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RESTRICTIONS OF CONVEX SUBSETS OF $C(X)$

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ABSTRACT. The main result of this paper is a theorem giving a measure-theoretic condition which is necessary and sufficient for a closed convex subset S of $C(X)$ to have the so-called bounded extension property with respect to a closed subset F of X. This theorem generalizes well-known results on closed subspaces by Bishop, Gamelin and Semadeni.

1. **Introduction**. In the following X denotes a compact Hausdorff space, F a closed subset of X and F' denotes $X \setminus F$. $C(X)$ and $C(F)$ denote the spaces of all continuous complex-valued functions on X and F, respectively. A and S denote a closed linear subspace and a closed convex subset of $C(X)$, respectively. The closure is with respect to the sup norm topology. Further, $A|F$ and $S|F$ denote the set of all restrictions to F of A and S, respectively. For $\tilde{f} \in A$ or $\tilde{f} \in S$, $\tilde{f}|F$ denotes the restriction of $\tilde{f}$ to F, and for $\mu \in M(X)$, the set of all complex Radon measures on X, μ_F denotes the restriction of μ to F. Likewise, $\mu_{F'}$ denotes the restriction of μ to F'. By $A^\perp$ we understand the set of all annihilating measures for A, i.e. the set of all $\mu \in M(X)$ such that $\int_X \tilde{f}\, d\mu = 0$ for all $\tilde{f} \in A$.

The following definition is a special case of a definition due to Michael and Pełczyński [11]:

1.1. DEFINITION. Let X, F, A and $A|F$ be as above. A is said to have the *bounded extension property* (BEP) with respect to F if for every $f \in A|F$ and each closed set $G \subseteq X$ with $G \cap F = \varnothing$, and for each $\varepsilon > 0$, there exists a $\tilde{f} \in A$ such that

(1) $\tilde{f}|F = f$,

(2) $\|\tilde{f}\| = \|f\|_F$,

(3) $\|\tilde{f}\|_G < \varepsilon$.

The following result is due to Gamelin [5]:

1.2. THEOREM. *With the same notations as above, the following two conditions are equivalent*:

(a) *A has the BEP w.r.t. F*,

(b) *$\mu \in A^\perp$ implies that $\mu_F \in A^\perp$ for all $\mu \in M(X)$.* □

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In [5] Gamelin introduced the linear operator

$$T_F : A^{\perp} \to M(F)/(A|F)^{\perp}$$

given by

$$T_F(\mu) = \mu_F + (A|F)^{\perp} \quad \text{for } \mu \in A^{\perp}.$$

It is immediately verified that $\|T_F\| = 0$ if and only if condition (b) is satisfied.

In other words, the following equivalence is valid:

(i) $\|T_F\| = 0$ if and only if A has the BEP w.r.t. F.

It is also proved in [5] that

(ii) $\|T_F\| < 1$ if and only if $A|F$ is closed in $C(F)$.

It is easy to give an example which proves that $\|T_F\| = 0$ is a strictly stronger condition than $\|T_F\| < 1$ [8]. In [2] Arenson studies the same type of problems for a closed convex subset S of $C(X)$. Arenson introduces the following notation:

$$H_S(\mu) = \sup_{\tilde{f} \in S} \operatorname{Re} \mu(\tilde{f}),$$

where μ is an arbitrary element of $M(X)$. (By $\operatorname{Re} \mu(\tilde{f})$ we understand $\operatorname{Re} \int_X \tilde{f}\, d\mu$.)

In [2] is proved that the condition

(b′) $H_S(\mu_F) + H_S(\mu_{F'}) = H_S(\mu)$ for all $\mu \in M(X)$,

implies that $S|F$ is closed in $C(F)$. It is easily seen that condition (b′) is equivalent to condition (b) above in the case when S is a closed subspace of $C(X)$. Therefore it is natural to ask the following two questions:

1° Is it possible to draw a stronger conclusion than that $S|F$ is closed in $C(F)$ from the assumption (b′)?

2° Is it possible to give a weaker condition than (b′) under which we may conclude that $S|F$ is closed in $S|F$?

The answers to both questions are positive. The main result of this paper is a theorem showing that a certain property of S relative to F, analogous to the BEP in the case of a subspace, is equivalent to condition (b′). This theorem generalizes Theorem 1.2 and, at the same time, Arenson's result [2] mentioned above.

An answer to the second question above is given in [8]. This result may appear in a different paper.

In §3 of the present paper we study the relations between the properties in §2 and dominating continuous and lower semicontinuous functions. The results obtained in this direction generalize results by Gamelin [6, Chapter II, §12], and Semadeni [13], which again are generalizations of well-known results due to Bishop [3] and Glicksberg [7]. At the same time we obtain a generalization of the main result of [9].

In §4 an example is given.

Finally, it should be mentioned that the methods used in §2 of the present paper are different from the methods used by Gamelin in [5] and Arenson in [2].

2. **The main result.** In light of Definition 1.1 we introduce the following generalization of the bounded extension property:

2.1. DEFINITION. Let X be a compact Hausdorff space, F a closed subset of X and let S be a closed convex subset of $C(X)$. We say that S has *the B.E.P. w.r.t. F* if for each pair $\tilde{f} \in S$, $g \in S|F$ and every closed set $G \subseteq X$ with $G \cap F = \emptyset$ and every $\varepsilon > 0$, there exists a $\tilde{g} \in S$ such that

(A) $\tilde{g}|F = g$,

(B) $\|\tilde{g} - \tilde{f}\| = \|g - f\|_F$,

(C) $\|\tilde{g} - \tilde{f}\|_G < \varepsilon$.

(Here and in the following we use the notation $f = \tilde{f}|F$, $g = \tilde{g}|F$, etc.)

It is easily seen that this property is equivalent to the property in Definition 1.1 in the case when $S = A$, a closed subspace of $C(X)$.

The next theorem is the main result of this paper.

2.2. THEOREM. *Let X, F and S be as in Definition* 2.1. *Then the following three conditions are equivalent*:

(a′) *S has the BEP w.r.t. F.*

(a″) *For each pair $\tilde{f} \in S$, $g \in S|F$, each closed set $G \subseteq X$ with $G \cap F = \emptyset$ and each $\varepsilon > 0$, there exists an $\tilde{h} \in S$ such that*

(A′) $\|h - g\|_F < \varepsilon$,

(B′) $\|\tilde{h} - \tilde{f}\| < \|g - f\|_F + \varepsilon$,

(C′) $\|\tilde{h} - \tilde{f}\|_G < \varepsilon$.

(b′) $H_S(\mu) = H_S(\mu_F) + H_S(\mu_{F'})$ *for all* $\mu \in M(X)$.

PROOF. (a′) $\Rightarrow$ (b′): Let μ be an arbitrary nontrivial element of $M(X)$ and let ε be any positive number. We first discuss the case when $H_S(\mu_F) < \infty$ and $H_S(\mu_{F'}) < \infty$. Then there exist $\tilde{f}, \tilde{g} \in S$ such that

$$\operatorname{Re} \mu_F(\tilde{f}) > H_S(\mu_F) - \varepsilon/4 \quad \text{and} \quad \operatorname{Re} \mu_{F'}(\tilde{g}) > H_S(\mu_{F'}) - \varepsilon/4.$$

If $\tilde{f}|F = f = g = \tilde{g}|F$, we obtain immediately that

$$H_S(\mu) = H_S(\mu_F) + H_S(\mu_{F'}) - \varepsilon/2.$$

Therefore, in the following we assume that $f \neq g$. Now let δ be a positive number such that $\delta < \varepsilon/(4 \cdot \|\mu\|)$ and let G be a closed subset of X, such that $G \cap F = \emptyset$ and such that

$$|\mu|(F' \setminus G) < \varepsilon/(4 \cdot \|f - g\|_F).$$

The existence of such a set G follows from the regularity of μ. By our assumption there exists an $\tilde{h} \in S$ such that

$$\tilde{h}|F = f, \quad \|\tilde{g} - \tilde{h}\| = \|g - f\|_F \quad \text{and} \quad \|\tilde{g} - \tilde{h}\|_G < \delta.$$

This implies that

$$\operatorname{Re} \mu(\tilde{h}) = \operatorname{Re} \mu_F(\tilde{f}\,) + \operatorname{Re} \mu_{F'}(\tilde{g}) + \operatorname{Re} \mu_{F'}(\tilde{h} - \tilde{g}).$$

We also have

$$|\operatorname{Re} \mu_{F'}(\tilde{h} - \tilde{g})| \leqslant |\operatorname{Re} \mu_{F' \setminus G}(\tilde{h} - \tilde{g})| + |\operatorname{Re} \mu_G(\tilde{h} - \tilde{g})|$$
$$< |\mu|(F' \setminus G) \cdot \|f - g\|_F + \|\mu\| \cdot \delta < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

From this we obtain

$$\operatorname{Re} \mu(\tilde{h}) > \operatorname{Re} \mu_F(\tilde{f}\,) + \operatorname{Re} \mu_{F'}(\tilde{g}) - \varepsilon/2 > H_S(\mu_F) + H_S(\mu_{F'}) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can conclude that

$$H_S(\mu) = \sup_{\tilde{h} \in S} \operatorname{Re} \mu(\tilde{h}) \geqslant H_S(\mu_F) + H_S(\mu_{F'}).$$

Since the opposite inequality is satisfied for all $\mu \in M(X)$, the result follows.

In the case when $H_S(\mu_F) = \infty$ or $H_S(\mu_{F'}) = \infty$ or both, we have to prove that $H_S(\mu) = \infty$. Assume first that $H_S(\mu_F) = \infty$. Then there exists, for any $T > 0$, an $\tilde{f} \in S$ such that $\operatorname{Re} \mu_F(\tilde{f}) > T$.

Now let $\tilde{g}$ be an arbitrary element of S. Then by the BEP it follows that there exists an $\tilde{h} \in S$ such that $\tilde{h}|F = \tilde{f}|F$ and such that

$$\operatorname{Re} \mu(\tilde{h}) > T - |\operatorname{Re} \mu_{F'}(\tilde{g})| - \varepsilon$$

for arbitrary small $\varepsilon > 0$. This is proved exactly as in the case above. Keeping $\tilde{g} \in S$ and $\varepsilon > 0$ fixed and letting $T \to \infty$ by choice of $\tilde{f}$, we conclude that $H_S(\mu) = \infty$. The case $H_S(\mu_{F'}) = \infty$ is treated in the same manner.

(b′) ⇒ (a″): We need the following lemma:

2.3. Lemma. *Let E be a Banach space and A and B two nonempty convex subsets of E with* $\operatorname{dist}(A,B) = d > 0$. *Then there exists a $\phi \in E^*$, the set of all continuous linear functionals on E, with $\|\phi\| \leqslant 1$ such that*

$$\operatorname{Re} \phi(x) \geqslant \lambda + d \qquad \textit{for all } x \in B, \textit{and}$$
$$\operatorname{Re} \phi(y) \leqslant \lambda \quad \textit{for all } y \in A,$$

where λ is some real constant.

Proof of Lemma 2.3. Let U be the open ball of E of radius d. By hypothesis, U is disjoint from the convex set $B - A$. By the Separation Theorem for Convex Sets, there exists $\phi \in E^*$ such that

$$\sup\{\operatorname{Re}\phi(x);\ \|x\| \leqslant d\} \leqslant d \leqslant \inf\{\operatorname{Re}\phi(y);\ y \in B - A\}.$$

The first estimate shows that $\|\phi\| \leqslant 1$.

The second estimate shows that

$$\sup\{\operatorname{Re}\phi(x); x \in A\} \leqslant d + \inf\{\operatorname{Re}\phi(y);\ y \in B\}. \quad \square$$

To prove (b′) ⇒ (a″), we assume that $H_S(\mu) = H_S(\mu_F) + H_S(\mu_{F'})$ for all $\mu \in M(X)$, and, for contradiction, that there exist a pair $\tilde{f} \in S$, $g \in S|F$, a constant $\varepsilon > 0$ and a closed set $G \subseteq X$ with $G \cap F = \varnothing$, such that each function $\tilde{h} \in S$ violates at least one of the three conditions (A′), (B′) and (C′) of Theorem 2.2.

Next we define the set

$$\Omega = \Big\{\tilde{k} \in C(X); \|k - g\|_F \leqslant \frac{\varepsilon}{2} \wedge \|\tilde{k} - \tilde{f}\| \leqslant \|f - g\|_F + \frac{\varepsilon}{2} \wedge \|\tilde{k} - \tilde{f}\|_G \leqslant \frac{\varepsilon}{2}\Big\}.$$

Since X is normal, it follows by Tietze's theorem that Ω is nonempty. It is easily checked that Ω is convex and that $\operatorname{dist}(\Omega, S) \geqslant \varepsilon/2$.

We apply Lemma 2.3 to the case where $E = C(X)$, $B = \Omega$, $A = S$. Hence we know that there exists a measure $\nu \in C(X)^* = M(X)$ such that $\|\nu\| \leqslant 1$, and

$$\operatorname{Re}\nu(\tilde{k}) \geqslant \lambda + d \quad \text{for all } \tilde{k} \in \Omega, \text{ and}$$

$$\operatorname{Re}\nu(\tilde{h}) \leqslant \lambda \quad \text{for all } \tilde{h} \in S,$$

where λ and d are real constants and $d > 0$. From our assumption we have

$$(*) \qquad \operatorname{Re}\nu_F(\tilde{h}) + \operatorname{Re}\nu_{F'}(\tilde{g}) \leqslant H_S(\nu) \leqslant \lambda$$

for all $\tilde{h}, \tilde{g} \in S$. Let β be a real number such that $0 < \beta < d/2$.

Next choose an open set $O \supseteq F$ such that

$$|\nu|(O \setminus F) < \beta/2(\|f - g\|_F + \varepsilon), \qquad X \setminus O \supseteq G.$$

Then define β_1 by

$$\beta_1 = \min(\varepsilon/2, \beta/2\|\nu\|).$$

Now let $\tilde{k}$ be an element of $C(X)$ having the following properties:

(A‴) $\tilde{k}|F = g$,

(B‴) $\|\tilde{k} - \tilde{f}\| \leqslant \|f - g\|_F + \varepsilon/2$,

(C‴) $\|\tilde{k} - \tilde{f}\|_{X\setminus O} < \beta_1$.

By Tietze's theorem the existence of such a $\tilde{k}$ is clear. It is also obvious that $\tilde{k} \in \Omega$, since $\beta_1 \leqslant \varepsilon/2$ and $G \subseteq X \setminus O$. Then we have

$$(**) \qquad \begin{aligned} \operatorname{Re}\nu(\tilde{k}) &= \operatorname{Re}\nu_F(\tilde{k}) + \operatorname{Re}\nu_{F'}(\tilde{k}) \\ &= \operatorname{Re}\nu_F(g) + \operatorname{Re}\nu_{F'}(\tilde{f}) + \operatorname{Re}\nu_{F'}(\tilde{k} - \tilde{f}). \end{aligned}$$

We also have the following estimate:

$$|\mathrm{Re}\, \nu_{F'}(\tilde{k} - \tilde{f})| \leq |\mathrm{Re}\, \nu_{O \setminus F}(\tilde{k} - \tilde{f})| + |\mathrm{Re}\, \nu_{X \setminus O}(\tilde{k} - \tilde{f})|$$

$$< (\|f - g\|_F + \varepsilon/2) \cdot |\nu|(O \setminus F) + (\beta/2\|\nu\|) \cdot \|\nu\| < \beta < d/2.$$

From (**) then follows:

$$\mathrm{Re}\, \nu(\tilde{k}) < \mathrm{Re}\, \nu_F(g) + \mathrm{Re}\, \nu_{F'}(\tilde{f}) + d/2.$$

But since $g \in S|F$ and $\tilde{f} \in S$ it follows from (*) that

$$\mathrm{Re}\, \nu_F(g) + \mathrm{Re}\, \nu_{F'}(\tilde{f}) \leq H_S(\nu) \leq \lambda.$$

On the other hand, since $\tilde{k} \in \Omega$ we have

$$\mathrm{Re}\, \nu(\tilde{k}) \geq \lambda + d.$$

Putting these inequalities together, we obtain

$$\lambda + d \leq \mathrm{Re}\, \nu(\tilde{k}) < \mathrm{Re}\, \nu_F(g) + \mathrm{Re}\, \nu_{F'}(\tilde{f}) + d/2 \leq \lambda + d/2,$$

a contradiction. Hence we conclude that there exists an $\tilde{h} \in S$ satisfying conditions (A′), (B′) and (C′) of Theorem 2.2.

(a″) ⇒ (a′): Fix $\tilde{f}_0 \in S$, $g_0 \in S|F$, $\varepsilon_0 > 0$ and G_0, a closed subset of X with $G_0 \cap F = \varnothing$, for the rest of this discussion. We define $\varepsilon_1 = \varepsilon_0/8$.

Applying (a″) to the case where $\tilde{f} = \tilde{f}_0$, $g = g_0$, $G = G_0$ and $\varepsilon = \varepsilon_1 = \varepsilon_0/8$, we conclude that there exists a $\tilde{h}_1 \in S$ satisfying the conditions:

$$\|h_1 - g_0\|_F \leq \varepsilon_1, \quad \|\tilde{h}_1 - \tilde{f}_0\| \leq \|f_0 - g_0\|_F + \varepsilon_1,$$

$$\|\tilde{h}_1 - \tilde{f}_0\|_{G_0} < \varepsilon_1.$$

Next we apply (a″) to the pair $\tilde{h}_1 \in S$, $g_0 \in S|F$ and the constant $\varepsilon = \varepsilon_1/2$, G arbitrary. We conclude that there exists an $\tilde{h}_2 \in S$ such that

$$\|h_2 - g_0\|_F \leq \varepsilon_1/2, \quad \|\tilde{h}_2 - \tilde{h}_1\| \leq \|h_1 - g_0\|_F + \varepsilon_1/2.$$

Continuing this process, by induction we obtain a sequence of functions, $\{\tilde{h}_n\}$, with $\tilde{h}_n \in S$, satisfying the conditions

$$\|h_n - g_0\|_F \leq \varepsilon_1/2^{n-1}, \quad \|\tilde{h}_n - \tilde{h}_{n-1}\| \leq \|h_{n-1} - g_0\|_F + \varepsilon_1/2^{n-1}.$$

We now claim that $\{\tilde{h}_n\}$ is a Cauchy sequence in S. Given any $\zeta > 0$, let m be a positive integer satisfying the condition $\varepsilon_1/2^{m-2} < \zeta$. Then for $p > q > m$, we have

$$\begin{aligned}
\|\tilde{h}_p - \tilde{h}_q\| &\leq \|\tilde{h}_p - \tilde{h}_{p-1}\| + \cdots + \|\tilde{h}_{q+1} - \tilde{h}_q\| \\
&\leq \|h_{p-1} - g_0\|_F + \varepsilon_1/2^{p-1} + \cdots + \|h_q - g_0\|_F + \varepsilon_1/2^q \\
&\leq \varepsilon_1/2^{p-2} + \varepsilon_1/2^{p-1} + \cdots + \varepsilon_1/2^{q-1} + \varepsilon_1/2^q \\
&< \varepsilon_1/2^{q-3} < \zeta.
\end{aligned}$$

Since S is closed, there exists a $\tilde{g}_0 \in S$ such that $\tilde{h}_n \to \tilde{g}_0$ uniformly on X. From the condition $\|h_n - g_0\|_F < \varepsilon_1/2^{n-1}$ for all n, it follows that $\tilde{g}_0|F = g$. Furthermore we have

$$\begin{aligned}\|\tilde{h}_n - \tilde{f}_0\| &< \|\tilde{h}_n - \tilde{h}_{n-1}\| + \cdots + \|\tilde{h}_2 - \tilde{h}_1\| + \|\tilde{h}_1 - \tilde{f}_0\| \\ &< \|h_{n-1} - g_0\|_F + \varepsilon_1/2^{n-1} + \cdots + \|f_0 - g_0\|_F + \varepsilon_1 \\ &< 2\varepsilon_1 + 2\varepsilon_1 + \|f_0 - g_0\|_F < \|f_0 - g_0\|_F + \varepsilon_0\end{aligned}$$

for all n. Hence we have

$$\|\tilde{g}_0 - \tilde{f}_0\| < \|g_0 - f_0\| + \varepsilon_0.$$

In the same manner we obtain

$$\|\tilde{h}_n - \tilde{f}_0\|_{G_0} < \|\tilde{h}_n - \tilde{h}_{n-1}\| + \cdots + \|\tilde{h}_2 - \tilde{h}_1\| + \|\tilde{h}_1 - \tilde{f}_0\|_{G_0} < \varepsilon_0/2.$$

Passing to the limit as $n \to \infty$, we have

$$\|\tilde{g}_0 - \tilde{f}_0\|_{G_0} < \varepsilon_0/2 < \varepsilon_0.$$

So far we have proved that for each pair $\tilde{f}_0 \in S$, $g_0 \in S|F$ and every closed set $G_0 \subseteq X$, with $G_0 \cap F = \emptyset$ and every $\varepsilon_0 > 0$, there exists a $\tilde{g}_0 \in S$ such that

(A″) $\tilde{g}_0|F = g_0$,
(B″) $\|\tilde{g}_0 - \tilde{f}_0\| < \|g_0 - f_0\|_F + \varepsilon_0$,
(C″) $\|\tilde{g}_0 - \tilde{f}_0\|_{G_0} < \varepsilon_0$.

It remains to prove that this implies that S has the BEP w.r.t. F, in other words that (B″) above can be replaced by condition (B) of Definition 2.1. To prove this step, we apply a method used by Gamelin [6, Chapter II, Theorem 12.5]. Now we fix $\tilde{f} \in S$, $g \in S|F$, $\varepsilon > 0$ and G a closed set with $G \cap F = \emptyset$. We can assume without loss of generality that $\tilde{f}|F = f \neq g$. We define $\varepsilon' = \min(\varepsilon, \|g - f\|_F)$. From our result above it follows that there exists a $\tilde{g}_1 \in S$ such that:

$$\tilde{g}_1|F = g, \quad \|\tilde{g}_1 - \tilde{f}\| < \|g - f\|_F + \varepsilon'/4$$

and

$$\|\tilde{g}_1 - \tilde{f}\|_G < \varepsilon'/4.$$

Assume as induction hypothesis the existence of $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_{n-1} \in S$ such that $\tilde{g}_j|F = g$ and such that

$$\|\tilde{g}_j - \tilde{f}\| < \|g - f\|_F + \varepsilon'/2^{j+1}$$

and also such that

$$|\tilde{g}_j(x) - \tilde{f}(x)| < \varepsilon'/2^{j+1} \quad \text{for all } x \in X \setminus U_j,$$

where

$$U_j = \left\{ x \in X;\ |\tilde{g}_k(x) - \tilde{f}(x)| < \|g - f\|_F + \frac{\varepsilon'}{2^{j+1}},\ 1 \leq k \leq j-1 \right\} \Big\backslash G.$$

We notice that all the sets U_j, $j = 2, 3, \ldots, n$, are open and that $F \subseteq U_j$ for all j. From our previous result we know that there exists a function $\tilde{g}_n \in S$ such that $\tilde{g}_n|F = g$, $\|\tilde{g}_n - \tilde{f}\| \leq \|g - f\|_F + \varepsilon'/2^{n+1}$ and such that $|\tilde{g}_n(x) - \tilde{f}(x)| \leq \varepsilon'/2^{n+1}$ for all $x \in X \setminus U_n$. This proves the existence of a sequence $\{\tilde{g}_n\}$ with the properties listed above.

Next we define

$$\tilde{g} = \sum_{j=1}^{\infty} \frac{1}{2^j} \tilde{g}_j$$

and claim that $\tilde{g} \in S$ and satisfies conditions (A), (B) and (C) of Definition 2.1. That $\tilde{g} \in S$ follows from the fact that the functions

$$\tilde{\gamma}_n = \left(1 - \frac{1}{2^n}\right)^{-1} \sum_{j=1}^{n} \frac{1}{2^j} \tilde{g}_j$$

are all in S and $\tilde{\gamma}_n \to \tilde{g}$ uniformly as $n \to \infty$. $\tilde{g}|F = g$ is obvious and, using the inequality

$$|\tilde{g}(x) - \tilde{f}(x)| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} |\tilde{g}_j(x) - \tilde{f}(x)|,$$

it follows that $\|\tilde{g} - \tilde{f}\| = \|g - f\|_F$ and $\|\tilde{g} - \tilde{f}\|_G < \varepsilon$.

3. **Dominating continuous and lower semicontinuous functions.** Let X and F be as in the last section and let A denote a closed linear subspace of $C(X)$.

We have the following result:

3.1. THEOREM. *Let X, F and A be as mentioned above. Then the following conditions are equivalent*:

(a) *A has the BEP w.r.t. F.*

(b) *$\mu \in A^{\perp}$ implies that $\mu_F \in A^{\perp}$ for all $\mu \in M(X)$.*

(c) *For every $f \in A|F$ and for each $p: X \to (0, \infty)$, which is continuous and such that $|f(x)| \leq p(x)$ for all $x \in F$, there exists an extension $\tilde{f} \in A$ of f such that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.*

(d) *For every $f \in A|F$ and for each $\phi: X \to (0, \infty]$ which is lower semicontinuous and such that $|f(x)| \leq \phi(x)$ for all $x \in F$, there exists an extension $\tilde{f} \in A$ of f such that $|\tilde{f}(x)| \leq \phi(x)$ for all $x \in X$.*

PROOF. The proof of (c) $\Rightarrow$ (d) is in [9] and [10]; the other implications are well known. For details, see [8].

We are able to generalize this result to the case when S is a closed convex subset of $C(X)$.

3.2. THEOREM. *Let X and F be as above and let S be a closed convex subset* *of $C(X)$. Then the following conditions are equivalent*:

(a′) *S has the BEP w.r.t. F.*

(b′) $H_S(\mu) = H_S(\mu_F) + H_S(\mu_{F'})$ *for all* $\mu \in M(X)$.

(c′) *Given a pair $\tilde{f} \in S$, $g \in S|F$ and a continuous function $q: X \to (0, \infty)$ such that $|f(x) - g(x)| < q(x)$ for all $x \in F$, then there exists an extension $\tilde{g} \in S$ of g such that $|\tilde{f}(x) - \tilde{g}(x)| < q(x)$ for all $x \in X$.*

(d′) *Given a pair $\tilde{f} \in S$, $g \in S|F$ and a lower semicontinuous function $\psi: X \to (0, \infty]$ such that $|f(x) - g(x)| \leq \psi(x)$ for all $x \in F$, then there exists an extension $\tilde{g} \in S$ of g such that $|\tilde{f}(x) - \tilde{g}(x)| < \psi(x)$ for all $x \in X$.*

PROOF. (a′) $\Leftrightarrow$ (b′) is part of Theorem 2.2.

(d′) $\Rightarrow$ (a′) is trivial by Tietze's theorem. Therefore, it suffices to prove the implications (b′) $\Rightarrow$ (c′) and (c′) $\Rightarrow$ (d′).

(b′) $\Rightarrow$ (c′): We introduce the set $S' = \{\tilde{f}/q; \tilde{f} \in S\}$. Since q is strictly positive and continuous, it follows that $S' \subseteq C(X)$. We claim that S' is closed and convex in $C(X)$. To prove this, we notice that the operator $M_{1/q}: C(X) \to C(X)$ given by $M_{1/q}(\tilde{f}) = \tilde{f}/q$ is an invertible bounded linear operator. From the linearity it follows that $S' = M_{1/q}(S)$ is a convex subset of $C(X)$ and from the invertibility that S' is closed in $C(X)$. Now let μ be an arbitrary element of $M(X)$. Then we have

$$H_{S'}(\mu) = \sup_{\gamma \in S'} \operatorname{Re} \mu(\gamma) = \sup_{\tilde{f} \in S} \operatorname{Re} \mu(\tilde{f}/q) = H_S(\mu/q).$$

In the same way we obtain that $H_{S'}(\mu_F) = H_S(\mu_F/q)$ and $H_{S'}(\mu_{F'}) = H_S(\mu_{F'}/q)$. We also notice that $(\mu/q)_F = \mu_F/q$ and $(\mu/q)_{F'} = \mu_{F'}/q$.

Applying our assumption (b′) to the measure μ/q, we obtain

$$H_S(\mu_F/q) + H_S(\mu_{F'}/q) = H_S(\mu/q),$$

or, equivalently,

$$H_{S'}(\mu_F) + H_{S'}(\mu_{F'}) = H_{S'}(\mu).$$

Since this condition holds for all $\mu \in M(X)$, (b′) $\Rightarrow$ (a′) applied to S' gives that S' has the BEP w.r.t. F. Let $\tilde{\zeta} = \tilde{f}/q \in S'$ and $\gamma = g/q|F \in S'|F$. Then there exists a $\tilde{\gamma} \in S'$ such that for all $x \in X$ we have

$$|\tilde{\gamma}(x) - \tilde{\zeta}(x)| \leq \|\gamma - \zeta\|_F = \sup_{x \in F} \left| \frac{f(x) - g(x)}{q(x)} \right| < 1,$$

i.e.

$$\left| \frac{\tilde{f}(x) - \tilde{g}(x)}{q(x)} \right| < 1 \quad \text{for all } x \in X,$$

or

$$|\tilde{f}(x) - \tilde{g}(x)| < q(x) \quad \text{for all } x \in X.$$

(This proof is based on a method used by Gamelin in [5] and [6].)

(c′) ⇒ (d′): This step is a modification of the proof of (c) ⇒ (d) of Theorem 3.1. and we leave out the details.

4. **Example.** Before we proceed to our example we need a definition and a lemma.

4.1. DEFINITION. Let S be a closed convex cone in a topological vector space E. The *dual cone* S^* is defined by

$$S^* = \left\{ \mu \in E^*;\ \operatorname{Re} \mu(\tilde{f}\,) \geq 0 \text{ for all } \tilde{f} \in S \right\}.$$

4.2. LEMMA. *Let X be as above and let S be a closed convex cone in $C(X)$. Then the following two conditions are equivalent*:

(i) $H_S(\nu) = H_S(\nu_F) + H_S(\nu_{F'}),\ \forall \nu \in M(X)$,

(ii) $\nu \in S^* \Rightarrow \nu_F \in S^*$ *and* $\nu_{F'} \in S^*,\ \forall \nu \in M(X)$.

(This lemma is in Arenson [2]. The proof is simple.)

Now let T denote the unit circle and let A denote the disc algebra considered as a closed subspace of $C(T)$. Let S be the closed convex cone given by

$$S = \left\{ \tilde{f} \in A;\ \operatorname{Re} \tilde{f}\,(x) \geq 0 \text{ for all } x \in T \right\}.$$

Then we have

4.3. PROPOSITION. *Let F be a closed subset of T. Then S has the BEP w.r.t. F if and only if F has Lebesgue measure* 0.

PROOF. Assume first that S has the BEP w.r.t. F. Since $0, 1 \in S$, it follows that F is a p-set for A. But from Glicksberg's Peak Set Theorem [6, Chapter II, Theorem 12.7] and the F. and M. Riesz theorem, it follows that F has Lebesgue measure 0.

To prove the opposite implication, we need the following lemma:

4.4. LEMMA. $S^* = M^+ + A^{\perp}$, *where M^+ denotes the set of all nonnegative measures in $M(T)$.*

PROOF OF LEMMA 4.4. The inclusion $M^+ + A^{\perp} \subseteq S^*$ is obvious.

Suppose $f \in C(T)$ satisfies $\operatorname{Re} \mu(f) \geq 0$ for all $\mu \in M^+ + A^{\perp}$. Since this estimate holds for all $\mu \in A^{\perp}$, we have $f \in A$. Since it holds for all $\mu \in M^+$, we have $\operatorname{Re} f \geq 0$. Consequently, $f \in S$. It follows from the Separation Theorem for Convex Sets that $M^+ + A^{\perp}$ is w^*-dense in S^*. It suffices now to show that $M^+ + A^{\perp}$ is w^*-closed. For this, let $\{\mu_\alpha + \nu_\alpha\}$ be a net in $M^+ + A^{\perp}$ which converges w^* to σ. Then $\|\mu_\alpha\| = \int d\mu_\alpha = \int d(\mu_\alpha + \nu_\alpha) \to \int d\sigma$, such that the μ_α are bounded. If μ is a w^*-cluster point of the μ_α, then $\mu \in M^+$, and $\sigma - \mu \in A^{\perp}$, so $\sigma \in M^+ + A^{\perp}$. □

Let F be any closed subset of T of Lebesgue measure 0 and let $\mu \in S^*$. Then μ can be written as $m + \nu$, where $m \in M^+$ and $\nu \in A^\perp$. From this follows

$$\operatorname{Re} \mu_F(\tilde{f}) = \operatorname{Re} m_F(\tilde{f}) + \operatorname{Re} \nu_F(\tilde{f}) = \operatorname{Re} m_F(\tilde{f}) = m_F(\operatorname{Re} \tilde{f}) \geqslant 0,$$

since $\nu_F \in A^\perp$ by the F. and M. Riesz theorem, m_F is a positive measure and $\operatorname{Re} \tilde{f} \geqslant 0$ if $\tilde{f} \in S$. Hence $\mu_F \in S^*$. We also have

$$\operatorname{Re} \mu_{F'}(\tilde{f}) = \operatorname{Re} m_{F'}(\tilde{f}) + \operatorname{Re} \nu_{F'}(\tilde{f}) = \operatorname{Re} m_{F'}(\tilde{f}) = m_{F'}(\operatorname{Re} \tilde{f}) \geqslant 0,$$

since $\nu_{F'} \in A^\perp$, $m_{F'}$ is positive and $\operatorname{Re} \tilde{f} \geqslant 0$ whenever $\tilde{f} \in S$. Hence, $\mu_{F'} \in S^*$ and S has the BEP w.r.t. F by the main result in §2 and Lemma 4.2. □

We have the following consequence of Proposition 4.3:

4.5. Corollary. *Let F be a closed subset of T of Lebesgue measure* 0 *and let f be an arbitrary continuous function on F with nonnegative real part. Assume also that p: $T \to (0, \infty)$ is lower semicontinuous and such that $|f(x)| < p(x)$ for all $x \in F$. Then there exists a function $\tilde{f} \in A$ such that*

(i) $\tilde{f}|F = f$,
(ii) $|\tilde{f}(x)| < p(x)$ *for all* $x \in T$,
(iii) $\operatorname{Re} \tilde{f}(x) \geqslant 0$ *for all* $x \in T$.

Proof. We first claim that

$$S|F = \{f \in C(F);\ \operatorname{Re} f(x) \geqslant 0 \text{ for all } x \in F\}.$$

That $S|F$ is a closed cone in $C(F)$ is proved in [8, Chapter II, pp. 30–31]. In fact, this is an easy consequence of the BEP. Let us assume that there exists a $f_0 \in C(F)$ with $\operatorname{Re} f_0 \geqslant 0$, but $f_0 \notin S|F$. Applying the Separation Theorem for Convex Sets, we conclude that there exists a measure μ_0 on F such that

$$\operatorname{Re} \mu_0(f_0) < 0 \quad \text{and} \quad \operatorname{Re} \mu_0(f) \geqslant 0 \quad \text{for all } f \in S|F.$$

But consider μ_0 as an element of $M(T)$, it follows that $\mu_0 \in S^*$. By Lemma 4.4, $\mu_0 = m_0 + \nu_0$, where $m_0 \in M^+$ and $\nu_0 \in A^\perp$. Since F is of Lebesgue measure 0, this implies that $\mu_{0F} = m_{0F}$, and, therefore, that $\operatorname{Re} \mu_0(f_0) = \operatorname{Re} \mu_{0F}(f_0) = \operatorname{Re} m_{0F}(f_0) \geqslant 0$, since m_{0F} is nonnegative and $\operatorname{Re} f_0 \geqslant 0$ on F. This is a contradiction; hence our claim is proved. The remaining part of the corollary follows immediately from our previous results.

5. **Open problems**. In [2], Arenson claims that the result: "$H_S(\mu) = H_S(\mu_F) + H_S(\mu_{F'})$ for all $\mu \in M(X)$ implies that $S|F$ is closed" is valid for general Banach spaces. (The meaning of $S|F$ in this context is given in [2].) A natural question is therefore whether or not our result Theorem 2.2 can be generalized to this case in the same manner.

One could also ask if it is possible to generalize our example in §4 to the cone $S = \{\tilde{f} \in A;\ \operatorname{Re} \tilde{f}(x) \geqslant 0 \text{ for all } x \in X\}$, where A is any uniform algebra on an arbitrary compact Hausdorff space X. It seems likely that this example could be generalized to the polydisc algebra, but so far this is not proved.

Another question is if in Corollary 4.5 we could have started with a lower semicontinuous function $p: \Delta \to (0, \infty)$ and concluded that there exists a function $\tilde{f}$ in the disc algebra such that $\operatorname{Re} \tilde{f}(x) > 0$ on all of Δ and $|\tilde{f}(x)| < p(x)$ for all $x \in \Delta$ (cf. [5, Theorem, pp. 284–285]).

Finally it should be mentioned that most of the results in this paper are also in [8]. New results are Theorem 3.2 and Corollary 4.5. The proof of Lemma 4.4 is a shorter one than the one given in [8].

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