

We shall also be concerned with a connection on the defining bundle $(G, G/H, H, \pi)$ over G/H . We define the canonical connection on this bundle by $\phi(X) = X_{\mathfrak{H}}$ where X is any left invariant vector field on G and $X_{\mathfrak{H}}$ is its \mathfrak{H} component relative to the decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$. This is easily seen to be a G invariant connection where G acts to the left.

4. Chern-Simons theory on reductive homogeneous spaces. In this section we establish our main result which relates the Simons character and forms TP of the canonical affine connection of the second kind to those of the canonical

connection on the defining bundle over G/H .

First we need some notation. Let $(B, G/H, Gl(n), \pi, \omega)$ be the bundle of bases over G/H with ω the canonical affine connection of the second kind and Ω the curvature form of ω . Let $(G, G/H, H, p, \phi)$ be the defining bundle over G/H with its canonical affine connection ϕ and Φ the curvature form of ϕ . Denote $i_s: H \rightarrow Gl(n)$ the isotropy representation. Choose left invariant vector fields X_i , $1 \leq i \leq n$, on G such that the X_i span \mathfrak{M} .

We define a map $F: G \rightarrow B$ which is the principal tool used in the sequel.

$$F(g) = (gH, P_*(X_1)_g, \dots, P_*(X_n)_g) = (gH, g_*(X_1)_{eH}, \dots, g_*(X_n)_{eH}).$$

LEMMA 4.1. $F(gh) = F(g)i_s(h)$.

PROOF.

$$\begin{aligned} F(gh) &= (gH, g_* h_*(X_1)_{eH}, \dots, g_* h_*(X_n)_{eH}) \\ &= (gH, g_*(X_1)_{eH}, \dots, g_*(X_n)_{eH})i_s(h) = F(g)i_s(h). \end{aligned}$$

LEMMA 4.2. Define a connection γ on B by choosing the horizontal subspace at $F(g)$ to be the image $F_*(\mathfrak{M}_g)$ and extending by right invariance under $Gl(n)$. Then γ is a well-defined G invariant connection on B .

PROOF. πF is onto G/H so we have a subspace $F_*(\mathfrak{M}_g)$ at a point on each fiber. If $\pi F(g_1) = \pi F(g_2)$ then $g_1 = g_2 h$ for some $h \in H$. Now by Lemma 4.1, $F(gh) = F(g)i_s(h)$ and $F_*(\mathfrak{M}_{gh}) = F_*(\mathfrak{M}_g)(i_s(h))_*$. So our definition can be extended to be right invariant on the fiber.

To show that we have defined a connection we need only verify that the subspace we have defined is complementary at each point to the tangent space of the fiber. Fix $g \in G$. Then $F_*(\mathfrak{M}_g)$ has dimension at most n and since $\pi_* F_*(\mathfrak{M}_g) = g_* P_*(\mathfrak{M}_e) = g_*(T_{eH})$ it is exactly of dimension n and is complementary to the tangent space of the fiber.

To check the invariance under the left action of G on G/H note that

$$\begin{aligned} F(g_1 g_2) &= (g_1 g_2 H, g_{1*} g_{2*}(X_1)_{eH}, \dots, g_{1*} g_{2*}(X_n)_{eH}) \\ &= g_{1*}(g_2 H, g_{2*}(X_1)_{eH}, \dots, g_{2*}(X_n)_{eH}) = g_{1*} F_*(\mathfrak{M}_{g_2}). \end{aligned}$$

Then $F_*(\mathfrak{M}_{g_1 g_2}) = F_*(g_{1*} \mathfrak{M}_{g_2}) = g_{1*} F_*(\mathfrak{M}_{g_2})$ giving the required result.

REMARK. Note that by definition \mathfrak{M}_g is the horizontal subspace at g for the canonical connection ϕ .

LEMMA 4.3. The connection γ defined in Lemma 4.2 is the canonical affine connection of the second kind ω on B .

PROOF. Since both γ and ω are G -invariant it suffices to work in a neighborhood N^* as described in §3. Let ∇ denote covariant differentiation with respect to γ and let σ be the local section of B given by X_1^*, \dots, X_n^* with notation as in §3.

Since by definition of γ , $\sigma_* X_1^*|_{p_0}$ are horizontal. By Theorem 3.1, γ is the canonical affine connection of the second kind ω .

LEMMA 4.4. *Let \hat{G} be the principal $Gl(n)$ bundle which is associated to G by the isotropy representation i_s . Then $\hat{G} = G \times Gl(n)/-$. Denote by $j: G \rightarrow \hat{G}$ the natural map $j(g, e) = (\bar{g}, e)$.*

Now define a map $F: \hat{G} \rightarrow B$ by the conditions

$$(i) \hat{F}_j(g, e) = F(g) \text{ and}$$

$$(ii) \hat{F}(\hat{g}\omega) = (\hat{F}(\hat{g}))\omega$$

for all $\omega \in Gl(n)$. Then F is a connection preserving bundle map.

PROOF. First we must check that \hat{F} is well defined. That is, if $j(\bar{g}, e) = j(g, e)\omega$ then $F(\bar{g}) = F(g)\omega$. Now $j(\bar{g}, e), j(g, e)$ lie in the same fiber if and only if \bar{g}, g lie in the same fiber. Then $\bar{g} = gh$ for some $h \in H$. $j(\bar{g}, e) = j(gh, e) = j(g, i_s(h)e) = j(g, e)i_s(h)$. Also $F(\bar{g}) = F(gh) = F(g)i_s(h)$ by Lemma 4.1. Therefore \hat{F} is well defined.

To show that \hat{F} is connection preserving note that since \hat{F} commutes with the action of $Gl(n)$ it is enough to check that the image of the horizontal space at each $j(g, e)$ is the horizontal space at $F(g)$. Now the horizontal space H at $j(g, e)$ is by definition $j_*(\mathcal{M}_g)$. So $\hat{F}_*(H(j(g, e))) = \hat{F}_*j_*\mathcal{M}_g = F_*\mathcal{M}_g$. The result then follows from Lemmas 4.2 and 4.3.

THEOREM 4.5. *Let $Q \in I^1(Gl(n)), u \in H^{2l}(B_{Gl(n)}, Z)$ such that $W(Q) = r(u)$, then*

$$(i) Q(\Omega) = (i_s^* Q)(\Phi),$$

$$(ii) F^* TQ(\omega) = T(i_s^* Q)(\phi),$$

$$(iii) S_{Q,u}(\omega) = S_{i_s^* Q, i_s^* u}(\phi).$$

PROOF. (i) follows from Lemmas 1.7 and 4.4 since F of Lemma 4.4 induces the identity map of the base spaces.

(ii) $F^* TQ(\omega) = j^* \hat{F}^* TQ(\omega)$ with notation as in Lemma 4.4. The result now follows from Lemmas 1.10 and 4.4 applying naturality of TP under connection preserving bundle maps.

(iii) follows from Lemmas 1.9, 4.4, the fact that \hat{F} induces the identity map on base spaces and naturality of the Simons character under connection preserving bundle maps.

5. Symmetric spaces. The symmetric spaces with their canonical Lie algebra decomposition are all reductive homogeneous spaces. Further on a symmetric space G/H the canonical affine connection of the second kind coincides with

the canonical affine connection of the first kind [17] and by Theorem 15.6 of [17] this is the connection induced by any G -invariant metric. Such metrics always exist if H is compact.

We proceed to do some computations on the defining bundle $(G, G/H, H, p, \phi)$ over a symmetric space G/H . Let $P \in I^d(H)$ be any H invariant polynomial.

$$\mathfrak{G} = \mathfrak{S} + \mathfrak{M} \quad \text{with } [\mathfrak{S}, \mathfrak{S}] \subset \mathfrak{S}, [\mathfrak{S}, \mathfrak{M}] \subset \mathfrak{M}, [\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{S}.$$

Let X, Y be left invariant vector fields on G .

$$\phi(X) = X_{\mathfrak{S}},$$

$$d\phi(X, Y) = -\phi(X, Y) = -[X, Y]_{\mathfrak{S}} = -[X_{\mathfrak{M}}, Y_{\mathfrak{M}}] - [X_{\mathfrak{S}}, Y_{\mathfrak{S}}],$$

$$\frac{1}{2}[\phi, \phi](X, Y) = [X_{\mathfrak{S}}, Y_{\mathfrak{S}}],$$

$$\Phi(X, Y) = (d\phi + \frac{1}{2}[\phi, \phi])(X, Y) = -[X_{\mathfrak{M}}, Y_{\mathfrak{M}}],$$

$$\begin{aligned} \Phi_t(X, Y) &= (t\Phi + \frac{1}{2}(t^2 - t)[\phi, \phi])(X, Y) \\ &= -t[X_{\mathfrak{M}}, Y_{\mathfrak{M}}] + (t^2 - t)[X_{\mathfrak{S}}, Y_{\mathfrak{S}}]. \end{aligned}$$

Define forms $\gamma_1: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{S}$ by $\gamma_1(X, Y) = [X_{\mathfrak{M}}, Y_{\mathfrak{M}}]$, $\gamma_2: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{S}$ by $\gamma_2(X, Y) = [X_{\mathfrak{S}}, Y_{\mathfrak{S}}]$. Then

$$\begin{aligned} TP(\phi) &= d \int_0^1 P(\phi, \Phi_t) dt \\ &= d \int_0^1 \sum_{r=0}^{d-1} (-1)^r t^r (t^2 - t)^{d-r-1} \binom{d-1}{r} P(\phi, \overbrace{\gamma_1, \dots, \gamma_1}^r, \gamma_2) dt \\ &= \sum_{r=0}^{d-1} P(\phi, \overbrace{\gamma_1, \dots, \gamma_1}^r, \gamma_2) \int_0^1 d(-1)^r t^r (t^2 - t)^{d-r-1} \binom{d-1}{r} dt. \end{aligned}$$

Set

$$\begin{aligned} W_{d,r} &= d \binom{d-1}{r} (-1)^r \int_0^1 t^{d-1} (1-t)^{d-r-1} dt (-1)^{d-r-1} \\ &= d \binom{d-1}{r} (-1)^{d-1} \frac{(d-1)! (d-r-1)!}{(2d-r-1)!} \\ &= (-1)^{d-1} \frac{d! (d-1)!}{r! (2d-r-1)!}. \end{aligned}$$

Then

$$TP(\phi) = \sum_{r=0}^{d-1} W_{d,r} P(\phi, \overbrace{\gamma_1, \dots, \gamma_1}^r, \gamma_2).$$

We have the following result which shows that in some cases the secondary invariants of Chern-Simons define cohomology classes on symmetric spaces.

THEOREM 5.1. *Suppose $P \in I(H)$ is the restriction of an invariant polynomial $P \in I(G)$ then $P(\Phi) = 0$.*

PROOF. Let $\omega_{\mathfrak{M}}: \mathfrak{G} \rightarrow \mathfrak{G}$ be the \mathfrak{G} -valued one form defined by projection of a left-invariant vector field on G along \mathfrak{G} . Then

$$\begin{aligned} P(\Phi) &= (-1)^d P([\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]) = \text{Ad}_{\omega_{\mathfrak{M}}} (-1)^d P(\omega_{\mathfrak{M}}, [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]) \\ &\quad + (d-1)(-1)^d P(\omega_{\mathfrak{M}}, [\omega_{\mathfrak{M}}, [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]], [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]) = 0 \end{aligned}$$

where the first term vanishes by $\text{Ad } G$ invariance and the second term vanishes by the Jacobi identity $[\theta, [\theta, \theta]] = 0$ for any \mathfrak{G} -valued one form θ .

For general P the author has been unable to determine the $TP(\phi)$ except on a case by case basis (§§6, 7). However if the degree of P is low we have

PROPOSITION 5.2. *Suppose the symmetry of the symmetric Lie algebra is induced by an inner automorphism of G . Then*

(i) *If $P \in I^1(H)$ is the restriction of an $I(G)$ invariant polynomial then $TP(\phi) = \tau p$ where $p \in H^2(B_G, R)$ is the element corresponding to P .*

(ii) *If $P \in I^2(H)$ is the restriction of an $I^2(G)$ invariant polynomial P then $TP(\phi) = \tau p$ where $p \in H^4(B_G, R)$ is the element corresponding to P .*

PROOF. $\omega_{\mathfrak{G}} = \phi$ the projection on H along \mathfrak{M} ; ω_G the Maurer-Cartan form of G .

$$\begin{aligned} TP(\phi) &= P(\phi) = P(\omega_{\mathfrak{G}}) = P(\omega_G - \omega_{\mathfrak{M}}) \\ \text{(i)} \quad &= P(\omega_G) - P(\omega_{\mathfrak{M}}) = P(\omega_G) = \tau p \end{aligned}$$

where $P(\omega_{\mathfrak{M}}) = P(-\omega_{\mathfrak{M}})$ by applying the symmetry and by the formula of (2).

$$\begin{aligned} TP(\phi) &= W_{2,0} P(\phi, \gamma_2) + W_{2,1} P(\phi, \gamma_1) \\ &= \frac{1}{3} P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{G}}]) - P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]), \\ \tau p &= -\frac{1}{3} P(\omega_G, [\omega_G, \omega_G]) \\ &= -\frac{1}{3} P(\omega_{\mathfrak{G}} + \omega_{\mathfrak{M}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{G}}] + [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}] + 2[\omega_{\mathfrak{G}}, \omega_{\mathfrak{M}}]) \\ \text{(ii)} \quad &= -\frac{1}{3} P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{G}}]) - \frac{1}{3} P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{M}}]) \\ &\quad - \frac{2}{3} P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]) - \frac{1}{3} P(\omega_{\mathfrak{M}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{G}}]) \\ &\quad - \frac{1}{3} P(\omega_{\mathfrak{M}}, [\omega_{\mathfrak{M}}, \omega_{\mathfrak{M}}]) - \frac{2}{3} P(\omega_{\mathfrak{M}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{M}}]) \\ &= -\frac{1}{3} P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{G}}]) - P(\omega_{\mathfrak{G}}, [\omega_{\mathfrak{G}}, \omega_{\mathfrak{M}}]). \end{aligned}$$

So $TP(\phi) = \tau p$. The last step uses $\text{Ad } G$ invariance in setting $P(\omega_{\mathfrak{M}}, [\omega_{\mathfrak{S}}, \omega_{\mathfrak{M}}]) = P(\omega_{\mathfrak{S}}, [\omega_{\mathfrak{S}}, \omega_{\mathfrak{M}}])$.

6. Lie groups. In this section we consider the defining bundle $(G \times G, G, \Delta G, p, \phi)$ exhibiting a Lie group as a symmetric space (ΔG denotes the diagonal in $G \times G$). If $s: G \rightarrow G \times G$ by $(\bar{g}, \bar{e}) \rightarrow (g, e)$ is the canonical section of this bundle we compute the forms $s^* TP(\phi)$. This permits we us recover the main result of [12] from a somewhat different viewpoint.

Let $(B, Gl(n), G, \pi, \omega)$ be the bundle of bases over G with ω the connection associated to a G bi-invariant metric. To say a metric is bi-invariant under the action of G is the same as saying it is $G \times G$ invariant when considered as a metric on a symmetric space.

THEOREM 6.1. *For any $Q \in I(G)$ we have $Q(\Phi) = 0$.*

PROOF. Let $\mathfrak{G} \oplus \mathfrak{G}$ be the Lie algebra of $G \times G$. Define $\bar{Q} \in I(G \times G)$ by $\bar{Q}(x_1 \oplus x_2) = \frac{1}{2}(Q(x_1) + Q(x_2))$. Then \bar{Q} restricts to Q on G . The result then follows from Theorem 5.1.

COROLLARY 6.2. *For any $Q \in I(Gl(n))$ we have $Q(\Omega) = 0$.*

PROOF. Immediate from Theorem 6.1 and Theorem 4.5(i).

We now do some calculations in the defining bundle. Let us denote the section $(\bar{g}, \bar{e}) \rightarrow (g, e)$ by s .

$$\begin{array}{ccc} G & \longrightarrow & G \times G \\ & \searrow s & \downarrow \\ & & \frac{G \times G}{\Delta G} \end{array}$$

Let $(X_1, X_2) \in \mathfrak{G} \oplus \mathfrak{G}$, the Lie algebra of $G \times G$.

$$(X_1, X_2) = \frac{1}{2}((X_1 + X_2, X_1 + X_2) + (X_1 - X_2, X_2 - X_1))$$

exhibits a decomposition of (X_1, X_2) into \mathfrak{S} and \mathfrak{M} components.

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

Let ω_G denote the Maurer-Cartan form on G .

$$(s^* \phi)(X) = \frac{1}{2}X = \frac{1}{2}\omega_G(X),$$

$$(s^* \Phi)(X, Y) = -[\frac{1}{2}X, \frac{1}{2}Y] = -\frac{1}{8}[\omega_G, \omega_G](X, Y),$$

$$(s^* \gamma_1)(X, Y) = \frac{1}{8}[\omega_G, \omega_G](X, Y),$$

$$(s^* \gamma_2)(X, Y) = \frac{1}{8}[\omega_G, \omega_G](X, Y).$$

THEOREM 6.3. For any $Q \in I^k(G)$ we have

$$s^* TQ(\phi) = \frac{1}{2} \tau q,$$

$q \in H^{2k}(BG, R)$ is the class associated to Q . If Q is an integral polynomial, that is, q is in the image of the coefficient homomorphism

$$r: H^{2k}(BG, Z) \rightarrow H^{2k}(BG, R),$$

then $\overline{S_{Q,q}(\phi)} \frac{1}{2} \tau q$ where the bar denotes the coefficient homomorphism $H^{2k-1}(G, R) \rightarrow H^{2k-1}(G, R/Z)$.

PROOF.

$$\begin{aligned} s^* TQ(\phi) &= s^* \sum_{r=0}^{k-1} W_{k,r} Q(\phi, \overbrace{\gamma_1, \dots, \gamma_1}^r, \gamma_2) \\ &= \sum_{r=0}^{k-1} W_{k,r} \frac{1}{2} \left(\frac{1}{8} \right)^{k-1} Q(\omega_G, [\omega_G, \omega_G]) \\ &= \frac{1}{2} \left(-\frac{1}{2} \right)^{k-1} \frac{k! (k-1)!}{(2k-1)!} Q(\omega_G, [\omega_G, \omega_G]) = \frac{1}{2} \tau q. \end{aligned}$$

This shows the first part.

$$S_{Q,q}(\phi) = 1^* S_{Q,q}(\phi) = s^* p^* S_{Q,q}(\phi) = s^* \overline{TQ(\phi)} = \overline{s^* TQ(\phi)} = \overline{\frac{1}{2} \tau q},$$

where the third step uses Theorem 1.2.

COROLLARY 6.4. For any $Q \in I^k(Gl(n))$ and any section σ of the bundle of bases of G by left-invariant vector fields $\sigma^*(TQ(\omega)) = \text{Ad}^*(\frac{1}{2} \tau q)$ where $\text{Ad}: G \rightarrow Gl(n)$ is the adjoint representation. If Q is integral then $S_{Q,q}(\omega) = 0$.

PROOF. If $F: G \times G \rightarrow B$ as in (4) then $\sigma = Fs$ is a section of B . Further by proper choice of the vector fields X_1, \dots, X_n as in the beginning of (4) any section σ by left invariant vector fields is given by some such F . $\sigma^*(TQ(\omega)) = S^* F^*(TQ(\omega)) = S^*(T(i_s^* Q)(\phi))$ where the last step follows from Theorem 4.5 (ii). Now i_s may be identified with Ad in this case.

$$\sigma^*(TQ(\omega)) = s^*(T(\text{Ad}^* Q)(\phi)) = \frac{1}{2} \tau (\text{Ad}^* q) = \text{Ad}^* (\frac{1}{2} \tau q)$$

where the last step follows by naturality of τ .

For the second part by Theorem 4.5 (iii) $S_{Q,q}(\omega) = S_{\text{Ad}^* Q, \text{Ad}^* q}(\phi) = \frac{1}{2} \tau (\text{Ad}^* q)$ using Theorem 6.3. So $S_{Q,q}(\omega) = \text{Ad}^*(\frac{1}{2} \tau q)$. But for $Gl(n)$ $\frac{1}{2} \tau q$ is always integral [6] so $S_{Q,q}(\omega) = 0$ by the long exact sequence of coefficient homomorphisms corresponding to $0 \rightarrow Z \rightarrow R \rightarrow R/Z \rightarrow 0$.

Corollary 6.4 is the main result of [12].

7. Secondary classes on symmetric spaces. In this section we compute some secondary classes in the defining bundle over the compact irreducible Riemannian symmetric spaces fibered by the classical groups other than the groups themselves which are treated in [12] and in §6. Specifically we consider canonical generating polynomials $Q \in I(G)$. Then by Theorem 5.1, $T(i^*Q) \cdot (\phi)$ where $i: H \rightarrow G$ is the inclusion always defines a cohomology class in G corresponding to the defining bundle over G/H . The following proposition shows that it is enough to treat such a generating set.

PROPOSITION 7.1. *If $S = RQ$ is the product of G -invariant polynomials R and Q then $T(i^*S)(\phi)$ is the zero class.*

PROOF. By [11], $[T(i^*S)(\phi)] = [T(i^*R)(\phi)Q(\Phi)] = 0$ since $Q(\Phi) = 0$ by Theorem 5.1.

The list of spaces considered in this section is: $F_n = SO(2n)/U(n)$, $n > 2$; $G_n = Sp(n)/U(n)$, $n > 2$; the Grassmannian $U(p+q)/U(p) \times U(q)$; the Grassmannian $Sp(p+q)/Sp(p) \times Sp(q)$, $p+q > 2$; the Grassmannian $SO(p+q)/SO(p) \times SO(q)$, $p+q > 4$, treated in three cases depending upon the parity of p and q ; $SU(n)/SO(n)$ treated in three cases depending upon the parity of n ; $SU(2n)/Sp(n)$, $n > 1$.

Before we begin the computations some preliminary remarks about notation and the devices used are required.

If $T \subset G$ is a maximal torus of a compact Lie group then the map $j^*: H^*(B_G, R) \rightarrow H^*(B_T, R)$ is known to be injective [4]. In the following calculation we will identify elements of $H^*(B_G, R)$ with their images in $H^*(B_T, R)$.

If x_1, \dots, x_n are variables and $\sigma_j(x_1, \dots, x_n)$ denotes the j th elementary symmetric function in the x_i and S_j the sum of j th powers then [19]:

$$S_l - S_{l-1}\sigma_1 + S_{l-2}\sigma_2 + \dots + (-1)^{l-1}S_1\sigma_{l-1} + (-1)^l l\sigma_l = 0.$$

This fact will often be used in conjunction with Lemma 2.1 in the sequel.

For notation we have $i: H \rightarrow G$ the inclusion and one of the maps $H^*(B_G) \rightarrow H^*(B_H)$, $H^*(G) \rightarrow H^*(H)$, $I(G) \rightarrow I(H)$ depending on the context. For $Q \in I(G)$, $TQ(\phi)$ means $T(i^*Q)(\phi)$. All cohomology groups are understood to be with real coefficients.

(a) $F_n = SO(2n)/U(n)$, $n > 2$. Then if P_i and χ are the Pontryagin forms and Euler form we have $TP_i(\phi) = \tau p_i$ for $i \leq [n/2]$ and $T\chi(\theta) = \tau\chi$. Let $T \subset U(n) \subset SO(2n)$ where T is the standard maximal torus of $SO(2n)$. Now the generators of $H^*(SO(n))$ are the transgressions of the generators of $H^*(B_{SO(n)})$ and these may be identified with $p_j = \sigma_j(\theta_1^2, \dots, \theta_n^2)$, $\chi = \theta_1, \dots, \theta_n$ for $j = 1, \dots, n-1$ where θ_i are the generators of $H^*(B_T)$. The generators of $H^*(U(n))$ are the transgressions of the generators of $H^*(B_{U(n)})$ and

these may be identified with $c_j = \sigma_j(\theta_1, \dots, \theta_n)$, $j = 1, \dots, n$, where θ_i are again the generators of $H^*(B_T)$. Consider the following diagram:

$$\begin{array}{ccc} H^*(U(n)) & \xleftarrow{i^*} & H^*(SO(2n)) \\ \uparrow \tau & & \uparrow \tau \\ H^*(B_{U(n)}) & \xleftarrow{i^*} & H^*(B_{SO(2n)}) \end{array}$$

$$i^* \tau \chi = \tau i^* \chi = \tau c_n,$$

$$\begin{aligned} i^* \tau p_j &= \tau p_j = \tau \sigma_j(\theta_1^2, \dots, \theta_n^2) \\ &= \tau (-1)^{j+1/j} S_j(\theta_1^2, \dots, \theta_n^2) \\ &= \tau (-1)^{j+1/j} (-1) 2j \sigma_{2j}(\theta_1, \dots, \theta_n) = 2(-1)^j \tau c_{2j}. \end{aligned}$$

This shows that in dimensions less than or equal to $2n - 1$ restriction to the fiber is an injection on the cohomology level. Now $i^* T\chi(\theta) = \tau i^* \chi = \tau c_n \cdot i^* TP_j(\theta) = \tau i^* p_j = 2(-1)^j \tau c_{2j}$. Therefore $T\chi(\theta) = \tau \chi$, $TP_j(\theta) = \tau p_j$ for $j \leq [n/2]$.

(b) $G_n = \text{Sp}(n)/U(n)$, $n > 2$. Then if P_i is the canonical symplectic Pontryagin form we have $TP_i(\theta) = \tau p_i$ for $i \leq [n/2]$. Now $T \subset U(n) \subset \text{Sp}(n)$ the generators of $H^*(U(n))$ are as described in §2. The generators of $H^*(U(n))$ are the transgressions of the generators of $H^*(B_{\text{Sp}(n)})$ which may be identified with $p_j = \sigma_j(\theta_1^2, \dots, \theta_n^2)$, $j = 1, \dots, n$, where θ_i are the generators of $H^*(B_T)$. Consider the commutative diagram:

$$\begin{array}{ccc} H^*(U(n)) & \xleftarrow{i^*} & H^*(\text{Sp}(n)) \\ \uparrow \tau & & \uparrow \tau \\ H^*(B_{U(n)}) & \xleftarrow{i^*} & H^*(B_{\text{Sp}(n)}) \end{array}$$

$$\begin{aligned} i^* \tau p_j &= \tau i^* p_j = \tau \sigma_j(\theta_1^2, \dots, \theta_n^2) = \tau (-1)^{j+1} / j \sigma_j(\theta_1^2, \dots, \theta_n^2) \\ &= \tau (-1)^{j+1} / j (-1) (2j) \sigma_{2j}(\theta_1, \dots, \theta_n) = (-1)^j 2 \tau c_{2j}. \end{aligned}$$

This shows that in dimensions less than or equal to $2n - 1$ restriction to the fiber is an injection on the cohomology level. Now $i^* TP_i(\phi) = \tau i^* p_i = (-1)^i \tau c_{2i}$. Therefore $TP_i(\phi) = \tau p_i$ for $i \leq [N_2]$.

(c) The Grassmannian $U(p+q)/U(p) \times U(q)$. Then if C_r are the Chern forms on $U(p+q)$ we have $TC_r(\phi) = \tau c_p$ for $r \in \max(p, q)$. Let $T \subset U(p) \times U(q) \subset U(p+q)$ be the standard maximal torus. Then the generators of $H^*(U(p+q))$ are the transgressions of the C_r which may be identified with $\sigma_r(\theta_1, \dots, \theta_{p+q})$, $r = 1, \dots, p+q$. The generators of $H^*(U(p) \times U(q))$ are the transgressions of the $c'_j \times 1, 1 \times c''_k$ which may be identified with

$$\sigma_j(\theta_1, \dots, \theta_p) \times 1, \quad 1 \times \sigma_k(\theta_{p+1}, \dots, \theta_{p+q})$$

for $j = 1, \dots, p$ and $k = 1, \dots, q$. Consider the commutative diagram:

$$\begin{array}{ccc} H^*(U(p) \times U(q)) & \xleftarrow{i^*} & H^*(U(p+q)) \\ \uparrow \tau & & \uparrow \tau \\ H^*(B_{U(p) \times U(q)}) & \xleftarrow{i^*} & H^*(B_{U(p+q)}) \end{array}$$

$$\begin{aligned} i^* \tau c_r &= i^* \tau \sigma_r(\theta_1, \dots, \theta_{p+q}) = i^* \frac{(-1)^{r+1}}{r} \tau S_r(\theta_1, \dots, \theta_{p+q}) \\ &= \frac{(-1)^{r+1}}{r} (\tau S_r(\theta_1, \dots, \theta_p) \otimes 1 + 1 \otimes \tau S_r(\theta_{p+1}, \dots, \theta_{p+q})) \\ &= \frac{(-1)^{r+1}}{r} \frac{r}{(-1)^{r+1}} (\tau \sigma_r(\theta_1, \dots, \theta_p) \otimes 1 + 1 \otimes \sigma_r(\theta_{p+1}, \dots, \theta_{p+q})) \\ &= \tau c'_r \otimes 1 + 1 \otimes \tau c''_r. \end{aligned}$$

This shows that in dimensions less than or equal to $2 \max(p, q) - 1$ the restriction to the fiber is an injection of the cohomology level.

$$i^* TC_r(\phi) = \tau i^* c_r = \tau c'_r \otimes 1 + 1 \otimes \tau c''_r = i^* \tau c_r$$

so

$$TC_r(\phi) = \tau c_r \quad \text{for } r \leq \max(p, q).$$

(d) $\text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q)$, $p+q > 2$. Then if P_r are the symplectic Pontryagin forms on $\text{Sp}(p+q)$ we have $TP_r(\phi) = \tau p_r$ for $r \leq \max(p, q)$. Let $T \subset \text{Sp}(p) \times \text{Sp}(q) \subset \text{Sp}(p+q)$ be the standard maximal torus. The generators of $H^*(\text{Sp}(p+q))$ are τp_r where p_r may be identified with $\sigma_r(\theta_1^2, \dots, \theta_{p+q}^2)$. The generators of $H^*(\text{Sp}(p) \times \text{Sp}(q))$ are the transgressions of $p'_j \times 1, 1 \times p''_k$ which may be identified with $\sigma_j(\theta_1^2, \dots, \theta_p^2) \times 1, 1 \times \sigma_k(\theta_{p+1}^2, \dots, \theta_{p+q}^2)$, $j = 1, \dots, p$ and $k = 1, \dots, q$. Consider the commutative diagram:

$$\begin{array}{ccc}
H^*(\mathrm{Sp}(p) \times \mathrm{Sp}(q)) & \xleftarrow{i^*} & H^*(\mathrm{Sp}(p+q)) \\
\uparrow \tau & & \uparrow \tau \\
H^*(B_{\mathrm{Sp}(p)} \times B_{\mathrm{Sp}(q)}) & \xleftarrow{i^*} & H^*(B_{\mathrm{Sp}(p+q)})
\end{array}$$

$$\begin{aligned}
i^* \tau p_r &= i^* \tau \sigma_r(\theta_1^2, \dots, \theta_{p+q}^2) = i^* \frac{(-1)^{r+1}}{r} \tau S_r(\theta_1^2, \dots, \theta_{p+q}^2) \\
&= \frac{(-1)^{r+1}}{r} \tau (S_r(\theta_1^2, \dots, \theta_p^2) \otimes 1 + 1 \otimes S_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2)) \\
&= \tau \sigma_r(\theta_1^2, \dots, \theta_p^2) \otimes 1 + 1 \otimes \tau \sigma_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2) \\
&= \tau p'_r \otimes 1 + 1 \otimes \tau p''_r.
\end{aligned}$$

This shows that in dimensions less than or equal to $4 \max(p, q) - 1$ restriction to the fiber is an injection on the cohomology level. $i^* TP_r(\phi) = \tau i^* p'_r \times 1 + 1 \times \tau p''_r$. So $TP_r(\phi) = \tau p_r$ for $r \leq \max(p, q)$.

(e) $SO(2p+2q)/SO(2p) \times SO(2q)$, $p+q > 2$. Then if P_i are the Pontryagin forms on $SO(2p+2q)$ we have $TP_i(\phi) = \tau p_i$ for $i \leq \max(p-1, q-1)$. Let $T \subset SO(2p) \times SO(2q) \subset SO(2p+2q)$ be the standard maximal torus of $SO(2p+2q)$. The generators of $H^*(SO(2p+2q))$ are the transgressions of the generators of $H^*(B_{SO(2p+2q)})$ and these may be identified with $p_r = \sigma_r(\theta_1^2, \dots, \theta_{p+q}^2)$, $r = 1, \dots, p+q-1$ and $\chi = \theta_1, \dots, \theta_{p+q}$. The generators of $H^*(SO(2p) \times SO(2q))$ are the transgressions of the generators of $H^*(B_{SO(2p)} \times B_{SO(2q)})$ and may be identified with $p'_i \times 1 = \sigma_i(\theta_1^2, \dots, \theta_p^2) \times 1$, $\chi' = \theta_1, \dots, \theta_p \otimes 1$, $1 \times p''_j = 1 \times \sigma_j(\theta_{p+1}^2, \dots, \theta_{p+q}^2)$, $\chi'' = 1 \otimes \theta_{p+1}, \dots, \theta_{p+q}$, $i = 1, \dots, p-1$ and $j = 1, \dots, q-1$. Consider the diagram:

$$\begin{array}{ccc}
H^*(SO(2p) \times SO(2q)) & \xleftarrow{i^*} & H^*(SO(2p+2q)) \\
\uparrow \tau & & \uparrow \tau \\
H^*(B_{SO(2p)} \times B_{SO(2q)}) & \xleftarrow{i^*} & H^*(B_{SO(2p+2q)})
\end{array}$$

$$\begin{aligned}
i^* \tau p_r &= i^* \tau \sigma_r(\theta_1^2, \dots, \theta_{p+q}^2) \\
&= i^* \tau (-1)^{r+1} / r S_r(\theta_1^2, \dots, \theta_{p+q}^2) \\
&= \tau (-1)^{r+1} / r (S_r(\theta_1^2, \dots, \theta_p^2) \times 1 + 1 \times S_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2)) \\
&= \tau \sigma_r(\theta_1^2, \dots, \theta_p^2) \times 1 + 1 \times \tau \sigma_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2) \\
&= \tau p'_r \times 1 + 1 \times \tau p''_r.
\end{aligned}$$

So restriction to the fiber is an injection in dimensions $\leq 4 \max(p-1, q-1) - 1$. $i^* TP_r(\phi) = i^* p_r = \tau p'_r \times 1 + 1 \times \tau p''_r$. Therefore $TP_r(\phi) = \tau p_r$ for $r \leq \max(p-1, q-1)$.

(f) $SO(2p+2q+1)/SO(2p) \times SO(2q+1)$, $p+q > 2$. Then if P_i are the Pontryagin forms on $SO(2p+2q+1)$, $TP_i(\phi) = \tau p_i$, $i \leq \max(p-1, q)$. Let $T \subset SO(2p) \times SO(2q+1) \subset SO(2p+2q+1)$ be the standard maximal torus. The generators of $H^*(SO(2p+2q+1))$ are the transgressions of the canonical Pontryagin forms which may be identified with $P_r = \sigma_r(\theta_1^2, \dots, \theta_{p+q}^2)$, $r = 1, \dots, p+q$. The generators of $H^*(SO(2p) \times SO(2q+1))$ are the transgressions of the generators of $H^*(B_{SO(2p) \times SO(2q+1)})$ which may be identified with $p'_j \times 1 = \sigma_j(\theta_1^2, \dots, \theta_p^2) \times 1$, $\chi \times 1 = \theta_1, \dots, \theta_p \times 1$, $1 \times p''_k = 1 \times \sigma_k(\theta_1^2, \dots, \theta_{p+q}^2)$, $j = 1, \dots, p-1$ and $k = 1, \dots, q$. Consider the commutative diagram:

$$\begin{array}{ccc}
 H^*(SO(2p) \times SO(2q)) & \xleftarrow{i^*} & H^*(SO(2p+2q)) \\
 \uparrow \tau & & \\
 H^*(B_{SO(2p) \times SO(2q)}) & \xleftarrow{i^*} & H^*(B_{SO(2p+q)})
 \end{array}$$

$$\begin{aligned}
 i^* \tau p_r &= i^* \tau \sigma_r(\theta_1^2, \dots, \theta_{p+q}^2) \\
 &= i^* \tau \frac{(-1)^{r+1}}{r} S_r(\theta_1^2, \dots, \theta_{p+q}^2) \\
 &= \tau \frac{(-1)^{r+1}}{r} (S_r(\theta_1^2, \dots, \theta_p^2) \times 1 + 1 \times S_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2)) \\
 &= \tau (\sigma_r(\theta_1^2, \dots, \theta_p^2) \times 1 + 1 \times \sigma_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2)) \\
 &= \tau p'_r \times 1 + 1 \times \tau p''_r.
 \end{aligned}$$

So restriction to the fiber is an injection in dimensions $\leq 4 \max(p-1, q) - 1$. $i^* TP_r(\theta) = \tau i^* p_r = \tau p'_r \times 1 + 1 \times \tau p''_r$. Therefore $TP_r(\theta) = \tau p_r$ for $r \leq \max(p-1, q)$.

(g) $SO(2p+2q+2)/SO(2p+1) \times SO(2q+1)$, $p+q > 1$. Then if P_i are the Pontryagin forms on $SO(2p+2q+2)$ we have $TP_i(\phi) = \tau p_i$ for $i \leq \max(p, q)$. Let $T' \times T'' \subset T$ be standard maximal tori of $SO(2p+1) \times SO(2q+1)$ and $SO(2p+2q+2)$ respectively. The generators of $H^*(SO(2p+2q+2))$ are the transgressions of the generators of $H^*(B_{SO(2p+2q+2)})$ which may be identified with $p_j = \sigma_j(\theta_1^2, \dots, \theta_{p+q+1}^2)$, $\chi = \theta_1, \dots, \theta_{p+q}$, $j = 1, \dots, p+q$. Furthermore, the generators of

$H^*(SO(2p+1) \times SO(2q+1))$ are the transgressions of the generators of $H^*(B_{SO(2p+1) \times SO(2q+1)})$ which may be identified with $p'_j \times 1 = \sigma_j(\theta_1^2, \dots, \theta_p^2) \times 1$, $1 \times p''_k = 1 \times \sigma_k(\theta_{p+1}^2, \dots, \theta_{p+q}^2)$, $j = 1, \dots, p$ and $k = 1, \dots, q$. Consider the commutative diagram:

$$\begin{array}{ccc}
 H^*(SO(2p+1) \times SO(2q+1)) & \xleftarrow{i^*} & H^*(SO(2p+2q+2)) \\
 \uparrow & & \uparrow \\
 H^*(B_{SO(2p+1) \times SO(2q+1)}) & \xleftarrow{i^*} & H^*(B_{SO(2p+2q+2)})
 \end{array}$$

$$\begin{aligned}
 i^* \tau p_r &= i^* \tau \sigma_r(\theta_1^2, \dots, \theta_{p+q+1}^2) = i^* \tau \frac{(-1)^{r+1}}{r} S_r(\theta_1^2, \dots, \theta_{p+q+1}^2) \\
 &= \tau \frac{(-1)^{r+1}}{r} (S_r(\theta_1^2, \dots, \theta_p^2) \times 1 + 1 \times S_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2)) \\
 &= \tau(\sigma_r(\theta_1^2, \dots, \theta_p^2) \times 1 + 1 \times \sigma_r(\theta_{p+1}^2, \dots, \theta_{p+q}^2)) \\
 &= \tau p'_r \times 1 + 1 \times \tau p''_r.
 \end{aligned}$$

So restriction to the fiber is an injection in dimensions $\leq 4 \max(p, q) - 1$. $i^* TP_r(\theta) = \tau i^* p_r = \tau p'_r \times 1 + 1 \times \tau p''_r$. Therefore $TP_r(\phi) = \tau p_r$ for $r \leq \max(p, q)$.

(h) $SU(2k+1)/SO(2k+1)$. Then if C_j denote the Chern forms on $SU(2k+1)TC_{2j}(\phi)$ has component τc_{2j} in the ring generated by the τc_{2i} for $j \leq k$. $H^*(SU(2k+1))$ is generated by the transgressions of the generators of $H^*(B_{SU(2k+1)})$ which may be identified with $c_j = \sigma_j(\delta_1, \dots, \delta_{2k}, -\sum \delta_i)$, $2 \leq j \leq 2k+1$. $H^*(SO(2k+1))$ is generated by the transgressions of the generators of $H^*(B_{SO(2k+1)})$ which may be identified with $p_j(\theta_1^2, \dots, \theta_k^2)$, $j = 1, \dots, k$. If $T' \subset T$ are maximal tori of $SO(2k+1)$, $SU(2k+1)$ respectively then the restriction map on cohomology is given by $\delta_i \rightarrow \theta_i$, $1 \leq i \leq k$; $\delta_i \rightarrow -\theta_i$, $k+1 \leq i \leq 2k$. Now consider the diagram:

$$\begin{array}{ccc}
 H^*(SO(2k+1)) & \xleftarrow{i^*} & H^*(SO(2k+1)) \\
 \uparrow \tau & & \uparrow \tau \\
 H^*(B_{SO(2k+1)}) & \xleftarrow{i^*} & H^*(B_{SU(2k+1)})
 \end{array}$$

$$\begin{aligned}
i^* \tau c_{2j} &= i^* \tau \frac{(-1)^{2j+1}}{2j} S_{2j}(\delta_1, \dots, \delta_{2k}, -\sum \delta_i) \\
&= \tau \frac{(-1)^{2j+1}}{j} S_j(\theta_1^2, \dots, \theta_k^2) \\
&= \tau(-1)^j \sigma_j(\theta_1^2, \dots, \theta_k^2) = (-1)^j \tau p_j.
\end{aligned}$$

Therefore restriction to the fiber is an injection restricted to the ring generated by the τc_{2i} in dimensions $4j - 1$ for $j \leq k$.

$$i^* TC_{2j}(\phi) = \tau i^* c_{2j} = (-1)^j \tau p_j.$$

Therefore $TC_{2j}(\phi)$ has component τc_{2j} for $j \leq k$.

(i) $SU(2k)/SO(2k)$. Then if C_{2j} denote the canonical Chern forms of $SU(2k)$, $TC_{2j}(\phi)$ has component τc_{2j} , $j \leq k - 1$, in the ring generated by the transgressions of the generators of $H^*(B_{SU(2k)})$ which may be identified with $c_j = \sigma_j(\delta_1, \dots, \delta_{2k-1}, -\sum \delta_i)$, $j = 2, \dots, 2k$. $H^*(SO(2k))$ is generated by the transgressions of the generators of $H^*(B_{SO(2k)})$ which may be identified with $p_j = \sigma_j(\theta_1^2, \dots, \theta_k^2)$, $j = 1, \dots, k - 1$ and $\chi = \theta_1, \dots, \theta_k$. If $T' \subset T$ are maximal tori of $SO(2k)$, $SU(2k)$ respectively then the restriction map on cohomology is given by $\delta_i \rightarrow \theta_i$, $i = 1, \dots, k$, and $\delta_i \rightarrow -\theta_i$, $i = k + 1, \dots, 2k - 1$. Now consider the diagram:

$$\begin{array}{ccc}
H^*(SO(2k)) & \xleftarrow{i^*} & H^*(SU(2k)) \\
\uparrow \tau & & \uparrow \tau \\
H^*(B_{SO(2k)}) & \xleftarrow{i^*} & H^*(B_{SU(2k)})
\end{array}$$

$$\begin{aligned}
i^* \tau c_{2j} &= i^* \tau \frac{(-1)^{2j+1}}{2j} S_{2j}(\delta_1, \dots, \delta_{2k-1}, -\sum \delta_i) \\
&= \tau(-1)^{2j+1} / j S_j(\theta_1^2, \dots, \theta_k^2) = (-1)^j \tau \sigma_j(\theta_1^2, \dots, \theta_k^2) \\
&= (-1)^j \tau p_j.
\end{aligned}$$

Therefore restriction to the fiber is an injection restricted to the ring generated by the τc_{2i} in dimensions $4j - 1$ for $j \leq k - 1$. $i^* TC_{2j}(\phi) = \tau i^* c_{2j} = (-1)^j \tau p_j$. Therefore $TC_{2j}(\phi)$ has component τc_{2i} in the ring generated by the τc_{2i} for $j \leq k - 1$.

(j) $SU(2k)/Sp(k)$, $k > 1$. If C_{2j} denotes the canonical Chern form on $SU(2k)$ $TC_{2j}(\phi)$ has component τc_{2j} in the ring generated by the τc_{2i} for j

$\leq k$. The generators of $H^*(SU(2k))$ are the transgressions of the generators of $H^*(B_{SU(2k)})$ which may be identified with $c_j = \sigma_j(\delta_1, \dots, \delta_{2k-1}, -\sum \delta_i)$, $j = 2, \dots, 2k$. The generators of $H^*(Sp(k))$ are the transgressions of the generators of $H^*(B_{Sp(k)})$ and these may be identified with $p_j = \sigma_j(\theta_1^2, \dots, \theta_k^2)$, $j = 1, \dots, k$. If $T' \subset T$ are respectively the maximal tori of $Sp(k)$, $SU(2k)$ then the restriction map on cohomology is given by $\delta_i \rightarrow \theta_i$, $i = 1, \dots, k$, and $\delta_i \rightarrow -\theta_i$, $i = 1, \dots, 2k - 1$. Consider the commutative diagram:

$$\begin{array}{ccc} H^*(SU(2k)) & \xleftarrow{i^*} & H^*(Sp(k)) \\ \uparrow \tau & & \uparrow \tau \\ H^*(B_{SU(2k)}) & \xleftarrow{i^*} & H^*(B_{Sp(k)}) \end{array}$$

$$\begin{aligned} i^* \tau c_{2j} &= i^* \tau \sigma_{2j}(\delta_1, \dots, \delta_{2k-1}, -\sum \delta_i) \\ &= i^* \tau \frac{(-1)^{2j+1}}{2j} S_{2j}(\delta_1, \dots, \delta_{2k-1}, -\sum \delta_i) \\ &= \tau \frac{(-1)^{2j+1}}{j} S_j(\theta_1^2, \dots, \theta_k^2) = \tau(-1)^j \sigma_j(\theta_1^2, \dots, \theta_k^2) \\ &= (-1)^j \tau p_j. \end{aligned}$$

Therefore restriction to the fiber is an injection for elements in the ring generated by the τc_{2i} for $i \leq k$. $i^* TC_{2j}(\phi) = \tau i^* c_{2j} = i^* \tau c_{2j} = (-1)^j / 2 \tau p_j$. Therefore $TC_{2j}(\phi)$ has component τc_{2j} in the ring generated by the τc_{2i} .

Our results are summarized in the table below:

$F_n = \frac{SO(2n)}{U(n)}, n > 2$	$TP_i(\phi) = \tau p_i$ $Tx(\phi) = \tau x$	$i \leq [n/2]$
$G_n = \frac{Sp(n)}{U(n)}, n > 2$	$TP_i(\phi) = \tau p_i$	$i \leq [n/2]$
$U(p+q)/[U(p) \times U(q)]$	$TC_r(\phi) = \tau c_r$	$r \leq \max(p, q)$
$Sp(p+q)/[Sp(p) \times Sp(q)]$	$TP_r(\phi) = \tau p_r$	$r \leq \max(p, q)$
$SO(p+q)/SO(p) \times SO(q)$	$TP_r(\phi) = \tau p_r$	$r \leq \max\left(\left[\frac{p+1}{2}\right] - 1, \left[\frac{q+1}{2}\right] - 1\right)$
$\frac{SU(n)}{SO(n)}$	$TC_{2j}(\phi)$ has component τc_{2j} in the ring generated by τc_{2i}	$j \leq \left[\frac{n+1}{2}\right] - 1$
$\frac{SU(2n)}{Sp(n)}$	TC_{2i} has component τc_{2j} in the ring generated by τc_{2i}	$j \leq n$

8. Chern-Simons invariants on symmetric spaces. The Chern-Simons invariants obstructing conformal immersions are by Theorems 1.1 and 4.5 associated to the polynomials $i_s^* p_i^\perp$. That is, a necessary condition that G/H admit a conformal immersion in R^{n+k} is that $i_s^* p_i^\perp(\Phi) = 0$ and $\frac{1}{2}T(i_s^* p_i^\perp)(\phi) \in H^{4i-1}(G/H, \mathbb{Z})$, $i > [k/2]$, where $n = \dim(G/H)$. Of the spaces considered in §7 in most cases $i_s^* p_i^\perp(\Phi) \neq 0$ and this may be deduced from the fact that the associated characteristic class does not vanish [5].

The cases where necessarily $i_s^* p_i^\perp(\Phi) = 0$ are considered below. In all spaces G/H the elements $i_s^* p_i^\perp$ are the restrictions of elements of $I(G)$. The spaces considered below are: $SU(k)$; $SU(n)/SO(n)$, with two calculations depending upon the parity of n ; $SU(2k)/Sp(k)$; $S^n = SO(n+1)/SO(n)$. On $SU(2k+1)$ and $SU(2k+1)/SO(2k+1)$ we obtain nonintegral invariants and consequently results on conformal immersions.

The calculations are based on the following result of [5]:

THEOREM 8.1. *Let $\rho: G \rightarrow GL(n)$ be a representation of a compact Lie group G and ρ^* the induced map on the cohomology of classifying spaces. Identifying the elements of $H^*(B_G)$ with their images in $H^*(B_T)$ where T is a maximal torus we have:*

$$\rho^*(p) = \prod_j (1 + \omega_j)$$

where p is the Pontryagin polynomial and ω_j are the weights of the representation.

(a) $SU(k)$. The weights of the adjoint representation are $\theta_i - \theta_j$, $1 \leq i \leq j \leq k$, where we define $\theta_k = -\sum_{i=1}^{k-1} \theta_i$.

$$\begin{aligned} \tau \operatorname{Ad}^* p_l^\perp &= -\tau \operatorname{Ad}^* p_l = -\tau \sigma_l((\theta_i - \theta_j)^2) \\ &= -\tau \frac{(-1)^{l+1}}{l} S_l((\theta_i - \theta_j)^2) \\ &= \frac{(-1)^l}{2l} \tau \left(\sum_{i,j} \theta_i^{2l} - 2l \theta_i^{2l-1} \theta_j + \cdots + \theta_j^{2l} \right) \\ &= \frac{(-1)^l}{l} \tau k S_{2l}(\theta_i) = \frac{(-1)^l}{l} 2kl(-1)^{2l+1} \tau \sigma_{2l}(\theta_i) \\ &= 2(-1)^{l+1} k \tau c_{2l}. \end{aligned}$$

Let σ be a section of the bundle of bases by left invariant vector fields. Corollary 6.4 gives $\sigma^* \frac{1}{2} T p_l^\perp(\theta) = \frac{1}{4} \tau \operatorname{Ad}^* p_l^\perp = \frac{1}{2} (-1)^{l+1} k \tau c_{2l}$. Since τc_{2l}

generate a direct summand in $H^*(SU(k), \mathbb{Z})$ [4] the Chern-Simons invariants do not vanish for k odd.

THEOREM 8.2. *$SU(2k + 1)$ with bi-invariant metric does not conformally immerse in Euclidean space of codimension $2k - 1$.*

PROOF. Apply Theorems 1.1, 4.5 and the calculation above.

(b) $SU(2k)/SO(2k)$. Every $SO(2k)$ invariant polynomial in the ring generated by the Pontryagin polynomials is the restriction of an $SU(2k)$ invariant polynomial. The weights of the isotropy representation are $2\theta_i, \theta_i - \theta_j, \theta_i + \theta_j, 1 \leq i \leq j \leq k$. $i_s^* p_l = \sigma_l(4\theta_i^2, (\theta_i - \theta_j)^2, (\theta_i + \theta_j)^2)$. This is seen to lie in the ring generated by the Pontryagin forms by an easy induction argument using the formula of (7) relating the S_i and σ_i .

$$\begin{aligned}
 \tau i_s^* p_l^\perp &= -\tau i_s^* p_l = -\tau \sigma_l(4\theta_i^2, (\theta_i - \theta_j)^2, (\theta_i + \theta_j)^2) \\
 &= -\tau \frac{(-1)^{l+1}}{l} S_l(4\theta_i^2, (\theta_i - \theta_j)^2, (\theta_i + \theta_j)^2) \\
 &= \frac{(-1)^l}{l} \tau \left(\frac{1}{2} \sum 2^{2l} \theta_i^{2l} + \frac{1}{2} \sum (\theta_i - \theta_j)^{2l} + (\theta_i + \theta_j)^{2l} \right) \\
 &= -\tau(2^{2l-1} \sigma_l(\theta_i^2)) \\
 &\quad - \frac{1}{2} \frac{(-1)^{l+1}}{l} \tau \left(\sum \theta_i^{2l} - 2l \theta_i^{2l-1} \theta_j + \dots \right. \\
 &\quad \left. + \theta_j^{2l} + \theta_i^{2l} + 2l \theta_i^{2l-1} \theta_j + \dots + \theta_j^{2l} \right) \\
 &= -2^{2l-1} \tau \sigma_l(\theta_i^2) - \frac{(-1)^{l+1}}{l} \left(\sum \theta_i^{2l} + \binom{2l}{2} \theta_i^{2l-2} \theta_j^2 + \dots + \theta_j^{2l} \right) \\
 &= -2^{2l-1} \tau \sigma_l(\theta_i^2) - \frac{(-1)^{l+1}}{l} 2k \tau S_l(\theta_i^2) \\
 &= -2^{2l-1} \tau p_l - 2k \tau p_l \\
 &= -(2^{2l-1} + 2k) \tau p_l.
 \end{aligned}$$

Now τc_{2l} restricts to $(-1)^l \tau p_l$. Thus the component of $\frac{1}{2} \tau i_s^* p_l^\perp(\theta)$ in the τc_{2l} summand is $(-1)^{l+1} (2^{2l-2} + k) \tau c_{2l}$. This is always integral.

(c) $SU(2k + 1)/SO(2k + 1)$. Every $SO(2k + 1)$ invariant polynomial is the restriction of an $SU(2k + 1)$ polynomial. Therefore the Chern-Simons invariants are well defined. The weights of the isotropy representation are $\theta_i - \theta_j, \theta_i + \theta_j, \theta_i, 2\theta_i, 1 \leq i < j \leq k$.

$$\begin{aligned}
\tau i_s^* p_l &= -\tau i_s^* p_l^\perp = -\tau \sigma_l(4\theta_i^2, \theta_i^2, (\theta_i + \theta_j)^2, (\theta_i - \theta_j)^2) \\
&= \frac{(-1)^l}{l} \tau S_l(4\theta_i^2, \theta_i^2, (\theta_i + \theta_j)^2, (\theta_i - \theta_j)^2) \\
&= \frac{(-1)^l}{l} \tau \left(\sum \theta_i^{2l} + \frac{1}{2} \sum ((\theta_i + \theta_j)^{2l} + (\theta_i - \theta_j)^{2l}) + \frac{1}{2} \sum (2\theta_i)^{2l} \right) \\
&= \frac{(-1)^l}{l} \tau \left((1 + 2^{2l-1}) S_l(\theta_i^2) + \sum \left(\theta_i^{2l} + \binom{2l}{2} \theta_i^{2l-2} \theta_j^2 + \dots + \theta_j^{2l} \right) \right) \\
&= -\tau \sigma_l(\theta_i^2)(1 + 2^{2l-1}) + \frac{(-1)^l}{l} 2k \tau S_l(\theta_i^2) \\
&= -(1 + 2^{2l-1}) \tau p_l - 2k \tau p_l \\
&= -(+2k + 1 + 2^{2l-1}) \tau p_l.
\end{aligned}$$

Now the τc_{2l} restrict to $(-1)^l \tau p_l$ on the fiber. Thus the component of $\frac{1}{2} T(i_s^* p_l^\perp)(\phi)$ in the τc_{2l} direction is $\frac{1}{2} (-1)^{l+1} (2k + 1 + 2^{2l-1}) \tau c_{2l}$. Since τc_{2l} generates a direct summand in $H^*(SU(2k + 1), \mathbb{Z})$ [4] this is not integral.

THEOREM 8.3. $SU(2k + 1)/SO(2k + 1)$ with $SU(2k + 1)$ invariant metric does not conformally immerse in codimension $2k - 1$.

PROOF. Apply calculation above and Theorems 1.1, 4.5.

(d) $SU(2k)/Sp(k)$, $k > 1$. Every $Sp(k)$ invariant polynomial is the restriction of an $SU(2k)$ invariant polynomial. Thus the Chern-Simons invariant are defined. The weights of the isotropy representation are $\theta_i - \theta_j$, $\theta_i + \theta_j$, $1 \leq i < j \leq k$.

$$\begin{aligned}
\tau i_s^* p_l^\perp &= -\tau i_s^* p_l = -\tau i_s^* p_l = -\tau \sigma_l((\theta_i - \theta_j)^2, (\theta_i + \theta_j)^2) \\
&= \frac{(-1)^l}{l} S_l((\theta_i - \theta_j)^2, (\theta_i + \theta_j)^2) \\
&= \tau \frac{(-1)^l}{l} \left(\frac{1}{2} \sum ((\theta_i - \theta_j)^{2l} + (\theta_i + \theta_j)^{2l}) - \frac{1}{2} \sum (2\theta_i)^{2l} \right) \\
&= \tau \frac{(-1)^l}{l} \left(\sum \theta_i^{2l} + \binom{2l}{2} \theta_i^{2l-2} \theta_j^2 + \dots + \theta_j^{2l} - 2^{2l-1} \sum \theta_i^{2l} \right) \\
&= \frac{(-1)^{l+1}}{l} 2^{2l-1} \tau S_l(\theta_i^2) + \tau \frac{(-1)^l}{l} (2k) S_l(\theta_i^2) \\
&= (-2k + 2^{2l-1}) \tau \sigma_l(\theta_i^2) = (-2k + 2^{2l-1}) \tau p_l.
\end{aligned}$$

Now τc_{2l} restricts to $(-1)^l \tau p_l$ on the fiber. Thus the component of $\frac{1}{2} T(i_s^* p_l^\perp) \cdot (\phi)$ in the τc_{2l} direction is $(-1)^l (-k + 2^{2l-2}) c_{2l}$. This is always integral.

(e) $S_n = SO(n+1)/SO(n)$, $n \geq 3$. The isotropy representation $i_s^*: SO(n) \rightarrow GL(n)$ is the canonical injection. Thus $i_s^* p_l^\perp = p_l^\perp$, $\frac{1}{2} T p_l^\perp(\phi) = -\frac{1}{2} \tau p_l$. By [6] this is a generator for the integral cohomology of $SO(n+1)$. This is to be expected since S^n isometrically embeds in R^{n+1} .

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