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ON THE INTEGRABLE AND SQUARE-INTEGRABLE REPRESENTATIONS OF Spin(1, 2m)

BY

ERNEST THIELEKER[1]

ABSTRACT. All the unitary equivalence classes of irreducible integrable and square-integrable representations of the groups Spin$(1,2m)$, $m \geqslant 2$, are determined. The method makes use of some elementary results on differential equations and the classification of irreducible unitary representations of these groups. In the latter classification, certain ambiguities resulting from possible equivalences not taken into account in a previous paper, are cleared up here.

1. Introduction. For a number of real semisimple Lie groups G, it is possible to classify the irreducible unitary representations of G by determining which of the irreducible components of the various (nonunitary) principal series representations of G can be made unitary by means of a redefined inner product. See for example [4], [12] and [16]. Let K be a maximal compact subgroup of G. It is a celebrated result of Harish-Chandra that G possesses a discrete series of unitary representations if and only if rank (G) = rank (K). (See [9, Theorem 13].) By definition, an irreducible unitary representation of G belongs to the discrete series if its matrix elements are square-integrable with respect to Haar measure. An important problem for several applications is to determine which of the irreducible unitary representations of G actually belong to the discrete series. In this paper this problem is solved for the cases G = Spin $(1,2m)$, for $m \geqslant 2$. The main result in this direction is Theorem 6, which confirms a conjecture made in Thieleker [16]. This theorem also determines the integrable representations as well as the square-integrable ones for these groups. In the special case when $m = 2$, Spin $(1,4)$ is isomorphic to the universal covering group of the deSitter group, and for this case our results can easily be deduced from those of Dixmier in [4]. See the comments in Thieleker [15, §13]. It would be inconvenient for our purposes to include the case $m = 1$. Thus it is omitted. However, since Spin$(1,2)$ is isomorphic to

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SL_2 (**R**), the results for this case are well known. See for example the book by Lang [14].

Before describing the contents of the paper in more detail, we indicate our notation and review some known facts. As is customary, let **C**, **R**, and **Z** denote the set of complex numbers, real numbers, and integers respectively. If L is a Lie group, **L** will denote the Lie algebra of L, $\mathbf{L}_\mathbf{C}$ will denote the complexification $\mathbf{L}_\mathbf{C} = \mathbf{C} \otimes \mathbf{L}$, and $U(\mathbf{L})$ will denote the complex universal enveloping algebra of **L**. If $[\Pi, \mathcal{H}]$ is a continuous representation of L on a Banach space $\mathcal{H}$, let $\mathcal{H}^\infty$ denote the linear space of differentiable vectors in $\mathcal{H}$. Then there is a uniquely determined representation of $U(\mathbf{L})$ on H^∞ which we denote by $[d\Pi, \mathcal{H}^\infty]$. As in [6] a quasi-simple representation of L is one in which the operators $d\Pi(z)$ and $\Pi(z)$ act as scalar multiplication for every z in the center of $U(\mathbf{L})$ and z in the center of L. If K is a compact group, we denote by $\Omega(K)$ the set of all equivalence classes of finite-dimensional irreducible representations of K, or equivalently, the set of classes of irreducible unitary representations of K. If, moreover, L is a closed subgroup of K, and $[\mu] \in \Omega(L)$, we denote by $\Omega_\mu(K)$ the set of μ-admissible classes in $\Omega(K)$, that is, the set of all classes in $\Omega(K)$ which contain the class $[\mu]$ in their restrictions to L.

Now suppose that G is a semisimple Lie group with a finite center, and suppose that K is a maximal compact subgroup of G. Let $[\Pi, \mathcal{H}]$ be a quasi-simple representation of G. The main technical problem in determining whether Π is square-integrable or not is to determine the asymptotic behaviour of certain matrix elements of $\Pi(g)$ as g approaches the boundary of G. Let $[\omega]$ and $[\omega']$ be elements of $\Omega(K)$ and let Z_G be a Casimir element of G, that is a second order central element in $U(\mathbf{G})$. Then, in the language of Harish-Chandra [9], the function $g \to E(\omega')\Pi(g)E(\omega)$ is an elementary, (ω', ω)-spherical function and satisfies a certain differential equation arising from the fact that $d\Pi(Z_G)$ acts as scalar multiplication. (If $[\omega] \in \Omega(K)$, then $E(\omega)$ is the projection on $\mathcal{H}$ corresponding to the type ω K-isotypic component.) When Π is a class 1 representation, a derivation of this differential equation is given by Harish-Chandra in [8]. Harish-Chandra's generalization to the case of general Π is given in Warner [18, vol. II]. It should be remarked that when Π is a nonunitary principal series representation of G, then the function $g \to E(\omega')\Pi(g)E(\omega)$ is essentially what Harish-Chandra calls an Eisenstein integral. (See [10] or [17].) For the case when the real rank of G is 1, the differential equation referred to above reduces to an ordinary differential equation with operator coefficients. (See §2 or [17].) If, moreover, $G = \text{Spin}(1, 2m)$, it is possible to choose the pair $[\omega]$, $[\omega']$ in such a manner that this equation reduces to one with scalar coefficients which may be solved in terms of hypergeometric functions, with a suitable change of variables. We call such a pair of K-types a locking pair. (See §5.) It is interesting to note that such a locking pair of K-types does not exist in general for the nonunitary principal series representations of Spin$(1, 2m + 1)$, but they do exist for most nonunitary principal series representations of the other real rank 1 groups. This fact will be discussed in a later paper. It turns out that when $G = \text{Spin}(1, 2m)$, that a pair of locking K-types exists not only for every nonunitary principal series representation of G, but also for those irreducible components that correspond to the square-integrable representations. However, such locking pairs do not exist for all irreducible components of nonunitary principal series representations. This circumstance makes it necessary to resort to additional arguments based on the integral representation of the Harish-Chandra c-function to exclude these "extraneous" irreducible components from the possible square-integrable representations. These arguments are given in §13. In §§10 and 11 we review the classification of the quasi-simple irreducible representations and the irreducible unitary representations of G. In [16] there were some ambiguities in this classification due to certain infinitesimal equivalences which were not taken into account in [15] and [16]. These additional equivalences are given correctly by Gavrilik and Klimyk [5]. In §10 we give a somewhat more condensed reformulation of these results. We thank these authors for making a preprint of their paper available to us.

2. The differential equation. Assume at this point that G is a real semisimple connected Lie group of arbitrary split rank l. Let $\mathbf{G} = \mathbf{P} + \mathbf{K}$ be a Cartan decomposition of $\mathbf{G}$, where $\mathbf{K}$ is the Lie algebra of a subgroup K such that Ad (K) is maximal compact in Ad (G), and $\mathbf{P}$ is a Cartan subspace of $\mathbf{G}$ corresponding to this choice of K. Let θ be the Cartan involution corresponding to this decomposition. Thus, θ fixes every element in $\mathbf{K}$ and reverses sign of every element in $\mathbf{P}$. It is known that θ extends uniquely to a compact involution θ' of the complexification $\mathbf{G}_\mathbf{C}$ of $\mathbf{G}$. Thus, θ' fixes a compact real form $\mathbf{G}_u$ elementwise. Let $\mathbf{A}$ be a maximal abelian subspace of $\mathbf{P}$. Fix a lexicographical ordering on $\mathbf{A}^*$, the real dual of $\mathbf{A}$. Let Δ be the set of restricted positive roots, where positive is defined by this ordering. Finally, let $\langle\ ,\ \rangle$ be the sesquilinear form on $\mathbf{G}_\mathbf{C}$ defined by the formula

$$\langle X, Y\rangle = -cB(X, \theta' Y),$$

for X, $Y \in \mathbf{G}_\mathbf{C}$, where B is the Killing form on $\mathbf{G}_\mathbf{C}$ and c is a positive constant to be adjusted later. Since B is negative definite on $\mathbf{G}_u$, it follows easily that the form $\langle\ ,\ \rangle$ is an inner product. It also follows easily from the invariance of the Killing form that we have, for all X, Y and Z in $\mathbf{G}_\mathbf{C}$,

$$\langle [X, Y], Z\rangle = -\langle Y, [\theta' X, Z]\rangle.$$

For $\alpha \in \Delta$ let $\mathbf{N}_\alpha$ be the subspace of G defined by $\mathbf{N}_\alpha = \{Z \in \mathbf{G} | \mathrm{ad}\,(H)Z = \alpha(H)Z$ for all $H \in \mathbf{A}\}$. Then, for $\alpha \neq \beta$, $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ are mutually orthogonal relative to the inner product $\langle\ ,\ \rangle$. Moreover, as α ranges over Δ, the subspaces $\mathbf{N}_\alpha$ generate a maximal nilpotent subalgebra $\mathbf{N}$. Let $m(\alpha)$ be the multiplicity of the root $\alpha \in \Delta$, and let $\{Z_{\alpha i} | i = 1, 2, \ldots, m(\alpha)\}$ be an orthonormal basis of $\mathbf{N}_\alpha$. Then the set $\{\theta Z_{\alpha i} | i = 1, 2, \ldots, m(\alpha)\}$ is an orthonormal basis of $\theta\mathbf{N}_\alpha$. For each $\alpha \in \Delta$ and index $i = 1, 2, \ldots, m(\alpha)$ define elements $Y_{\alpha i}$ and $X_{\alpha i}$ by the formulas

$$X_{\alpha i} = (1/\sqrt{2})(Z_{\alpha i} + \theta Z_{\alpha i}), \qquad Y_{\alpha i} = (1/\sqrt{2})(Z_{\alpha i} - \theta Z_{\alpha i}). \tag{1}$$

Then since obviously $\theta X_{\alpha i} = X_{\alpha i}$ and $\theta Y_{\alpha i} = -Y_{\alpha i}$, we have $X_{\alpha i} \in \mathbf{K}$, and $Y_{\alpha i} \in \mathbf{P}$. For $\alpha \in \Delta$, let H_α be the element of A defined by $\langle H_\alpha, H\rangle = \alpha(H)$ for all $H \in \mathbf{A}$. Then we have for $1 \leqslant \iota \leqslant m(\alpha)$

$$[Z_{\alpha\iota}, \theta Z_{\alpha\iota}] = -H_\alpha. \tag{2}$$

In fact, $[Z_{\alpha\iota}, \theta Z_{\alpha\iota}] \in \mathbf{A}$, and for all $H \in \mathbf{A}$,

$$\langle[\theta Z_{\alpha\iota}, Z_{\alpha\iota}], H\rangle = -\langle Z_{\alpha\iota}, [Z_{\alpha\iota}, H]\rangle = \langle Z_{\alpha\iota}, \alpha(H) Z_{\alpha\iota}\rangle = \alpha(H).$$

Hence, the assertion follows.

Let $[\Pi, \mathcal{H}]$ be any differentiable representation of G on a topological vector space $\mathcal{H}$, and let $d\Pi$ be the corresponding action of the Lie algebra G. We also use the same symbol to denote the uniquely defined extension of this action to the universal enveloping algebra $U(\mathbf{G})$ of $\mathbf{G}$. For the time being we will simplify the notation by writing $q_1 g q_2 = d\Pi(q_1)\Pi(g) d\Pi(q_2)$ for all $q_1, q_2 \in U(\mathbf{G})$ and $g \in G$.

LEMMA 1. *For each* $\alpha \in \Delta$, $t \in \mathbf{R}$, $t \neq 0$, *and* $H \in \mathbf{A}$ *such that* $\alpha(H) \neq 0$, *write* $h(t) = \exp tH$. *Then*

$$h(t) \sum_{\iota=1}^{m(\alpha)} (Y_{\alpha\iota}^2 - X_{\alpha\iota}^2) = m(\alpha) \coth(\alpha(H)t) h(t) H_\alpha$$
$$+ [\sinh \alpha(H)t]^{-2} \sum_{\iota=1}^{m(\alpha)} [X_{\alpha\iota}^2 h(t) + h(t) X_{\alpha\iota}^2 - 2\cosh(\alpha(H)t) X_{\alpha\iota} h(t) X_{\alpha\iota}].$$

PROOF. For $1 \leqslant \iota \leqslant m(\alpha)$ we have the following computations:

$$h(t) Y_{\alpha\iota} = (\mathrm{Ad}\,(h(t)) Y_{\alpha\iota}) h(t) = [\cosh(\alpha(H)t) Y_{\alpha\iota} + \sinh(\alpha(H)t) X_{\alpha\iota}] h(t),$$
$$h(t) X_{\alpha\iota} = (\mathrm{Ad}\,(h(t)) X_{\alpha\iota}) h(t) = [\cosh(\alpha(H)t) X_{\alpha\iota} + \sinh(\alpha(H)t) Y_{\alpha\iota}] h(t).$$

Under the assumptions of the lemma we may and do eliminate the element $Y_{\alpha\iota} h(t)$ from these equations and obtain

(3) $\qquad h(t)Y_{\alpha\iota} = \coth(\alpha(H)t)h(t)X_{\alpha\iota} - [1/\sinh(\alpha(H)t)]X_{\alpha\iota}h(t).$

Now from (1) and (2) we have $[X_{\alpha\iota}, Y_{\alpha\iota}] = (1/2)[Z_{\alpha\iota} + \theta Z_{\alpha\iota}, Z_{\alpha\iota} - \theta Z_{\alpha\iota}] = -[Z_{\alpha\iota}, \theta Z_{\alpha\iota}] = H_\alpha$. Hence applying (3) twice yields

$$\begin{aligned} h(t)Y_{\alpha\iota}^2 &= \coth(\alpha(H)t)h(t)(Y_{\alpha\iota}X_{\alpha\iota} + H_\alpha) - [1/\sinh(\alpha(H)t)]X_{\alpha\iota}h(t)Y_{\alpha\iota} \\ &= \coth(\alpha(H)t)h(t)H_\alpha + [\coth(\alpha(H))t]^2 h(t)X_{\alpha\iota}^2 \\ &\quad + [\sinh(\alpha(H)t)]^{-2}[X_{\alpha\iota}^2 h(t) - 2\cosh(\alpha(H)t)X_{\alpha\iota}h(t)X_{\alpha\iota}]. \end{aligned}$$

The lemma follows from the observation that $[\coth(\alpha(H)t)]^2 - 1 = [\operatorname{csch}(\alpha(H)t)]^2 = [\sinh(\alpha(H)t)]^{-2}$. Q.E.D.

Now assume that G has real rank 1. Then either $\Delta = \{\alpha\}$ or $\Delta = \{\alpha, 2\alpha\}$. In the second case set $q = m(2\alpha)$, and in the first case set $q = 0$. In both cases let $p = m(\alpha)$. We adjust the constant c in the definition of the form $\langle\, ,\, \rangle$ so that $\langle H, H\rangle = 1$, where $H \in \mathbf{A}$ such that $\alpha(H) = 1$. More explicitly, c is given by $c = 1/B(H,H)$. Hence, also $H = H_\alpha$.

Let M be the centralizer of $\mathbf{A}$ in K. If $\mathbf{M} \neq \{0\}$, let r be the dimension of M and let $\{W_1, \ldots, W_r\}$ be an orthonormal basis of $\mathbf{M}$. Define the following elements of the universal enveloping algebra $U(\mathbf{G})$:

(4a) $\qquad Z_M = -\sum_{i=1}^{r} W_i^2, \quad \text{if } \mathbf{M} \neq \{0\}, \text{ and, } \quad Z_M = 0, \text{ if } \mathbf{M} = \{0\}.$

(4b) $\qquad Z_K = -\sum_{i=1}^{p} X_{\alpha i}^2 - \sum_{i=1}^{q} X_{2\alpha i}^2 + Z_M,$

(4c) $\qquad Z_G = H^2 + \sum_{i=1}^{p} Y_{\alpha i}^2 + \sum_{i=1}^{q} Y_{2\alpha i}^2 + Z_K,$

where in (4b) and (4c) the second summation symbol is to be interpreted as 0 in case $q = 0$. Note that Z_G, Z_K, and Z_M are Casimir elements of the Lie algebras $\mathbf{G}$, $\mathbf{K}$, and $\mathbf{M}$ respectively. Also let Z_L be the element of the universal enveloping algebra $U(\mathbf{G})$ defined by

(4d) $\qquad Z_L = -\sum_{i=1}^{q} X_{2\alpha i}^2 + Z_M, \quad \text{if } q \neq 0 \text{ and } Z_L = Z_M, \quad \text{if } q = 0.$

Now let $[\Pi, \mathcal{H}]$ be a quasi-simple representation of G on a Banach space, and let $[\Pi^\infty, \mathcal{H}^\infty]$ be the differentiable representation associated with $[\Pi, \mathcal{H}]$. (See [18, Vol. I, p. 254].) Then for every element Z in the center of $U(\mathbf{G})$ we have $d\Pi^\infty(Z) = \gamma(Z)1_{\mathcal{H}^\infty}$, and $Z \to \gamma(Z)$ is a homomorphism of the center of $U(\mathbf{G})$ into $\mathbf{C}$. Let us write $\Gamma = \gamma(Z_G)$. As in Lemma 1 we write $h(t) = \Pi^\infty(\exp tH)$ with $H = H_\alpha$, and follow the notational convention of that lemma. Thus we have $dh(t)/dt = Hh(t) = h(t)H$, and $d^2h(t)/dt^2 = H^2h(t) = h(t)H^2$. From Lemma 1 we then immediately have the following result:

LEMMA 2. *On* $(0,\infty)$ *the operator-valued function* $t \to h(t)$ *satisfies the differential equation*

$$\frac{d^2}{dt^2}h(t) + (p\coth t + 2q\coth 2t)\frac{d}{dt}h(t)$$

$$-[\sinh t]^{-2}\Big((Z_K - Z_L)h(t) + h(t)(Z_K - Z_L) + 2\cosh t\sum_{i=1}^{p} X_{\alpha i}h(t)X_{\alpha i}\Big)$$

$$-[\sinh 2t]^{-2}\Big((Z_L - Z_M)h(t) + h(t)(Z_L - Z_M) + 2\cosh 2t\sum_{i=1}^{q} X_{2\alpha i}h(t)X_{2\alpha i}\Big)$$

$$= \Gamma h(t) = h(t)Z_M,$$

where the last term on the left-hand side is to be replaced by 0 *in case* $q = 0$. *Note, that in that case the last three terms on the left-hand side are equal to* 0.

3. **Operator-valued spherical functions.** For G any connected semisimple real Lie group let $[\Pi, \mathcal{H}]$ be a quasi-simple representation of G. Let $\Omega(K)$ be the set of equivalence classes of finite-dimensional irreducible representations of K. For each class $[\omega]$ of $\Omega(K)$ let $E(\omega)$ be the projection on $\mathcal{H}$ such that $E(\omega)\mathcal{H}$ is the subspace of vectors that transform like $[\omega]$ under the restriction $\Pi|_K$. Then it is known that $[\Pi, \mathcal{H}]$ is K-finite; that is, $E(\omega)\mathcal{H}$ has finite dimension for all $[\omega] \in \Omega(K)$. (See [6].) Moreover, it is known [6] that $E(\omega)\mathcal{H}$ consists of analytic vectors for the representation Π, for all $[\omega] \in \Omega(K)$. Thus, in particular, $E(\omega)\mathcal{H} \subset \mathcal{H}^\infty$. Hence, $\Pi(g)E(\omega) = \Pi^\infty(g)E(\omega)$ for all $[\omega] \in \Omega(K)$.

Now assume that G has real rank 1. Then in the notation of Lemma 2 we have, for any pair of classes $[\omega]$ and $[\omega']$ in $\Omega(K)$, $E(\omega')\Pi(\exp tH)E(\omega) = E(\omega')h(t)E(\omega)$, and the function $t \to E(\Pi, t; \omega', \omega)$ defined by $E(\Pi, t; \omega', \omega) = E(\omega')h(t)E(\omega)$ is an analytic function from $(-\infty, \infty)$ to the finite-dimensional space HOM $(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$. This function is by definition the *elementary* (ω,ω')*-spherical function associated with the representation* Π.

If $[\omega, \mathcal{H}_\omega]$ and $[\omega', \mathcal{H}_{\omega'}]$ are two finite-dimensional K modules, and K' any subgroup of K, let $\mathrm{HOM}_{K'}(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$ denote the subspace of HOM $(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$ consisting of K' intertwining maps. In other terms,

$$\mathrm{HOM}_{K'}(\mathcal{H}_\omega, \mathcal{H}_{\omega'}) = \{T \in \mathrm{HOM}(\mathcal{H}_\omega, \mathcal{H}_{\omega'}) | T\omega(k) = \omega'(k)T, \quad \text{all } k \in K'\}.$$

Since $mh(t) = h(t)m$ for all $t \in \mathbf{R}$ and $m \in M$, it follows that for any pair (ω, ω') of K-types, one has $E(\Pi, t; \omega', \omega) \in \mathrm{HOM}_M(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$ for all $t \in \mathbf{R}$. In particular, the (ω, ω')-elementary spherical function $E(\Pi, \cdot; \omega', \omega)$ is

0 unless there is an M-type $[\mu] \in \Omega(M)$ which occurs in both restrictions $\omega|_M$ and $\omega'|_M$.

Define an action $H(\omega, \omega')$ of K on HOM $(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$ by the formula

$$H(\omega, \omega')(k)T = \omega'(k)T\omega(k^{-1})$$

for all $k \in K$, and $T \in$ HOM $(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$. Then it is a straightforward matter to check that $H(\omega, \omega')$ is a representation of K on the linear space $\mathrm{Hom}(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$. Let ω^* denote the representation of K contragredient to ω. Then the representation $[H(\omega, \omega'), \mathrm{HOM}(\mathcal{H}_\omega, \mathcal{H}_{\omega'})]$ is equivalent to the representation $[\omega' \otimes \omega^*, \mathcal{H}_\omega \otimes \mathcal{H}_{\omega^*}]$. More precisely, there is a canonical linear isomorphism $\phi\colon \mathcal{H}_{\omega'} \otimes \mathcal{H}_{\omega^*} \to \mathrm{HOM}(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$ with the property that $\phi(v \otimes f) \cdot (w) = f(w)v$ for all $v \in \mathcal{H}_{\omega'}, w \in \mathcal{H}_\omega$, and $f \in \mathcal{H}_{\omega^*}$. Then we have

LEMMA 3. *For all* $k \in K$,

$$\phi \circ (\omega'(k) \otimes \omega^*(k)) = H(\omega, \omega')(k) \circ \phi.$$

PROOF. This is a straightforward computation on elements in $\mathcal{H}_{\omega'} \otimes \mathcal{H}_{\omega^*}$ of the form $v \otimes f$. Then extend the result of this computation by linearity. Q.E.D.

We remark further that if α is an automorphism of K, then for any finite-dimensional representation $[\omega, \mathcal{H}_\omega]$ of K, one may define the representation $[\alpha\omega, \mathcal{H}_\omega]$ by $\alpha\omega(k) = \omega(\alpha(k))$ for all $k \in K$. Then the canonical map ϕ of the above lemma also has the intertwining property:

$$\phi \circ (\omega'(k) \otimes \alpha\omega^*(k)) = H(\alpha\omega, \omega')(k) \circ \phi,$$

with $k \in K$.

Let $\mathbf{L}$ denote the subspace of $\mathbf{K}$ defined by $\mathbf{L} = \mathbf{M}$, if $q = 0$, and if $q \neq 0$, $\mathbf{L} = \mathbf{M} + \mathrm{span}\,\{X_{2\alpha i} | i = 1, \ldots, q\}$. Then we have

LEMMA 4. $\mathbf{L}$ *is a subalgebra of* $\mathbf{K}$. *Let* $\mathbf{Q}$ *be the orthogonal complement of* $\mathbf{L}$ *in* $\mathbf{K}$. *Then we have* $[\mathbf{Q}, \mathbf{Q}] \subset \mathbf{L}$. *In particular,* $\mathbf{L}$ *is reductive in* $\mathbf{K}$.

PROOF. The conclusion of the lemma is clear if $q = 0$. Assume that $q \neq 0$. Then if i and j are integers between 1 and q, we have by (1) and (2),

$$[X_{2\alpha i}, X_{2\alpha j}] = (1/2)[Z_{2\alpha i}, \theta Z_{2\alpha j}] + (1/2)[\theta Z_{2\alpha i}, Z_{2\alpha j}],$$

since $\pm 4\alpha$ are not restricted roots. Now the element $[\theta Z_{2\alpha i}, Z_{2\alpha j}]$ centralizes $\mathbf{A}$. Hence, $[\theta Z_{2\alpha i}, Z_{2\alpha j}] \in \mathbf{A} + \mathbf{M}$. But the $\mathbf{A}$-component of this element changes sign under the involution θ. Hence $[X_{2\alpha i}, X_{2\alpha j}] \in \mathbf{M}$. Hence $\mathbf{L}$ is a subalgebra of $\mathbf{K}$. Next note that $\mathbf{Q} = \mathrm{span}_{\mathbf{R}}\,\{X_{\alpha i} | i = 1, \ldots, p\}$. By (1) and (2) and an argument similar to the above one we have $[X_{\alpha i}, X_{\alpha j}] \in \mathbf{M} + \mathbf{L}$. Hence, the lemma follows. Q.E.D.

By the last lemma, there exists an involution τ uniquely defined by the property that $\tau(Y) = -Y$ if $Y \in \mathbf{Q}$, and $\tau(Y) = Y$ if $Y \in \mathbf{L}$.

In the course of proving the above lemma the following was also shown.

LEMMA 5. *Assume that* $q \neq 0$. *Let* $\mathbf{S}$ *be the subspace of* $\mathbf{L}$ *defined by* $\mathbf{S} = \mathrm{span}_{\mathbf{R}}\{X_{2\alpha i} \mid i = 1, \ldots, q\}$. *Then* $\mathbf{L} = \mathbf{S} + \mathbf{M}$ *and* $[\mathbf{S}, \mathbf{S}] \subset \mathbf{M}$.

Hence, there also exists a uniquely defined involution β such that $\beta(Y) = -Y$, if $Y \in \mathbf{S}$ and $\beta(Y) = Y$ if $Y \in \mathbf{M}$.

Let L be the analytic subgroup of $\mathbf{K}$ determined by the Lie algebra $\mathbf{L}$. Then β determines an involution on L which fixes the subgroup M elementwise. We also denote this involution by the symbol β. Similarly, we also denote by τ the involution on K determined by the involution τ defined on $\mathbf{K}$.

We state the results of some simple calculations in the following lemma.

LEMMA 6. *Let* Z_M, Z_K, *and* Z_L *be the elements of* $U(\mathbf{G})$ *defined by the formulas* (4). *Let* $T \in HOM(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$. *Then*

(i) $dH(\omega, \omega')(Z_L)T = dH(\tau\omega, \omega')(Z_L)T$.

(ii)

$$d\omega'(Z_K - Z_L)T + Td\omega(Z_K - Z_L) = (1/2)[dH(\omega, \omega')(Z_K - Z_L)T + dH(\tau\omega, \omega')(Z_K - Z_L)T].$$

(iii)

$$-2 \sum_{i=1}^{p} d\omega'(X_{\alpha i})Td\omega(X_{\alpha i}) = (1/2)[dH(\tau\omega, \omega')(Z_K) - dH(\omega, \omega')(Z_K)]T.$$

(iv) *If* $q \neq 0$,

$$-2 \sum_{i=1}^{q} d\omega'(X_{2\alpha i})Td\omega(X_{2\alpha i}) = (1/2)[dH(\beta\omega, \omega')(Z_L) - dH(\omega, \omega')(Z_L)]T.$$

(v) *If* T *is also in* $\mathrm{HOM}_M(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$, *then*

$$d\omega'(Z_L - Z_M)T + Td\omega(Z_L - Z_M) = (1/2)[dH(\beta\omega, \omega')(Z_L) + dH(\omega, \omega')(Z_L)]T.$$

Note that if $q = 0$, *both sides of the last equation are* 0.

PROOF. If $Z \in U(\mathbf{L})$, then $d\omega(Z) = d(\tau\omega)(Z)$. Hence, (i) follows. If $Z \in \mathbf{Q}$, then for all $T \in \mathrm{HOM}(\mathcal{H}_\omega, \mathcal{H}_{\omega'})$, $dH(\omega, \omega')(Z)T = d\omega'(Z)T - Td\omega(Z)$, and $dH(\tau\omega, \omega')(Z)T = d\omega'(Z)T + Td\omega(Z)$, since $d(\tau\omega)(Z) = d\omega(\tau(Z)) = -d\omega(Z)$. Similarly, if $Z \in \mathbf{S}$, then $dH(\omega, \omega')(Z)T = d\omega'(Z)T - Td\omega(Z)$, and $dH(\beta\omega, \omega')(Z)T = d\omega'(Z)T + Td\omega(Z)$. Hence, if $Z \in \mathbf{Q}$, then

$$(1/2)[dH(\omega,\omega')(Z^2)T + dH(\tau\omega,\omega')(Z^2)T] = d\omega'(Z^2)T + Td\omega(Z^2),$$

and (ii) follows from (4b) and (4d). Also if $Z \in \mathbf{Q}$, then

$$(1/2)[dH(\tau\omega,\omega')(Z^2) - dH(\omega,\omega')(Z^2)] = 2d\omega'(Z)Td\omega(Z).$$

Hence, (iii) follows from (4b) and (i). If $Z \in \mathbf{S}$, then

$$d\omega'(Z^2)T + Td\omega(Z^2) = (1/2)[dH(\omega,\omega')(Z^2)T + dH(\beta\omega,\omega')(Z^2)T.$$

Hence (iv) follows from (4d) and the fact that $dH(\beta\omega,\omega') = dH(\omega,\omega')$ on $U(\mathbf{M})$. Moreover (v) also follows from this and the further observation that $dH(\omega,\omega')(m)T = 0$, if $T \in \mathrm{HOM}_M(\mathcal{H}_\omega,\mathcal{H}_{\omega'})$ and $m \in U(\mathbf{M})$. Q.E.D.

Now pick two irreducible K-types $[\omega]$, $[\omega'] \in \Omega(K)$. To simplify the notation let us write

$$\mathfrak{Z}_K = dH(\omega,\omega')(Z_K),\quad \mathfrak{Z}_K^\tau = dH(\tau\omega,\omega')(Z_K),$$

$$\mathfrak{Z}_L = dH(\omega,\omega')(Z_L),\quad \text{and}\quad \mathfrak{Z}_L^\beta = dH(\beta\omega,\omega')(Z_L).$$

Here ω and ω' are the primary representations of K in $\Pi|_K$ on the subspaces $E(\omega)\mathcal{H}$ and $E(\omega')\mathcal{H}$ respectively. Lemmas 2 and 6 may now be combined to yield the following result.

LEMMA 7. *The elementary* (ω,ω')*-spherical function* $E(t) = E(\Pi,t;\omega,\omega')$ *satisfies the following differential equation on* $(0,\infty)$.

$$\begin{aligned}\left(\frac{d^2}{dt^2} + (p\coth t + 2q\coth 2t)\frac{d}{dt}\right)&E(t)\\ &-(1/2)[\sinh t]^{-2}(\mathfrak{Z}_K + \mathfrak{Z}_K^\tau - 2\mathfrak{Z}_L)E(t)\\ &+(1/2)[\sinh t]^{-2}\cosh t(\mathfrak{Z}_K^\tau - \mathfrak{Z}_K)E(t)\\ &-(1/2)[\sinh 2t]^{-2}(\mathfrak{Z}_L + \mathfrak{Z}_L^\beta)E(t)\\ &+(1/2)[\sinh 2t]^{-2}\cosh 2t(\mathfrak{Z}_L^\beta - \mathfrak{Z}_L)E(t)\\ &= \Gamma E(t) - E(t)\omega(Z_M).\end{aligned}$$

PROOF. Observe that for q_1 and $q_2 \in U(\mathbf{K})$ we have

$$d\omega'(q_1)E(\Pi,t;\omega,\omega')d\omega(q_2) = E(\omega')q_1h(t)q_2E(\omega).$$

Hence, in the equation of Lemma 2 we premultiply by $E(\omega')$ and postmutıply by $E(\omega)$. The result then follows easily from Lemma 6. Q.E.D.

4. The case $q = 0$. Now assume that $q = 0$. Hence G is locally isomorphic to Spin$(1, n)$ with $n \geqslant 2$. Since $U(\mathbf{M})$ acts trivially on

$$\mathrm{HOM}_M\,(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$$

under the actions $dH(\omega, \omega')$, and $dH(\tau\omega, \omega')$, and since for all $t \in (0, \infty)$, we have $E(\Pi, t; \omega, \omega') \in \mathrm{HOM}_M\,(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$, the differential equation in Lemma 7 becomes

$$\left(\frac{d^2}{dt^2} + p\coth t\frac{d}{dt}\right)E(t) - (1/2)[\sinh t]^{-2}(\mathfrak{T}_K + \mathfrak{T}_K^\tau)E(t) \tag{5}$$

$$+(1/2)[\sinh t]^{-2}\cosh t(\mathfrak{T}_K^\tau - \mathfrak{T}_K)E(t)\Gamma E(t) - E(t)\omega(Z_M),$$

where as in Lemma 7 we write for brevity $E(t) = E(\Pi, t; \omega, \omega')$.

We make the substitution $x = (\tanh(t/2))^2$ and write

$$F(\Pi, x; \omega, \omega') = E(\Pi, t; \omega, \omega') \quad \text{for } t \in \mathbf{R},$$

or if no ambiguity results, we write more briefly $F(x) = F(\Pi, x; \omega, \omega')$. Note that the above substitution maps $\mathbf{R}\backslash\{0\}$ onto the interval (0, 1) and maps the singularities of the above differential equation as follows: 0 to 0 and $\pm\infty$ to 1. We also have then $(\mathrm{sech}(t/2))^2 = 1 - x$, $\coth t = (1 + x)/2\sqrt{x}$, $(\sinh t)^{-2} = (1 - x)^2/4x$, $\cosh t = (1 + x)/(1 - x)$, $d/dt = \sqrt{x}(1 - x)d/dx$, and hence also $d^2/dt^2 = x(1 - x)^2 d^2/dx^2 + (1/2)(1 - 3x)\cdot(1 - x)d/dx$. Thus, by straightforward manipulations, equation (5) results in the following lemma.

LEMMA 8. *On the interval* (0,1) *the operator-valued function $x \to F(x)$ satisfies the differential equation*

$$\begin{aligned} &x(1 - x)\frac{d^2}{dx^2}F(x) + (1/2)[p + 1 + x(p - 3)]\frac{d}{dx}F(x) + (1/4)\mathfrak{T}_K^\tau F(x) \\ &\qquad -(1/4x)\mathfrak{T}_K F(x) \\ &\qquad = (1 - x)^{-1}(\Gamma F(x) - F(x)\omega(Z_M)). \end{aligned} \tag{6}$$

Now assume that in addition to $q = 0$, one also has $\mathrm{rank}(G) = \mathrm{rank}(K) = m \geqslant 2$. Then G is locally isomorphic to Spin$(1, 2m)$. Let $\mathbf{T}$ be a Cartan subalgebra of $\mathbf{K}$ with the property that the intersection $\mathbf{T} \cap \mathbf{M}$ is a Cartan subalgebra of M. Identify the dual space of $\mathbf{T_C}$ with $\mathbf{T_C}$ itself by means of the form $\langle\ ,\ \rangle$. It is known that there exists an orthonormal basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$ of $\mathbf{T_C}$ with the following properties:

(1) $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\} \subset \sqrt{-1}\,\mathbf{T}$.

(2) The set $\{\varepsilon_1, \ldots, \varepsilon_{m-1}\}$ is a basis of $\sqrt{-1}\,\mathbf{T} \cap \sqrt{-1}\,\mathbf{M}$.

(3) Let Δ_K be the set of roots of $\mathbf{K_C}$ relative to the Cartan subalgebra T. Then Δ_K is given by $\Delta_K = \{\pm\varepsilon_i + \varepsilon_j, \pm\varepsilon_i - \varepsilon_j | 1 \leqslant i < j \leqslant m\}$.

(4) Let Δ_n be the set of noncompact roots of $\mathbf{G_C}$ relative to $\mathbf{T}$. By definition these are the roots of $\mathbf{G_C}$ not in Δ_K. They are given by $\Delta_n = \{\pm\varepsilon_i | i = 1, \ldots, m\}$.

(5) Let Δ_M denote the set of roots of $\mathbf{M_C}$ relative to $\mathbf{T} \cap \mathbf{M}$. Linear forms on this subspace are to be thought of as linear forms on $\mathbf{T_C}$ by extending them by 0 on the vector ε_m. Then Δ_M is given by $\Delta_M = \{\pm\varepsilon\}$, if $m = 2$, and if $m > 2$, $\Delta_M = \{\pm\varepsilon_k, \pm\varepsilon_i + \varepsilon_j, \pm\varepsilon_i - \varepsilon_j | 1 \leqslant i < j \leqslant m-1, 1 \leqslant k \leqslant m-1\}$.

We put a lexicographic order $<$ on the real vector space $\sqrt{-1}\,\mathbf{T}$ such that $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_m$. For each $[\omega] \in \Omega(K)$, let $\Lambda(\omega)$ be the highest weight of ω in the above basis and ordering of $\sqrt{-1}\,\mathbf{T}$. Then $\Lambda(\omega) = \sum_1^m \Lambda(\omega)_i \varepsilon_i$, where the components $\Lambda(\omega)_i$ are either all integers or all half odd integers and satisfy the inequalities

$$\Lambda(\omega)_1 \geqslant \Lambda(\omega)_2 \geqslant \cdots \geqslant \Lambda(\omega)_{m-1} \geqslant |\Lambda(\omega)|_m. \tag{7}$$

Let $[\mu] \in \Omega(M)$, and let $M(\mu)$ be the highest weight of $[\mu]$. Then $M(\mu) = \sum_1^{m-1} M(\mu)_i \varepsilon_i$, and the components $M(\mu)_i$ are either all integers or all half odd integers and satisfy the inequalities

$$M(\mu)_1 \geqslant M(\mu)_2 \geqslant \cdots \geqslant M(\mu)_{m-1} \geqslant 0. \tag{8}$$

A necessary and sufficient set of conditions that $[\mu]$ occurs in the restriction of $[\omega]$ to the subgroup M is the following (see for example [19]):

(9a) For $i = 1, \ldots, m-1$, $M(\mu)_i - \Lambda(\omega)_i$ is an integer.

$$\begin{aligned} \Lambda(\omega)_1 \geqslant M(\mu)_1 \geqslant \Lambda(\omega)_2 \geqslant M(\mu)_2 \geqslant \cdots \geqslant M(\mu)_{m-2} \geqslant \Lambda(\omega)_{m-1} \\ \geqslant M(\mu)_{m-1} \geqslant |\Lambda(\omega)_m|. \end{aligned} \tag{9b}$$

Moreover, if the class $[\omega]$ does occur in this restriction then it occurs with multiplicity 1.

LEMMA 9. *For the involution τ we have $\Lambda(\tau\omega)_i = \Lambda(\omega)_i$, if $i < m$, and $\Lambda(\tau\omega)_m = -\Lambda(\omega)_m$, and $[\tau\omega] \in \Omega(K)$ with highest weight $\Lambda(\tau\omega)$.*

PROOF. For $j \leqslant m-1$, $\sqrt{-1}\,\varepsilon_j \in \mathbf{M}$. Hence, $\tau(\sqrt{-1}\,\varepsilon_j) = \sqrt{-1}\,\varepsilon_j$. However, the element $\sqrt{-1}\,\varepsilon_m$ lies in the orthogonal complement of $\mathbf{M}$ in $\mathbf{K}$. Hence $\tau(\sqrt{-1}\,\varepsilon_m) = -\sqrt{-1}\,\varepsilon_m$. In particular, the involution τ fixes the Cartan subalgebra $\mathbf{T}$. Hence the dual action of this involution takes the weight $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{m-1}, \Lambda_m)$ into the weight $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{m-1}, -\Lambda_m)$. Here we identify $\mathbf{T_C}$ with $\mathbf{C}^m$ by means of the basis defined above. Thus τ maps positive roots into positive roots by item (3) above, and maps dominant forms into dominant forms by the inequalities (7). The lemma follows by the simplicity of the highest weight of an irreducible representation of K. Q.E.D.

As in §3, let $[\Pi, \mathcal{H}]$ be a quasi-simple representation of G and let $[\omega]$, $[\omega']$ be a pair of K-types. Let us now write $V = \text{HOM}\,(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$ and $V_M = \text{HOM}_M\,(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$. Assume now that the latter space is nontrivial. Let $\Omega(\omega, \omega')$ denote the set of K-types that occur in the complete reduction of the K-module $[H(\omega, \omega'), V]$, and which contain nontrivial M-invariant subspaces, and if $[\Psi] \in \Omega(\omega, \omega')$ we write V^{Ψ} for the K-primary component of V corresponding to $[\Psi]$. Similarly, let $\Omega(\tau\omega, \omega')$ denote the set of K-types that occur in the complete reduction of the K-module $[H(\tau\omega, \omega'), V]$ and which contain nontrivial M-invariant subspaces, and if $[\Psi] \in \Omega(\tau\omega, \omega')$, we write $V^{\tau\Psi}$ for the K-primary component of V corresponding to $[\Psi]$. Let V_M^{Ψ} and $V_M^{\tau\Psi}$ denote the intersections $V_M^{\Psi} = V_M \cap V^{\Psi}$ and $V_M^{\tau\Psi} = V_M \cap V^{\tau\Psi}$, defined for each K-type in $\Omega(\omega, \omega')$ and $\Omega(\tau\omega, \omega')$ respectively. Then clearly, we have the direct sum decompositions

$$V_M = \oplus\{V_M^{\Psi} \| [\Psi] \in \Omega(\omega, \omega')\} = \oplus\{V_M^{\tau\Psi} | [\Psi] \in \Omega(\tau\omega, \omega')\}.$$

We remark that even if a K-type $[\Psi]$ occurs in both $\Omega(\omega, \omega')$ and $\Omega(\tau\omega, \omega')$ the subspaces V_M^{Ψ} and $V_M^{\tau\Psi}$ do not coincide in general. This is because the operators $\mathfrak{Z}_K^{\tau}$ and $\mathfrak{Z}_K$ do not commute in general. We do have the following result however.

LEMMA 10. (1) *Let* $[\Psi]$ *be a class in* $\Omega(\omega, \omega')$ *or in* $\Omega(\tau\omega, \omega')$. *Let* $\Lambda(\Psi)$ *be its highest weight. Then* $\Lambda(\Psi) = \psi\varepsilon_1$, *where* ψ *is a nonnegative integer.*

(2) *If* $[\Psi] \in \Omega(\omega, \omega')$ *or* $\Omega(\tau\omega, \omega')$, $\mathfrak{Z}_K$ *or* $\mathfrak{Z}_K^{\tau}$ *acts as multiplication by the eigenvalue* $l(\Psi) = (\psi + m - 1)^2 - (m - 1)^2$ *on the subspace* V_M^{Ψ} *or* $V_M^{\tau\Psi}$ *respectively.*

PROOF. Since V_M consists entirely of M-fixed vectors for both actions $H(\omega, \omega')$ and $H(\tau\omega, \omega')$, so do the subspaces V_M^{Ψ} and $V_M^{\tau\Psi}$. Hence the branching rule (9b) requires that $\Lambda(\omega)$ have the form indicated in the lemma.

Next note that one-half the sum of the positive roots of K is given by the formula $\delta_K = \sum_1^{m-1} (m - j)\varepsilon_j$. Also by a standard result, the eigenvalue of Z_K acting in a K-primary module of type $[\omega]$ is given by

$$\langle \Lambda(\omega) + 2\delta_K, \Lambda(\omega)\rangle = \langle \Lambda(\omega) + \delta_K, \Lambda(\omega) + \delta_K\rangle - \langle \delta_K, \delta_K\rangle.$$

(See for example, Jacobson [13, p. 247].) Statement 2 of the lemma follows from these considerations and statement 1. Q.E.D.

5. **Locked M-types.** The main difficulty in solving the differential equation of Lemma 2 results from the fact that the operators $\mathfrak{Z}_K$, $\mathfrak{Z}_K^{\tau}$, $\mathfrak{Z}_L$ and $\mathfrak{Z}_L^{\beta}$ do not

commute in general. This difficulty is avoided in the locked M-type situation which we now define.

Let $\Omega(M)$ be the set of equivalence classes of finite-dimensional irreducible representations of M. Let $[\omega]$ and $[\omega']$ be two K-types in $\Omega(K)$. We say that an M-type $[\mu] \in \Omega(M)$ is *locked* between $[\omega]$ and $[\omega']$ if $[\mu]$ is the only M-type that occurs in the complete reduction of both restrictions $[\omega]|_M$ and $[\omega']|_M$. If there exists a locked M-type between two K-types $[\omega]$ and $[\omega']$ we shall also say that $[\omega]$ and $[\omega']$ are *locking* K-types.

Assume now that $q = 0$. Then for $[\Pi, \mathcal{H}]$ a quasi-simple representation of G, it is known that each K-type occurs with multiplicity of at most 1 in the restriction of this representation to K. Hence if $[\omega]$ and $[\omega']$ is a locking pair of K-types which occur in $[\Pi, \mathcal{H}]$, then the space $\mathrm{HOM}_M\,(E(\omega)\mathcal{H}, E(\omega')\mathcal{H})$ has dimension 1. In fact, this space is spanned by a single intertwining map which takes the unique type μ invariant subspace of $E(\omega)\mathcal{H}$ into the unique type μ invariant subspace $E(\omega')\mathcal{H}$.

The next lemma gives examples of locking pairs of K-types. We use the notation and assumptions of the last section.

LEMMA 11. *Let $q = 0$, and assume that* $\mathrm{rank}(G) = \mathrm{rank}(K) = m$. *Let $[\omega]$ be any K-type. Let $\Lambda(\omega) = \sum_1^m \Lambda(\omega)_i\varepsilon_i$ be its highest weight. Let $[\omega']$ be another K-type whose highest weight components satisfy the conditions $\Lambda(\omega')_i = |\Lambda(\omega)_{i+1}|$, $i = 1, 2, \ldots, m-1$. (Note that the first $m-1$ components of $\Lambda(\omega')$ then satisfy the inequalities* (7) *since all the components of $\Lambda(\omega)$ satisfy these inequalities. Note also that the component $\Lambda(\omega')_m$ is unspecified here except for the condition that it satisfies the last inequality in* (7).) *Let $[\mu] \in \Omega(M)$. Assume that $M(\mu)_i = |\Lambda(\omega)_{i+1}| = \Lambda(\omega')_i$, $i = 1, 2, \ldots, m-1$. Then $[\omega]$ and $[\omega']$ are locking K-types, and $[\mu]$ is locked between $[\omega]$ and $[\omega']$.*

PROOF. If $M(\mu')$ is the highest weight of an M-type $[\mu']$ which occurs in both restrictions $\omega|_M$ and $\omega'|_M$, then the branching rule requires that $\Lambda(\omega)_i \geqslant M(\mu')_i \geqslant |\Lambda(\omega)_{i+1}|$, and $\Lambda(\omega')_i \geqslant M(\mu')_i \geqslant |\Lambda(\omega')_{i+1}|$ for $1 \leqslant i \leqslant m-1$. But, for those indices we have $\Lambda(\omega')_i = |\Lambda(\omega)_{i+1}|$. Hence the only M-type $[\mu']$ which satisfies these conditions is $[\mu'] = [\mu]$ with μ defined in the statement of the lemma. Q.E.D.

Assume that $[\omega]$ and $[\omega']$ are a locking pair of K-types as in the last lemma. Since $\mathrm{HOM}_M\,(E(\omega')\mathcal{H}, E(\omega)\mathcal{H})$ has dimension 1, the operators $\mathfrak{T}_K$ and $\mathfrak{T}_K^\tau$ act as multiplication by scalars l and l^τ respectively, where the general form of the parameters l and l^τ is given in Lemma 10. We now determine them in terms of the highest weights given in the last lemma.

LEMMA 12. *Make the assumptions of Lemma* 11, *and let $\Lambda(\omega)$ and $\Lambda(\omega')$ be the highest weights of locking K-types as given in that lemma. If the component*

$\Lambda(\omega)_m \neq 0$, *let* $\sigma = \Lambda(\omega)_m/|\Lambda(\omega)_m|$, *and if the component* $\Lambda(\omega)_m = 0$, *let* $\sigma = 0$. *Then we have*

$$l = (\Lambda(\omega)_1 - \sigma\Lambda(\omega')_m + (m-1))^2 - (m-1)^2,$$

$$l^\tau = (\Lambda(\omega)_1 + \sigma\Lambda(\omega')_m + (m-1))^2 - (m-1)^2.$$

PROOF. Since $\mathrm{HOM}_M\,(E(\omega')\mathcal{H}, E(\omega)\mathcal{H})$ has dimension 1, there must be a unique class $[\Psi]$ in $\Omega(\omega', \omega)$ and a unique class $[\Psi^\tau]$ in $\Omega(\tau\omega', \omega)$. Here we use the notation of Lemma 10, By statement 1 of that lemma we have $\Lambda(\Psi) = \psi\varepsilon_1$, and $\Lambda(\Psi^\tau) = \psi^\tau\varepsilon_1$, where ψ and ψ^τ are certain nonnegative integers which we now determine. Write $\mathcal{H}_\omega = E(\omega)\mathcal{H}$, and $\mathcal{H}_{\omega'} = E(\omega')\mathcal{H}$. Then by Lemma 3 we have the following isomorphisms of K-modules: $[\omega \otimes \omega'^*, \mathcal{H}_\omega \otimes \mathcal{H}_{\omega'^*}] \simeq [H(\omega', \omega), \mathrm{HOM}\,(\mathcal{H}_{\omega'}, \mathcal{H}_\omega)]$, and $[\omega \otimes (\tau\omega')^*, \mathcal{H}_\omega \otimes \mathcal{H}_{\omega'^*}] \simeq [H(\tau\omega', \omega), \mathrm{HOM}\,(\mathcal{H}_{\omega'}, \mathcal{H}_\omega)]$. Let $W(K)$ be the Weyl group of $\mathbf{K}_\mathbf{C}$ defined for the choice of Cartan subalgebra $\mathbf{T}$ in the last section. We identify $\sqrt{-1}\,\mathbf{T}$ with $\mathbf{R}^m$ by means of the basis given in that section. Let $s \in W(K)$ be defined by $s(\lambda_1, \ldots, \lambda_m) = (\sigma\lambda_m, \lambda_1, \ldots, \lambda_{m-2}, \sigma\lambda_{m-1})$ for $(\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m$. By standard results, the element defined above is an element of $W(K)$. (See for example [1, pp. 209–210].) Hence, the vectors $-s\Lambda(\omega')$ and $-s\Lambda(\tau\omega')$ are weights in the representations ω'^* and $(\tau\omega')^*$ respectively. Let Ψ' and $\Psi^{\tau\prime}$ be vectors defined by $\Psi' = \Lambda(\omega) + \delta_K - s\Lambda(\omega')$, and $\Psi^{\tau\prime} = \Lambda(\omega) + \delta_K - s\Lambda(\tau\omega')$. Explicitly, using Lemma 9 and the formula for δ_K,

$$\Psi^{\tau\prime} = (\Lambda(\omega)_1 + m - 1 + \sigma\Lambda(\omega')_m, m-2, \ldots, 1, 0),$$

$$\Psi' = (\Lambda(\omega)_1 + m - 1 - \sigma\Lambda(\omega')_m, m-2, \ldots, 1, 0).$$

Hence, if $s \in W(K)$, and $s \neq e$, $s\Psi' < \Psi'$, and $s\Psi^{\tau\prime} < \Psi^{\tau\prime}$ since these are both regular dominant forms. Hence, it follows from a standard formula for tensor products of irreducible finite-dimensional modules over simple Lie algebras that the forms $\Psi = \Psi' - \delta_K$ and $\Psi^\tau = \Psi^{\tau\prime} - \delta_K$ are highest weights of irreducible K-modules occuring in the tensor products $\omega \otimes \omega'^*$ and $\omega \otimes (\tau\omega')^*$ respectively. By the formula for δ_K in §4, they have the form given in statement 1 of Lemma 10. Hence, the irreducible K-submodules to which these highest weights correspond contain nontrivial M-invariant subspaces of dimension 1. Since the space $\mathrm{HOM}_M\,(\mathcal{H}_{\omega'}, \mathcal{H}_\omega)$ has dimension 1, these are the unique K-submodules with this property. Lemma 12 now follows from statement 2 of Lemma 10. Q.E.D.

6. The spherical function for a locking pair of K-types. We continue the assumption that $q = 0$ and $\mathrm{rank}(G) = \mathrm{rank}(K)$. Let $[\omega']$ and $[\omega]$ be a locking pair of K-types as given in Lemma 11. Let $[\mu]$ be the M-type locked between these K-types. Let $[\Pi, \mathcal{H}]$ be a quasi-simple representation of G, and let $E(\omega', \omega, \mu)$ be a nonzero intertwining map which intertwines the single type M-invariant subspace of $\mathcal{H}_{\omega'}$ with the single type μ M-invariant subspace of $\mathcal{H}_{\omega}$. Let $x \to F(\Pi, x; \omega', \omega)$ be the function defined at the beginning of §4. (For notational convenience, we have interchanged the roles of ω and ω'.) For brevity we shall sometimes denote this function simply by $x \to F(x)$. Then there exists a scalar-valued function $x \to \Phi(\Pi, x; \omega', \omega)$, or more briefly $x \to \Phi(x)$, such that, for all $x \in [0, 1)$, $F(x) = \Phi(x)E(\omega', \omega, \mu)$, since $\mathrm{HOM}_M(\mathcal{H}_{\omega'}, \mathcal{H}_{\omega})$ has dimension 1.

Consider the expression $F(x)\omega'(Z_M)$ occuring in equation (6). (The roles of ω' and ω are interchanged there!) For each $[\mu'] \in \Omega(M)$ let $P(\mu')$ be the projection that projects onto the type μ M-invariant subspace of $\mathcal{H}_{\omega'}$ and corresponds to the direct sum decomposition of this space into M-irreducible components. Then clearly, since $[\mu]$ is locked between $[\omega]$ and $[\omega']$, we have $F(\cdot)P(\mu') = 0$, unless $[\mu'] = [\mu]$. Hence

$$\begin{aligned} F(\cdot)\omega'(Z_M) &= \sum_{\mu'} F(\cdot)P(\mu')\omega'(Z_M) \\ &= F(\cdot)P(\mu)\omega'(Z_M) = F(\cdot)\Gamma_M(\mu), \end{aligned}$$

where $\Gamma_M(\mu)$ is the eigenvalue of the Casimir operator Z_M which corresponds to the class $[\mu]$.

Next we note that for the case under consideration we have $p = 2m - 1$. We also introduce the following parameters. Let β be either one of the roots of the equation $\beta^2 - p\beta = \Gamma - \Gamma_M(\mu)$. Further, let a, b, c, and α be given by

$$\begin{aligned} a &= -(m-1) - \sigma\Lambda(\omega')_m + \beta, \qquad b = \Lambda(\omega)_1 + \beta, \\ c &= m + \Lambda(\omega)_1 - \sigma\Lambda(\omega')_m, \\ \alpha &= (\Lambda(\omega)_1 - \sigma\Lambda(\omega')_m)/2. \end{aligned}$$

LEMMA 13. *There exists a complex number $B(\beta, \omega', \omega)$ such that on the interval $[0, 1)$ the function $x \to \Phi(x) = \Phi(\Pi, x; \omega', \omega)$ is given by*

$$\Phi(x) = B(\beta, \omega', \omega)x^{\alpha}(1 - x)^{\beta}F(a, b; c, x),$$

where $x \to F(a, b; c, x)$ is the hypergeometric function defined by the series

$$F(a, b; c, x) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!\,(c)_j} x^j.$$

(We recall the definition of the Legendre symbol $(a)_j$, etc.: $(a)_0 = 1$, $(a)_j = a(a+1)\cdots(a+j-1)$, if $j > 0$.)

PROOF. By introducing the parameters l and l^τ of Lemma 12 into equation (6) of Lemma 8, it follows that the function Φ satisfies the differential equation

$$x(1-x)\Phi''(x) + \tfrac{1}{2}(p+1+(p-3)x)\Phi'(x) + [\tfrac{1}{4}l^\tau - \tfrac{1}{4}l/x]\Phi(x) \\ - 1/(1-x)[\Gamma - \Gamma_M(\mu)]\Phi(x) = 0.$$

The singularities of this equation are at 0 and 1. This circumstance suggests the definition of a function F by the formula $\Phi(x) = x^\alpha(1-x)^\beta F(x)$, where the parameters α and β are to be adjusted so that the differential equation satisfied by F does not contain terms involving $1/x$ and $1/(1-x)$ respectively, thus weakening the singularities to regular ones. A straightforward but slightly messy calculation shows that for this to happen, α and β must satisfy the quadratic equations

$$\beta^2 - p\beta = \Gamma - \Gamma_M(\mu),$$

$$(\alpha + (m-1)/2)^2 = [(\Lambda(\omega)_1 - \sigma\Lambda(\omega')_m + (m-1)/2]^2,$$

and F must satisfy the differential equation

$$x(1-x)F''(x) + \tfrac{1}{2}[p+1+4\alpha + x(p-3+4(\alpha+\beta))]F'(x)$$

$$+([\tfrac{1}{2}\Lambda(\omega)_1 + \tfrac{1}{2}\sigma\Lambda(\omega')_m + (m-1)/2]^2 - [\alpha+\beta-(m-1)/2]^2)F(x) = 0.$$

Here we have used the values of the parameters l and l^τ given in the statement of Lemma 12. The last equation is a hypergeometric differential equation. Since the identity element in G corresponds to $x = 0$, it must hold that the limit $\Phi(x)$, as $x \to 0+$ exists. This dictates the choice of positive root for α. Thus with α, a, b and c given above the lemma follows. Q.E.D.

7. **The nonunitary principal series.** At this point we review some known facts about the nonunitary principal series of representations of a semisimple Lie group of arbitrary real rank l. We use the notation of §1. Let M be the centralizer of $\mathbf{A}$ in K, and let $[\mu] \in \Omega(M)$. Let $[\mu, \mathcal{H}]$ be a finite-dimensional irreducible representation of M of type $[\mu]$, and let $\langle\ ,\ \rangle_\mu$ be an inner product defined on the space $\mathcal{H}$ such that the representation μ is unitary with respect to this inner product. Since M is compact, such an inner product is known to exist by a well-known elementary arguement. We denote by $L^2_\mu(K)$ the linear space of $\mathcal{H}$-valued measurable functions on K, measurable with respect to Haar measure on K, and which satisfy the following conditions:

$$(10) \qquad F(mk) = \mu(m)F(k),$$

for all $F \in L^2_\mu(K)$, $m \in M$, and $k \in K$, and

$$\int_K \langle F(k), F(k)\rangle_\mu dk < \infty,$$

where dk is the Haar measure of K normalized such that $\int dk = 1$. Identifying functions that differ only on a set of measure zero, $L^2_\mu(K)$ becomes a Hilbert space with the inner product $(\,,\,)$ defined by $(F, G) = \int_K \langle F(k), G(k)\rangle_\mu dk$.

Let N and A be the analytic subgroups of G corresponding to the subalgebras **N** and **A** respectively. Then N is maximal nilpotent in G and NA is maximal solvable with N normal in NA. Moreover $NAK = G$ is an Iwasawa decomposition of G. Let κ: $G \to K$, σ: $G \to NA$ be the analytic maps defined by this Iwasawa decomposition by the property that for each $g \in G$, $\sigma(g)$ and $\kappa(g)$ are the unique elements in NA and K respectively such that $g = \sigma(g)\kappa(g)$.

Let Λ be a complex character of NA; that is, Λ is a one-dimensional representation of the group NA into the multiplicative subgroup of $\mathbf{C}\backslash\{0\}$. Then Λ is the identity on N. Let P be the real character defined on NA by $P^2(s) = \det[\mathrm{Ad}|_{NA}(s)]$ for $s \in NA$. Then define an action on the space $L^2_\mu(K)$ by the formula

(11) $$\Pi_\Lambda(g)F(k) = \Lambda(\sigma(kg))P(\sigma(kg))F(\kappa(kg)),$$

for all $g \in G$, and $k \in K$, and for $F \in L^2_\mu(K)$. It is known that $g \to \Pi_\Lambda(g)$ is a continuous representation of G on $L^2_\mu(K)$. (See [6, §12].) It is easily shown that this representation is linearly equivalent to an induced representation. (See the proof of the next lemma.) It is also known that for each pair Λ, $[\mu] \in \Omega(M)$, this representation is quasi-simple. (See [3].) We shall need the explicit form of the infinitesimal character of these quasi-simple representations on the Casimir element of G. For each complex character Λ let λ be its differential. Hence λ is a complex linear form on $\mathbf{N} + \mathbf{A}$ which is zero on $\mathbf{N}$ and such that $\Lambda(n \exp h) = e^{\lambda(h)}$, for all $n \in N$ and $h \in \mathbf{A}$. If ρ denotes the differential of P, then it is given by the formula $2\rho = \sum \{m(\alpha)\alpha | \alpha \in \Delta\}$, on $\mathbf{A}$ and by zero on $\mathbf{N}$. Let $\{H_i | i = 1, \ldots, l\}$ be an orthonormal basis of $\mathbf{A}$ with respect to the inner product $\langle\,,\,\rangle$. Then a Casimir element in $U(\mathbf{G})$ is given by

(12a) $$Z_G = \sum_{i=1}^{l} H_i^2 + \sum_{\alpha\in\Delta} \sum_{i=1}^{m(\alpha)} (Y_{\alpha i}^2 - X_{\alpha i}^2) + Z_M,$$

where Z_M is the Casimir element of M defined as in (4a). By formulas (1) and (2) of §1, the last formula may be rewritten as

(12b) $$Z_G = \sum_{i=1}^{l} H_i^2 - \sum_{\alpha\in\Delta} m(\alpha)H_\alpha + 2 \sum_{\alpha\in\Delta} \sum_{i=1}^{m(\alpha)} Z_{\alpha i}\theta Z_{\alpha i}.$$

We then have the following lemma.

LEMMA 14. *Let Λ be a complex character of NA. Let $\Gamma_M(\mu)$ be the eigenvalue of Z_M corresponding to the class $[\mu] \in \Omega(M)$. Let γ be the infinitesimal character of $U(G)$ corresponding to the representation Π_Λ. Then* $\gamma(Z_G) = \langle\lambda,\lambda\rangle + \Gamma_M(\mu)$

$-\langle\rho,\rho\rangle$, *where we identify a linear form* ν *on* $\mathbf{A}$ *with the uniquely defined element* $H_\nu \in \mathbf{A}$ *such that* $\nu(h) = \langle h, H_\nu\rangle$ *for all* $h \in \mathbf{A}$.

PROOF. Let $C_\mu^\infty(K)$ be the space of $\mathcal{H}$-valued infinitely differentiable functions which have the transformation property (10). Let $C_{\Lambda\mu}^\infty(G)$ be the set of functions F defined on G by $F(g) = (\Lambda P)(\sigma(g))F(\kappa(g))$, for $F \in C_\mu^\infty(K)$ and $g \in G$. Then it is easy to see that the representation $[\Pi_\Lambda, C_\mu^\infty(K)]$ is linearly equivalent to the representation of G on $C_{\Lambda\mu}^\infty(G)$ by right translations; the restriction-to-K map provides a linear equivalence. Let R and L denote, respectively, the action of G on C^∞ functions by right and left translations. Also denote the corresponding differential actions of $U(\mathbf{G})$ by the same symbols. Let $F \in C_{\Lambda\mu}^\infty(G)$. Then, since Z_G is in the center of $U(\mathbf{G})$, we have $R(Z_G)F = L(Z_G)F$. Now let $\mathfrak{N}$ be the right ideal in $U(\mathbf{G})$ given by $\mathfrak{N} = \mathbf{N}U(\mathbf{G})$. Then $L(q)F = 0$ for all $q \in \mathfrak{N}$, and $L(h)F = (\lambda + \rho)(h)F$ for all $h \in \mathbf{A}$, and $L(Z_M)F = d\mu(Z_M)F$, by (10), and the fact that M is a normal subgroup of NAM. The lemma now follows easily from formula (12b). Q.E.D.

REMARK. For the case when $l = 1$, the above argument implies that the nonunitary principal series representations are quasi-simple, a fact which is true for general l as remarked above.

Next, we take note of the following.

LEMMA 15. *For* $g \in G$, F *and* $G' \in L_\mu^2(K)$,

$$(\Pi_\Lambda(g)F, G') = (F, \Pi_{\bar{\Lambda}^{-1}}(g^{-1})G').$$

Hence, the representation contragredient to $[\Pi_\Lambda, L_\mu^2(K)]$ *is* $[\Pi_{\bar{\Lambda}^{-1}}, L_\mu^2(K)]$.

PROOF. See for example [15, Lemma 8]. Q.E.D.

For the rank 1 case a complex character Λ is determined by a single complex number ν defined by $\nu = \lambda(H)$, with $H = H_\alpha$ as in §1. Recall that the Hermitian form $\langle\ ,\ \rangle$ has been normalized so that $\langle H, H\rangle = 1$. Then for the case $l = 1$ the formula in Lemma 14 reduces to

$$\gamma(Z_G) = \nu^2 - ((p + 2q)/2)^2 + \Gamma_M(\mu). \tag{13}$$

For this case it is also convenient to write Π_ν for the group action in place of Π_Λ. In this notation, the action contragedient to Π_ν is $\Pi_{-\bar{\nu}}$.

We make some remarks concerning the action of K on $L_\mu^2(K)$ by right translations. For each $[\omega] \in \Omega(K)$ let $[\omega, \mathcal{H}_\omega]$ be a finite-dimensional irreducible representation of K of type $[\omega]$. Let $\langle\ ,\ \rangle_\omega$ be an inner product on $\mathcal{H}_\omega$ which makes the restriction of ω to the derived subgroup of K unitary. (Actually, it can be shown that for the rank 1 case the last condition implies that ω is unitary, unless G is a multiple covering of SU $(1,n)$.) Let $\mathrm{HOM}_M(\mathcal{H}_\omega,\mathcal{H})$ be the space of intertwining maps from the M-module $[\omega|_M,\mathcal{H}_\omega]$ to the M-module $[\mu,\mathcal{H}]$. Let $(\mu\colon \omega)$ denote the multiplicity of μ in $\omega|_M$. Then $\mathrm{HOM}_M(\mathcal{H}_\omega,\mathcal{H}) \neq \{0\}$ if and only if $(\mu\colon \omega) \neq 0$. Assume that $(\mu\colon \omega) \neq 0$. Then decompose $\mathcal{H}_\omega$ into an orthogonal direct sum of M-irreducible subspaces. With j an integer with $1 \leqslant j \leqslant (\mu\colon \omega)$, let $\mathcal{H}^\mu_{\omega j}$ be a type μ irreducible subspace that occurs in this decomposition, and assume that distinct j correspond to distinct subspaces. Then for each such index j there corresponds an intertwining map $T_j^\omega \in \mathrm{HOM}_M(\mathcal{H}_\omega,\mathcal{H})$ such that $T_j^\omega \neq 0$ and T_j^ω is zero on the orthogonal complement of the subspace $\mathcal{H}^\mu_{\omega j}$. In fact, if T is any map in $\mathrm{HOM}_M(\mathcal{H}_\omega,\mathcal{H})$ which does not vanish on $\mathcal{H}^\mu_{\omega j}$, one may define T_j^ω as the composition of T with the orthogonal projection onto $\mathcal{H}^\mu_{\omega j}$.

LEMMA 16. *Assume* $(\mu\colon \omega) \neq 0$. *Let* $\{e_1,\ldots,e_N\}$ *be an orthonormal basis of* $\mathcal{H}$. *Then the functions* F_{ji}^ω, *defined on* K *for* $1 \leqslant j \leqslant (\mu\colon \omega)$, $1 \leqslant i \leqslant N$, *by* $F_{ji}^\omega(k) = T_j^\omega\,\omega(k)e_i$, *constitute an orthogonal basis of* $E(\omega)L^2_\mu(K)$. *Moreover, for each such function we have*

$$(F_{ji}^\omega, F_{ji}^\omega) = \mathrm{Trace}(T_j^{\omega *}\, T_j^\omega),$$

where $T_j^{\omega *}$ *is the map from* $\mathcal{H}$ *to* $\mathcal{H}_\omega$ *which is adjoint to* T_j^ω.

PROOF. One has, for each $k \in K$, $F_{ji}^\omega(k) = \sum_1^N \omega_{li}(k)T_j e_l$. Hence, by the Schur orthogonality relations,

$$\begin{aligned}(F_{ji}, F_{j'i'}) &= \int_K \sum_{l'=1}^N \sum_{l=1}^N \omega_{li}(k)\overline{\omega_{l'i'}(k)}\langle T_j^\omega e_l, T_{j'}^\omega e_{l'}\rangle_\mu\, dk \\ &= \sum_{l=1}^N \langle T_j^\omega e_l, T_{j'}^\omega e_{l'}\rangle_\mu \delta_{ii'} = \sum_{l=1}^N \langle T_{j'}^{\omega *}\, T_j^\omega e_l, e_l\rangle_\omega \delta_{ii'},\end{aligned}$$

where the last expression results from the definition of $T_{j'}^{\omega *}$. However, if $y \in \mathcal{H}$ and $x \in (\mathcal{H}^\mu_{\omega j'})^\perp$, $\langle T_{j'}^{\omega *} y, x\rangle_\omega = \langle y, T_{j'}^\omega x_\mu = 0$. Hence, the last summation is equal to zero, unless $j = j'$ and $i = i'$. If $j = j'$ and $i = i'$, this expression is equal to Trace $(T_{j'}^{\omega *}\, T_j^\omega)$. The lemma now follows by counting dimensions and the Frobenius reciprocity theorem. Q.E.D.

8. **The constant** $B(\beta,\omega',\omega)$. We return now to the assumptions of §5, and take the quasi-simple representation there to be one of the nonunitary principal series representations $[\Pi_\nu, L^2_\mu(K)]$ corresponding to the complex parameter ν and the irreducible representation μ of M. By equation (13), the quadratic equation for the parameter β takes the form $(\beta - p/2)^2 = \nu^2$. Hence $\beta = p/2 \pm \nu$.

We need to consider the action of the Lie algebra $\mathbf{G}$ on the space of K-finite vectors in $L^2_\mu(K)$. Since the action of K is by right translations, it is sufficient to consider the action of $\mathbf{P}$. For each $Y \in \mathbf{P}$ let $\psi(Y)$ denote the operator on $L^2_\mu(K)$ defined by multiplication by the function $k \to \Psi(Y,k) = \langle \mathrm{Ad}\,(k)Y, H\rangle$. Then we have the following result.

LEMMA 17. *Let* $[\omega'], [\omega] \in \Omega_\mu(K)$. *Let* $0 \neq Y \in \mathbf{P}$. *Then*

(1) $E(\omega')\psi(Y)E(\omega) \neq 0$ *if and only if* $\Lambda(\omega') - \Lambda(\omega)$ *is one of the noncompact roots* $\{\varepsilon_i | i = 1, \ldots, m\}$. $E(\omega')d\Pi_\nu(Y)E(\omega) = 0$, *unless* $\Lambda(\omega') - \Lambda(\omega)$ *is a noncompact root.*

(2) *Assume that* $\Lambda(\omega') - \Lambda(\omega)$ *is a noncompact root. Then*

$$E(\omega')\,d\Pi_\nu(Y)E(\omega) = Q_\nu(\omega',\omega)E(\omega')\psi(Y)E(\omega),$$

where $Q_\nu(\omega',\omega)$ *is the complex number given by*

$$Q_\nu(\omega',\omega) = \nu + \Lambda(\omega)_j + m - j + 1/2, \quad \text{if } \Lambda(\omega') - \Lambda(\omega) = \varepsilon_j,$$

or

$$Q_\nu(\omega',\omega) = \nu - \Lambda(\omega)_j - m + j + 1/2, \quad \text{if } \Lambda(\omega') - \Lambda(\omega) = -\varepsilon_j.$$

PROOF. The first part of statement 1 results immediately from Lemma 5 in [15], and the proof of the rest of the lemma is contained in the proof of Theorem 2 in [15]. Q.E.D.

It is convenient to extend the definition of the function $[\omega'] \to Q_\nu(\omega',\omega)$ to all μ-admissible K-types, not just those for which $\Lambda(\omega') - \Lambda(\omega)$ is a noncompact root. For this purpose, let Ξ denote the lattice in $\sqrt{-1}\,\mathbf{T}$ generated by the set of vectors $\{\varepsilon_1, \ldots, \varepsilon_m\}$. Thus, $\Xi = \mathbf{Z}\varepsilon_1 \oplus \mathbf{Z}\varepsilon_2 \oplus \cdots \oplus \mathbf{Z}\varepsilon_m$. Let R_ν denote the function on $\Xi \times \frac{1}{2}\Xi$ defined as follows. Let $\tau = \sum_1^m \tau_j\varepsilon_j$ and $\lambda = \sum_1^m \lambda_j\varepsilon_j$ with $\tau_j \in \mathbf{Z}$ and $\lambda_j \in \frac{1}{2}\mathbf{Z}$. Let $R_\nu(\tau,\lambda)$ be given by

$$R_\nu(\tau,\lambda) = \prod_{j=1}^{m} R_{\nu j}(\tau_j,\lambda_j), \tag{14}$$

where

$$R_{\nu j}(\tau_j,\lambda_j) = (\nu + \lambda_j + m - j + 1/2)_{\tau_j} \quad \text{if } \tau_j \geqslant 0, \tag{15a}$$

$$= (\nu - \lambda_j - m + j + 1/2)_{-\tau_j} \quad \text{if } \tau_j \leqslant 0. \tag{15b}$$

Recall that, as in Lemma 13, $(a)_n$ denotes the Legendre symbol. Then the function $Q_\nu(\cdot,\omega)$ defined by $Q_\nu(\omega',\omega) = R_\nu(\Lambda(\omega') - \Lambda(\omega), \Lambda(\omega))$ is the desired extension.

Next, let $\|\cdot\|_1$ denote the l^1-norm on $\mathbf{T}_{\mathbf{C}}$ defined by means of the basis $\{\varepsilon_1, \ldots, \varepsilon_m\}$. Thus, for $x = \sum_1^m x_i \varepsilon_i$, $\|x\|_1 = \sum_1^m |x|_i$. Then we have the following extension of the above lemma.

LEMMA 18. *Let* $[\omega'], [\omega] \in \Omega_\mu(K)$. *Let* $n = \|\Lambda(\omega') - \Lambda(\omega)\|_1$. *Then for all nonzero* $Y \in \mathbf{P}$,

(1) $E(\omega')\psi(Y)^n E(\omega) \neq 0$,

(2) $E(\omega')d\Pi_\nu(Y)^n E(\omega) = Q_\nu(\omega', \omega)E(\omega')\psi(Y)^n E(\omega)$, *and if* $0 \leqslant N < n$,

(3) $E(\omega')d\Pi_\nu(Y)^N E(\omega) = 0 = E(\omega')\psi(Y)^N E(\omega)$.

PROOF. We prove statement (3) first. It is known that a K-type $[\omega'']$ occurs in the complete reduction of the tensor product K-module $[\mathrm{Ad} \otimes \omega, \mathbf{P} \otimes E(\omega)L^2_\mu(K)]$, only if $\Lambda(\omega'') - \Lambda(\omega)$ is a noncompact root; hence, only if $|\Lambda(\omega'') - \Lambda(\omega)|_1 = 1$. An easy induction argument shows that if $[\omega'']$ occurs in the complete reduction of the tensor product K-module $[(\mathrm{Ad})^N \otimes \omega, (\otimes^N P) \otimes E(\omega)L^2_\mu(K)]$, then $|\Lambda(\omega'') - \Lambda(\omega)|_1 \leqslant N$. Since the spaces $d\Pi_\nu(\mathbf{P})E(\omega)L^2_\mu(K)$ and $\psi(\mathbf{P})E(\omega)L^2_\mu(K)$ are K-module homomorphic images of this last tensor product, statement (3) follows.

Statement (2) is proved by induction. It is obvious for $n = 0$. Assume that it is true for all n with $0 \leqslant n \leqslant N$, for some $N \geqslant 0$. In the subsequent argument we shall for notational convenience identify K-types with their highest weights. Assume that $[\omega']$ and $[\omega]$ are μ-admissible K-types with $|\omega' - \omega|_1 = N + 1$. By statement (1) of Lemma 17,

$$E(\omega')d\Pi_\nu(Y)^{N+1}E(\omega) = \sum_\sigma E(\omega')d\Pi_\nu(Y)E(\omega' + \sigma)d\Pi_\nu(Y)^N E(\omega), \tag{16}$$

where the sum is over noncompact roots. By statement (3), the only terms which contribute to this sum are those for which $|\omega' + \sigma - \omega|_1 \leqslant N$. On the other hand, by the triangle inequality $|\omega' + \sigma - \omega|_1 \geqslant |\omega' - \omega|_1 - |\sigma|_1 = N$. Hence the only terms which contribute to the above sum are those for which $|\omega' + \sigma - \omega|_1 = N$. Now let σ be a noncompact root that contributes to the above sum. Then $\sigma = \pm\varepsilon_j$ for some j with $1 \leqslant j \leqslant m$. Write $\tau = \omega' - \omega = \sum \tau_i \varepsilon_i$, $\omega = \sum \omega_i \varepsilon_i$, and $\omega' = \sum \omega'_i \varepsilon_i$.

Case 1. $\sigma = \varepsilon_j$. Since $|\tau|_1 = N + 1$, and $|\tau + \varepsilon_j|_1 = N$, we must have $\tau_j < 0$. By Lemma 17 and the induction hypothesis,

$$\begin{aligned} &E(\omega')d\Pi_\nu(Y)(E(\omega' + \sigma)d\Pi_\nu(Y)^N E(\omega) \\ &\qquad = (\nu - \omega'_j + \varepsilon_j - m + j + 1/2) \\ &\qquad\quad \cdot R_\nu(\tau + \varepsilon_j, \omega)E(\omega')\psi(Y)E(\omega' + \varepsilon_j)\psi(Y)^N E(\omega). \end{aligned}$$

By (15b),

$$(\nu - \omega_j' - m + j + 1/2)R_{\nu j}(\tau_j + 1, \omega_j)$$
$$= (\nu - \omega_j - \tau_j + 1 - m + j + 1/2)R_{\nu j}(\tau_j + 1, \omega_j) = R_{\nu j}(\tau_j, \omega_j).$$

Hence, by (14),

$$E(\omega')d\Pi_\nu(Y)E(\omega' + \sigma)d\Pi_\nu(Y)^N E(\omega)$$
$$= R_\nu(\tau, \omega)E(\omega')\psi(Y)E(\omega' + \varepsilon_j)\psi(Y)^N E(\omega).$$

Notice that the factor $R_\nu(\tau, \omega)$ is independent of the index j.

Case 2. $\sigma = -\varepsilon_j$. Then since $|\tau|_1 = N + 1$ and $|\tau - \varepsilon_j|_1 = N$, we have $\tau_j > 0$. Then by the induction hypothesis and Lemma 17,

$$E(\omega')d\Pi_\nu(Y)E(\omega' - \varepsilon_j)d\Pi_\nu(Y)^N E(\omega)$$
$$= (\nu + \omega_j' + m - j + 1/2)$$
$$\cdot R_\nu(\tau - \varepsilon_j, \omega)E(\omega')\psi(Y)E(\omega' - \varepsilon_j)\psi(Y)^N E(\omega).$$

By (15a), $(\nu + \omega_j' + m - j + 1/2)R_{\nu j}(\tau_j - 1, \omega_j) = R_{\nu j}(\tau_j, \omega_j)$. Hence, again by (14),

$$E(\omega')d\Pi_\nu(Y)E(\omega' + \sigma)d\Pi_\nu(Y)^N E(\omega)$$
$$= R_\nu(\tau, \omega)E(\omega')\psi(Y)E(\omega' + \sigma)\psi(Y)^N E(\omega).$$

Statement (2) now follows from (16) and the definition of the factor $Q_\nu(\omega', \omega)$. Q.E.D.

Now let $[\omega']$ and $[\omega]$ be locking K-types as in §5. The coefficient $B(\beta, \omega', \omega)$ in Lemma 13 depends on the choice of the nonzero intertwining map $E(\omega', \omega, \mu)$. A direct check using Lemma 11 shows that the integer $2\alpha = \Lambda(\omega)_1 - \sigma\Lambda(\omega')_m$ is equal to $|\Lambda(\omega') - \Lambda(\omega)|_1$. Thus, by the first statement of Lemma 18 we may and do take $E(\omega', \omega, \mu) = E(\omega')\psi(H)^{2\alpha}E(\omega)$. Note also that, by statement (3) of Lemma 18, $E(\omega')d\Pi_\nu(Y)^N E(\omega) = 0$, unless $N \geqslant 2\alpha$. This is as it should be since, as follows easily from the formula for Φ in Lemma 13, $t \to \Phi((\tanh(t/2))^2)$ has the leading term $t^{2\alpha}$ in its power series expansion.

From the formula in Lemma 13 we have

$$E(\omega', \omega, \mu)B(\beta, \omega', \omega) = \frac{1}{(2\alpha)!}\frac{d^{2\alpha}}{dt^{2\alpha}}\left(\tanh\frac{t}{2}\right)^{2\alpha}\Bigg|_{t=0} E(\omega', \omega, \mu)$$
$$= \frac{1}{(2\alpha)!}\frac{d^{2\alpha}}{dt^{2\alpha}}E(\Pi_\nu, t; \omega', \omega)\Bigg|_{t=0} = \frac{1}{(2\alpha)!}E(\omega')\,d\Pi_\nu(H)^{2\alpha}E(\omega).$$

By the second statement of Lemma 18, the last expression is equal to $(1/(2\alpha)!)Q_\nu(\omega',\omega)E(\omega',\omega,\mu)$. Thus we have proved the following statement.

LEMMA 19. *Let*

$$E(\omega',\omega,\mu) = E(\omega')\psi(H)^{2\alpha}E(\omega).$$

Then the coefficient $B(\beta,\omega',\omega)$ *of Lemma* 13 *is given by*

$$B(\beta,\omega',\omega) = \frac{1}{(2\alpha)!}Q_\nu(\omega',\omega).$$

9. **Some remarks on integrable and square-integrable representations.** Let $[\Pi,\mathcal{H}]$ be an irreducible unitary representation of G on a Hilbert space with inner product (,). Let Z denote the center of G, and let $G^* = G/Z$. The representation $[\Pi,\mathcal{H}]$ is said to be *square-integrable* if there exist nonzero vectors $x, y \in \mathcal{H}$ such that

$$\int_{G^*} |(\Pi(g)x,y)|^2 dg^* < \infty.$$

Here dg^* denotes a Haar measure on G^*. The representation $[\Pi,\mathcal{H}]$ is said to be *integrable* if

$$\int_{G^*} |(\Pi(g)x,y)| dg^* < \infty,$$

for some pair of vectors $x, y \in \mathcal{H}$.

The following characterizations of integrable and square-integrable representations are known.

LEMMA 20. *Let* $[\Pi,\mathcal{H}]$ *be an irreducible unitary representation of* G.

(1) *If* $[\Pi,\mathcal{H}]$ *is a square-integrable representation, then*

$$\int_{G^*} |(\Pi(g)x,y)|^2 dg^* < \infty$$

for all $x, y \in \mathcal{H}$.

(2) *Let* $d\mathcal{H}$ *denote the linear subspace of K-finite elements of* $\mathcal{H}$. *If* $[\Pi,\mathcal{H}]$ *is integrable then*

$$\int_{G^*} |(\Pi(g)x,y)| dg^* < \infty$$

for all $x, y \in d\mathcal{H}$.

(3) *Let* $[\Pi,\mathcal{H}]$ *be an irreducible quasi-simple representation of* G *on a Hilbert space. Assume that* Π *is unitary on* Z. *Let* $d\mathcal{H}$ *be the linear subspace of K-finite vectors. Suppose that for some elements* x *and* y, $x \neq 0, y \neq 0, x, y \in d\mathcal{H}$,

$$\int_{G^*} |(\Pi(g)x,y)|^2 dg^* < \infty.$$

Then $[\Pi, \mathcal{H}]$ *is infinitesimally equivalent to a square-integrable unitary representation of* G.

PROOF. See Harish-Chandra [7, Lemmas 3 and 4 and the corollary to Lemma 3].

We recall another well-known result concerning integration on G. Let G be an arbitrary connected simply connected semisimple Lie group. Then we have the polar decomposition $G = KAK = KA_0K$, where $A_0 = \text{Exp}\,(\mathbf{A}_0)$, and $\mathbf{A}_0$ is the positive Weyl chamber corresponding to the given ordering in $\mathbf{A}$. Let $h \to D(h)$ be the function on $\mathbf{A}_0$ defined by

$$D(h) = (\sinh \alpha(h))^{m(\alpha)}(\sinh 2\alpha(h))^{m(2\alpha)}.$$

Here we use the notation of §1. Then there exists a normalization of Haar measure dg such that, for all $f \in L^1(G)$,

$$\int_G f(g)dg = \int_K\int_K\int_{A_0} f(k_1 \exp(h)k_2)D(h)dk_1\,dk_2\,dh.$$

For a proof of the last statement, see Helgason [11, §10.1].

It is convenient to have another form for the above integrability and square-integrability conditions. Assume now that $l = 1$. We use the notation of §2. Let $[\Pi, \mathcal{H}]$ be a quasi-simple representation of G, and let $[\omega]$ and $[\omega']$ be two K-types which occur in the complete reduction of the restriction $[\Pi|_K, \mathcal{H}]$. Recall that, for all $t \in (-\infty, \infty)$, $E(\Pi, t; \omega', \omega) = E(\omega)\Pi(\exp t)E(\omega')$. Let $E(\Pi, t; \omega', \omega)^*$ denote the adjoint of $E(\Pi, t; \omega', \omega)$. Then, for each t,

$$E(\Pi, t; \omega', \omega)^* \in \text{HOM}_M\,(E(\omega)\mathcal{H}, E(\omega')\mathcal{H}).$$

Since the space $E(\omega')$ has finite dimension, the trace of $E(\Pi, t; \omega', \omega)E(\Pi, t; \omega', \omega)^*$ exists for each t and is equal to the trace of $E(\Pi, t; \omega', \omega)^*E(\Pi, t; \omega', \omega)$. Denote this trace by $\mathcal{E}(\Pi, t; \omega', \omega)$. Then we have the following lemma.

LEMMA 21. *Let* $[\Pi, \mathcal{H}]$ *be an irreducible component of* $[\Pi_\nu, L^2_\mu(K)]$ *for some pair* (ν, μ). *Then the function* $t \to \mathcal{E}(\Pi, t; \omega', \omega)$ *is positive for each pair* $[\omega']$, $[\omega]$. *Moreover,* $[\Pi, \mathcal{H}]$ *is infinitesimally equivalent to a square-integrable representation if for some pair* $[\omega]$, $[\omega']$ *of irreducible unitary K-types which occur in the complete reduction of* $\Pi|_K$ *the integral*

$$\int_0^\infty \mathcal{E}(\Pi, t; \omega', \omega)(\sinh t)^p(\sinh 2t)^q\,dt$$

converges. Conversely, if $[\Pi, \mathcal{H}]$ *is square-integrable, then the above integral converges for all pairs* $[\omega'], [\omega]$ *of irreducible unitary K-types. This representation is infinitesimally equivalent to an integrable unitary representation if the integral*

$$\int_0^\infty \mathfrak{S}(\Pi, t; \omega', \omega)^{1/2} (\sinh t)^p (\sinh 2t)^q \, dt$$

converges for a pair of irreducible unitary K-types $[\omega'], [\omega]$ *which occur in the complete reduction of* $\Pi|_K$. *Conversely, if the representation is integrable, then this integral converges for all such K-types.*

PROOF. For $1 \leqslant j \leqslant (\mu{:}\ \omega)$, $1 \leqslant j' \leqslant (\mu{:}\ \omega')$, let F_{ji}^{ω} and $F_{j'i'}^{\omega'}$ be the functions of Lemma 16. For $t \in (0, \infty)$ consider the function on $K \times K$: $(k_1, k_2) \to |(\Pi(k_1 h(t) k_2) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|^2$. Since the restrictions $\omega|_Z$ and $\omega'|_Z$ are unitary, so is the restriction $\omega|_Z$, and moreover this function depends only on the cosets in $K^* \times K^* = K/Z \times K/Z$. Hence,

$$k_1 h(t) k_2 \to |(\Pi(k_1 h(t) k_2) F_{ji}, F_{j'i'})|^2$$

depends only on $G^* = G/Z$. By the definition of the functions F_{ji}^{ω}, $F_{j'i'}^{\omega'}$, and the Schur relations,

$$\begin{aligned} &\int_{K^*} \int_{K^*} |(\Pi(k_1 h(t) k_2) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|^2 \, dk_1 \, dk_2 \\ &\quad = \int_{K^*} \int_{K^*} \left| \sum_m \sum_{m'} \omega(k_2)_{mi} \overline{\omega(k_1^{-1})} (\Pi(h(t)) F_{ji}^{\omega}, F_{j'i'}^{\omega'}) \right|^2 dk_1 \, dk_2 \\ &\quad = \sum_m \sum_{m'} |(\Pi(h(t)) F_{jm}^{\omega}, F_{j'm'}^{\omega'})|^2 . \end{aligned}$$

Hence, by summing on j and j' we have

$$\mathfrak{S}(\Pi, t; \omega', \omega) = \int_{K^*} \int_{K^*} \sum_{j=1}^{(\mu:\omega)} \sum_{j'=1}^{(\mu:\omega')} |(\Pi(k_1 h(t) k_2) F_{ji}^{\omega} F_{j'i'}^{\omega'})|^2 . \tag{17}$$

Assume that $\mathfrak{S}(\Pi, t; \omega', \omega) \in L^1(D(Ht)dt)$. Then by (17) and the dominated convergence theorem, for each k_1 and $k_2 \in K^*$ the function $t \to |(\Pi(k_1 h(t) k_2) F_{ij}^{\omega}, F_{j'i'}^{\omega'})|^2$ is integrable with respect to the measure $D(Ht)\, dt$. Since this is a positive function, it follows from Tonelli's theorem that the function $(k_1 h(t) k_2) \to |(\Pi(k_1 h(t) k_2) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|^2$ is integrable with respect to Haar measure $dk_1 \, dk_2 D(tH)\, dt$. If $t \to \mathfrak{S}(\Pi, t; \omega', \omega)^{1/2} \in L^1(D(Ht)\, dt)$, then again, by (17) and the dominated convergence theorem,

$$t \to |(\Pi(k_1 h(t) k_2) F_{ji}^{\omega}, F_{j'i'}^{\omega'})| \in L^1(D(Ht)\, dt)$$

for each $k_1, k_2 \in K$. Hence, the integrability of the function $k_1 h(t) k_2 \to |(\Pi(k_1 h(t) k_2) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|$ follows from Tonelli's theorem. Hence the first condition in the lemma implies square-integrability, and the second condition implies integrability, by Lemma 20.

Conversely, suppose that the representation Π is infinitesimally equivalent to a square-integrable representation. Then the function

$$g \to \sum_{j} \sum_{j'} |(\Pi(g) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|^2$$

is integrable. Hence Fubini's theorem and formula (17) imply the first condition of the lemma. If Π is integrable, then Lemma 20 implies that, for each index i, i', j, and j', $|(\Pi(\cdot) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|$ is integrable. Hence, by the continuity of this function and by Fubini's theorem it follows that the functions $t \to |(\Pi(h(t)) F_{ji}^{\omega}, F_{j'i'}^{\omega'})|$ are in $L^1(D(Ht)dt)$. Now the sum over all indices of these functions dominates $\mathfrak{S}(\Pi, \cdot; \omega', \omega)^{1/2}$. Hence the latter function is integrable, and the lemma follows. Q.E.D.

10. **The structure of the nonunitary principal series.** In this section and the next one we review some known results concerning the structure of the nonunitary principal series representations of Spin $(1, n)$. In this section we review the results on the composition series of these representations and the infinitesimal equivalences among the various irreducible components. In the next section we review the results on the unitarizability of the irreducible components. These results are essentially a reformulation of results in [15], [16], and [5].

As in §7, let $[\mu, \mathcal{H}_\mu]$ be an irreducible finite-dimensional representation of M, and let ν be the parameter $\nu = \lambda(H)$. Let $\mathbf{T}_1$ be the maximally split Cartan subalgebra of $\mathbf{G}$ given by $\mathbf{T}_1 = \mathbf{A} \oplus \mathbf{T}_M$, where $\mathbf{T}_M = \mathbf{T} \cap \mathbf{M}$, and $\mathbf{T}$ is the compact Cartan subalgebra of §3. Define the elements $f_1 = H$ and $f_i = \varepsilon_{i-1}$ for $i = 2, \ldots, m$. Then these elements constitute a basis for the real space $\mathbf{A} \oplus_{\mathbf{R}} \sqrt{-1}\,\mathbf{T} \cap \mathbf{M}$, which we denote by $\mathbf{T}_{1\mathbf{R}}$. This space contains the roots of $\mathbf{G}_{\mathbf{C}}$ under the identification of $\mathbf{G}_{\mathbf{C}}$ with its dual space under the form $\langle\, , \rangle$. We notice that under this identification, linear forms on $\mathbf{T}_M$ are thought of as forms on $\mathbf{T}_1$ which vanish on $\mathbf{A}$. We introduce a lexicographic ordering on the real space $\mathbf{T}_{1\mathbf{R}}$ such that $f_1 > f_2 > \cdots > f_m$. Corresponding to this ordering the simple roots of $\mathbf{G}_{\mathbf{C}}$ are given by

$$\Sigma = \{f_1 - f_2, f_2 - f_3, \ldots, f_{m-1} - f_m, f_m\}.$$

Recall that a linear form on $\mathbf{T}_{1\mathbf{R}}$ is called *dominant integral* if $2\langle\Lambda, \alpha\rangle / \langle\alpha, \alpha\rangle \in \mathbf{Z}^+$, for Λ the linear form and $\alpha \in \Sigma$. This linear form is called dominant integral and *special* if, in addition, $\langle\Lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Sigma \backslash \{f_m\}$. In terms of the components Λ_i of Λ in the basis $\{f_1, \ldots, f_m\}$, these conditions become $\Lambda_1 \geqslant \Lambda_2 \geqslant \cdots \geqslant \Lambda_m \geqslant 0$, for Λ to be dominant, with the components either all integers or all half odd integers. The condition for Λ to be dominant integral and special is that $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m \geqslant 0$, for Λ dominant and integral. The usual condition that Λ be dominant integral and *regular* is that Λ be dominant integral and that $\langle \Lambda, \alpha \rangle \neq 0$ for *all* $\alpha \in \Sigma$. In terms of components, this last condition is $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m > 0$. The purpose of introducing this weakening of the notion "regular" to that of "special" is to account for the reducible cases of the unitary principal series in the statement of the next theorem.

Let δ_G be one-half the sum of the positive roots of $G_{\mathbf{C}}$. Explicitly,

$$\delta_G = \sum_1^m \delta_{Gj} f_j = \sum_{j=1}^m (m - j + 1/2) f_j,$$

with $\delta_{Gj} = (m - j + 1/2)$. Equip the real vector space $\sqrt{-1}\,\mathbf{T}_M$ with the ordering obtained by restricting the ordering on $\mathbf{T}_{1\mathbf{R}}$. Then δ_M, one-half the sum of the positive roots of $\mathbf{M}_{\mathbf{C}}$, is the restriction of δ_G to $\sqrt{-1}\,\mathbf{T}_M$. Let $M(\mu)$ be the highest weight of $[\mu]$ in this ordering. Define a linear form on $\mathbf{T}_1$ by the formula

$$\Lambda(\nu, \mu) = -\nu f_1 + \sum_{j=2}^m (M(\mu)_{j-1} + \delta_{Mj}) f_j,$$

where, as in §4, the highest weight components of $M(\mu)$ are given by $M(\mu) = \sum_1^{m-1} M(\mu)_j \varepsilon_j = \sum_2^m M(\mu)_{j-1} f_j$. Let $W(\mathbf{G}_{\mathbf{C}})$ be the Weyl group of $\mathbf{G}_{\mathbf{C}}$ relative to the Cartan subalgebra $\mathbf{T}_1$. This group is described explicitly as the group of linear maps on $\mathbf{T}_1$ defined by the finite group of transformations on the set $\{\pm f_1, \ldots, \pm f_m\}$ generated by the permutations on the set $\{f_1, \ldots, f_m\}$ and the maps σ_j given by $\sigma_j(f_k) = f_k$ for $j \neq k$, and $\sigma_j(f_j) = -f_j$. The irreducibility criterion may be stated as follows.

THEOREM 1. *The representation* $[\Pi_\nu, L^2_\mu(K)]$ *is irreducible if and only if the form* $\Lambda(\nu, \mu)$ *is not on the* $W(\mathbf{G}_{\mathbf{C}})$*-orbit of a dominant special integral form. If this representation is irreducible, then it is equivalent to the representation* $[\Pi_{-\bar{\nu}}, L^2_\mu(K)]$, *and these are the only equivalences.*

Now suppose that irreducibility fails. Then the form $\Lambda(\nu, \mu)$ lies on the Weyl group orbit of a uniquely determined dominant special integral form. We denote this form by $\Delta(\nu, \mu)$. Thus, in terms of components,

$$\Delta(\nu, \mu) = \sum_1^m \Delta(\nu, \mu)_j f_j,$$

where the components $\Delta(\nu, \mu)_j$ are all either integers or half odd integers which satisfy the inequalities

$$\Delta(\nu,\mu)_1 > \Delta(\nu,\mu)_2 > \cdots > \Delta(\nu,\mu)_m \geqslant 0.$$

Since $M(\mu) + \delta_M = \Lambda(\nu,\mu)|_{T_M}$, this restriction must also be dominant integral and regular. Hence, the condition that $\Delta(\nu,\mu)$ be dominant integral and special is equivalent to the condition that for some index i, $1 \leqslant i \leqslant m$, we have $\nu = \Delta(\nu,\mu)_i$ or $-\nu = \Delta(\nu,\mu)_i$. In the latter case, $\Lambda(\nu,\mu) = (1,2,\ldots,i)\Delta(\nu,\mu)$, and in the former case, $\Lambda(\nu,\mu) = \sigma_1(1,2,\ldots,i)\Delta(\nu,\mu)$. Here we use standard cycle notation for the elements of the Weyl group that correspond to permutations on the basis vectors.

THEOREM 2. *Suppose $\Lambda(\nu,\mu)$ is on the Weyl group orbit of the dominant integral special form $\Delta(\nu,\mu)$. Then as remarked above, $\pm\nu = \Delta(\nu,\mu)_i$ for some index i with $1 \leqslant i \leqslant m$. Three cases are distinguished*:

Case 1. *$i < m$. Then the following are proper supplementary subspaces*:

$$D_i^+(\nu,\mu) = \oplus\{E(\omega)L_\mu^2(K)|\Lambda(\omega)_i \geqslant |\nu| - m + (1/2) + i\},$$

$$D_i^-(\nu,\mu) = \oplus\{E(\omega)L_\mu^2(K)|\Lambda(\omega)_i < |\nu| - m + (1/2) + i\}.$$

The subspace $D_i^+(\nu,\mu)$ is invariant and irreducible under the action $\Pi_{|\nu|}$, and the subspace $D_i^-(\nu,\mu)$ is invariant and irreducible under the action $\Pi_{-|\nu|}$. These are the only irreducible components. In particular, any nonzero vector in $D_i^+(\nu,\mu)$ $(D_i^-(\nu,\mu))$ is cyclic for $L_\mu^2(K)$ under the action $\Pi_{-|\nu|}$ $(\Pi_{|\nu|})$.

Case 2. *$i = m$, and $\nu \neq 0$. There are three proper subspaces*:

$$D_m^+(\nu,\mu) = \oplus\{E(\omega)L_\mu^2(K)|\Lambda(\omega)_m \geqslant |\nu| + \tfrac{1}{2}\},$$

$$D_m^-(\nu,\mu) = \oplus\{E(\omega)L_\mu^2(K)|\Lambda(\omega)_m \leqslant -|\nu| - \tfrac{1}{2}\},$$

$$D_m^F(\nu,\mu) = \oplus\{E(\omega)L_\mu^2(K)|-|\nu| - \tfrac{1}{2} < \Lambda(\omega)_m < |\nu| + \tfrac{1}{2}\}.$$

The spaces $D_m^+(\nu,\mu)$ and $D_m^-(\nu,\mu)$ and invariant and irreducible under the action $\Pi_{|\nu|}$, and the subspace $D_m^F(\nu,\mu)$ is invariant and irreducible under the action $\Pi_{-|\nu|}$. There is the orthogonal direct sum decomposition: $L_\mu^2(K) = D_m^+(\nu,\mu) \oplus D_m^-(\nu,\mu) \oplus D_m^F(\nu,\mu)$.

Case 3. *$\nu = 0$. Then, necessarily, $i = m$ and $\nu = \Delta(\nu,\mu)_m = 0$. There are two supplementary subspaces*:

$$D_m^+(0,\mu) = \oplus\{E(\mu)L_\mu^2(K)|\Lambda(\omega)_m \geqslant 1/2\},$$

$$D_m^-(0,\mu) = \oplus\{E(\mu)L_\mu^2(K)|\Lambda(\omega)_m \leqslant -1/2\}.$$

Both of these subspaces are proper invariant and irreducible subspaces under the action Π_0 of G.

REMARK. Case 3 above is the only case of reducible unitary principal series representation.

In order to discuss the equivalences between irreducible components in the nonirreducible cases, we fix a dominant integral special form Δ on $\mathbf{T}_{1\mathbf{R}}$. For each index i, $1 \leqslant i \leqslant m$, define an M-type $[\mu_i]$ such that $\Lambda(-\Delta_i, \mu_i) = (1,2,\ldots,i)\Delta$. Explicitly, this means the following. If $i = 1$, $M(\mu_1)_{j-1} = \Delta_j - \delta_{Gj}$, for $2 \leqslant j \leqslant m$, and if $i > 1$, then

$$\begin{aligned} M(\mu_i)_j &= \Delta_j - \delta_{Gj+1} = \Delta_j - m + j + \tfrac{1}{2}, & j \leqslant i-1, \\ &= \Delta_{j+1} - \delta_{Gj+1} = \Delta_{j+1} - m + j + \tfrac{1}{2}, & j \geqslant i. \end{aligned}$$

THEOREM 3. *Let Δ be a dominant integral special form on $\mathbf{T}_{1\mathbf{R}}$. Let the symbol $\simeq$ denote infinitesimal equivalence. Then there are the following infinitesimal equivalences of irreducible components of nonunitary principal series representations*:

$$[\Pi_{\Delta_i}, D_i^+(\Delta_i, \mu_i)] \simeq [\Pi_{-\Delta_{i+1}}, D_{i+1}^-(\Delta_{i+1}, \mu_{i+1})], \qquad 1 \leqslant i < m-1,$$

$$[\Pi_{\Delta_{m-1}}, D_{m-1}^+(\Delta_{m-1}, \mu_{m-1})] \simeq [\Pi_{-\Delta_m}, D_m^F(\Delta_m, \mu_m)].$$

In addition, there are the following equivalences between quotient representations and subrepresentations:

$$[\Pi_{\Delta_m}, L^2_{\mu_m}(K)]/[\Pi_{\Delta_m}, D_m(\Delta_m, \mu_m) + D_m(\Delta_m, \mu_m)] \simeq [\Pi_{-\Delta_m}, D_m^F(\Delta_m, \mu_m)],$$

$$[\Pi_{-\Delta_m}, L^2_{\mu_m}(K)]/[\Pi_{-\Delta_m}, D_m^F(\Delta_m, \mu_m)] \simeq [\Pi_{\Delta_m}, D_m^+(\Delta_m, \mu_m) + D_m^-(\Delta_m, \mu_m)],$$

if $\Delta_m \neq 0$, and if $i < m$,

$$[\Pi_{\Delta_i}, L^2_{\mu_i}(K)]/[\Pi_{\Delta_i}, D_i^+(\Delta_i, \mu_i)] \simeq [\Pi_{-\Delta_i}, D_i^-(\Delta_i, \mu_i)],$$

$$[\Pi_{-\Delta_i}, L^2_{\mu_i}(K)]/[\Pi_{-\Delta_i}, D_i^-(\Delta_i, \mu_i)] \simeq [\Pi_{\Delta_i}, D_i^+(\Delta_i, \mu_i)].$$

Moreover, two irreducible components are equivalent if and only if they have the same K-module structure and the same infinitesimal character.

PROOF OF THEOREMS 1, 2 AND 3. By the proofs of Theorems 3, 4 and 5 of [15], the nonunitary principal series fails to be irreducible if and only if the following condition R holds:

(R): There exist two μ-admissible K-types $[\omega]$ and $[\omega'] \in \Omega_\mu(K)$ such that $\Lambda(\omega) - \Lambda(\omega')$ is a noncompact root, and $Q_\nu(\omega', \omega) = 0$.

Now suppose that condition **R** holds. Let $[\omega]$ and $[\omega'] \in \Omega_\mu(K)$ such that $Q_\nu(\omega', \omega) = 0$, and $\Lambda(\omega') - \Lambda(\omega) = \pm\varepsilon_i$, for $1 \leqslant i \leqslant m$. Then if $i < m$, the proofs referred to above also show that the index i is unique as in the pair $[\omega]$ and $[\omega']$. If $i = m$, this index is still unique, while now there are two pairs $[\omega], [\omega'] \in \Omega_\mu(K)$ such that $Q_\nu(\omega', \omega) = 0$. First suppose that $i < m$. If $\Lambda(\omega') - \Lambda(\omega) = \varepsilon_i$, then $0 = Q_\nu(\omega', \omega) = \nu + \Lambda(\omega)_i + m - i + \frac{1}{2}$; that is, $-\nu = \Lambda(\omega)_i + \delta_{Gi}$. Moreover, the branching rule (9) requires that $\Lambda(\omega)_i > M(\mu)_i - 1$, and $M(\mu)_{i-1} > \Lambda(\omega)_i$, if also $i > 1$. Thus, if one sets $M(\mu)_0 = \infty$, $-\nu \in (M(\mu)_i + \delta_{Gi+1}, M(\mu)_{i-1} + \delta_{Gi})$. A similar argument shows that if $\Lambda(\omega') - \Lambda(\omega) = -\varepsilon_i$, then $\nu \in (M(\mu)_i + \delta_{Gi+1}, M(\mu)_{i-1} + \delta_{Gi})$. Hence, with either sign for ε_i, $|\nu| \in (\Lambda(\nu,\mu)_{i+1}, \Lambda(\nu,\mu)_i)$, where in case $i = 1$, the limit on the right of the interval is to be replaced by ∞.

Next, suppose $i = m$. Then if $\Lambda(\omega') - \Lambda(\omega) = \varepsilon_m$, the definition of $Q_\nu(\omega', \omega)$ requires that $-\nu = \Lambda(\omega)_m + \frac{1}{2}$, and if $\Lambda(\omega') - \Lambda(\omega) = -\varepsilon_m$, $\nu = \Lambda(\omega)_m + \frac{1}{2}$. Since both conditions obtain for in general distinct K-types $[\omega] \in \Omega_\mu(K)$, the branching rule implies that

$$0 \leqslant |\nu| < M(\mu)_{m-1} + \tfrac{1}{2} = \Lambda(\nu,\mu)_{m-1}.$$

Thus, in all cases, condition **R** implies that $\nu - \Lambda(\nu,\mu)_i \in \mathbf{Z}$ and that $\Lambda(\nu,\mu)$ is Weyl conjugate to a dominant special integral form. Conversely, if the latter is the case, then by retracing the steps in the above argument, it can be shown that condition **R** obtains. Thus, Theorems 1 and 2 follow immediately from the results in [15] mentioned above.

Next we prove the necessity of the condition in the last sentence of Theorem 3. First, we recall the definition of a homomorphism first defined by Harish-Chandra. (See for example [3, §7.4].) Let $U(\mathbf{G})_0$ denote the centralizer of $\mathbf{T}_1$ in $U(\mathbf{G})$. Let $\mathfrak{N}_+$ denote the right ideal $\mathfrak{N}_+ = \mathbf{N}_+ U(\mathbf{G})$, where $\mathbf{N}_+$ is the nilpotent subalgebra of $\mathbf{G_C}$ spanned by the root vectors belonging to the positive roots of $\mathbf{G_C}$. Then it is known that $U(\mathbf{G})_0 = U(\mathbf{T}_1) + \mathfrak{N}_+ U(\mathbf{G})_0$, and the sum is direct. If $z \in Z(\mathbf{G})$, the center of $U(\mathbf{G})$, let $\phi(z)$ be the projection of z onto $U(\mathbf{G})_0$ according to the above direct sum. $z \to \phi(z)$ is an algebra homomorphism. If $u \in U(\mathbf{T}_1)$, define a polynomial function $\gamma(u)$ by the requirement that if u is the monomial $u = u_1 u_2 \cdots u_p$, $u_i \in \mathbf{T}_1$, then $\gamma(u)(\lambda) = \sum_{i=1}^{p} (\delta_G - \lambda)(u_i)$. Then it is known from a basic result of Harish-Chandra that the map $z \to \chi(z) = (\gamma \circ \phi)(z)$ is an algebra isomorphism from $Z(\mathbf{G})$ to the algebra of Weyl group invariant polynomial functions on $\mathbf{T}_1$. (See [3, §7.4].) Then the differentiable representation of G associated with the nonunitary principal series representation $[\Pi_\nu, L^2_\mu(K)]$ has the infinitesimal character $z \to \chi_{\Lambda(\nu,\mu)}(z)$. In fact, let v be a lowest weight vector for $[\mu, \mathcal{H}_\mu]$. Let F be a differentiable vector in $[\Pi_\nu, L^2_\mu(K)]$. Recall that $\langle\ ,\ \rangle_\mu$ is an inner product on $\mathcal{H}_\mu$ that makes the representation μ unitary. Let $g \to f(g)$ be the function defined by $f(g) = \langle \Pi_\nu(g)F(e), v\rangle_\mu$. It is a straightforward matter to show that $dL(h)f = (\delta_G - \Lambda(\nu,\mu))(h)f$, for $h \in \mathbf{T}_1$, and $dL(n)f = 0$, for $n \in \mathbf{N}_+$. Here

$dL(X+\sqrt{-1}\,Y)f=[dL(X)+\sqrt{-1}\,dL(Y)]f$, $X, Y \in \mathbf{G}$, and $dL(X)f(g) = (d/dt)f(\exp Xg)|_{t=0}$. Let $C_{\Lambda(\nu,\mu)}(G)$ denote the space of continuous complex valued functions on G with the above transformation properties This space is a G-module under the action of right translations. Moreover, the linear subspace of K-finite vectors is a $U(\mathbf{G})$-module under the differential of this action. Finally, the map $F \to f$ is a 1:1 map from the subspace of K-finite vectors in $L^2_\mu(K)$ to the subspace of K-finite vectors in $C_{\Lambda(\nu,\mu)}(G)$. It is easy to see that this map is also an intertwining map. Hence, the assertion follows.

It follows that two nonunitary principal series representations $[\Pi_\nu, L^2_\mu(K)]$ and $[\Pi_{\nu'}, L^2_{\mu'}(K)]$ have the same infinitesimal character if and only if the forms $\Lambda(\nu,\mu)$ and $\Lambda(\nu',\mu')$ lie on the same Weyl group orbit. Hence, this condition is necessary that the irreducible components be infinitesimally equivalent. Another necessary condition is that two such irreducible components have the same K-type. A glance at the branching rule (9) shows that the latter condition obtains precisely in the cases indicated in the theorem.

The sufficiency of the condition stated in the theorem is proved by explicitly constructing the intertwining operators between the indicated representations. This is done in [5] for the first set of equivalences and in [15] for the indicated equivalences between subrepresentations and subquotients.

11. **The unitary representations of** Spin $(1,2m)$. In [16] a method was given for constructing all unitary representations of Spin $(1,n)$ by using intertwining operators. The final result involved some ambiguities because the equivalences expressed in Theorem 3 were not taken into account. In this section the results of [16] are reviewed, and Theorem 3 is used to complete the classification.

First let us review the procedure for constructing all the unitary representations other than the unitary principal series. Fix an irreducible representation of M, $[\mu, \mathcal{H}_\mu]$. Let (,) denote the inner product on $L^2_\mu(K)$ defined in §6. Suppose the parameter ν is real and nonnegative. Let $[\Pi_\nu, D^+]$ be an irreducible component of $[\Pi_\nu, L^2_\mu(K)]$ and let $[\Pi_{-\nu}, D^-]$ be an irreducible component of the G-module $[\Pi_{-\nu}, L^2_\mu(K)]$. Of course, in the irreducible cases $D^+ = L^2_\mu(K) = D^-$. Let $\mathcal{V}$ denote the linear subspace of $L^2_\mu(K)$ consisting of K-finite vectors, and let $\mathcal{V}^\pm$ denote the intersections $\mathcal{V}^\pm = D^\pm \cap \mathcal{V}$. Let $[d\Pi_\nu, \mathcal{V}]$, $[d\Pi_\nu, \mathcal{V}^+]$, and $[d\Pi_{-\nu}, \mathcal{V}^-]$ denote the $U(\mathbf{G})$-modules that correspond in the usual way to the G-modules $[\Pi_\nu, L^2_\mu(K)]$, $[\Pi_\nu, D^+]$, and $[\Pi_{-\nu}, D^-]$ respectively. Suppose that $T(\pm\nu)$ are $U(\mathbf{G})$-module homomorphisms from $[d\Pi_{\pm\nu}, \mathcal{V}]$ to $[d\Pi_{\mp\nu}, \mathcal{V}]$. Then one can consruct Hermitian forms $A(\pm\nu, \mu; \, , \,)$ defined as follows.

$$A(\pm\nu, \mu; F, G) = (T(\pm\nu)F, G),$$

for F, $G \in L^2_\mu(K)$. Then it follows from Lemma 15 that for each $Y \in \mathbf{G}$, the linear map $d\Pi_{\pm\nu}(Y)$ is skew symmetric with respect to the Hermitian forms

$A(\pm\nu, \mu; \cdot, \cdot)$. In fact,

$$A(\pm\nu, \mu; d\Pi_{\pm\nu}(Y)F, G) = (T(\pm\nu)d\Pi_{\pm\nu}(Y)F, G) = (d\Pi_{\mp\nu}(Y)T(\pm\nu)F, G)$$
$$= -(T(\pm\nu)F, d\Pi_{\pm\nu}(Y)G) = -A(\pm\nu, \mu; F, d\Pi_{\pm\nu}(Y)G),$$

for all $F, G \in L^2_\mu(K)$. Moreover, if the Hermitian forms $A(\pm\nu, \mu; \cdot, \cdot)$ are positive, then their radicals, that is, the subsets $\mathfrak{N}$ defined by $\mathfrak{N} = \{f \in \mathcal{V} | A(\pm\nu, \mu; f, f) = 0\}$, are $U(\mathbf{G})$-submodules of $[\Pi_{\pm\nu}, \mathcal{V}]$. These radicals are actually the kernels of the maps $T(\pm\nu)$. If these kernels consist of the trivial subspace $\{0\}$, then the nonunitary principal series representation is irreducible, and is unitarizable by the form $A(\pm\nu, \mu; \cdot, \cdot)$; that is, the infinitesimally unitary representation resulting from these forms extends to a unitary representation of G on the completion of the inner product space $(\mathcal{V}, A(\pm\nu, \mu; \cdot, \cdot))$. If on the other hand these kernels are maximal proper submodules of $[\Pi_{\pm\nu}, \mathcal{V}]$, then by the second part of Theorem 3, the quotient modules $[\Pi_{\pm\nu}, \mathcal{V}]/[\Pi_{\pm\nu}, \mathfrak{N}]$ are isomorphic to irreducible components $[\Pi_{\mp\nu}, D]$ or $[\Pi_{\mp\nu}, \mathcal{V}]$. The intertwining maps $T(\pm\nu)$ implement this isomorphism. The inner product induced on these quotients by the form $A(\pm\nu, \mu; \cdot, \cdot)$ makes these modules infinitesimally unitary. Again, these infinitesimally unitary representations of $U(\mathbf{G})$ may be "lifted" to unitary representations of G. It is shown in §6, and §7 of [16], that all unitary representations of G other than the principal series arise in this way. In [16, §7] the maps $T(\pm\nu)$ which lead to unitary representations are constructed explicitly. If $[\Pi_{\pm\nu}, D]$ is an irreducible component of a nonunitary principal series representation, which is unitarizable in the manner described above, we shall call the resulting unitary representation of G the *unitary representation associated* with the representation $[\Pi_{\pm\nu}, D]$. The classification of unitary representations is as follows.

THEOREM 4. I. *An irreducible unitary representation of* $G = \mathrm{Spin}(1, 2m)$, $m \geqslant 2$, *is unitarily equivalent to at least one of the following representations.*

(1) *The unitary principal series*: $[\Pi_\nu, L^2_\mu(K)]$ *for* $\nu \in \sqrt{-1}\,\mathbf{R}$. (*If* $M(\mu)_{m-1}$ *is half an odd integer, assume that* $\nu \neq 0$.)

(2) *The irreducible complementary series*: *If* $M(\mu)_{m-1} > 0$, *let* $i = m$, *otherwise, let i be the least positive integer for which* $M(\mu)_i = 0$. *Then for all real* ν *for which* $0 < |\nu| < (m + \frac{1}{2} - i)$ *there is an irreducible unitary representation associated with* $[\Pi_\nu, L^2_\mu(K)]$.

(3) *Unitary representations associated with the following proper irreducible components of nonunitary principal series representations*:

Case 1. *If* $1 \leqslant i < m$, *and* $M(\mu)_i = 0$, $[\Pi_{|\nu|}, D_i^+(\nu, \mu)]$. *If in addition*, $|\nu| = m + \frac{1}{2} - i$, *then also* $[\Pi_{-|\nu|}, D_i^-(\nu, \mu)]$. (*Note*: *i here is the smallest integer such that* $M(\mu)_i = 0$.)

Cases 2,3. $i = m$, $M(\mu)_{m-1} > 0$, $[\Pi_{|\nu|}, D_m^{\pm}(\nu, \mu)]$, *and if in addition*, $|\nu| = \frac{1}{2}$, $[\Pi_{-1/2}, D_m^F(\pm\frac{1}{2}, \mu)]$.

II. *In the above list there are the following equivalences, and these are the only ones.*

(1) *If* $\nu \neq 0$, $\nu \in \sqrt{-1}\mathbf{R}$, *then* $[\Pi_\nu, L_\mu^2(K)]$ *is unitarily equivalent to* $[\Pi_{-\nu}, L_\mu^2(K)]$.

Let $\sim$ *denote unitary equivalence of associated unitary representations.*

(2) *Then for the irreducible complementary series above* $[\Pi_\nu, L_\mu^2(K)] \sim [\Pi_{-\nu}, L_\mu^2(K)]$.

(3) *For* $M(\mu)_{m-1} > 0$, $[\Pi_{-1/2}, D_m^F(\pm\frac{1}{2}, \mu)] \sim [\Pi_{|\nu'|}, D_{m-1}^-(\nu', \mu')]$, *where* $|\nu'|$ *and* μ' *are defined by* $M(\mu)_j = M(\mu')_j$ *if* $j < m - 1$, $M(\mu')_{m-1} = 0$, *and* $|\nu'| = M(\mu)_{m-1} + \frac{1}{2}$. *If* $M(\mu)_{m-1} = 0$, *let* i *be the least integer such that* $M(\mu)_i = 0$. *Assume also that* $1 < i \leqslant m - 1$, *and* $|\nu| = m + \frac{1}{2} - i$. *Then* $[\Pi_{|\nu'|}, D_{i-1}^+(\nu', \mu')] \sim [\Pi_{-|\nu|}, D_i^-(\nu, \mu)]$, *where* $|\nu'|$ *and* μ' *are defined by* $M(\mu')_j = M(\mu)_j$, *if* $j \leqslant i - 2$, $M(\mu')_j = 0$, *if* $j \geqslant i - 1$, *and* $|\nu'| = M(\mu)_{i-1} + m - i + \frac{1}{2}$.

Remark 1. There is a misprint in statement II B of Theorem 3 in [16]. It is corrected here in statement I(3), Case 1, above.

Remark 2. The trivial one-dimensional unitary representation is included in I(3), Case 1, above by setting $i = 1$, $|\nu| = m - \frac{1}{2}$, and μ equal to the trivial representation of M. It is then the representation $[\Pi_{-|\nu|}, D_1^-(\nu, \mu)]$.

Proof. Statement I of the theorem is a restatement of that portion of Theorem 3 in [16] that applies to the situation under consideration. To prove statement II, recall that for G any real connected semisimple Lie group, two irreducible unitary representations of G are infinitesimally equivalent if and only if they are unitarily equivalent. (See [6, Theorem 8].) Statement II follows from Theorem 1 for the case of the irreducible principal series and the case of the irreducible complementary series representations. For the other unitary representations, statement II follows from Theorem 3 and the following observations.

For $M(\mu)_{m-1} > 0$, the form $\Delta(\nu, \mu)$ has the components $\Delta(\nu, \mu)_j = M(\mu)_j + \delta_{Gj+1} = M(\mu)_j + m - j + \frac{1}{2}$, for $1 \leqslant j \leqslant m - 1$, and $\Delta(\nu, \mu)_m = |\nu| = \frac{1}{2}$. Hence, in Theorem 3, $\mu_m = \mu$, and $\mu_{m-1} = \mu'$ with $M(\mu')_j = M(\mu)_j$, if $j \leqslant m - 2$, $M(\mu')_{m-1} = |\nu| - \delta_{Gm} = \frac{1}{2} - \frac{1}{2} = 0$, and $|\nu'| = \Delta(\nu, \mu)_{m-1} = M(\mu)_{m-1} + \frac{1}{2}$.

In case $M(\mu)_{m-1} = 0$, we have $\Delta(\nu, \mu)_j = M(\mu)_{j-1} + \delta_{Gj+1} = M(\mu)_{j-1} + m - j = \frac{1}{2}$, if $j \leqslant i - 1$, $\Delta(\nu, \mu)_i = |\nu| = m + \frac{1}{2} - i$, and $\Delta(\nu, \mu)_j = \delta_{Gj} = m - j + \frac{1}{2}$, if $j \geqslant i + 1$. Then $\mu' = \mu_i$ and $|\nu'| = \Delta_i$. Hence, by the discussion preceding the statement of Theorem 3, $M(\mu')_j = M(\mu)_j$ if $j < i - 1$, $M(\mu')_{i-1} = \Delta_i - \delta_{Gi} = m + \frac{1}{2} - i - (m - i + \frac{1}{2}) = 0$, and $M(\mu')_j = 0$ also for $j \geqslant i$.

Also, $|\nu'| = \Delta_{i-1} = M(\mu)_{i-1} + m - i + \frac{1}{2}$. With these observations, statement 2 and the theorem follows. Q.E.D.

12. Integrable and square-integrable representations. In this section we will show that the unitary representations corresponding to the irreducible components $[D_m^+(\nu,\mu),\Pi_{|\nu|}]$ and $[D_m^-(\nu,\mu),\Pi_{|\nu|}]$ for Case 2 in Theorem 2 are all square-integrable.

Let $[\omega']$ be an arbitrary K-type. According to Lemma 11 one may find another K-type $[\omega]$ such that the pair ω', ω is a locking pair. Let $[\mu]$ denote the M-type which is locked between $[\omega]$ and $[\omega']$ when the latter pair of K-types are chosen according to Lemma 11. In order to apply Lemma 21 we need the asymptotic behaviour of the function $t \to \mathcal{E}(\Pi_\nu, t; \omega', \omega)$. For the locked K-type situation under consideration,

$$\mathcal{E}(\Pi_\nu, t; \omega', \omega) = \operatorname{Trace} E(\omega)\Pi_\nu(\exp t)E(\omega') \cdot \Pi_{-\bar{\nu}}(\exp(-t))E(\omega)T|\Phi((\tanh(t/2))^2)|^2,$$

where $T = \operatorname{Trace} E(\omega',\omega,\mu)^* E(\omega',\omega,\mu)$. Then, by Lemma 13,

$$\mathcal{E}(\Pi_\nu, t; \omega', \omega) = T|B(\beta,\omega',\omega)|^2 x^{2\alpha}(1-x)^{2\beta}|F(a,b;c,x)|^2,$$

with $x = (\tanh(t/2))^2$, $c = m + \Lambda(\omega)_1 - \sigma\Lambda(\omega')_m$, $a = -m + 1 - \sigma\Lambda(\omega')_m + \beta$, $b = \Lambda(\omega)_1 + \beta$, and $\alpha = \frac{1}{2}[\Lambda(\omega)_1 - \sigma\Lambda(\omega')_m]$. We recall that β is one of the two roots $\beta = m - \frac{1}{2} \pm \nu$. The choice of root is determined by the behaviour of the hypergeometric function at the point $x = 1$, or more generally, its analytic continuation about the point $x = 1$ in the complex x-plane. These results are classical. (See for example [2].) For future reference, we consider all the cases of ν, those that lead to square-integrable representations as well as those that do not.

Case 1. $\operatorname{Re}\nu \neq 0$, and $\frac{1}{2} + \nu - \sigma\Lambda(\omega')_m \in \mathbf{Z}$. Notice that this condition implies that (,) is not Weyl group equivalent to a dominant integral regular form. Hence, by Theorem 1, this is an irreducible case. By [2, p. 57], the hypergeometric series converges absolutely at $x = 1$, provided $\operatorname{Re}(a + b - c) < 0$. This condition is equivalent to the condition that $-2m + 1 + 2\operatorname{Re}\beta < 0$. Write $\nu = \nu_R + \sqrt{-1}\,\nu_I$, with $\nu_R, \nu_I \in \mathbf{R}$. By assumption, $\nu_R \neq 0$. Thus, one may choose β to be the root $\beta = m - \frac{1}{2} - (\nu_R/|\nu_R|)\nu$. With this choice of β, $\operatorname{limit}_{x\to 1-} (1-x)^{-\beta}\Phi(x)$ exists and is given by

$$\operatorname*{limit}_{x\to 1-} (1-x)^{-\beta}\Phi(x) = B(\beta,\omega',\omega)F(a,b;c,1) = B(\beta,\omega',\omega)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Case 2. $\nu \neq 0$, $\operatorname{Re}\nu = 0$. This case corresponds to the irreducible unitary principal series representations. In this case for either choice of root for β the hypergeometric series fails to converge at $x = 1$. The appropriate analytic continuation formula is given in [2, p. 108] and results in the following expansion for $\Phi(x)$ valid at $x = 1$:

$$\Phi(x) = x^{[\Lambda(\omega)_1 - \Lambda(\omega')_m]/2}\{(1-x)^{m-1/2+\nu} c_1(\omega', \omega, \nu) F(a_1, b_1; c_1, 1-x) + (1-x)^{m-1/2-\nu} c_2(\omega', \omega, \nu) F(a_2, b_2; c_2, 1-x)\},$$

where, recalling Lemma 19,

$$c_1(\omega', \omega, \nu) = \frac{\Gamma(m + \Lambda(\omega)_1 - \sigma\Lambda(\omega')_m)\Gamma(2\nu)}{[\Lambda(\omega)_1 - \sigma\Lambda(\omega')_m]!\,\Gamma(m - \frac{1}{2} + \Lambda(\omega)_1 + \nu)\Gamma(\frac{1}{2} - \nu - \sigma\Lambda(\omega')_m)} Q_\nu(\omega', \omega),$$

$$c_2(\omega', \omega, \nu) = \frac{\Gamma(m + \Lambda(\omega)_1 - \sigma\Lambda(\omega')_m)\Gamma(-2\nu)}{[\Lambda(\omega)_1 - \sigma\Lambda(\omega')_m]!\,\Gamma(m - \frac{1}{2} + \Lambda(\omega)_1 + \nu)\Gamma(\frac{1}{2} + \nu - \sigma\Lambda(\omega')_m)} Q_\nu(\omega', \omega),$$

$$a_1 = \nu + \tfrac{1}{2} - \sigma\Lambda(\omega')_m, \qquad a_2 = -\nu + \tfrac{1}{2} - \sigma\Lambda(\omega')_m,$$
$$b_1 = \nu - \tfrac{1}{2} + m + \Lambda(\omega)_1, \qquad b_2 = -\nu - \tfrac{1}{2} + m + \Lambda(\omega)_1,$$
$$c_1 = 1 - 2\nu, \qquad c_2 = 1 + 2\nu.$$

Case 3. $\nu + \frac{1}{2} - \sigma\Lambda(\omega') \in \mathbf{Z}$. We split this case into two subcases:

Case 3A. $|\nu| + \frac{1}{2} - \sigma\Lambda(\omega')_m \leqslant 0$. We remark that under the assumption of this case, $[\omega]$ and $[\omega']$ occur in either $D_m^+(\nu, \mu)$ or in $D_m^-(\nu, \mu)$, in the notation of Theorem 2. Take the root β to be $\beta = m - \frac{1}{2} + |\nu|$. Then in Lemma 13, $a = \frac{1}{2} + |\nu| - \sigma\Lambda(\omega')_m$ and $b = m - \frac{1}{2} + |\nu| + \Lambda(\omega)_m$. Hence a is a nonpositive integer, and thus the hypergeometric series reduces to a polynomial. Thus, with this choice of root β the following expression for $\Phi(x)$ is valid for all x:

$$\Phi(x) = B(\beta, \omega', \omega) x^\alpha (1-x)^{m-1/2+|\nu|} F(a, b; c, x).$$

Case 3B. $|\nu| + \frac{1}{2} - \sigma\Lambda(\omega')_m \geqslant 1$. Take β to be the root $\beta = m - \frac{1}{2} = |\nu|$. Then $a = \frac{1}{2} - |\nu| - \sigma\Lambda(\omega')_m$ and $b = m - \frac{1}{2} - |\nu| + \Lambda(\omega)_1$. Thus, $a \leqslant -2\sigma\Lambda(\omega')_m$. If the component $M(\mu)_{m-1} = 0$ in Lemma 11, then also $\Lambda(\omega')_m = 0 = \Lambda(\omega)_m$. If $M(\mu)_{m-1} > 0$, then we may choose the mth component of $\Lambda(\omega')$ to have the same sign as $\Lambda(\omega)_m$. In both these cases, a is a nonpositive integer. Hence, with the above choice of β the following expression for $\Phi(x)$ is valid for all x:

$$\Phi(x) = B(\beta, \omega', \omega) x^{\alpha} (1 - x)^{m-1/2-|\nu|} F(a, b; c, x).$$

We note that $x = (\tanh(t/2))^2$ remains bounded at $t = \pm\infty$, while $1 - x = (\cosh(t/2))^{-2} \sim e^{-|t|}$, as $t \to \pm\infty$, and $(\sinh t)^p = (\sinh t)^{2m-1} \sim e^{(2m-1)|t|}$ as $t \to \pm\infty$. In Case 3A we have $|\mathcal{E}(\Pi, t; \omega', \omega)| \sim e^{-(2m-1+2|\nu|)|t|}$ as $t \to \pm\infty$. Hence, from Lemma 21 we obtain the following result.

THEOREM 5. *The representations* $[\Pi_{|\nu|}, D_m^{\pm}(\nu, \mu)]$ *of Theorem* 2 *are infinitesimally equivalent to square-integrable unitary representations provided that* $\nu \neq 0$. *They are infinitesimally equivalent to integrable unitary representations provided that* $|\nu| > m - \frac{1}{2}$.

13. **Exhaustion of the discrete series for** Spin $(1, 2m)$. In this section we intend to show that the unitary representations listed in Theorem 5 exhaust the discrete series, that is the square-integrable representations of Spin $(1, 2m)$ for $m \geqslant 2$.

Let $[\Pi, \mathcal{H}]$ be an irreducible unitary representation of G. To show that this representation is not square-integrable, it is sufficient to show that one of the functions $t \to (\Pi(h(t))F, y)$ for $0 \neq F \in E(\omega)\mathcal{H}$, $0 \neq y \in E(\omega')\mathcal{H}$, and $[\omega]$, $[\omega']$ certain irreducible K-types fails to be square-integrable on the interval $(0, \infty)$ with respect to the measure $(\sinh t)^{2m-1}\, dt$. In fact, since F and y may be embedded in an orthogonal basis of $E(\omega)\mathcal{H}$ and $E(\omega')\mathcal{H}$ respectively, there must exist a constant $C > 0$ such that

$$|(\Pi(h(t))F, y)|^2 \leqslant C\mathcal{E}(\Pi, t; \omega', \omega), \qquad t \in (0, \infty).$$

Hence the remark follows from Lemma 21.

Now, for the unitary principal series and the irreducible supplementary series, one may take $[\omega]$ and $[\omega']$ to be locking K-types as in Lemma 11. Then the fact that the function $t \to (\Pi(h(t))F, y)$ is not square-integrable follows the explicit formulas in Cases 1 and 2 of the last section. According to these formulas, the asymptotic behaviour is given by $|(\Pi(h(t))F, y)|^2 \sim [\cosh \frac{1}{2}t]^{-4\beta} \sim e^{-t(2m-1-\nu_R)}$, as $t \to \infty$, where $\nu_R = \operatorname{Re} \nu \geqslant 0$. Hence, these matrix elements are not square-integrable in these cases.

It should be remarked at this point that the asymptotic behaviour of matrix elements for the irreducible nonunitary principal series is known for more general G. (See for example [18, Chapter 9] or [17, Chapter 8].)

Next, we turn our attention to those irreducible unitary representations that come from proper subrepresentations of the nonunitary principal series representations. Because of the equivalences expressed in part II of Theorem 4, it is sufficient to consider the following representations of item 3 in the list given in Theorem 4:

(1) For $1 \leqslant i < m$, $M(\mu)_i = 0$, and $\nu = m + \frac{1}{2} - i$, the representation $[\Pi_{-\nu}, D_i^-(\nu,\mu)]$.

(2) For $M(\mu)_{m-1} > 0$, $\nu = \frac{1}{2}$, the representation $[\Pi_{-\frac{1}{2}}, D_i^F(\nu,\mu)]$.

For the sake of brevity, let us denote any of the above representations by $[\Pi_{-\nu}, E]$, with ν the appropriate positive value of ν, and let $[d\Pi_{-\nu}, dE]$ be the corresponding module of K-finite vectors.

The following is known. (See Theorem 2 above or Theorems 3, 4, and 5 of [15].) If v is any nonzero K-finite vector in $E^\perp$, the orthogonal complement of E, then the cyclic $U(\mathbf{G})$-module it generates under the action $d\Pi_{-\nu}$ includes the module dE. Actually, for the cases listed under (1) above, this cyclic module is the entire space of K-finite vectors. By means of Lemma 11 we pick a locking pair of K-types $[\omega]$ and $[\omega']$ as follows. Let $i = m$ for Cases 2 above; otherwise, let i be the index in Cases 1. Then set $\Lambda(\omega)_j = \Lambda(\omega')_{j+1} = M(\mu)_j$, if $1 \leqslant j < i$. ($\Lambda(\omega')_1$ is unspecified.) Set $\Lambda(\omega)_j = M(\mu)_j = 0$, if $i \leqslant j$, and set $\Lambda(\omega')_j = 0$, if $i < j$. Then the K-type $[\omega']$ occurs in E, but the K-type $[\omega]$ occurs in the orthogonal complement $E^\perp$. However, by the remark made above, for any element $0 \neq v \in E(\omega)E^\perp$, we may and do choose an element $q \in U(\mathbf{G})$ such that $0 \neq d\Pi_{-\nu}(q)v \in dE$. We set $y = d\Pi_{-\nu}(q)v$.

By the discussion at the beginning of §11, a unitary representation infinitesimally equivalent to $[\Pi_{-\nu}, E]$ has matrix functions constructed as follows. Let $T(\nu)$ be the intertwining operator from $[d\Pi_\nu, \mathcal{V}]$ to $[d\Pi_{-\nu}, \mathcal{V}]$. It has the kernel $E^\perp \cap \mathcal{V}$. (Recall that $\mathcal{V}$ denotes the linear subspace of all K-finite vectors in $L^2_\mu(K)$.) Now, since it is known and easy to show that $T(\nu)$ acts as a scalar operator on subspaces transforming under irreducible K-types, the matrix function of the unitary representation in question is

$$g \to A(\nu,\mu;\Pi_\nu(g)F,y) = (T(\nu)\Pi_\nu(g)F,y),$$

which is proportional to $(\Pi_\nu(g)F,y)$, provided that y is chosen to lie in a K-irreducible subspace of E. We assume that the latter is the case. In order to show that the restriction function $t \to (\Pi_\nu(h(t))F,y)$ is not square-integrable with respect to the measure $(\sinh t)^{2m-1}dt$, we avail ourselves of the following lemma.

LEMMA 22. *Let G have split rank* 1, *and assume that* $\operatorname{Re}\nu > 0$, *where* $\nu = \lambda(H)$. (*Recall the notion of* §6.) *Assume that $X \in \mathcal{V}$, and let X denote the element of $\mathcal{V}$ defined by $\tilde{X} = X(e)$. Then there exists a linear map C from $\mathcal{V}$ to $\mathcal{H}_\mu$ such that*

$$\lim_{t\to\infty} e^{(\rho-\lambda)(H)t}(\Pi_\nu(h(t))F,X) = \langle CF,\tilde{X}\rangle_\mu.$$

This result is essentially known, and its proof will be outlined in the next section. We apply this lemma as follows. For the cases under consideration, we have $(\rho - \lambda)(H) = m - \frac{1}{2} - \nu = i - 1$. If one sets $X = v \in E(\omega)E^{\perp}$, then one may use the explicit form of the matrix function given in §12 under Case 3B to conclude that $\text{limit}_{t\to\infty} e^{(i-1)t}(\Pi_\nu(h(t))F, v) = \langle CF, \tilde{v}\rangle_\mu \neq 0$ for some F. On the other hand, applying the lemma again, we conclude that for $x = y \in E$ we also have $\text{limit}_{t\to\infty} e^{(i-1)t}(\Pi_\nu(h(t))F, y) = \langle CF, \tilde{y}\rangle_\mu \neq 0$ for some F. Hence the asymptotic behaviour of the matrix function $t \to (\Pi_\nu(h(t))F, y)$ is given by $(\Pi_\nu(h(t))F, y) \sim e^{-(i-1)t}$, as $t \to \infty$. Hence we have the following result.

THEOREM 6. *The square-integrable representations listed in Theorem 5 exhaust all the square-integrable representations of* Spin(1, 2m), $m \geqslant 2$, *up to unitary equivalence.*

14. **Proof of Lemma 22.** Since the main steps in the proof of Lemma 22 are known, we only sketch its proof here, and refer to the bibliography for the proof of some of the details.

Let V be the maximal nilpotent subgroup of G defined by $V = \theta N$. Here θ is the Cartan involution of G corresponding to the Lie algebra involution of §1. Then by the Bruhat double coset lemma, the set NAMV is an open dense subset of G. Let $g \to H(g)$, and $g \to n(g)$ be analytic projections onto the analytic manifolds $\mathbf{A}$ and N respectively such that for all $g \in G$,

$$g = n(g)\exp(H(g))\kappa(g),$$

where κ is the function from G to K defined in §7. Now it follows from Lemma 43 in [8] that for all $v \in V$, $-H(v) \in \mathbf{A}_0$, and $-H(v) + H(h(t)vh(-t)) \in \mathbf{A}_0$, with $t \in (0, \infty)$. One must modify the result and proof of [8] to take into account the fact that we are using the Iwasawa decomposition $G = NAK$, instead of the decomposition $G = KAN$ as in [8]. In particular $v \to \rho(H(v))$ is bounded on V. Moreover, it follows from Lemma 44 of [8] that $v \to e^{2\rho(H(v))}$ is integrable on V with respect to a Haar measure dv. Hence, this measure may be normalized such that $\int_V e^{2\rho(H(v))}dv = 1$. With this normalization, this lemma also states that the following integration formula is valid for all continuous f on K:

$$\int_K f(k)\,dk = \int_M \int_V f(m\kappa(v))e^{2\rho(H(v))}\,dv\,dm. \tag{18}$$

Recall that dm and dk are normalized such that $\int dm = 1 = \int dk$.

Next, for F, X continuous functions in $L^2_\mu(K)$, and for $k, k' \in K$, and $m \in M$, we have $\langle F(k), X(k')\rangle_\mu = \langle F(mK), X(mk')\rangle_\mu$. This follows from the fact that μ is unitary with respect to the inner product $\langle\ ,\ \rangle_\mu$, and from the transformation properties of X and F with respect to left translations by m. Hence from Lemma 15 and formula (18) above,

$$
(19)\quad \begin{aligned} e^{t(\rho-\lambda)(H)}(\Pi_\nu(h(t))F, X) &= e^{t(\rho-\lambda)(H)}(F, \Pi_{-\bar{\nu}}(h(-t))X) \\ &= \int_V \langle F(\kappa(v)), X(\kappa(\kappa(v)h(-t)))\rangle_\mu e^{(\rho-\lambda)(tH+H[\kappa(v)h(-t)])}\,dv. \end{aligned}
$$

A key point in the proof of Lemma 22 is to justify the interchange of the limit in the statement of the lemma and the integration in equation (19). First note that the integrand can be simplified by taking into account the Iwasawa decomposition and the fact that N is normal in NA. To simplify notation write for $t \in (0, \infty)$, $x = h(t)$, and for $a, g \in G$ write $g^a = aga^{-1}$. Then in $N\backslash G$, $\exp(H(v)) \exp[tH + H(\kappa(v)x^{-1})]\kappa(\kappa(v)x^{-1}) = x \exp(H(v))\kappa(v)x^{-1}(\mathrm{mod}\, N) = xvx^{-1}(\mathrm{mod}\, N) = \exp(H(v^x))\kappa(v^x)(\mathrm{mod}\, N)$. Hence, $[tH + H(\kappa(v)x^{-1})] = H(v^x) - H(v)$, and $\kappa(\kappa(v)x^{-1}) = \kappa(v^x)$.

Pick an ε, $0 < \varepsilon < 1$, sufficiently small such that $\lambda' = \lambda - \varepsilon$ has nonnegative real part on the Weyl chamber A_0. Thus the exponent in the integrand of equation (19) may be written as

$$
\begin{aligned} (\rho - \lambda)(tH + H(\kappa(v)h(-t))) = (1 - \varepsilon)\rho(H(v^x)) + (1 + \varepsilon)\rho(H(v)) \\ + \lambda'[H(v) - H(v^x)]. \end{aligned}
$$

Since the elements $-H(v)$, $-H(v^x)$, and $-H(v) + H(v^x)$ all lie in $\mathbf{A}_0$, the exponent is equal to or less than $(1 + \varepsilon)\rho(H(v))$. Now, by the corollary to Lemma 44 in [8], the function $v \to e^{(1+\varepsilon)\rho(H(v))}$ is integrable on V. Hence, the Lebesgue dominated convergence theorem applies, and the limit may be interchanged with the integration. Now, it is easy to show that $\mathrm{limit}_{t\to\infty}\, \kappa(h(t)vh(-t)) = e$. (See [18, Vol. II, p. 319] for details.) Hence, by the continuity of X and H,

$$
\operatorname*{limit}_{t\to\infty} X(\kappa(v)^{h(t)}) = X(e),
$$

and

$$
\operatorname*{limit}_{t\to\infty} H(h(t)vh(-t)) = 0.
$$

Thus,

$$
\operatorname*{limit}_{t\to\infty} e^{t(\rho-\lambda)(H)}(\Pi_\nu(h(t))F, X) = \int_V \langle F(\kappa(v)), X(e)\rangle_\mu e^{(\lambda+\rho)(H(v))}\,dv.
$$

Let C denote the map from $\mathcal{V}$ to $\mathcal{H}_\mu$ defined by the integral

$$
F \to CF = \int_V F(\kappa(v))e^{(\lambda+\rho)(H(v))}\,dv.
$$

The lemma now follows immediately.

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Department of Mathematics, University of South Florida, Tampa, Florida 33620