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FINITENESS IN THE MINIMAL MODELS OF SULLIVAN

BY

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ABSTRACT. Let X be a 1-connected topological space such that the vector spaces $\Pi_*(X) \otimes \mathbf{Q}$ and $H^*(X; \mathbf{Q})$ are finite dimensional. Then $H^*(X; \mathbf{Q})$ satisfies Poincaré duality. Set $\chi_\Pi = \Sigma(-1)^p \dim \Pi_p(X) \otimes \mathbf{Q}$ and $\chi_c = \Sigma(-1)^p \dim H^p(X; \mathbf{Q})$. Then $\chi_\Pi \leq 0$ and $\chi_c \geq 0$. Moreover the conditions: (1) $\chi_\Pi = 0$, (2) $\chi_c > 0$, $H^*(X; \mathbf{Q})$ evenly graded, are equivalent. In this case $H^*(X; \mathbf{Q})$ is a polynomial algebra truncated by a Borel ideal.

Finally, if X is a finite 1-connected C.W. complex, and an r-torus acts continuously on X with only finite isotropy, then $\chi_\Pi \leq -r$.

1. **Introduction.** In this paper all vector spaces are defined over a field, Γ, of characteristic zero. We shall consider positively graded *finite* dimensional vector spaces $R = \Sigma_{k \geq 0} R^k$ (R^k is the subspace of elements of degree k) with homogeneous bases $x_1, \ldots, x_n$. The free commutative algebra over R is written $F(R)$ or $F(x_1, \ldots, x_n)$. $[F^l(R)]^k$ denotes the subspace spanned by elements of the form $x_{i_1} \cdot \cdots \cdot x_{i_l}$ with $\Sigma_\nu \deg x_{i_\nu} = k$. Such elements are called homogeneous of degree k.

Write $R = Q \oplus P$ where Q (respectively P) is the space spanned by the elements of even (respectively odd) degree. Then $F(R) = \bigvee Q \otimes \bigwedge P$ is the tensor product of the symmetric algebra $\bigvee Q$ over Q with the exterior algebra $\bigwedge P$ over P. We can also write $F(R) = F(x_1) \otimes \cdots \otimes F(x_n)$.

Now suppose (A, d_A) is a graded commutative differential algebra (positively graded, associative, with identity $1 \in A^0$) and suppose $\tau: R \to A \otimes F(R)$ is a linear map, homogeneous of degree 1. Then τ extends to a unique derivation, d_τ, of degree 1 in $A \otimes F(R)$ such that $d_\tau(a \otimes 1) = 0$. Extend d_A to $A \otimes F(R)$ by writing $d_A(a \otimes z) = d_A a \otimes z$.

DEFINITION. $(A, d_A; \tau; x_1, \ldots, x_n)$ will be called a *finite tower* over A if

(1) $$\tau(x_1) \in A, \quad \tau(x_i) \in A \otimes F(x_1, \ldots, x_{i-1}) \qquad (i \geq 2)$$

and

(2) $$(d_\tau + d_A)^2 = 0.$$

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The graded differential algebra $(A \otimes F(R), d_\tau + d_A)$ is called the *Koszul complex of the tower*, and the cohomology algebra $H(A \otimes F(R))$ is called the *cohomology* of the tower.

If $A = \Gamma$ then $(\tau; x_1, \ldots, x_n)$ will be called simply a *finite tower*. (In this case $\tau(x_1) = 0$.) If $\deg x_i > 0$ (respectively $> k$) for all i then the tower is called connected (respectively k-connected).

Let $(F(R), d_\tau)$ be the Koszul complex of a finite tower. The number

$$\chi_\Pi = \sum_k (-1)^k \dim R^k$$

is called the *homotopy Euler characteristic* of the tower. If $\dim H(F(R), d_\tau) < \infty$ the tower is called *c-finite*; in this case

$$\chi_c = \sum_k (-1)^k \dim H^k(F(R))$$

is called the *cohomology Euler characteristic*. Finally, if

$$\tau(x_i) \in F^+(x_1, \ldots, x_{i-1}) \cdot F^+(x_1, \ldots, x_{i-1}), \qquad i = 2, 3, \ldots,$$

then the tower is called *minimal*.

Among the principal results of this paper is the following:

THEOREM 1. *Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, c-finite minimal tower. Then $\chi_\Pi \leqslant 0$ and $\chi_c \geqslant 0$. Moreover, the following conditions are equivalent*:

(1) $\chi_\Pi = 0$.

(2) $\chi_c > 0$.

(3) $H(F(x_1, \ldots, x_n))$ *is evenly graded.*

(*In fact we shall show that for each p, $\Sigma_{i \geqslant p}(-1)^i \dim R^i \leqslant 0$ (Corollary 2 in §6) where R^i is the span of the x_j with $\deg x_j = i$.*)

The proof of Theorem 1 is contained in the next six sections. Then, in §8, we show that under the hypotheses of Theorem 1, $H(F(x_1, \ldots, x_n))$ satisfies Poincaré duality, and that the degree m of the top dimensional cohomology class is given by

$$m = r - \sum_{i=1}^{n} (-1)^{\deg x_i} \deg x_i,$$

where r is the number of x_i of even degree.

In §§9 and 10, we show that if m is even and $\chi_\Pi < 0$, then the Poincaré inner product in $\Sigma_j H^{2j}(F(x_1, \ldots, x_n))$ is hyperbolic. Finally in §11, we show that if $\chi_\Pi = 0$ then $(F(x_1, \ldots, x_n), d_\tau)$ is isomorphic with a Koszul complex of the form $(\bigvee Q \otimes \bigwedge P, d)$ with $d(Q) = 0$ and $d(P) \subset \bigvee Q$. In this case $H(\bigvee Q \otimes \bigwedge P) \cong \bigvee Q/I$, where I is the ideal generated by $d(P)$.

Now consider a connected topological space X and let $A(X)$ be the graded commutative differential algebra of rational differential forms on the singular complex of X (cf. Sullivan [5, §D]): in particular, $H(A(X)) \cong H^*(X; \mathbf{Q})$ (singular cohomology). There is a commutative connected graded differential algebra $(F(R), d)$ (over $\mathbf{Q}$) and a homomorphism $\phi: F(R) \to A(X)$ of graded differential algebras such that

(1) ϕ induces an isomorphism of cohomology.

(2) There is a homogeneous basis $\{x_\alpha\}_{\alpha \in \mathfrak{T}}$ of R, where $\mathfrak{T}$ is *well ordered*, such that dx_α is a polynomial in those x_β with $\beta < \alpha$ and $\deg x_\beta \leqslant \deg x_\alpha$.

Moreover, $(F(R), d)$ is determined up to isomorphism by these conditions.

We shall call the spaces R^k the *pseudo dual rational homotopy spaces of* X, and denote them by $\Pi_\psi^k(X)$. If $H^1(X; \mathbf{Q}) = 0$ and $H^*(X; \mathbf{Q})$ has finite type, then these spaces are finite dimensional. If, in addition, X is simply connected then there are natural isomorphisms [5, §Z] $[\Pi_k(X) \otimes \mathbf{Q}]^* \cong \Pi_\psi^k(X)$.

Write $\Pi_\psi^*(X) = \Sigma_k \Pi_\psi^k(X)$ and $\Pi_*(X) = \Sigma_k \Pi_k(X)$. Then the remarks above, together with Theorem 1, yield:

THEOREM 1′. *Let X be a connected topological space such that $\Pi_\psi^*(X)$ and $H^*(X; \mathbf{Q})$ are finite dimensional. Then*

$$\sum_k (-1)^k \dim \Pi_\psi^k(X) \leqslant 0 \quad \textit{and} \quad \sum_k (-1)^k \dim H^k(X; \mathbf{Q}) \geqslant 0.$$

Moreover, the following conditions are equivalent:

(1) $\Sigma_k (-1)^k \dim \Pi_\psi^k(X) = 0$.

(2) $\Sigma_k (-1)^k \dim H^k(X; \mathbf{Q}) > 0$.

(3) $H^p(X; \mathbf{Q}) = 0$, *p odd.*

COROLLARY 1. *If X is simply connected then the theorem remains true if $\Pi_\psi^k(X)$ is replaced by $\Pi_k(X) \otimes \mathbf{Q}$ everywhere in the statement.*

Finally, we have the following application to transformation groups (see Remark 3 below):

THEOREM T. *Let a compact Lie group G of rank r act on a simply connected finite C.W. complex X with only finite isotropy. Assume that $\Pi_*(X) \otimes \mathbf{Q}$ is finite dimensional. Then $\Sigma(-1)^k \dim \Pi_k(X) \otimes \mathbf{Q} \leqslant -r$.*

PROOF. According to Allday [1, Theorem 2.1.1, p. 177] this follows from Theorem 1′(1).

REMARKS. 1. The special case of a finite tower over A with R oddly graded and $\tau(R) \subset A$ was first considered by Koszul [4] in 1950. Cartan [2] showed that the cohomology of a homogeneous space can be calculated via a Koszul complex of this form, where, in addition, A is a symmetric algebra and $d_A = 0$. Cartan also obtains a special case of Theorem 1; indeed the general theorem will be established by reduction to this earlier result.

The Koszul complex of a minimal connected tower is a nilpotent minimal model as defined by Sullivan [5].

2. Historically, this paper begins with Theorem T which was conjectured by W. Y. Hsiang in 1969 or earlier. Then in 1971 Allday [1] reduced Hsiang's conjecture to Theorem 1′ ($\chi_\Pi \leqslant 0$) in the simply connected case. The translation from Theorem 1′ (1) to Theorem 1(1) was observed by Sullivan who poses it as question 5 in [5, §Q].

3. Theorem T remains valid for a much wider class of spaces, X. In particular it is sufficient to assume that X is connected (but not necessarily simply connected) if we replace $\Pi_*(X) \otimes \mathbf{Q}$ by $\Pi_\psi^*(X)$ everywhere in the statement. Precise statements and details of the proof will appear elsewhere. As a special case of this generalized Theorem T, however, we have

THEOREM H. *Let $K \subset G$ be compact Lie groups and suppose a torus, T, acts on G/K continuously, with only finite isotropy. Then* $\dim T \leqslant \operatorname{rank} G - \operatorname{rank} K$.

When $K = (e)$ this is proved by Allday [1]. If G/K is 1-connected then Theorem H follows from the "ungeneralized" Theorem T.

2. **Notation.** By a graded commutative differential algebra (A, d_A) we mean a positively graded associative algebra $A = \sum_{k \geqslant 0} A^k$ with identity $1 \in A^\circ$ such that $ab = (-1)^{rs} ba$, $a \in A^r$, $b \in A^s$. Here d_A denotes a derivation of degree 1 with $d_A^2 = 0$. The cohomology algebra $\ker d_A / \operatorname{Im} d_A$ is written $H(A) = \sum_k H^k(A)$. A homomorphism ϕ: $(A, d_A) \to (B, d_B)$ of graded differential algebra induces a homomorphism ϕ^*: $H(A) \to H(B)$.

The tensor product of graded algebras A and B is given the multiplication defined by $(a \otimes b)(a' \otimes b') = (-1)^{qp'} aa' \otimes bb'$, $b \in B^q$, $a' \in A^{p'}$.

The subspace of a vector space spanned by elements $u_1, \ldots$ is denoted by $(u_1, \ldots)$. If U and V are subspaces of a vector space W, $U + V$ is the subspace spanned by U and V. If W is an algebra, $U \cdot V$ is the subspace spanned by elements of the form uv, $u \in U$, $v \in V$; $U \cdot U$ is written U^2.

An evenly (respectively oddly) graded space is a space with no nonzero elements of odd (respectively even) degree.

The identity map of any set is denoted by ι.

Let $R = (x_1, \ldots, x_n)$ be as in the introduction, and suppose $(A, d_A; \tau; x_1, \ldots, x_n)$ is a tower over A with Koszul complex $(A \otimes F(R), d)$. Then for each m, $(A, d_A; \tau; x_1, \ldots, x_m)$ is a tower over A with Koszul complex the subdifferential algebra $(A \otimes F(x_1, \ldots, x_m), d)$. Write this (B, d_B).

Then $A \otimes F(R) = B \otimes F(x_{m+1}, \ldots, x_n)$ and so we may regard τ as a linear map τ: $(x_{m+1}, \ldots, x_n) \to B \otimes F(x_{m+1}, \ldots, x_n)$. Clearly $(B, d_B; \tau; x_{m+1}, \ldots, x_n)$ is a tower over B whose Koszul complex coincides with the Koszul complex $(A \otimes F(R), d)$.

Next, let $(\tau; x_1, \ldots, x_n)$ be a finite tower and denote by (B, d_B) the subdifferential algebra $F(x_1, \ldots, x_m)$ of $(F(R), d)$. Then, as above, $(F(R), d)$ is also the Koszul complex of the tower $(B, d_B; \tau; x_{m+1}, \ldots, x_n)$ over B. The projection $\rho\colon B \to \Gamma$ satisfies $\rho \circ d_B = 0$. Hence by Lemma 1, below, it determines a tower $(\bar{\tau}; x_{m+1}, \ldots, x_n)$ with

$$\bar{\tau}(x_i) = (\rho \otimes \iota)(\tau x_i) \in F(x_{m+1}, \ldots, x_n), \qquad i = m+1, \ldots, n.$$

The maps

$$F(x_1, \ldots, x_m) \to F(x_1, \ldots, x_n)$$

and

$$\rho \otimes \iota\colon F(x_1, \ldots, x_n) \to F(x_{m+1}, \ldots, x_n)$$

are homomorphisms of graded differential algebras. They will be called, respectively, a *base inclusion* and a *fibre projection*.

Finally, suppose $(A, d_A; \tau; x_1, \ldots, x_n)$ is a tower over A. Let $\omega \in S_n$ be some permutation such that for each i, $\tau(x_{\omega(i)}) \in A \otimes F(x_{\omega(1)}, \ldots, x_{\omega(i-1)})$. Then $(A, d_A; \tau; x_{\omega(1)}, x_{\omega(2)}, \ldots, x_{\omega(n)})$ is again a tower over A; it is called a *rearrangement* of the original tower, and has the same Koszul complex.

Observe that the following properties of a tower $(\tau; x_1, \ldots, x_n)$: c-finiteness, k-connectivity, minimality depend only on the Koszul complex, and so hold for any rearrangement. (In particular, the tower is minimal if and only if $\tau(R) \subset F^+(R) \cdot F^+(R)$.) If $(\tau; x_1, \ldots, x_n)$ is a minimal connected tower then there is a permutation, ω, such that $\deg x_{\omega(1)} \leq \deg x_{\omega(2)} \leq \ldots$, and $(\tau; x_{\omega(1)}, \ldots, x_{\omega(n)})$ is again a tower.

LEMMA 1. *Suppose $(A, d_A; \tau; x_1, \ldots, x_n)$ is a tower, and let $\phi\colon (A, d_A) \to (B, d_B)$ be a homomorphism of graded commutative differential algebras. Define $\sigma\colon R \to B \otimes F(R)$ by $\sigma(x_i) = (\phi \otimes \iota)(\tau x_i)$.*

Then $(B, d_B; \sigma; x_1, \ldots, x_n)$ is a tower over B and $\phi \otimes \iota\colon A \otimes F(R) \to B \otimes F(R)$ is a homomorphism of graded differential algebras. Moreover if $\phi^*\colon H(A) \to H(B)$ is an isomorphism then $(\phi \otimes \iota)^*$ is an isomorphism.

PROOF. Clearly $\sigma(x_i) \in B \otimes F(x_1, \ldots, x_{i-1})$. Moreover

$$d_\sigma \circ (\phi \otimes \iota)(1 \otimes x_i) = \sigma(x_i) = (\phi \otimes \iota) \circ d_\tau(1 \otimes x_i)$$

and

$$d_\sigma \circ (\phi \otimes \iota)(a \otimes 1) = d_\sigma(\phi a \otimes 1) = 0 = (\phi \otimes \iota) \circ d_\tau(a \otimes 1).$$

Since $d_\sigma \circ (\phi \otimes \iota) - (\phi \otimes \iota) \circ d_\tau$ is a $(\phi \otimes \iota)$-derivation these equations imply that it is zero:

$$d_\sigma \circ (\phi \otimes \iota) = (\phi \otimes \iota) \circ d_\tau.$$

Hence also $(d_\sigma + d_B) \circ (\phi \otimes \iota) = (\phi \otimes \iota) \circ (d_\tau + d_A)$.

Now we obtain

$$(d_\sigma + d_B)^2(1 \otimes x_i) = (d_\sigma + d_B)^2(\phi \otimes \iota)(1 \otimes x_i)$$
$$= (\phi \otimes \iota)(d_\tau + d_A)^2(1 \otimes x_i) = 0.$$

Since $(d_\sigma + d_B)^2(b \otimes 1) = d_B^2(b) \otimes 1 = 0$, it follows that $(d_\sigma + d_B)^2 = 0$. Thus $(B, d_B; \sigma; x_1, \ldots, x_n)$ is a tower and $\phi \otimes \iota$ is a homomorphism of graded differential algebras.

Finally, suppose ϕ^* is an isomorphism. We shall show (by induction on m) that the restrictions

$$(\phi \otimes \iota)_m: A \otimes F(x_1, \ldots, x_m) \to B \otimes F(x_1, \ldots, x_m)$$

induce isomorphisms of cohomology.

Suppose first that $m = 1$. Filter $A \otimes F(x_1)$ and $B \otimes F(x_1)$ by the subspaces

$$L^{-p} = \sum_{j=0}^{p} A \otimes F^j(x_1) \quad \text{and} \quad \hat{L}^{-p} = \sum_{j=0}^{p} B \otimes F^j(x_1), \qquad p = 0, 1, \ldots.$$

Then $\phi \otimes \iota$ is filtration preserving, and so it induces a homomorphism α_i: $(E_i, d_i) \to (\hat{E}_i, \hat{d}_i)$ of spectral sequences. In particular, α_1 is given by

$$\alpha_1 = \phi^* \otimes \iota: H(A) \otimes F(x_1) \xrightarrow{\cong} H(B) \otimes F(x_1).$$

Thus each α_i $(1 \leqslant i < \infty)$ is an isomorphism. Since $E_i^{p,q} = 0 = \hat{E}_i^{p,q}$ for $p > 0$ we have $E_\infty^{p,q} = \text{ind lim } E_i^{p,q}$ (i large). It follows that α_∞ is an isomorphism. Hence $(\phi \otimes \iota)^*$ induces an isomorphism in the bigraded algebra determined by the filtrations in $H(A \otimes F(x_1))$ and $H(B \otimes F(s_1))$. This implies that $(\phi \otimes \iota)^*$ is an isomorphism.

Finally, assume by induction that $(\phi \otimes \iota)^*_{m-1}$ is an isomorphism. Write $(\phi \otimes \iota)_{m-1} = \psi$, $A \otimes F(x_1, \ldots, x_{m-1}) = A'$, $B \otimes F(x_1, \ldots, x_{m-1}) = B'$. Apply the argument above to

$$\phi_m \otimes \iota = \psi \otimes \iota: A' \otimes F(x_m) \to B' \otimes F(x_m)$$

to obtain that $(\phi_m \otimes \iota)^*$ is an isomorphism. Q.E.D.

Example. Let $(A, d_A; \tau; x_1)$ be a tower with deg x_1 odd. Its Koszul complex is given by $(A \otimes \bigwedge x_1, d)$, where

$$d(a \otimes x_1 + b \otimes 1) = d_A a \otimes x_1 + \left((-1)^{\deg a} a \cdot \tau(x_1) + d_A b\right) \otimes 1.$$

In particular $\tau(x_1)$ is a cocycle representing a class $\alpha \in H(A)$.

A short exact sequence $0 \to A \to^{\phi} A \otimes \bigwedge x_1 \to^{\Psi} A \to 0$ is given by $\phi a = a \otimes 1$, $\Psi(a \otimes x_1 + b \otimes 1) = a$. The ensuing long exact (Gysin) sequence in cohomology has connecting homomorphism $\partial: H(A) \to H(A)$ given by $\partial\beta = \alpha \cdot \beta$. This sequence yields the short exact sequence

$$0 \to \operatorname{Coker} \partial \xrightarrow{\bar{\phi}^*} H(A \otimes \wedge x_1) \xrightarrow{\Psi^*} \operatorname{Ker} \partial \to 0.$$

(Cf. [3, Chapter III] for details.)

3. **Pure towers.** Let $(\sigma; x_1, \ldots, x_n)$ be a finite tower. As in §1 write $R = (x_1, \ldots, x_n) = Q \oplus P$ where Q is evenly graded and P is oddly graded. The tower will be called *pure* if $\sigma(P) \subset \vee Q$ and $\sigma(Q) = 0$. Koszul complexes of pure towers were studied by Koszul [4] and H. Cartan [2]; we recall here some of their results.

Let $(\vee Q \otimes \wedge P, d)$ be the Koszul complex of a pure tower $(\sigma; x_1, \ldots, x_n)$. Then d: $\vee Q \otimes \wedge^i P \to \vee Q \otimes \wedge^{i-1} P$, and thus the gradation $\vee Q \otimes \wedge P = \Sigma_k \vee Q \otimes \wedge^k P$ leads to a gradation of $H(\vee Q \otimes \wedge P)$, written $H(\vee Q \otimes \wedge P) = \Sigma_k H_k(\vee Q \otimes \wedge P)$. Let $\vee Q \circ P$ be the ideal in $\vee Q$ generated by $\sigma(P)$; then the inclusion l: $\vee Q \to \vee Q \otimes \wedge P$ induces an isomorphism l^*: $\vee Q / \vee Q \circ P \xrightarrow{\cong} H_0(\vee Q \otimes \wedge P)$.

If $P_1 \subset P$ is any graded subspace then $\vee Q \otimes \wedge P_1$ is a subdifferential algebra of $\vee Q \otimes \wedge P$.

LEMMA 2. *If* $H_k(\vee Q \otimes \wedge P_1) \neq 0$ *then* $H_k(\vee Q \otimes \wedge P) \neq 0$.

PROOF. By considering a sequence of spaces $P_1 \subset P_2 \subset \cdots \subset P_m = P$ we can reduce to the case $P = P_1 \oplus (x)$. Set $(A, d_A) = (\vee Q \otimes \wedge P_1, d)$; then $\vee Q \otimes \wedge P = A \otimes \wedge x$ is the Koszul complex of the tower $(A, d_A; \sigma; x)$.

Now apply the example of §2 to obtain a Gysin sequence in which the connecting homomorphism ∂: $H(A) \to H(A)$ is multiplication by the class $\alpha \in H(A)$ represented by $\sigma(x)$. Since $\sigma(x) \in \vee Q$ it follows that for some $p > 0$, $\alpha \in H_0^p(\vee Q \otimes \wedge P_1)$. This implies that ∂ restricts to linear maps ∂_i: $H_i(\vee Q \otimes \wedge P_1) \to H_i(\vee Q \otimes \wedge P_1)$ of positive degree. In particular, since $H_k(\vee Q \otimes \wedge P_1) \neq 0$, then $\operatorname{Coker} \partial_k \neq 0$.

Finally note that the inclusion $\operatorname{Coker} \partial \to H(A \otimes \wedge x)$ of the example in §2 is the direct sum of inclusions $\operatorname{Coker} \partial_i \to H_i(\vee Q \otimes \wedge P)$. Thus since $\operatorname{Coker} \partial_k \neq 0$ we have $H_k(\vee Q \otimes \wedge P) \neq 0$. Q.E.D.

The following is due to Cartan [2]. A detailed proof is given in [3, Chapter 2].

THEOREM 2. *Let* $(\vee Q \otimes \wedge P, d)$ *be the Koszul complex of a connected pure tower such that* $\dim H(\vee Q \otimes \wedge P) < \infty$. *Then* $H(\vee Q \otimes \wedge P)$ *has nonnegative Euler characteristic* χ. *Moreover* $\dim P - \dim Q$ *is the nonnegative integer k with the property*

$$(3) \qquad H_k(\vee Q \otimes \wedge P) \neq 0, \quad H_{k+p}(\vee Q \otimes \wedge P) = 0, \qquad p \geqslant 1.$$

Finally, the following conditions are equivalent:

(i) $\dim P = \dim Q$.

(ii) $\chi > 0$.

(iii) $H(\bigvee Q \otimes \bigwedge P)$ *is evenly graded.*
(iv) $H(\bigvee Q \otimes \bigwedge P) = H_0(\bigvee Q \otimes \bigwedge P)$.

REMARK. $\dim H(\bigvee Q \otimes \bigwedge P) < \infty$ if and only if $\dim \bigvee Q/\bigvee Q \circ P < \infty$. In fact note that $\ker d$ is a $\bigvee Q$-submodule of the finitely generated $\bigvee Q$-module $\bigvee Q \otimes \bigwedge P$. Because $\bigvee Q$ is noetherian $\ker d$ is finitely generated. Thus $H(\bigvee Q \otimes \bigwedge P)$ is a finitely generated $\bigvee Q$ module.

This implies (clearly) that $H(\bigvee Q \otimes \bigwedge P)$ is a finitely generated module over $H_0(\bigvee Q \otimes \bigwedge P)$. Thus $\dim H(\bigvee Q \otimes \bigwedge P) < \infty$ if and only if $\dim H_0(\bigvee Q \otimes \bigwedge P) < \infty$; i.e., if and only if $\dim \bigvee Q/\bigvee Q \circ P < \infty$.

4. **The S-spectral sequence.** As in §1 let $R = (x_1, \ldots, x_n)$. Assume $\deg x_i > 0$, $i = 1, \ldots, n$. Let S be a subspace spanned by some of the x_i and let T be the subspace spanned by the remaining x_j. (Then $R = T \oplus S$.)

Now suppose $(A, d_A; \tau; x_1, \ldots, x_n)$ is a tower over A. Then $A \otimes F(R) = A \otimes F(T) \otimes F(S)$ and so a bigradation of $A \otimes F(R)$ is given by

$$[A \otimes F(R)]^{p,q} = [A \otimes F(T) \otimes F^{-q}(S)]^{p+q}.$$

Write $(A \otimes F(R), d_\tau + d_A) = (C, d_c)$: $C = \sum_{p,q} C^{p,q}$.

Clearly $C^{p,q} \cdot C^{r,s} \subset C^{p+r,q+s}$ and so C is filtered by the ideals $I^p = \sum_{j \geqslant p} C^{j,*}$. (Note $C^{p,q} = 0$ if $p < 0$.)

Now let σ: $R \to A \otimes F(T)$ be the unique linear map such that $\sigma(x) = 0$, $x \in T$ and $\sigma(x) - \tau(x) \in A \otimes F(T) \otimes F^+(S)$, $x \in S$. Extend σ to a derivation d_σ in C such that $d_\sigma(A) = 0$. Clearly $d_\sigma^2 = 0$.

LEMMA 3. (i) *d_σ is homogeneous of bidegree* $(0, 1)$.
(ii) $d_c - d_\sigma$: $I^p \to I^{p+1}$.

PROOF. Clear.

The lemma shows that the I^p filter the graded differential algebra (C, d_c), and that the first term of the resulting spectral sequence (of graded differential algebras) is given by

$$(E_0, d_0) \cong (C, d_\sigma). \tag{4}$$

Moreover, because the elements in $F^q(S)$ have degree at least q, it follows that $C^{p,q} = 0$ unless $0 \leqslant -2q \leqslant p$. This implies that the spectral sequence converges to $H(C, d_c)$. This spectral sequence will be called the *S-spectral sequence*.

In particular, if P denotes the subspace of R of elements of odd degree then the P-spectral sequence will be called the *odd spectral sequence*.

5. **The odd spectral sequence of a tower.** Let $R = (x_1, \ldots, x_n)$ and suppose $(\tau; x_1, \ldots, x_n)$ is a connected finite tower. As usual write $F(R) = \bigvee Q \otimes \bigwedge P$.

Let $\sigma: R \to \bigvee Q$ be the linear map defined by $\sigma(Q) = 0$ and $\sigma(x) - \tau(x) \in \bigvee Q \otimes \bigwedge^+ P$, $x \in P$. Then $(\sigma; x_1, \ldots, x_n)$ is a pure tower, called the *associated pure tower* for $(\tau; x_1, \ldots, x_n)$.

Observe as well that $\tau(Q) \subset F(R)^{\text{odd}} \subset \bigvee Q \otimes \bigwedge^+ P$. It follows that

(5) $$d_\tau - d_\sigma: \bigvee Q \otimes \bigwedge P \to \bigvee Q \otimes \bigwedge^+ P.$$

If (E_i, d_i) is the odd spectral sequence for the original tower then

(6) $$(E_0, d_0) \cong (\bigvee Q \otimes \bigwedge P, d_\sigma)$$

(cf. formula (4), §4). This isomorphism restricts to isomorphisms $E_0^{p,q} \cong (\bigvee Q \otimes \bigwedge^{-q} P)^{p+q}$. Thus there is an algebra isomorphism

(7) $$E_1 \cong H(\bigvee Q \otimes \bigwedge P, d_\sigma)$$

which restricts to isomorphisms

(8) $$E_1^{p,q} \cong H_{-q}^{p+q}(\bigvee Q \otimes \bigwedge P, d_\sigma).$$

Now we show that $d_1 = 0$, so that $E_2 \cong E_1$. In fact by (5), $(d_\tau - d_\sigma)(Q) \subset \bigvee Q \otimes \bigwedge^+ P$ while

$$(d_\tau - d_\sigma)(P) \subset (\bigvee Q \otimes \bigwedge^+ P) \cap \left(\sum_{r \text{ even}} (\bigvee Q \otimes \bigwedge P)^r \right)$$
$$\subset \sum_{j \geq 2} \bigvee Q \otimes \bigwedge^j P.$$

Thus

$$(d_\tau - d_\sigma): (\bigvee Q \otimes \bigwedge^{-q} P)^{p+q} \to \sum_{j \geq 1} (\bigvee Q \otimes \bigwedge^{-q+j} P)^{p+q+1}.$$

It follows that $d_\tau - d_\sigma: I^p \to I^{p+2}$ (the I^p are the ideals defining the spectral sequence) and hence the differential $d_1 = 0$.

Similarly, it follows at once from the definition that

$$I^p \cap F(R)^r = \sum_{k \geq p-r} (\bigvee Q \otimes \bigwedge^k P)^r.$$

Thus if $J = \bigvee Q \otimes \bigwedge^+ P$ and $J_k = J \cdot \cdots \cdot J$ (k factors), then $I^p \cap F(R)^r = \sum_{k \geq p-r} J_k \cap F(R)^r$.

Now suppose $F(R') = \bigvee Q' \otimes \bigwedge P'$ is the Koszul complex of a second finite tower and assume $\phi: F(R) \to F(R')$ is a homomorphism of graded differential algebras. Since J and J' are the ideals generated by elements of odd degree, $\phi(J_k) \subset J'_k$, $k = 1, 2, \ldots$.

Now the formula above for I^p shows that ϕ is filtration preserving. Hence it induces a homomorphism of spectral sequences. In particular, if ϕ is an isomorphism then ϕ^{-1} is also filtration preserving and so ϕ and ϕ^{-1} induce inverse isomorphisms of the odd spectral sequences.

6. **Proof of Theorem 1.** Recall from §2 that a tower $(\tau;\ x_1, \ldots, x_n)$ determines towers $(\bar{\tau};\ x_p, \ldots, x_n)$. Since $\bar{\tau}(x_p) = 0$, x_p is a cocycle in $(F(x_p, \ldots, x_n))$. Let $[x_p] \in H(F(x_p, \ldots, x_n), d_{\bar{\tau}})$ be the class represented by x_p.

PROPOSITION 1. *Let $(\tau;\ x_1, \ldots, x_n)$ be a connected, finite, minimal tower. Write $R = (x_1, \ldots, x_n)$, $F(R) = \bigvee Q \otimes \bigwedge P$. Suppose (E_i, d_i) denotes the odd spectral sequence and $(\sigma;\ x_1, \ldots, x_n)$ is the associated pure tower. Then the following are equivalent*:

(1) *The tower is c-finite*: $\dim H(\bigvee Q \otimes \bigwedge P, d_\tau) < \infty$.

(2) *For each p the class $[x_p] \in H(F(x_p, \ldots, x_n), d_{\bar{\tau}})$ is nilpotent*: $[x_p]^k = 0$, *some k*.

(3) $\dim H(\bigvee Q \otimes \bigwedge P, d_\sigma) < \infty$.

(4) $\dim E_1 < \infty$.

PROOF. (1) $\Rightarrow$ (2). This is deferred until §7 (Lemma 5).

(2) $\Rightarrow$ (3). Denote by Q_p the subspace of Q spanned by the x_i of even degree with $i \leqslant p$, and set $Q_0 = 0$. We show first by induction on p that the elements of Q_p represent nilpotent classes in $H(\bigvee Q \otimes \bigwedge P, d_\sigma)$.

This is certainly true for $p = 0$. Suppose it is true for $p - 1$. If x_p has odd degree then $Q_p = Q_{p-1}$ and our claim is true for p. If x_p has even degree our hypothesis shows that for some $u, v_i \in \bigvee Q \otimes \bigwedge P$ and some $k \geqslant 1$:

$$x_p^k = d_\tau u - \sum_{i=1}^{p-1} x_i \cdot v_i.$$

Hence $d_\tau u - x_p^k \in Q_{p-1} \cdot \bigvee Q + \bigvee Q \otimes \bigwedge^+ P$. Thus formula (5) yields

$$d_\sigma u - x_p^k \in Q_{p-1} \cdot \bigvee Q + \bigvee Q \otimes \bigwedge^+ P. \tag{9}$$

Now write $u = \sum u_i$, $u_i \in \bigvee Q \otimes \bigwedge^i P$. Since $d_\sigma\colon \bigvee Q \otimes \bigwedge^i P \to \bigvee Q \otimes \bigwedge^{i-1} P$, it follows from (9) that

$$d_\sigma u_1 - x_p^k \in Q_{p-1} \cdot \bigvee Q.$$

Since the elements of $\bigvee Q$ are d_σ-cocycles and the elements of Q_{p-1} represent nilpotent classes in $H(\bigvee Q \otimes \bigwedge P, d_\sigma)$, it follows that the elements of $Q_{p-1} \cdot \bigvee Q$ represent nilpotent classes. Hence the equation above implies that x_p represents a nilpotent class. The induction is now closed.

We have now shown that the elements in Q represent nilpotent classes in $H(\bigvee Q \otimes \bigwedge P, d_\sigma)$. This implies that $\bigvee Q$ has finite dimensional image in $H(\bigvee Q \otimes \bigwedge P, d_\sigma)$. The remark in §3 now implies that

$$\dim H(\bigvee Q \otimes \bigwedge P, d_\sigma) < \infty.$$

(3) $\Rightarrow$ (4). Apply formula (7).

(4) $\Rightarrow$ (1). Recall that the spectral sequence converges to

$$H(\bigvee Q \otimes \bigwedge P, d_\tau). \quad \text{Q.E.D.}$$

COROLLARY. *If* $(\tau; x_1, \ldots, x_n)$ *is a connected, finite, c-finite, minimal tower, then for each p the tower* $(\bar{\tau}; x_p, \ldots, x_n)$ *is also c-finite.*

THEOREM 1. *Let* $(\tau; x_1, \ldots, x_n)$ *be a connected, finite, c-finite, minimal tower. Then* $\chi_\Pi \leqslant 0$ *and* $\chi_c \geqslant 0$. *Moreover, the following conditions are equivalent.*

(1) $\chi_\Pi = 0$.

(2) $\chi_c > 0$.

(3) $H(F(x_1, \ldots, x_n), d_\tau)$ *is evenly graded.*

PROOF. We adopt the notation of Proposition 1. Then according to Proposition 1, $H(\bigvee Q \otimes \bigwedge P, d_\sigma)$ has finite dimension. Denote its Euler characteristic by χ. Since $H(\bigvee Q \otimes \bigwedge P, d_\sigma) \cong E_1$ and since (E_i, d_i) converges to $H(\bigvee Q \otimes \bigwedge P, d_\tau)$ it follows that $\chi = \chi_c$.

Moreover, since $H(\bigvee Q \otimes \bigwedge P, d_\sigma)$ has finite dimension we can apply Theorem 2, §3 to obtain $\chi_\Pi = \dim Q - \dim P \leqslant 0$ and $\chi_c = \chi \geqslant 0$.

The equivalence of conditions (i), (ii) and (iii) in Theorem 2 implies that conditions (1) and (2) are equivalent, and hold if and only if

$$H(\bigvee Q \otimes \bigwedge P, d_\sigma)$$

is evenly graded. But in this case E_1 is evenly graded and so the odd spectral sequence collapses at the E_1-term. In particular, $H(\bigvee Q \otimes \bigwedge P, d_\tau)$ is evenly graded. Thus (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). But clearly (3) $\Rightarrow$ (2). Q.E.D.

COROLLARY 1. *Let* (E_i, d_i) *be the odd spectral sequence of a connected, finite, c-finite minimal tower. Then for* $i \geqslant 1$: $E_i^{p,q} = 0$, $q < \chi_\Pi$.

PROOF. By formula (8), $E_1^{p,q} \cong H_{-q}^{p+q}(\bigvee Q \otimes \bigwedge P, d_\sigma)$. If $q < \chi_\Pi$ then $H_{-q}(\bigvee Q \otimes \bigwedge P, d_\sigma) = 0$ by formula (3), Theorem 2. Thus $E_1^{p,q} = 0$, $q < \chi_\Pi$ and so $E_i^{p,q} = 0$, $q < \chi_\Pi$. Q.E.D.

COROLLARY 2. *Let* $(\tau; x_1, \ldots, x_n)$ *be a connected, finite, c-finite minimal tower. Then for each p,*

$$\sum_{i=p}^{n} (-1)^{\deg x_i} \leqslant 0. \tag{10}$$

In particular, if $R = (x_1, \ldots, x_n)$ *then for each p,*

$$\sum_{i>p} (-1)^i \dim R^i \leqslant 0. \tag{11}$$

PROOF. In view of the corollary to Proposition 1 we may apply Theorem 1 to the tower $(\bar{\tau}; x_p, \ldots, x_n)$ to obtain (10). Next note (cf. §2) that we may rearrange the x_i so that $\deg x_{\omega(1)} \leqslant \deg x_{\omega(2)} \leqslant \dots$. Now (11) is a special case of (10) (with x_i replaced by $x_{\omega(i)}$). Q.E.D.

COROLLARY 3. *Let X be a connected topological space such that $H^*(X; \mathbf{Q})$ and $\Pi^*_\psi(X)$ are finite dimensional. Then for each p, $\sum_{i \geqslant p}(-1)^i \dim \Pi^i_\psi(X) \leqslant 0$. If X is simply connected then for each p, $\sum_{i \geqslant p}(-1)^i \dim \Pi_i(X) \otimes \mathbf{Q} \leqslant 0$.*

PROOF. This follows from Corollary 2 in the same way Theorem 1′ followed from Theorem 1 (cf. §1). Q.E.D.

COROLLARY 4. *The odd spectral sequence for a connected, finite, c-finite, minimal tower with $\chi_\Pi = 0$ collapses at the E_1-term.*

Proposition 2 below and its proof are due to C. Allday (private communication). It is a special case of his conjecture ** in [1]; the general case remains open.

Let $A = \sum_{k \geqslant 0} A^k$ be a graded vector space of finite type. Its *Poincaré series* is the formal series $f_A(t) = \sum_k \dim A^k t^k$. Following Hsiang set

$$\rho_0(A) = \inf\{\alpha \in \mathbf{R} \mid (1-t)^\alpha f_A(t) \to 0 \quad \text{as} \quad t \to 1-\}.$$

If $g = \sum a_k t^k$ and $h = \sum b_k t^k$ are two formal series with integer coefficients we write $g \leqslant h$ to mean $a_k \leqslant b_k$, each k.

PROPOSITION 2. *Suppose $(F(R), d_\tau)$ is the Koszul complex of a connected, finite, minimal tower $(\tau; x_1, \dots, x_n)$ with homotopy Euler characteristic χ_Π. Assume $H(F(R))$ is finitely generated as an algebra over Γ. Then $\chi_\Pi \leqslant \rho_0(H(F(R)))$.*

REMARK. As will appear in the proof, $\rho_0(H(F(R)))$ is the Krull dimension of the commutative algebra $\sum_k H^{2k}(F(R))$.

PROOF. Denote the commutative subalgebra $\sum_k H^{2k}(F(R))$ by A. Using the argument of [7, p. 201] we construct a sequence $z_0, \dots, z_l$ of homogeneous elements in A^+ as follows: assume $z_0, \dots, z_i$ are constructed with $z_0 = 0$, and generate an ideal Z_i with isolated prime ideals $J_1, \dots, J_k$ (cf. [6, p. 211]). These are necessarily graded, and hence in A^+. Thus either $k = 1$ and $J_1 = A^+$ or there is a homogeneous element z_{i+1} in A^+ such that $z_{i+1} \notin \bigcup_\lambda J_\lambda$.

The sequence $Z_1, Z_2, \dots$ terminates at some Z_l because A is noetherian; in particular, A^+ is the unique prime ideal for Z_l and so A/Z_l has finite dimension.

Choose a sequence $K_l \supset \cdots \supset K_1 \supset K_0$ with K_i an isolated prime ideal for Z_i; then $z_i \notin K_{i-1}$, $i \geqslant 1$. Thus an easy induction on $l - i$ shows that the obvious homomorphism $\bigvee(z_{i+1}, \dots, z_l) \to A/Z_i$ is injective. In particular we have an inclusion $\bigvee(z_1, \dots, z_l) \to A$. On the other hand, if F is a (finite dimensional) graded space such that $F \oplus Z_l = A$, then the obvious map $\bigvee(z_1, \dots, z_l) \otimes F \to A$ is surjective. It follows that (denoting $\bigvee(z_1, \dots, z_l)$

by B) that $f_B \leqslant f_A \leqslant f_B f_F$, whence $\rho_0(A) = \rho_0(B) = l$.

Moreover, if $S \subset H^{\text{odd}}(F(R))$ is a finite dimensional graded subspace, which, together with A generates $H(F(R))$, then $f_A \leqslant f_{H(F(R))} \leqslant f_A \cdot f_{\wedge S}$. Hence $\rho_0(H(F(R))) = \rho_0(A) = l$.

On the other hand, let $y_1, \ldots, y_l$ be cocycles representing $z_1, \ldots, z_l$ and let $U = (u_1, \ldots, u_l)$ be a graded space with $\deg u_i = \deg z_i - 1$. Define Koszul complexes $(F(R) \otimes \wedge U, d)$ and $(H(F(R)) \otimes \wedge U, \bar{d})$ by

$$d(\Phi \otimes 1) = d_\tau \Phi \otimes 1, \quad d(1 \otimes u_i) = y_i \otimes 1, \quad \bar{d}(1 \otimes u_i) = z_i \otimes 1.$$

According to [4] there is a spectral sequence converging to

$$H(F(R) \otimes \wedge U, d)$$

with E_2-term $H(H(F(R)) \otimes \wedge U, \bar{d})$. (See [3, Chapter III] for details.) (This spectral sequence, introduced by Koszul, is a special case of the Eilenberg-Moore spectral sequence.) Since A/Z_l has finite dimension, the argument in the remark of §3 shows that, so does $H(H(F(R)) \otimes \wedge U)$.

It follows that $H(F(R) \otimes \wedge U)$ has finite dimension. Its homotopy Euler characteristic is given by $\chi_\Pi - l$ (χ_Π the homotopy Euler characteristic of $F(R)$). Now apply Theorem 1 to get $\chi_\Pi - l \leqslant 0$; i.e. $\chi_\Pi \leqslant l = \rho_0(H(F(R)))$. Q.E.D.

COROLLARY. *Let X be a connected topological space with* $\dim \Pi_\psi^*(X) < \infty$ *and $H^*(X; \mathbf{Q})$ finitely generated. Set* $\chi_\Pi(X) = \Sigma(-1)^k \dim \Pi_\psi^k(X)$ *and* $\rho_0(X) = \rho_0(H^*(X; \mathbf{Q}))$. *Then* $\chi_\Pi(X) \leqslant \rho_0(X)$.

7. Two lemmas.

LEMMA 4. *Suppose $(\tau; x_1, \ldots, x_n)$ is a connected, finite, minimal tower. Then there is a tower $(\sigma; x_1, \ldots, x_n, y_1, \ldots, y_n)$ with* $\deg y_i = \deg x_i - 1$ *and with the following properties for each i $(1 \leqslant i \leqslant n)$:*

(i) $\sigma(x_i) = \tau(x_i)$.

(ii) $\sigma(y_i) - x_i \in (x_1, \ldots, x_{i-1}) \cdot F(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$.

(iii) $H(F^+(x_1, \ldots, x_i, y_1, \ldots, y_i), d_\sigma) = 0$. (*Note that* (ii) *implies that* $F(x_1, \ldots, x_i, y_1, \ldots, y_i)$ *is stable under* d_σ.)

(iv) *If $i < n$ then for some* $w \in (x_1, \ldots, x_i) \cdot F(x_1, \ldots, x_i, y_1, \ldots, y_i)$, $d_\sigma(x_{i+1} - w) = 0$.

PROOF. We use induction on p to define elements

$$\sigma(y_p) \in F(x_1, \ldots, x_p, y_1, \ldots, y_p)$$

so that conditions (i)–(iv) hold for $i \leqslant p$.

If $p = 1$ set $\sigma(y_1) = x_1$. Since $\tau(x_1) = 0$ it follows that $(\sigma; x_1, y_1)$ and $(\sigma; x_1, \ldots, x_n, y_1)$ are towers. Condition (ii) is obvious, while (iii) asserts that $H(F^+(x_1, y_1)) = 0$; this is a simple and classical calculation. (If $\deg x_1$ is odd it is essential that Γ have characteristic 0!.)

Finally, since the original tower was minimal, for some $a \in F^+(x_1)$, $\tau(x_2) = ax_1$. Set $w = (-1)^{\deg a} a y_1$. Then in $F(x_1, x_2, y_1)$, $d_\sigma(x_2 - w) = ax_1 - ax_1 = 0$.

Suppose now that $\sigma(y_j)$ is constructed for $j < p$ so that (i)–(iv) hold for $j < p$. Then $(\sigma; x_1, \ldots, x_p, y_1, \ldots, y_{p-1})$ is a tower, and by (iv) there is an element $w \in (x_1, \ldots, x_{p-1}) \cdot F(x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1})$ such that $d_\sigma(x_p - w) = 0$. Set $\sigma(y_p) = x_p - w$. Then $d_\sigma^2(y_p) = d_\sigma(x_p - w) = 0$ and so $(\sigma; x_1, \ldots, x_p, y_1, \ldots, y_p)$ is a tower. Hence so is $(\sigma; x_1, \ldots, x_n, y_1, \ldots, y_p)$.

Moreover (ii) (for $i = p$) is immediate from the definition. To check (iii) write $(F(x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1}), d_\sigma) = (A, d_A)$. Then

$$(F(x_1, \ldots, x_p, y_1, \ldots, y_p), d_\sigma)$$

is the Koszul complex of the tower $(A, d_A; \sigma, x_p, y_p)$ over (A, d_A).

Let $\rho: A \to \Gamma$ and $\rho \otimes \iota: A \otimes F(x_p, y_p) \to F(x_p, y_p)$ be the projections. By (i) and (ii) (for $i = p$) $(\rho \otimes \iota)(\sigma x_p) = 0$ and $(\rho \otimes \iota)(\sigma y_p) = x_p$. Thus if we define $(\bar{\sigma}; x_p, y_p)$ by $\bar{\sigma}(y_p) = x_p$, $\bar{\sigma}(x_p) = 0$, then $(\rho \otimes \iota) \circ d_\sigma = d_{\bar{\sigma}} \circ (\rho \otimes \iota)$.

By our induction hypothesis (iii) (for $i = p - 1$), ρ^* is an isomorphism. Hence (cf. Lemma 1, §2) $(\rho \otimes \iota)^*$ is an isomorphism. Thus

$$H(F^+(x_1, \ldots, x_p, y_1, \ldots, y_p), d_\sigma) \cong H(F^+(x_p, y_p), d_{\bar{\sigma}}) = 0.$$

It remains to prove (iv). Since $\tau(x_{p+1})$ is a cocycle in $F(x_1, \ldots, x_p)$, it is a cocycle in $F(x_1, \ldots, x_p, y_1, \ldots, y_p)$. By (iii) (for $i = p$) we can write $\tau(x_{p+1}) = d_\sigma(w)$ for some $w \in F(x_1, \ldots, x_p, y_1, \ldots, y_p)$. In view of (i) this gives

$$d_\sigma(x_{p+1} - w) = 0. \tag{12}$$

Write $w = u + v$, $u \in (x_1, \ldots, x_p) \cdot F(x_1, \ldots, x_p, y_1, \ldots, y_p)$, $v \in F(y_1, \ldots, y_p)$.

We prove (iv) by showing that $v = 0$. If $v \neq 0$ then for some q,

$$v = \sum_{k=0}^{m} y_q^k b_k,$$

where $b_k \in F(y_1, \ldots, y_{q-1})$, $b_m \neq 0$ and $m \geqslant 1$. Suppose this is the case.

Let $I = (x_1, \ldots, x_p) \cdot F(x_1, \ldots, x_p, y_1, \ldots, y_p)$ and set $J = (x_1, \ldots, x_{q-1}) \cdot F(x_1, \ldots, x_p, y_1, \ldots, y_p)$. (If $q = 1$ set $J = 0$.) Then we have the short exact sequence

$$0 \to I \cdot I + J \to F(x_1, \ldots, x_p, y_1, \ldots, y_p)$$
$$\xrightarrow{\Pi} [\Gamma \oplus (x_q, \ldots, x_n)] \otimes F(y_1, \ldots, y_p) \to 0.$$

It follows from (ii) that $\sigma(y_i) \in I$ $(i \leqslant p)$. The minimality of $(\tau; x_1, \ldots, x_n)$ implies that $\sigma(x_i) = \tau(x_i) \in I \cdot I$ $(i \leqslant p + 1)$. Hence $d_\sigma(I) \subset I \cdot I$ and $d_\sigma(x_{p+1}) \in I \cdot I$. Thus applying Π to equation (12) we find that

$$\Pi d_\sigma v = \Pi d_\sigma (x_{p+1}) - \Pi d_\sigma u = 0.$$

Moreover it follows from (ii) that $d_\sigma y_i = \sigma(y_i) \in J (i < q)$ and $d_\sigma y_q - x_q \in J$. Hence

$$\Pi d_\sigma v = \sum_{k=1}^{m} k\Pi\left(x_q y_q^{k-1} b_k\right) = \sum_{k=1}^{m} k x_q y_q^{k-1} b_k \neq 0.$$

This contradiction shows that $v = 0$. The induction is now closed and the proof is complete. Q.E.D.

COROLLARY. *For each* p $(\sigma; x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1}, x_p, \ldots, x_n)$ *is a tower, and the induced fibre projection*

$$\Pi: \left(F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}), d_\sigma\right) \to \left(F(x_p, \ldots, x_n), d_{\bar{\tau}}\right)$$

induces an isomorphism in cohomology.

LEMMA 5. *Let* $(\tau; x_1, \ldots, x_n)$ *be a connected, finite, c-finite minimal tower. Let* $[x_p] \in H(F(x_p, \ldots, x_n))$ *be the class represented by* x_p. *Then for each* p $(1 \leqslant p \leqslant n)$, $[x_p]$ *is nilpotent.*

PROOF. Let $(\sigma; x_1, \ldots, x_n, y_1, \ldots, y_n)$ be the tower of Lemma 4, and fix p. Part (iv) of Lemma 4 implies that for some w of the form

$$w = \sum_{i=1}^{p-1} x_i u_i, \qquad u_i \in F(x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1}),$$

we have $d_\sigma(x_p - w) = 0$ in $F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1})$.

Thus if Π is the projection in the corollary to Lemma 4, $\Pi(x_p - w) = x_p$. Let $\alpha \in H(F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}), d_\sigma)$ be the cohomology class represented by $x_p - w$. Then $\Pi^*\alpha = [x_p]$. Since Π^* is an isomorphism we need only prove that $\alpha^k = 0$, for some k.

Denote by (X, d_X) the Koszul complex of $(\tau; x_1, \ldots, x_n)$. Since $H(X)$ is finite dimensional there is an integer $k \geqslant 2$ such that $H^j(X) = 0, j \geqslant k$. Let $C \subset X^{k-1}$ be a subspace such that $X^{k-1} = C \oplus (\ker d_X)^{k-1}$. Define a graded, d_X-stable ideal $I \subset X$ by $I^j = 0$ $(j < k - 1)$, $I^{k-1} = C$, $I^j = X^j$ $(j \geqslant k)$.

Let $A = X/I$ and let d_A be the derivation induced by d_X in A. Then the projection $\rho: X \to A$ is a homomorphism of graded differential algebras, and ρ^* is an isomorphism.

On the other hand, $(F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}), d_\sigma)$ is the Koszul complex of the tower $(X, d_X; \sigma; y_1, \ldots, y_{p-1})$ over (X, d_X). Thus by Lemma 1, §2 there is a tower $(A, d_A; \lambda; y_1, \ldots, y_{p-1})$ such that

$$(\rho \otimes \iota): \left(X \otimes F(y_1, \ldots, y_{p-1}), d_\sigma\right) \to \left(A \otimes F(y_1, \ldots, y_{p-1}), d_A + d_\lambda\right)$$

is a homomorphism of graded differential algebras. Moreover $(\rho \otimes \iota)^*$ is an isomorphism.

But $x_p - w \in X^+ \otimes F(y_1, \ldots, y_{p-1})$ and so $(\rho \otimes \iota)(x_p - w) \in A^+ \otimes F(y_1, \ldots, y_{p-1})$. Since $A^j = 0$, $j \geqslant k$ this implies that

$$(\rho \otimes \iota)(x_p - w)^k = \left[(\rho \otimes \iota)(x_p - w)\right]^k = 0.$$

Hence $(\rho \otimes \iota)^*(\alpha^k) = 0$. Since $(\rho \otimes \iota)^*$ is an isomorphism, $\alpha^k = 0$. Q.E.D.

8. **Poincaré duality.** A finite dimensional graded algebra $A = \sum_{p=0}^n A^p$ is said to have *formal dimension n* if $A^n \neq 0$ (and $A^p = 0, p > n$). A *Poincaré duality algebra* (P.d.a.) is a finite dimensional graded commutative algebra $A = \sum_{p=0}^n A^p$ such that $\dim A^n = 1$ and such that multiplication defines nondegenerate bilinear maps $A^p \times A^{n-p} \to A^n (\cong \Gamma), p = 0, 1, \ldots$. If ε is a nonzero element in A^n then the scalar product $\langle\ ,\ \rangle$ in A given by

$$\langle \alpha, \beta \rangle = 0, \quad \deg \alpha + \deg \beta \neq n, \quad \langle \alpha, \beta \rangle \varepsilon = \alpha \cdot \beta, \quad \deg \alpha + \deg \beta = n$$

induces isomorphisms $A^{n-p} \cong (A^p)^*$. These are called, respectively, the *Poincaré scalar product* and the *Poincaré isomorphism.*

The tensor product of two graded commutative algebras is a P.d.a. if and only if each factor is a P.d.a. If (A, d_A) is a graded differential algebra such that A and $H(A)$ both have formal dimension n, and if A is a P.d.a., then so is $H(A)$.

In this section we establish

THEOREM 3. *Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, c-finite minimal tower with odd spectral sequence (E_i, d_i). Then*

(i) *$H(F(x_1, \ldots, x_n), d_\tau)$ and each E_i $(i \geqslant 1)$ have the same formal dimension m, given by*

$$m = r - \sum_{i=1}^{n} (-1)^{\deg x_i} \deg x_i,$$

where r is the number of x_i of even degree.

(ii) *$H(F(x_1, \ldots, x_n), d_\tau)$ and each E_i are P.d.a.'s.*

(iii) *For $i \geqslant 1$ $E_i^{*,q} = 0$, $q < \chi_\Pi$ and $E_i^{*,q} \neq 0$, $q = \chi_\Pi$.*

Exactly as in §1 (Theorem 1 $\Rightarrow$ Theorem 1′), Theorem 3 yields

THEOREM 3′. *Let X be a connected topological space such that $H^*(X; \mathbf{Q})$ and $\Pi_\psi^*(X)$ are finite dimensional. Then $H^*(X; \mathbf{Q})$ is a P.d.a. of formal dimension m given by*

$$m = \sum_i \dim \Pi_\psi^{2i}(X) - \sum_k (-1)^k k \dim \Pi_\psi^k(X).$$

If X is simply connected the theorem remains true if $\Pi_\psi^*(X)$ is replaced by $\Pi_*(X) \otimes \mathbf{Q}$ everywhere in the statement.

LEMMA 6. *Theorem 3 is correct for pure towers.*

PROOF. Suppose $(\tau; x_1, \ldots, x_n)$ is a pure tower satisfying the hypotheses of the theorem. Write $R = (x_1, \ldots, x_n)$; $F(R) = \bigvee Q \otimes \bigwedge P$. Let $z_1, \ldots, z_r$ be a homogeneous basis of Q and choose an integer $k \geqslant 2$ so that $z_i^k = d_\tau w_i$, $i = 1, \ldots, r$.

Let $U = (u_1, \ldots, u_r)$ be a graded vector space with $\deg u_i = k \deg z_i - 1$. Define a graded differential algebra $(\bigvee Q \otimes \bigwedge P \otimes \bigwedge U, d)$ by

$$d\Phi = d_\tau \Phi, \quad \Phi \in \bigvee Q \otimes \bigwedge P \quad \text{and} \quad du_i = z_i^k, \qquad i = 1, \ldots, r.$$

Then an isomorphism ϕ: $(\bigvee Q \otimes \bigwedge P \otimes \bigwedge U, d_\tau \otimes \iota) \xrightarrow{\cong} (\bigvee Q \otimes \bigwedge P \otimes \bigwedge U, d)$ is given by

$$\phi\Phi = \Phi, \quad \Phi \in \bigvee Q \otimes \bigwedge P, \quad \phi u_i = u_i - w_i, \qquad i = 1, \ldots, r.$$

This yields an isomorphism of graded algebras

$$(13) \qquad H(\bigvee Q \otimes \bigwedge P, d_\tau) \otimes \bigwedge U \xrightarrow{\cong} H(\bigvee Q \otimes \bigwedge P \otimes \bigwedge U, d).$$

On the other hand, let A_i be the truncated polynomial algebra $\bigvee(z_i)/z_i^k$ and set $A = A_1 \otimes \cdots \otimes A_r$. The projection Π: $\bigvee Q \to A$ determines a graded differential algebra $(A \otimes \bigwedge P, \bar{d})$.

Moreover, Π extends to the homomorphism Π: $(\bigvee Q \otimes \bigwedge P \otimes \bigwedge U, d) \to (A \otimes \bigwedge P, \bar{d})$ of graded differential algebras given by $\Pi(y) = y$, $y \in P$ and $\Pi(u) = 0$, $u \in U$. Since the restriction of Π to $\bigvee Q \otimes \bigwedge U$ induces an isomorphism $H(\bigvee Q \otimes \bigwedge U) \xrightarrow{\cong} A$, Lemma 1, §2 shows that Π induces an isomorphism of graded algebras

$$(14) \qquad \Pi^*: H(\bigvee Q \otimes \bigwedge P \otimes \bigwedge U, d) \xrightarrow{\cong} H(A \otimes \bigwedge P, \bar{d}).$$

Now A and $\bigwedge P$ are obviously P.d.a.'s of formal dimensions a and d given by

$$a = \sum_{i=1}^{r} (k-1) \deg z_i \quad \text{and} \quad d = \sum_{i=1}^{s} \deg y_i,$$

where $y_1, \ldots, y_s$ is any homogeneous basis of P. Hence $A \otimes \bigwedge P$ is a P.d.a. of formal dimension $a + d$, and $(A \otimes \bigwedge P)^{a+d} = A^a \otimes \bigwedge^s P$.

Since $\operatorname{Im} \bar{d} \subset \sum_{j<s} A \otimes \bigwedge^j P$ it follows that the elements in $A^a \otimes \bigwedge^s P$ are not coboundaries; hence $H(A \otimes \bigwedge P)$ is a P.d.a. of formal dimension $a + d$. Now the isomorphisms (13) and (14) show that $H(\bigvee Q \otimes \bigwedge P, d_\tau)$ is a P.d.a. of formal dimension

$$m = a + d - \sum_{i=1}^{r} \deg u_i.$$

A simple calculation shows now that m is given by the formula of Theorem 3(i).

This proves parts (i) and (ii) of Theorem 3 for pure towers. (The odd spectral sequence collapses in this case!) Part (iii) follows at once from formula (3) of Theorem 2. Q.E.D.

LEMMA 7. *Let* $(\tau;\ x_1, \ldots,\ x_n)$ *be a tower satisfying the hypotheses of Theorem* 3. *Then* $H(F(x_1, \ldots,\ x_n), d_\tau)$ *has formal dimension* m, *where* m *is given by Theorem* 3(i).

PROOF. By induction on n. For $n = 1$, $\tau = 0$, x_1 has odd degree and the lemma is trivial. Assume it holds for $n - 1$ and distinguish two cases:

Case 1. $\deg x_1$ is odd. Write $F(x_1, \ldots,\ x_n) = \bigwedge x_1 \otimes F(x_2, \ldots,\ x_n)$ and filter by the ideals $I^p = \sum_{j \geqslant p}(\bigwedge x_1)^j \otimes F(x_2, \ldots,\ x_n)$. The resulting spectral sequence $\check{E}_i$ satisfies (if $\deg x_1 > 1$)

$$\check{E}_2^{p,q} = (\bigwedge x_1)^p \otimes H^q(F(x_2, \ldots, x_n), d_{\bar{\tau}}).$$

According to the corollary to Proposition 1, §6, the tower $(\bar{\tau};\ x_2, \ldots, x_n)$ also satisfies the hypotheses of Theorem 3. Thus by the induction hypothesis $H(F(x_2, \ldots, x_n))$ has formal dimension

$$l = r - \sum_{i=2}^{n} (-1)^{\deg x_i} \deg x_i = m - \deg x_1.$$

The formal dimension of $\bigwedge x_1$ is simply $\deg x_1 = m - l$. Hence $\check{E}_2^{p,q} = 0$ if $p > m - l$ or $q > l$, while $\check{E}_2^{m-l,l} \neq 0$. It follows that $\check{E}_\infty^{p,q} = 0$ if $p > m - l$ or $q > l$ and $\check{E}_\infty^{m-l,l} \neq 0$. Hence $\check{E}_\infty$, and so $H(F(x_1, \ldots, x_n), d_\tau)$ have formal dimension m. The case $\deg x_1 = 1$ is left to the reader.

Case 2. $\deg x_1$ is even. Choose k so that $x_1^k = d_\tau w$ and let $U = (u)$ be a 1-dimensional graded space with $\deg u = k \deg x_1 - 1$. Let A be the truncated polynomial algebra $\bigvee(x_1)/x_1^k$.

The projection $\bigvee(x_1) \to A$ defines a tower $(A, 0;\ \rho, x_2, \ldots,\ x_n)$. Moreover a slight modification of the proof of Lemma 6 yields an isomorphism of graded algebras:

$$H(F(x_1, \ldots,\ x_n), d_\tau) \otimes \bigwedge(u) \cong H(A \otimes F(x_2, \ldots,\ x_n), d_\rho).$$

Now filter $A \otimes F(x_2, \ldots,\ x_n)$ and repeat the argument of Case 1 (with A replacing $\bigwedge(x_1)$) to complete the proof. Q.E.D.

PROOF OF THEOREM 3. (i) Let $(\sigma;\ x_1, \ldots,\ x_n)$ be the associated pure tower; according to Proposition 1, §6 it is c-finite. Hence Lemma 6 applies and shows that $H(F(x_1, \ldots, x_n), d_\sigma)$ is a P.d.a. of formal dimension m. This is therefore true of E_1 as well (cf. §5).

On the other hand by Lemma 7, $H(F(x_1, \ldots,\ x_n), d_\tau)$ also has formal dimension m. Since for each i

$$\text{formal dim } E_1 \geqslant \text{formal dim } E_i \geqslant \text{formal dim } H(F(x_1, \ldots,\ x_n), d_\tau),$$

it follows that all the E_i have formal dimension m.

(ii) By Lemma 6, E_1 is a P.d.a. Since $E_i = H(E_{i-1})$ and since E_i and E_{i-1} have the same formal dimension, an inductive argument shows that each E_i is a P.d.a. Hence E_∞ is a P.d.a. and so $H(F(x_1, \ldots, x_n), d_\tau)$ is a P.d.a.

(iii) The statement $E_i^{*,q} = 0$, $q < \chi_\Pi$, is Corollary 1 to Theorem 1, §6. Let $q = \chi_\Pi$. Then $E_1^{*,q}$ $(= H_{-q}(\bigvee Q \otimes \bigwedge P, d_\sigma))$ is a nonzero ideal in E_1 (cf. Theorem 2, §3). Since E_1 is a P.d.a. of formal dimension m (cf. Lemma 6) we have $E_1^{(m)} \subset E_1^{*,q}$. It follows that if $i \geqslant 1$, $0 \neq E_i^{(m)} \subset E_i^{*,q}$. Q.E.D.

9. **Hyperbolic towers.** Suppose A is a P.d.a. of even formal dimension $2m$. Then the Poincaré scalar product restricts to a symmetric inner product in the subspace $\Sigma_j A^{2j}$. An inner product space $(X, \langle\ ,\ \rangle)$ is called *hyperbolic* if there is a subspace Y such that $\langle y_1, y_2 \rangle = 0$ $(y_i \in Y)$ (then Y is called *isotropic*) and such that $\dim X = 2 \dim Y$. If $\Sigma_j A^{2j}$ is hyperbolic we say A is a *hyperbolic* P.d.a. Note that this is independent of the choice of basis vector in A^{2m}. In this section we prove

THEOREM 4. *Let $(\tau;\ x_1, \ldots, x_n)$ be a connected, finite, c-finite, minimal tower such that $H(F(x_1, \ldots, x_n), d_\tau)$ has formal dimension $2m$.*

Assume $\chi_\Pi < 0$. Then $H(F(x_1, \ldots, x_n), d_\tau)$ is a hyperbolic P.d.a. In particular (if $\Gamma \subset \mathbf{R}$) the inner product space $\Sigma_j H^{2j}(F(x_1, \ldots, x_n), d_\tau)$ has zero signature.

COROLLARY. *Let M be a simply connected, compact oriented $4k$-manifold such that $\Pi_*(M; \mathbf{Q})$ is finite dimensional. Assume that $H^j(M; \mathbf{Q}) \neq 0$ for some odd j. Then* $\operatorname{sign}(M) = 0$.

PROOF. It follows from Theorem 1′ that $\chi_\Pi < 0$. Now apply Theorem 4.

PROOF OF THEOREM 4. In the next section we show (Proposition 3) that the theorem holds for pure towers. Hence it holds for the E_1-term of the odd spectral sequence. On the other hand since E_i and $H(E_i)$ have the same formal dimension $2m$ it follows that there is an isometry $\Sigma_j E_i^{(2j)} \cong \Sigma_j H^{2j}(E_i) \oplus X$ where X is a hyperbolic inner product space and $\oplus$ means orthogonal direct sum. If $\Sigma_j E_i^{(2j)}$ is hyperbolic this implies that $\Sigma_j H^{2j}(E_i)$ is hyperbolic.

Thus an induction argument shows that $\Sigma_j E_\infty^{(2j)}$ is hyperbolic; the same must then be true for $\Sigma_j H^{2j}(F(x_1, \ldots, x_n), d_\tau)$. Q.E.D.

REMARK. Theorem 4 shows that the only "interesting" inner products arise when $\dim P = \dim Q$. In this case (cf. Theorem 5, §11) the Koszul complex is the Koszul complex of a pure tower, totally determined by a linear map σ: $P \to \bigvee Q$.

It would be interesting and useful to have an explicit means of calculating invariants of the inner product (e.g. signature) directly from σ.

10. **The pure case.** In this section we prove

PROPOSITION 3. *Let* $(\bigvee Q \otimes \bigwedge P, d_\sigma)$ *be the Koszul complex of a connected finite, c-finite, pure tower. Suppose* $H(\bigvee Q \otimes \bigwedge P)$ *has formal dimension* $2m$, *and assume* $\dim P > \dim Q$. *Then* $\Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P)$ *is hyperbolic.*

LEMMA 8. *There is a basis* $u_1, \dots, u_s$ *of* P *(not necessarily homogeneous) with the following properties: Let* $I_i \subset \bigvee Q$ *be the ideal generated by* $\sigma(u_1), \dots, \sigma(u_i)$. *Let* $I_0 = 0$. *Then*

(i) $\overline{\sigma(u_i)} \in \bigvee Q/I_{i-1}$ *is not a zero divisor*, $1 \leqslant i \leqslant r$, *where* $r = \dim Q$.

(ii) $\dim \bigvee Q/I_r < \infty$.

PROOF. We construct u_k $(1 \leqslant k \leqslant r)$ by induction on k and extend to any basis of P. If $k = 1$, let u_1 be any nonzero element of P. Now suppose (for some $k \leqslant r$) $u_1, \dots, u_k$ are constructed, and that (i) holds for $i \leqslant k$.

By the Noether decomposition theorem I_k is the finite irredundant intersection of primary ideals in $\bigvee Q$; denote the associated prime ideals by $J_1, \dots, J_l$ (cf. [6, Chapter 4]). Let $d(J_i)$ be the transcendence degree of $\bigvee Q/J_i$.

Suppose J_i is not contained in any J_j. Then according to [7, p. 394, Appendix 6], J_i has height k. Hence by [7, Theorem 20, Chapter 7], $d(J_i) = r - k$. Thus Macaulay's theorem [7, Theorem 26, Chapter 7] applies and asserts that I_k is unmixed; i.e., $d(J_i) = r - k$, $i = 1, \dots, l$. We now distinguish two cases:

Case 1. For some element u_{k+1} of P, $\overline{\sigma(u_{k+1})} \in \bigvee Q/I_k$ is not a zero divisor. In this case we have constructed a sequence $u_1, \dots, u_{k+1}$ satisfying (i); repeating the argument above yields ideals J with $d(J) = r - k - 1$, and so $r \geqslant k + 1$.

Case 2. Every $u \in P$ yields a zero divisor $\overline{\sigma(u)}$ in $\bigvee Q/I_k$. Choose an infinite sequence $w_1, w_2, \dots$ of elements in P such that any subsequence of length s is a basis (possible because char $\Gamma = 0$ and so Γ is infinite). Each $\overline{\sigma(w_i)}$ is a zero divisor in $\bigvee Q/I_k$. Hence by [6, Theorem 11, Chapter 4] $\sigma(w_i) \in \bigcup_j J_j$. By renumbering the J_j we can arrange that infinitely many $\sigma(w_i) \in J_1$.

It follows that J_1 contains $\sigma(P)$ and so $\bigvee Q \cdot \sigma(P) \subset J_1$. Thus

$$\dim \bigvee Q/J_1 \leqslant \dim \bigvee Q/\bigvee Q \cdot \sigma(P) \leqslant \dim H(\bigvee Q \otimes \bigwedge P) < \infty.$$

It follows that $d(J_1) = 0$ and so $k = r$.

Thus $u_1, \dots, u_r$ are constructed. Moreover, $d(J_j) = 0, j = 1, \dots, l$, and so $\dim \bigvee Q/J_j < \infty$, $j = 1, \dots, l$. This implies that $\dim \bigvee Q/I_r < \infty$. Q.E.D.

Now define differential algebras $(A_{p,t}, d)$, $p \leqslant t \leqslant s$, by $(A_{p,p}, d) = (\bigvee Q/I_p, 0)$ and

$$A_{p,t} = (\bigvee Q/I_p) \otimes \bigwedge(u_{p+1}, \ldots, u_t),$$

$$d(\Phi \otimes u_{\alpha_0} \wedge \cdots \wedge u_{\alpha_q}) = \sum_{j=0}^{q} (-1)^j \Phi \cdot \overline{\sigma(u_{\alpha_j})} \otimes u_{\alpha_0} \wedge \cdots \wedge \hat{u}_{\alpha_j} \wedge \cdots \wedge u_{\alpha_q}, \quad 1 \leqslant p \leqslant r.$$

Note that these are *not* graded differential algebras. Lemma 8 has the following corollary.

COROLLARY. *There is an isomorphism of algebras*

$$H(\bigvee Q \otimes \bigwedge P, d_\sigma) \xrightarrow{\cong} H(\bigvee Q/I_r \otimes \bigwedge(u_{r+1}, \ldots, u_s))$$

which restricts to isomorphisms

$$H_k(\bigvee Q \otimes \bigwedge P) \xrightarrow{\cong} H_k(\bigvee Q/I_r \otimes \bigwedge(u_{r+1}, \ldots, u_s)).$$

PROOF. Extend the projections $\Pi\colon \bigvee Q/I_p \to \bigvee Q/I_{p+1}$ to homomorphisms $\Pi\colon \bigvee Q/I_p \otimes \bigwedge(u_{p+1}) \to \bigvee Q/I_{p+1}$ by setting $\Pi(u_{p+1}) = 0$. It follows directly from Lemma 8(i) that Π^* is an isomorphism from

$$H(\bigvee Q/I_p \otimes \bigwedge(u_{p+1}))$$

onto $\bigvee Q/I_{p+1}$.

Write $\Pi_k = \Pi \otimes \iota\colon A_{p,p+k} \to A_{p+1,p+k}$. Assume by induction on k that Π_k^* is an isomorphism. Write Π_{k+1} in the form

$$\Pi_{k+1} = \Pi_k \otimes \iota\colon A_{p,p+k} \otimes \bigwedge(u_{p+k+1}) \to A_{p+1,p+k} \otimes \bigwedge(u_{p+k+1}).$$

Both sides have a Gysin sequence (cf. the example in §2) and so the 5-lemma implies that Π_{k+1}^* is an isomorphism.

In this way we obtain a sequence of isomorphisms

$$H(\bigvee Q/I_p \otimes \bigwedge(u_{p+1}, \ldots, u_s)) \xrightarrow{\cong} H(\bigvee Q/I_{p+1} \otimes \bigwedge(u_{p+2}, \ldots, u_s)), \quad 0 \leqslant p < r.$$

Composing them gives the desired isomorphism. Q.E.D.

LEMMA 9. *There is an ideal $I \subset \bigvee^+ Q$ and a basis (not necessarily homogeneous) $u_1, \ldots, u_s$ of P with the following properties: Let $\rho(u_i) \in \bigvee Q/I$ be the image of $\sigma(u_i)$ under the projection $\bigvee Q \to \bigvee Q/I$. Then*

(i) *$\bigvee Q/I$ has finite dimension and the elements in $\bigvee^+ Q/I$ are nilpotent.*

(ii) *There is an isomorphism of algebras,*

$$\Psi\colon H(\bigvee Q \otimes \bigwedge P, d_\sigma) \cong H(\bigvee Q/I \otimes \bigwedge(u_{r+1}, \ldots, u_s), d_\rho)$$

which restricts to isomorphisms

$$H_k(\bigvee Q \otimes \bigwedge P) \cong H_k(\bigvee Q/I \otimes \bigwedge(u_{r+1}, \dots, u_s)).$$

PROOF. Let $u_1, \dots, u_s$ be the basis of Lemma 8, and write $B = \bigvee Q/I_r$. Multiplication by $\overline{\sigma(u_i)}$ is a linear transformation ϕ_i of the finite dimensional commutative algebra B; in particular ϕ_i commutes with multiplication by elements of B. Let $d = \dim B$.

Then ideals $K_{r+1}, \dots, K_s, L_{r+1}, \dots, L_s$ are defined by the equations:

$$K_{r+1} = \phi_{r+1}^d(B), \quad \phi_{r+1}^d(L_{r+1}) = 0, \quad B = K_{r+1} \oplus L_{r+1},$$

and for $i > r + 1$,

$$K_i = \phi_i^d(L_{i-1}), \quad \phi_i^d(L_i) = 0, \quad L_{i-1} = K_i \oplus L_i.$$

ϕ_i restricts to an automorphism of K_i while each ϕ_i is nilpotent in L_s. Moreover $B = K_{r+1} \oplus \cdots \oplus K_s \oplus L_s$.

Now let I be the inverse image of $K_{r+1} \oplus \cdots \oplus K_s$ under the canonical projection $\bigvee Q \to B$. For $z \in Q$ we know that some $z^k \in \bigvee Q \cdot \sigma(P)$. It follows that if $\bar{z}$ is the image of z in B then multiplication by $\bar{z}$ is nilpotent in L_s. Hence $\bar{z}^j \in K_{r+1} \oplus \cdots \oplus K_s$ for some j and so $z^j \in I$. Thus the elements of $\bigvee^+ Q$ determine nilpotent elements in $\bigvee Q/I$. This implies (clearly) that $I \subset \bigvee^+ Q$, and (i) is proved.

To prove (ii) we need only show that projection

$$\bigvee Q/I_r \otimes \bigwedge(u_{r+1}, \dots, u_s) \to \bigvee Q/I \otimes \bigwedge(u_{r+1}, \dots, u_s)$$

induces an isomorphism in cohomology. Since K_i, L_i are ideals, we have

$$H(B \otimes \bigwedge(u_{r+1}, \dots, u_s)) = \sum_{i=r+1}^{s} H(K_i \otimes \bigwedge(u_{r+1}, \dots, u_s)) \oplus H(L_s \otimes \bigwedge(u_{r+1}, \dots, u_s)). \tag{15}$$

Now ϕ_i is multiplication by a coboundary, hence it induces zero in cohomology. On the other hand each K_k, L_j is stable under ϕ_i and ϕ_i is an isomorphism in K_i. Hence ϕ_i induces an isomorphism in

$$H(K_i \otimes \bigwedge(u_{r+1}, \dots, u_s)).$$

This implies that $H(K_i \otimes \bigwedge(u_{r+1}, \dots, u_s)) = 0$, $i = r + 1, \dots, s$, and so (ii) follows from formula (15). Q.E.D.

Denote $\bigvee Q/I$ by A, $\bigvee^+ Q/I$ by A^+. (But note that A is not graded!) Let $K \subset A$ be the subspace of elements x such that $x \cdot A^+ = 0$. Finally, denote $(u_{r+1}, \dots, u_s)$ by U. Then clearly

$$K \otimes u_{r+1} \wedge \cdots \wedge u_s \subset (A \otimes \bigwedge^{s-r} U) \cap \ker d_p \subset H_{s-r}(A \otimes \bigwedge U).$$

(The last inclusion is an inclusion because $\operatorname{Im} d_p \subset \Sigma_{j<s-r} A \otimes \bigwedge^j U$.)

COROLLARY. *The space K satisfies* $\dim K = 1$. *If $0 \neq a \in K$ then the class $[a \otimes u_{r+1} \wedge \cdots \wedge u_s]$ corresponds under the isomorphism Ψ of Lemma 9 to* *an element of* $H^{2m}(\bigvee Q \otimes \bigwedge P)$. (*Recall that* $2m$ *is the formal dimension of* $H(\bigvee Q \otimes \bigwedge P)$.)

PROOF. It follows from Lemma 9(i) that $K \neq 0$. Let $a \in K$. Then $(a \otimes u_{r+1} \wedge \cdots \wedge u_s) \cdot (A^+ \otimes \bigwedge U + A \otimes \bigwedge^+ U) = 0$. If $\alpha \in H(\bigvee Q \otimes \bigwedge P)$ is defined by $\Psi(\alpha) = [a \otimes u_{r+1} \wedge \cdots \wedge u_s]$, then this equation implies that $\alpha \cdot H^+(\bigvee Q \otimes \bigwedge P) = 0$. But this condition characterizes

$$H^{2m}(\bigvee Q \otimes \bigwedge P),$$

and $\dim H^{2m}(\bigvee Q \otimes \bigwedge P) = 1$ by Lemma 6, §8. Q.E.D.

LEMMA 10. *There is a subspace* $X \subset \Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P)$ *with the following properties*:

(i) $2 \dim X = \dim(\Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P))$.

(ii) $X \cdot X \cap H^{2m}(\bigvee Q \otimes \bigwedge P) = 0$.

PROOF. Clearly $\Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P) = \Sigma_j H_{2j}(\bigvee Q \otimes \bigwedge P)$. Thus we need only find a subspace $Z \subset \Sigma_j H_{2j}(A \otimes \bigwedge U)$ such that $2 \dim Z = \dim(\Sigma_j H_{2j}(A \otimes \bigwedge U))$ and $Z \cdot Z \cap K \otimes \bigwedge^{s-r} U = 0$ (cf. Lemma 9 and its corollary).

Furthermore, since $H^{2m}(\bigvee Q \otimes \bigwedge P) \subset H_{s-r}(\bigvee Q \otimes \bigwedge P)$, it follows that $s - r = 2k$.

Choose a subspace $N \subset A$ so that $A = N \oplus K = N \oplus (a)$ (a, a basis vector for K). Define a bilinear function $\langle\ ,\ \rangle: A \times A \to \Gamma$ by $a_1 a_2 - \langle a_1, a_2\rangle a \in N$. Since the elements of A^+ are nilpotent, and since a is a basis for K, it follows easily that this is a nondegenerate inner product in A.

Next assign $\bigwedge U$ the standard Poincaré scalar product determined by the basis vector $u_{r+1} \wedge \cdots \wedge u_s$ in $\bigwedge^{s-r} U$:

$$\Phi \wedge \Psi - \langle \Phi, \Psi \rangle u_{r+1} \wedge \cdots \wedge u_s \in \sum_{j<s-r} \bigwedge^j U.$$

These two scalar products define a scalar product $\langle\ ,\ \rangle$ in $A \otimes \bigwedge U$, for which $\langle A \otimes \bigwedge^j U, A \otimes \bigwedge^l U\rangle = 0$ unless $j + l = s - r$. In particular, $\langle \operatorname{Im} d_\rho, 1\rangle = 0$.

A simple calculation shows as well that

$$\langle \Phi, \Psi \rangle = \langle \Phi \cdot \Psi, 1 \rangle, \qquad \Phi, \Psi \in A \otimes \bigwedge U, \tag{16}$$

whence

$$\langle d_\rho \Phi, \Psi \rangle + (-1)^p \langle \Phi, d_\rho \Psi \rangle = 0, \qquad \Phi \in A \otimes \bigwedge^p U, \Psi \in A \otimes \bigwedge U. \tag{17}$$

Thus the scalar product of two cocycles depends only on their respective cohomology classes, and so a scalar product is induced in $H(A \otimes \bigwedge U)$. It satisfies

$$(18) \qquad \langle \alpha, \beta \rangle = \langle \alpha \cdot \beta, 1 \rangle, \qquad \alpha, \beta \in H(A \otimes \wedge U).$$

Moreover $\langle H_j(A \otimes \wedge U), H_l(A \otimes \wedge U)\rangle = 0$ if $j + l \neq s - r$; since $s - r = 2k$ the spaces $\Sigma_j A \otimes \wedge^{2j} U$ and $\Sigma_j H_{2j}(A \otimes \wedge U)$ are inner product spaces.

Now choose subspaces $C_j \subset A \otimes \wedge^j U$ such that $C_j \oplus d_\rho(A \otimes \wedge^{j+1} U) = \ker d_\rho \cap (A \otimes \wedge^j U)$. Then the restriction of $\langle\, ,\rangle$ to $\Sigma_j C_{2j}$ is nondegenerate and the inner product spaces $\Sigma_j C_{2j}$ and $\Sigma_j H_{2j}(A \otimes \wedge U)$ are isometric. Write

$$\sum A \otimes \wedge^{2j} U = \sum_j C_{2j} \oplus \Big(\sum_j C_{2j}\Big)^{\perp}.$$

The left-hand side is obviously hyperbolic. Moreover, $\Sigma_j d_\rho(A \otimes \wedge^{2j+1} U)$ is an isotropic subspace of $(\Sigma_j C_{2j})^\perp$ and

$$2 \dim \sum_j d_\rho(A \otimes \wedge^{2j+1} U) = \dim\Big(\sum C_{2j}\Big)^{\perp}.$$

Hence $(\Sigma C_{2j})^\perp$ is hyperbolic.

It follows that $\Sigma_j C_{2j}$ is hyperbolic; hence so is $\Sigma_j H_{2j}(A \otimes \wedge U)$. Choose an isotropic subspace $Z \subset \Sigma_j H_{2j}(A \otimes \wedge U)$ such that $2 \dim Z = \dim(\Sigma_j H_{2j}(A \otimes \wedge U))$. Formula (18) implies that $Z \cdot Z \cap (K \otimes u_{r+1} \wedge \cdots \wedge u_s) = 0$. Q.E.D.

Proof of Proposition 3. Let X be the subspace of Lemma 10. Then there is a basis, $\alpha_1, \ldots, \alpha_N$ of X with the following property: There are linearly independent elements $\beta_1, \ldots, \beta_N$ in $\Sigma_j H^{2j}(\vee Q \otimes \wedge P)$ such that

(i) β_i is homogeneous.

(ii) $\alpha_i - \beta_i \in \Sigma_{j > |\beta_i|} H^j(\vee Q \otimes \wedge P)$ ($|\beta_i| = \deg \beta_i$).

Now let $\langle\, ,\rangle$ denote the Poincaré scalar product in $H(\vee Q \otimes \wedge P)$. Then $\langle \beta_i, \beta_j \rangle = 0$ if $|\beta_i| + |\beta_j| < 2m$. On the other hand, if $|\beta_i| + |\beta_j| = 2m$ then (ii) implies that $\langle \beta_i, \beta_j \rangle \varepsilon = \beta_i \cdot \beta_j = \alpha_i \cdot \alpha_j$. Since

$$X \cdot X \cap H^{2m}(\vee Q \otimes \wedge P) = 0$$

this equation implies $\alpha_i \alpha_j = 0$; i.e. $\langle \beta_i, \beta_j \rangle = 0$ if $|\beta_i| + |\beta_j| = 2m$. Thus the β_j span an isotropic space $Y \subset \Sigma_j H^{2j}(\vee Q \otimes \wedge P)$. Since

$$\dim Y = \dim X = \tfrac{1}{2} \sum_j \dim H^{2j}(\vee Q \otimes \wedge P),$$

the inner product space $\Sigma_j H^{2j}(\vee Q \otimes \wedge P)$ is hyperbolic. Q.E.D.

11. **The case that $\chi_\Pi = 0$.** The object of this section is to establish

Theorem 5. *Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, c-finite, minimal tower. Assume $\chi_\Pi = 0$. Then the Koszul complex of the tower and the Koszul complex of the associated pure tower are isomorphic as graded differential* *algebras*: $(\bigvee Q \otimes \bigwedge P, d_\tau) \cong (\bigvee Q \otimes \bigwedge P, d_\sigma)$.

Throughout the section $(\tau; x_1, \ldots, x_n)$ denotes a fixed tower satisfying the hypotheses of the theorem; $R = (x_1, \ldots, x_n)$; $F(R) = \bigvee Q \otimes \bigwedge P$; $(\sigma; x_1, \ldots, x_n)$ is the associated pure tower. To establish the theorem we may assume without loss of generality that

$$\deg x_1 \leqslant \deg x_2 \leqslant \cdots. \tag{19}$$

We assume this throughout the section.

LEMMA 11. *Suppose x_i has even degree. Then for some*

$$u_i \in F^+(x_1, \ldots, x_{i-1}) \cdot F^+(x_1, \ldots, x_{i-1}), \quad d_\tau(x_i + u_i) = 0.$$

PROOF. By Corollary 4, §6, the odd spectral sequence collapses at the E_1-term. Moreover $d_0(x_i) = 0$. Thus x_i represents an element in $E_\infty^{p,0}$ ($p = \deg x_i$). It follows that $d_\tau(x_i + u_i) = 0$ for some $u_i \in \Sigma_{j>0} F(R)^{j+p,-j}$. But $F(R)^{j+p,-j} \subset \bigvee Q \otimes \bigwedge^j P$. Since u_i has even degree p this gives

$$u_i \in \sum_{j \geqslant 2} \bigvee Q \otimes \bigwedge^j P \subset F^+(R) \cdot F^+(R).$$

It follows now from (19) that $u_i \in F^+(x_1, \ldots, x_{i-1}) \cdot F^+(x_1, \ldots, x_{i-1})$. Q.E.D.

Now define an automorphism ϕ of the graded algebra $F(R)$ by setting $\phi x_i = x_i$ ($x_i \in P$) and $\phi x_i = x_i + u_i$ ($x_i \in Q$). Then ϕ restricts to automorphisms of each $F(x_1, \ldots, x_i)$. Hence a tower $(\rho; x_1, \ldots, x_n)$ is defined by

$$\rho(x_i) = \phi^{-1} d_\tau \phi(x_i), \qquad i = 1, \ldots, n.$$

Lemma 11 yields

$$\rho(x) = 0, \qquad x \in Q. \tag{20}$$

Clearly ϕ: $(F(R), d_\rho) \xrightarrow{\cong} (F(R), d_\tau)$ is an isomorphism of graded differential algebras. It follows (cf. §2) that $(\rho; x_1, \ldots, x_n)$ is a connected, finite, c-finite, minimal tower, with zero homotopy Euler characteristic. In view of (20) this tower can be rearranged (cf. §2) in the form $(\rho; z_1, \ldots, z_m, y_1, \ldots, y_m)$, where the z_i are a basis of Q and the y_i a basis of P. ($z_1, \ldots, y_m$ is a permutation of $x_1, \ldots, x_n$.)

Let $(\lambda; z_1, \ldots, z_m, y_1, \ldots, y_m)$ be the associated pure tower. Proposition 1, applied to $(\rho; x_1, \ldots, x_n)$ shows that $H(\bigvee Q \otimes \bigwedge P, d_\lambda)$ has finite dimension. Hence Theorem 2 implies that $H_+(\bigvee Q \otimes \bigwedge P, d_\lambda) = 0$. Let P_i be the subspace of P spanned by $y_1, \ldots, y_i$. Lemma 2, §3 implies now that for each i, $H_+(\bigvee Q \otimes \bigwedge P_i, d_\lambda) = 0$.

LEMMA 12. *For each i the inclusion θ: $\bigvee Q \to \bigvee Q \otimes \bigwedge P_i$ induces a surjective homomorphism θ^*: $\bigvee Q \to H(\bigvee Q \otimes \bigwedge P_i, d_\rho)$.*

PROOF. Since $H_+(\bigvee Q \otimes \bigwedge P_i, d_\lambda) = 0$, $H(\bigvee Q \otimes \bigwedge P_i, d_\lambda)$ is evenly graded. Thus if (F_k, d_k) is the odd spectral sequence for $(\rho;\ z_1, \ldots, z_m, y_1, \ldots, y_i)$, $F_1 = H(\bigvee Q \otimes \bigwedge P_i, d_\lambda)$ is evenly graded. Hence $F_1 = F_\infty$.

Now filter $\bigvee Q$ by the ideals $\hat{I}^p = \Sigma_{j \geqslant p} (\bigvee Q)^j$. The corresponding spectral sequence is given $\hat{F}_k = \bigvee Q$, $\hat{d}_k = 0$.

Observe that θ is filtration preserving and so induces a homomorphism θ_k: $\hat{F}_k \to F_k$ of spectral sequences. In particular, θ_1: $\bigvee Q \to H(\bigvee Q \otimes \bigwedge P_i, d_\lambda)$ is surjective, since $H_+(\bigvee Q \otimes \bigwedge P_i, d_\lambda) = 0$. But $F_1 = F_\infty$, $\hat{F}_1 = \hat{F}_\infty$; thus $\theta_\infty = \theta_1$ and so θ_∞ is surjective. This implies at once that θ^* is surjective. Q.E.D.

PROOF OF THEOREM 5. We continue the notation developed above. Since $(\rho;\ z_1, \ldots, z_m, y_1, \ldots, y_m)$ is a tower it follows that

$$\rho(y_i) \in \ker d_\rho \cap (\bigvee Q \otimes \bigwedge P_{i-1}), \qquad i = 1, 2, \ldots, m.$$

In view of Lemma 12 above we can write

$$\rho(y_i) = v_i + d_\rho w_i, \quad v_i \in \bigvee Q, \quad w_i \in \bigvee Q \otimes \bigwedge P_{i-1}.$$

Define an automorphism Ψ of the graded algebra $\bigvee Q \otimes \bigwedge P$ by setting

$$\Psi(z_i) = z_i, \quad \Psi(y_i) = y_i - w_i, \qquad i = 1, \ldots, m.$$

Then define γ: $R \to \bigvee Q \otimes \bigwedge P$ by

$$\gamma(y_i) = v_i \quad \text{and} \quad \gamma(z_i) = 0. \tag{21}$$

It follows from the definition that $\Psi d_\gamma = d_\rho \Psi$.

In particular, $\operatorname{Im} \gamma \subset F^+(R) \cdot F^+(R)$. In view of (21) this implies that $(\gamma;\ x_1, \ldots, x_n)$ is a pure, minimal tower. Thus it coincides with the associated pure tower.

Now consider the isomorphism $\phi \circ \Psi$: $(F(R), d_\gamma) \xrightarrow{\cong} (F(R), d_\tau)$ of Koszul complexes. According to §5 it induces an isomorphism of the odd spectral sequences. The isomorphism of the E_0-terms can be written α: $(F(R), d_\gamma) \xrightarrow{\cong} (F(R), d_\sigma)$. Thus $\phi \circ \psi \circ \alpha^{-1}$: $(F(R), d_\sigma) \xrightarrow{\cong} (F(R), d_\tau)$ is the desired isomorphism. Q.E.D.

COROLLARY 1. $H^*(F(R), d_\tau) \cong \bigvee Q / \bigvee Q \cdot \sigma(P)$.

PROOF. Apply Theorem 2.

COROLLARY 2. *Suppose the bases y_i of P and z_i of Q satisfy* $\deg y_i = g_i$, $\deg z_i = k_i$. *Then*

$$\sum_p \dim H^p(F(R), d_\tau) t^p = \prod_{i=1}^{m} (1 - t^{g_i+1}) \prod_{i=1}^{m} (1 - t^{k_i})^{-1}.$$

Moreover the Euler characteristic, χ_c (equals the dimension of the cohomology) is given by formula $\chi_c = ((g_i + 1) \ldots (g_m + 1))/(k_1 \ldots k_m)$.

PROOF. See [4] or [3, Chapter 2].

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