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RELATIVIZED WEAK DISJOINTNESS AND RELATIVELY INVARIANT MEASURES

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ABSTRACT. In this paper we study the relativized weak disjointness and the relativized regionally proximal relation for homomorphisms of point-transitive transformation groups, under the assumption of a relativized invariant measure. We also include a proof of a Folner-type result for syndetic subsets of an amenable group.

Introduction. In [P] R. Peleg proves the following:

If (X, T) and (Y, T) are metric minimal transformation groups supporting invariant measures then (X, T) and (Y, T) are weakly disjoint ($(X \times Y, T)$ has a point with dense orbit) iff their maximal equicontinuous factors $(X/S, T)$ and $(Y/S, T)$ are disjoint.

Motivated by this result Glasner conjectures at the end of [G] that

If $(X, T) \to^{\varphi} (Y, T)$ supports a relatively invariant measure where X is minimal and metric, then $R(\varphi) = \{(x, x') \in X \times X | \varphi(x) = \varphi(x')\}$ has a point with dense orbit iff the only almost periodic extension of Y which is a factor of X is Y itself.

The only if part of the above is easily seen to be true. Example 3.2.1 of this paper shows that the if part is not true without some further restriction on φ.

One possible restriction on φ is to assume that it is open. With this additional condition we prove the following variation of Glasner's conjecture (see 1.9). Suppose X and Y are minimal and metric, the natural projection $X \to X/S(\varphi)$ is open (where $S(\varphi)$ is the relativized equicontinuous structure relation), θ has a relative invariant measure, and $X/S(\varphi)$ and $Y/S(\theta)$ are relatively disjoint ($X/S(\varphi) \circ^{Z} Y/S(\theta) = \{(x/S(\varphi), y/S(\theta))\colon \varphi(x) = \theta(y)\}$ is a minimal set). Fix x in X. Then for some nonempty compact set, $B(x)$, contained in Y, the set $D(x) = \{y \in B(x)\colon (x, y)$ has a dense orbit in $X \circ^{Z} Y\}$ is a dense G_δ subset of $B(X)$.

When Z is a singleton, the restriction that $X \to X/S(\varphi)$ is open can be dropped, thus giving us a generalization of Peleg's theorem.

We also show that if X is minimal and $\varphi\colon X \to Z$ has a relatively invariant measure and if $z \in Z$, then there exists a compact set $B_z \subseteq X$ such that for x

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in B_z, $S(\varphi)(x) = Q(\varphi)(x)$ and moreover for $x' \in S(\varphi)(x)$, there exist nets t_n in T and x_n in B_z such that $x_n \to x$, $x_n t_n \to x$, and $x' t_n \to x$ (see 1.5).

These results are contained in §1 where we study relative invariant measures for homomorphisms between point-transitive transformation groups. The main technique used in §1 was motivated by Lemma 3 of Pelleg's paper [P] that shows that if (X, μ), (Y, υ) are probability spaces and (Y, υ) is separable then the following conditions on a measurable subset $E \subseteq X \times Y$ are equivalent.

(1) E is a rectangle (a.e. $\mu \times \upsilon$) of the form $X \times B$.

(2) $\upsilon(E_x) = C$ a.e. μ and $\upsilon(E_x \cap E_s) = C'$ a.e. $\mu \times \mu$ (E_x denotes the section at x).

§2 contains some results about the relativized equicontinuous structure relation, $S(\varphi)$, under the assumption that the almost periodic points in $R(\varphi)$ are dense in $R(\varphi)$. Some of these results are known (see [MW]), however, the proofs are new and provide a nice application of §1.

§3 contains examples. §4 contains a result for syndetic subsets of amenable groups similar to results in [EK] and [F], see [EK] and [F] for a more thorough discussion of the results.

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The referee pointed out that in a paper he was refereeing at the same time, Furstenberg and Glasner gave an entirely different proof of the Glasner conjecture (with the "open" hypothesis). Their proof is from the viewpoint of ergodic theory, [FG].

In response to a question to the referee I have included a proof of the Glasner conjecture (with the "open" hypothesis) in the nonmetric case, the conclusion being that $R(\varphi)$ is topologically transitive–that is, every nonempty open invariant set is dense (this is equivalent to having a point with dense orbit in the metric case), see 1.8 and 1.9.

Preliminaries. Let (X, T) be a transformation group with compact Hausdorff phase space. We will write X for both the transformation group (X, T) and the phase space. If X is point-transitive, let X_m denote the set of transitive points in X; when X is metric, X_m is a dense G_δ set. For a homomorphism φ of X onto Y, $R_m(\varphi) = \{(x, x') \in X_m \times X_m: \varphi(x) = \varphi(x')\}$, $P_m(\varphi) = \{(x, x') \in X_m$: there exists a net t_n in T with $xt_n \to x_0$, $x' t_n \to x_0\}$, for any x_0 in X_n; $Q_m(\varphi) = \{(x, x') \in R_m(\varphi)$: there exist nets t_n in T and (x_n, x_n') in $R_m(\varphi)$ such that $(x_n, x_n') \to (x, x')$ and $(x_n, x_n')t_n \to (x_0, x_0')\}$ for any x_0 in X_m; $S_m(\varphi)$ is the smallest closed (in $R_m(\varphi)$) invariant equivalence relation containing $Q_m(\varphi)$.

Let $D_m(\varphi)$ be the set of almost periodic points in $R_m(\varphi)$. When X is minimal, the subscript m is omitted. When Y is a singleton, φ is omitted.

Consider T with the discrete topology, let M be a minimal set in βT and let

J be the set of idempotents in M, see [E] or [G_b] for properties of βT and M.

Let $M(X)$ be the set of Borel probability measures on X. For μ in $M(X)$ define μt by $\mu t(A) = \mu(At^{-1})$ for every measurable set A.

DEFINITION. A section λ for $\varphi: X \to Y$ is a homomorphism $\lambda: Y \to M(X)$ such that $\hat{\varphi}(\lambda_y) = \delta_y$ where $\hat{\varphi}(\lambda_y)(A) = \lambda_y \varphi^{-1}(A)$ for every Borel subset A of Y and δ_y is the point mass at y.

Notation. $(\varphi, \lambda) \in \mathcal{S}$ will denote that λ is a section for φ, and $B(x)$ will denote the support of $\lambda_{\varphi(x)}$, for x in X.

REMARK. (1) Since λ is a homomorphism $\lambda_y t = \lambda_{yt}$, so $\lambda_{yt}(At) = (\lambda_y t)(At) = \lambda_y(Att^{-1}) = \lambda_y(A)$. (2) By the continuity of λ, if $y_n \to y$ then $\lambda_{y_n} \to \lambda_y$ weakly, which means $\int f\, d\lambda_{y_n} \to \int f\, d\lambda_y$ for every continuous real-valued function. This is equivalent to $\lambda_{y_n}(A) \to \lambda_y(A)$ for every Borel set A with $\lambda_y(\partial A) = 0$, where $\partial A = \mathrm{cls}(A) \setminus \mathrm{int}(A)$ is the boundary of A. Note also if F is a closed set, $\lambda_y(F) = \inf\{\lambda_y(V): V$ is an open set containing F with $\lambda_y(\partial V) = 0\}$. A set V with $\lambda_y(\partial V) = 0$ is called a continuity set (for λ_y). We will use the notation $\mathfrak{N}(F) = \mathfrak{N}_{\lambda_y}(F) = \mathfrak{N}_y(F)$ to denote the set of open continuity sets (for λ_y) containing F. Note that if $(\theta, \lambda) \in \mathcal{S}$, $\varepsilon > 0$, $z \in Z$, and V is a continuity set of λ, then there is a neighborhood U of z such that $|\lambda_z(V) - \lambda_{z'}(V)| < \varepsilon$ for z' in U.

We will be considering the following situation. X, Y, Z are point-transitive transformation groups and are compact Hausdorff; X_m, Y_m, Z_m are the sets of transitive points in X, Y, Z respectively, they are nonempty by assumption, $\varphi: X \to Z$ and $\theta: Y \to Z$ are homomorphisms onto Z. Let $X \circ^Z Y = \{(x,y): \varphi(x) = \theta(y)\}$; we write $X \perp^Z Y$ and say X is disjoint from Y over Z, if $X \circ^Z Y$ is minimal; we write $X \dot{-}^Z Y$ and say X is weakly disjoint from Y over Z, if $X \circ^Z Y$ is point-transitive. If N is a subset of $X \circ^Z Y$ let $N_x = \{y: (x,y) \in N\}$.

1. It may be helpful to first read the results in this section assuming X and Y are minimal, in which case the subscript "m" may be dropped.

(1.1) LEMMA. *Given $(\theta, \lambda) \in \mathcal{S}$. If N is a closed invariant subset of $X \circ^Z Y$, then $\lambda_{\varphi(x)}(N_x) = \lambda_{\varphi(x')}(N_{x'})$ for every x, x' in X_m.*

PROOF. For any $\varepsilon > 0$, take $V \in \mathfrak{N}(N_x)$ such that $\lambda_{\varphi(x)}(V) < \lambda_{\varphi(x)}(N_x) + \varepsilon$. Take a neighborhood U of $z = \phi(x)$ such that $|\lambda_z(V) - \lambda_{z'}(V)| < \varepsilon$ for z' in U. Let t_n be a net in T with $x' t_n \to x$. Then $N_{x'} t_n \subseteq V$ and $\varphi(x') t_n \in U$ for some t_n, since otherwise there would exist y_n in $N_{x'}$ such that $y_n t_n \notin V$ and some subnet $y_m t_m$ converges to $y \notin V$ since Y is compact, but $(x', y_m) t_m \in N$ and so $(x,y) \in N$ and $y \in N_x \subseteq V$, a contradiction. For such a t_n, $\lambda_{\varphi(x')}(N_{x'}) = \lambda_{\varphi(x') t_n}(N_{x'} t_n) < \lambda_{\varphi(x') t_n}(V) < \lambda_{\varphi(x)}(v) + \varepsilon < \lambda_{\varphi(x)}(N_x) + 2\varepsilon$. Thus $\lambda_{\varphi(x')}(N_{x'}) < \lambda_{\varphi(x)}(N_x)$, since ε was arbitrary. The lemma follows by symmetry.

(1.2) LEMMA. *Given* $(\theta, \lambda) \in \mathbb{S}$ *and* N *a closed invariant subset of* $X \circ^Z Y$. *Let* $K = K(N) = \{(x, x') \in R_m(\varphi): \lambda_{\varphi(x)}(N_x \triangle N_{x'}) = 0\}$. *Then* K *is a closed (in* $X_m \times X_m$*) invariant equivalence relation on* X_m *containing* $Q_m(\varphi)$. *In addition if* $(x_0, x_n) \in K$, $n = 1, 2, 3, \ldots$, *then*

$$\lambda_{\varphi(x_0)}\left(\bigcap \{N_{x_n}: n = 0, 1, 2, \ldots\}\right) = \lambda_{\varphi(x_0)}(N_{x_0}) = \lambda_{\varphi(x_0)}\left(\bigcup \{N_{x_n}: n = 0, 1, \ldots\}\right).$$

Note $(x, x') \in K$ *iff* $\lambda_{\varphi(x)}(N_x \cap N_{x'}) = \lambda_{\varphi(x)}(N_x)$ *iff* $\lambda_{\varphi(x)}(N_x \cup N_{x'}) = \lambda_{\varphi(x)}(N_x)$.

PROOF. We first prove the following two sublemmas.

LEMMA. *Suppose* N *is a closed invariant subset of* $X \circ^Z Y$, $(\theta, \lambda) \in \mathbb{S}$, $p \in X_m$, *and* $U \in \mathfrak{N}(N_p)$ *with* $\lambda_{\varphi(p)}(U \setminus N_p) < \varepsilon$. *Then for* q *in* X_m *sufficiently close to* p, $N_q \subseteq U$ *and* $\lambda_{\varphi(q)}(U \setminus N_q) < \varepsilon$.

PROOF. Clearly $N_q \subseteq U$ for q sufficiently close to p. Now take δ so that $0 < \delta < \varepsilon - \lambda_{\varphi(p)}(U \setminus N_p)$ and note that for q sufficiently close to p, $|\lambda_{\varphi(q)}(U) - \lambda_{\varphi(p)}(U)| < \delta$. So

$$\lambda_{\varphi(q)}(U \setminus N_q) = \lambda_{\varphi(q)}(U) - \lambda_{\varphi(q)}(N_q) < \lambda_{\varphi(p)}(U) + \delta - \lambda_{\varphi(p)}(N_p) = \lambda_{\varphi(p)}(U \setminus N_p) + \delta < \varepsilon.$$

LEMMA. *Suppose* λ *is a probability measure,* $A \subseteq B$, $C \subseteq D$, $\lambda(B \setminus A) < \varepsilon$ *and* $\lambda(D \setminus C) < \varepsilon$. *Then* $|\lambda(C \setminus A) - \lambda(D \setminus B)| < \varepsilon$.

PROOF. Note $B = A \cup (B \setminus A)$ so $B \setminus D = (A \setminus D) \cup [B \setminus (A \cup D)]$ and $0 < \lambda(B \setminus D) - \lambda(A \setminus D) = \lambda(B \setminus (A \cup D)) < \varepsilon$. Also note $A \setminus C = (A \setminus D) \cup (D \setminus C)$, so $0 < \lambda(A \setminus C) - \lambda(A \setminus D) = \lambda(D \setminus C) < \varepsilon$. Thus $|\lambda(B \setminus D) - \lambda(A \setminus C)| < \varepsilon$.

PROOF OF LEMMA 1.2. Clearly K is an equivalence relation. Note that $\lambda_{\varphi(p)}(N_p \triangle N_q) = 2\lambda_{\varphi(p)}(N_p \setminus N_q)$ for $(p, q) \in R_m(\varphi)$.

Now we will show that K contains $Q_m(\varphi)$. Suppose $(p, q) \in Q_m(\varphi)$, note $\lambda_{\varphi(p)} = \lambda_{\varphi(q)}$. Given any $\varepsilon > 0$, take $U \in \mathfrak{N}(N_p)$, $V \in \mathfrak{N}(N_q)$ such that $\lambda_{\varphi(p)}(U \setminus N_p) < \varepsilon$ and $\lambda_{\varphi(p)}(V \setminus N_q) < \varepsilon$, and so (i) $|\lambda_{\varphi(p)}(U \setminus V) - \lambda_{\varphi(p)}(N_p \setminus N_q)| < \varepsilon$. Now we can take $(p', q') \in R_m(\varphi)$ and t in T such that $N_{p'} \subseteq U$, $N_{q'} \subseteq V$, $\lambda_{\varphi(p')}(U \setminus N_{p'}) < \varepsilon$, $\lambda_{\varphi(p')}(V \setminus N_{q'}) < \varepsilon$, (and so (ii) $|\lambda_{\varphi(p')}(U \setminus V) - \lambda_{\varphi(p')}(N_{p'} \setminus N_{q'})| < \varepsilon$); (iii) $|\lambda_{\varphi(p)}(U \setminus V) - \lambda_{\varphi(p')}(U \setminus V)| < \varepsilon$; and $N_{p't} \subseteq U$, $N_{q't} \subseteq U$, $\lambda_{\varphi(p't)}(U \setminus N_{p't}) < \varepsilon$, $\lambda_{\varphi(p't)}(U \setminus N_{q't}) < \varepsilon$, (and so (iv) $\lambda_{\varphi(p')}(N_{p'} \setminus N_{q'}) = \lambda_{\varphi(p't)}(N_{p't} \setminus N_{q't}) = |\lambda_{\varphi(p't)}(U \setminus U) - \lambda_{\varphi(p't)}(N_{p't} \setminus N_{q't})| < \varepsilon$). Thus $\lambda_{\varphi(p)}(N_p \setminus N_q) < 4\varepsilon$ and since ε was arbitrary, $\lambda_{\varphi(p)}(N_p \setminus N_q) = 0$. So $(p, q) \in K$.

To show that K is closed in $X_m \times X_m$, let (p_n, q_n) be a net in K converging to a point (p, q) in $X_m \times X_m$. Given any $\varepsilon > 0$, take $U \in \mathfrak{N}(N_p)$ and $V \in \mathfrak{N}(N_q)$ such that $\lambda_{\varphi(p)}(U \setminus N_p) < \varepsilon$ and $\lambda_{\varphi(p)}(V \setminus N_q) < \varepsilon$, and so (i) $|\lambda_{\varphi(p)}(U \setminus V) - \lambda_{\varphi(p)}(N_p \setminus N_q)| < \varepsilon$. Now take (p_n, q_n) in K with $N_{p_n} \subseteq U$, $N_{q_n} \subseteq V$, $\lambda_{\varphi(p_n)}(U \setminus N_{p_n}) < \varepsilon$, $\lambda_{\varphi(p_n)}(V \setminus N_{q_n}) < \varepsilon$, (and so (ii) $|\lambda_{\varphi(p_n)}(U \setminus V) - \lambda_{\varphi(p_n)}(N_{p_n} \setminus N_{q_n})| < \varepsilon$); (iii) $|\lambda_{\varphi(p)}(U \setminus V) - \lambda_{\varphi(p_n)}(U \setminus V)| < \varepsilon$. Since $(p_n, q_n) \in K$, (iv) $\lambda_{\varphi(p_n)}(N_{p_n} \setminus N_{q_n}) = 0$. Thus $\lambda_{\varphi(p)}(N_p \setminus N_q) < 3\varepsilon$, and since ε was arbitrary, $\lambda_{\varphi(p)}(N_p \setminus N_q) = 0$. So $(p, q) \in K$.

The proof of the additional remark is trivial.

(1.3) Corollary. *If X is metric, then*

$$\lambda_{\varphi(x_0)}(N_{x_0}) = \lambda_{\varphi(x_0)}(\cap \{N_x : x \in K(x_0)\}),$$

for $x_0 \in X_m$.

Proof. Let $\{x_i\}_1^\infty$ be a countable dense subset of $K(x_0)$. Then since N is closed $\cap \{N_x : x \in K(x_0)\} = \cap \{N_{x_i} : i = 1, 2, \ldots\}$.

(1.4) Corollary. *Suppose $x_0 \in X_m$, A is a Borel set contained in $\theta^{-1}(\varphi(x_0))$, and $N = \mathrm{cls}((\{x_0\} \times A)T)$. Then for $x \in K(x_0)$, $\lambda_{\varphi(x_0)}(A \cap N_x) = \lambda_{\varphi(x_0)}(A)$ (i.e. $\lambda_{\varphi(x_0)}(A \setminus N_x) = 0$). If in addition $A = B \cap C$ where B is a Borel set contained in the support of $\lambda_{\varphi(x_0)}$ with $\lambda_{\varphi(x_0)}(B) = 1$ and C is a nonempty open set, then $A \subseteq N_x$.*

Proof. Note $N_{x_0} \supseteq A$, so $\lambda_{\varphi(x_0)}(A \setminus N_x) \leq \lambda_{\varphi(x_0)}(N_{x_0} \setminus N_x) = 0$ and since $(A \cap N_x) \cup (A \setminus N_x) = A$, we see that $\lambda_{\varphi(x_0)}(A \cap N_x) = \lambda_{\varphi(x_0)}(A)$. If $A = B \cap C$, then $0 = \lambda_{\varphi(x_0)}(A \setminus N_n) = \lambda_{\varphi(x_0)}(B \cap (C \setminus N_x)) = \lambda_{\varphi(x_0)}(C \setminus N_x)$, but $C \setminus N_x$ is open and so must be disjoint from the support of $\lambda_{\varphi(x_0)}$ and thus $(B \cap C) \setminus N_x = \varnothing$.

(1.5) Theorem. *Suppose X_m is a Borel subset of X, $(X = Y, \theta = \varphi)$, $(\varphi, \lambda) \in \mathfrak{S}$, and for all x in $X_m \cap B(x)$, $\lambda_{\varphi(x)}(X_m \cap B(x)) = 1$, where $B(x)$ is the support of $\lambda_{\varphi(x)}$. Then for $x \in X_m \cap B(x)$, $Q_m(\varphi)(x) = S_m(\varphi)(x)$ and has the form $x' \in Q_m(\varphi)(x)$ iff there exist nets t_n in T and x_n in $X_m \cap B(x)$ such that $x_n \to x$, $x_n t_n \to x$, $x' t_n \to x$. So for each x' in X_m,*

$$\begin{aligned} B(x') \cap S_m(\varphi)(x') &= B(x') \cap Q_m(\varphi)(x') \\ &= X_m \cap \big(\cap \{\mathrm{cls}(\alpha T(x') \cap X_m \cap B(x')) : \\ &\qquad \alpha = V \times V \text{ where } V \text{ is a neighborhood of } x_0 \text{ in } X\}\big) \end{aligned}$$

for any fixed x_0 in X_m.

Proof. Let $x \in X_m \cap B(x)$, $x' \in S_m(\varphi)(x)$, and V an open neighborhood of x in X. Let

$$N = \mathrm{cls}\big([\{x'\} \times (V \cap X_m \cap B(x))]T\big).$$

Then $N_x \supseteq V \cap X_m \cap B(x)$ by Corollary 1.4, since $x \in K(x')$ and $B(x) = B(x')$. So $(x, x) \in N$ and there exist t_V in T and x_V in $V \cap X_m \cap B(x)$ such that $x't_V \in V$ and $x_V t_V \in V$. This gives the desired net on the directed set consisting of open sets directed by inclusion. Clearly then $(x, x') \in Q_m(\varphi)$.

(1.6) COROLLARY. *Suppose in addition that $x_0 \in X_m$, X has a countable neighborhood base at x_0, X_m is a dense G_δ subset of X, and $S_m(\varphi) = R_m(\varphi)$. Then for each x' in X_m, the set $P_m(x) \cap X_m \cap B(x')$ is a dense G_δ subset of $X_m \cap B(x')$.*

PROOF. The corollary follows easily from the observation that $P_m(x') \cap X_m \cap B(x') = \bigcap \{\alpha T(x') \cap X_m \cap B(x'): \alpha = V \times V$ where V is a neighborhood of x_0 in $X\}$ for any fixed x_0 in X_m.

(1.7) PROPOSITION. *Suppose $(\varphi, \lambda) \in \mathfrak{S}$, X is metric, $S_m(\varphi) = R_m(\varphi)$, and $\lambda_{\varphi(x)}(X_m) = 1$ for $x \in X_m$. Then $\lambda_{\varphi(x)}(P_m(x)) = 1$ and $\lambda_{\varphi(x)} \times \lambda_{\varphi(x)}(P_m) = 1$ for x in X_m.*

PROOF. Fix x_0 in X_m. Since X is metric, X has a countable neighborhood base, $\mathcal{V}$, at x_0, and $R(\varphi)(x) \cap P_m(x) = \bigcap \{\alpha T(x) \cap R_m(\varphi)(x): \alpha = V \times V$ with $V \in \mathcal{V}\}$. Note $P_m(x)$ and P_m are G_δ sets and so are measurable.

Now $\alpha T \cap R(\varphi)$ is an open invariant set in $R(\varphi)$, so $N = R(\varphi) \setminus \alpha T$ is a closed invariant set in $R(\varphi)$. By 1.2 and 1.3, since X is metric

$$\lambda_{\varphi(x)}(N_x) = \lambda_{\varphi(x)}(\bigcap \{N_{x'}: x' \in S_m(\varphi)(x) = R_m(\varphi)(x)\})$$

for x in X_m. Note $\lambda_{\varphi(x)}(A) = \lambda_{\varphi(x)}(A \cap X_m) - \lambda_{\varphi(x)}(A \setminus X_m) = \lambda_{\varphi(x)}(A \cap X_m)$ for any Borel subset A of X, since $\lambda_{\varphi(x)}(X_m) = 1$. So if $\lambda_{\varphi(x)}(N_x) \neq 0$, there exists x^* in $X_m \cap (\bigcap \{N_{x'}: x' \in R_m(\varphi)(x)\})$. So $x^* \in N_{x^*}$, that is $(x^*, x^*) \in N = R(\varphi) \setminus \alpha T$, which is a contradiction since αT contains the diagonal of $X_m \times X_m$. Thus

$$\lambda_{\varphi(x)}(\alpha T(x) \cap R_m(\varphi)(x)) = \lambda_{\varphi(x)}(R_m(\varphi)(x) \setminus N_x) = 1.$$

So $\lambda_{\varphi(x)}(R(\varphi)(x) \cap P_m(x)) = 1$. Since the support of $\lambda_{\varphi(x)}$ is contained in $R(\varphi)(x)$, there is no loss in writing $\lambda_{\varphi(x)}(P_m(x)) = 1$ for x in X_m.

By Fubini's Theorem, $\lambda_{\varphi(x)} \times \lambda_{\varphi(x)}(P_m) = 1$ for x in X_m.

(1.8) LEMMA. *Suppose $(\theta, \lambda) \in \mathfrak{S}$, X is minimal (so $X = X_m$, $S(\varphi) = S_m(\varphi)$, Z is minimal), X and Y are metric, and the natural projection $\pi: X \to X/S(\varphi)$ is open. Fix x in X. Let $E(x) = \{y \in B(x): (x/S(\varphi), y)$ has dense orbit in $(X/S(\varphi)) \circ^Z Y\}$ where $B(x)$ is the support of $\lambda_{\varphi(x)}$, and let $D(x) = \{y \in B(x): (x, y)$ has dense orbit in $X \circ^Z Y\}$. If $E(x)$ is a dense subset of $B(x)$, then $D(x)$ is a dense G_δ subset of $B(x)$. If $\lambda_{\varphi(x)}(E(x)) = 1$, then $\lambda_{\varphi(x)}(D(x)) = 1$. If $E(x)$ is a dense subset of $B(x)$ but X and Y are not necessarily metric, then we can conclude that $X \circ^Z Y$ is topologically transitive.*

PROOF. Fix x in X. Let $\{U_i\}$, $\{V_i\}$ be countable families of open sets in X, Y respectively such that the set of $U_i \circ^Z V_i = (U_i \times V_i) \cap (X \circ^Z Y)$ is a countable base of nonempty open sets for the topology on $X \circ^Z Y$. Fix i. Let W be any open subset of $B(x)$ and $N = \text{cls}((\{x\} \times x)T)$. Then $\{x'\} \times W \subseteq N$ for all $x' \in S(\varphi)(x)$ by Corollary 1.4. So $S(\varphi)(W) \times W \subseteq N$. Now there exist w in W with $(x/S(\varphi), w)$ having dense orbit. Also $(U_i/S(\varphi)) \circ^Z V_i$ is open by assumption and nonempty since $U_i \circ^Z V_i \neq \emptyset$. So there exists t in T with $(x/S(\varphi), w)t \in (U_i/S(\varphi)) \circ^Z V_i$. Then for some $x' \in S(\varphi)(x)$, $x't \in U_i$. Thus $N \cap U_i \times V_i \neq \emptyset$, so there exist s in T and w' in W with $(x, w')s \in U_i \circ^Z V_i$. Now since W was an arbitrary open set in $B(x)$, the set $A_i = \{a \in B(x)\colon (x, a)t \in U_i \times V_i$ for some t in $T\}$ is dense in $B(x)$, clearly it is open in $B(x)$. Let $y \in \bigcap_1^\infty A_i$, then (x, y) has dense orbit in $X \circ^Z Y$. So $D(x) = \bigcap_1^\infty A_i$ is a dense G_δ subset of $B(x)$.

Now suppose $\lambda_{\varphi(x)}(E(x)) = 1$ and $\lambda_{\varphi(x)}(A_i) \neq 1$ for some i. Then $\lambda_{\varphi(x)}(Y \setminus A_i) \neq 0$. Let $N = \text{cls}([\{x\} \times (B(X) \setminus A_i)]T)$ and $C = \bigcap\{N_{x'}\colon x' \in S(\varphi)(x)\}$, then $\lambda_{\varphi(x)}(C) \neq 0$, by 1.3. Now $S(\varphi)(x) \times C \subseteq N$ and $(x/S(\varphi), c)t \in (U_i/S(\varphi)) \times V_i$ for some t in T and c in C since $\lambda_{\varphi(x)}(C) \neq 0$ and $\lambda_{\varphi(x)}(E(x)) = 1$. So there exists x' in $S(\varphi)(x)$ with $(x', c)t \in U_i \circ^Z V_i$. So $N \cap (U_i \times V_i) \neq \emptyset$ and there exist s in T and b in $B(x) \setminus A_i$ with $(x, b)s \in U_i \times V_i$. But then $b \in A_i$ by the definition of A_i, a contradiction. Thus $\lambda_{\varphi(x)}(A_i) = 1$ and $\lambda_{\varphi(x)}(D(x)) = 1$.

To prove the last statement, let $U \circ^Z V$ be a nonempty open invariant set where U and V are open in X and Y respectively, we wish to show that $(U \circ^Z V)T$ is dense. Let W be any relative open subset of $B(X)$, then $N = \text{cls}((\{x\} \times W)T)$ has nonempty intersection with $U \circ^Z V$ and there exist s in T and w' in W with $(x, w')s \in U \circ^Z V$. So there exists an open neighborhood W' of w' relative to $B(x)$ with $(\{x\} \times W')s \subset U \circ^Z V$. Then $X \circ^Z Y = \text{cls}((\{x\} \times W')T) \subset \text{cls}((U \circ^Z V)T)$.

(1.9) COROLLARY. *Suppose X and Y are minimal and metric, the natural projection $X \to X/S(\varphi)$ is open, $(\theta, \lambda) \in \mathfrak{S}$, and $X/S(\varphi) \perp^Z Y/S(\theta)$. Fix x in X and let $D(x) = \{y \in B(x)\colon (x, y)$ has a dense orbit in $X \circ^Z Y\}$ where $B(x)$ is the support of $\lambda_{\varphi(x)}$. Then $D(x)$ is a dense G_δ subset of $B(x)$ and $\lambda_{\varphi(x)}(D(x)) = 1$. In the nonmetric case we can conlcude that $X \circ^Z Y$ is topologically transitive.*

PROOF. Note $X/S(\varphi) \circ^Z Y$ is a disjoint union of minimal sets and, by 3.7 of [G], $X/S(\varphi) \to Z$ has a section; so by 1.8, $X/S(\gamma) \circ^Z Y$ has a point with dense orbit and thus is minimal (or see 3.8 of [W]). This implies $E(x) = B(x)$. So the hypothesis of 1.8 is satisfied and the corollary folllows.

Example 3.2.1 shows that the assumption that $X \to X/S(\varphi)$ is open cannot always be dropped. Example 3.2.2 shows one case in which 1.9 holds without

$X \to X/S(\varphi)$ being open. Note this answers a conjecture of Glasner, [G],

(1.10) LEMMA. *Suppose Z is a singleton, Y supports an invariant Borel probability measure, λ, X and Y are metric, and $S_m(\varphi) = X_m \times X_m$. Then for x in X_m, the set $D(x) = \{y \in Y\colon (x, y)$ has dense orbit in $X \times Y\}$ is a dense G_δ subset of Y. If also $\lambda(Y_m) = 1$, then $\lambda(D(x)) = 1$.*

PROOF. (This proof is similar to that of Lemma 1.8.) Fix x in X_m. Let $\{U_i\}$, $\{V_i\}$ be countable families of open sets in X, Y respectively such that the set of $U_i \times V_i$ is a countable base of nonempty sets for the topology on $X \times Y$. Fix i. Let W be any open subset of Y and $N = \text{cls}((\{x\} \times W)T)$. Then $\{x'\} \times W \subseteq N$ for all $x' \in S_m(\varphi)(x)$ by Corollary 1.4. So $X_m \times W \subseteq N$. So $X \times W \subseteq N$. Now there exists w in W with dense orbit, so for some t in T and x' in X, $(x', w)t \in U_i \times V_i$. So $N \cap (U_i \times V_i) \neq \emptyset$ and there exist s in T and w' in W such that $(x, w')s \in U_i \times V_i$. Now since W was an arbitrary open set in Y, the set $A_i = \{a \in Y\colon (x, a)t \in U_i \times V_i$ for some t in $T\}$ is dense in Y, clearly it is open in Y. Then $D(x) = \bigcap_1^\infty A_i$ is a dense G_δ subset of Y.

Now suppose $\lambda(Y_m) = 1$ and $\lambda(A_i) \neq 1$. Then $\lambda(Y \setminus A_i) \neq 0$. Let $N = \text{cls}([\{x\} \times (Y \setminus A_i)]T)$ and $C = \bigcap\{N_{x'}\colon x' \in S_m(\varphi)(x)\} = \bigcap\{N_{x'}\colon x' \in X_m\} = \bigcap\{N_{x'}\colon x' \in X\}$ since X_m is dense in X and N is closed. Then $\lambda(C) \neq 0$. Now $X \times C \subseteq N$, and $ct \in V_i$ for some t and c in C since $\lambda(C) \neq 0$ and $\lambda(Y_m) = 1$. So there exists x' in X with $(x', c)t \in U_i \times V_i$ and thus $N \cap (U_i \times V_i) \neq \emptyset$. So there exist s in T and y in $Y \setminus A_i$ with $(x, y)s \in U_i \times V_i$. But then $y \in A_i$ by definition, a contradiction. Thus $\lambda(A_i) = 1$ and $\lambda(D(x)) = 1$.

The following generalizes a result of Peleg [P].

(1.11) PROPOSITION. *Suppose Z is a singleton, X and Y are minimal and metric, Y has an invariant Borel probability measure, λ, and $X/S(\varphi) \perp Y/S(\theta)$. Fix x in X. Then the set $D(x) = \{y \in Y\colon (x, y)$ has dense orbit in $X \times Y\}$ is a dense G_δ subset of Y and $\lambda(D(x)) = 1$.*

PROOF. Note $X/S \perp Y$, by 1.8 since X/S has an invariant measure (or see 3.8 of [W] or 14 of [P]).

The remainder of the proof is similar to that of Lemma 1.8. Fix x in X. Let $\{U_i\}$, $\{V_i\}$ be countable families of open sets in X, Y respectively such that the set of $U_i \times V_i$ is a countable base of nonempty open sets for the topology on $X \times Y$. Fix i. Let W be any open set of Y and $N = \text{cls}((\{x\} \times W)T)$. Then $\{x'\} \times W \subseteq N$ for all $x' \in S(\varphi)(x)$ by Corollary 1.4. So $S(\varphi)(x) \times W \subseteq N$. Now $U_i/S(\varphi)$ has nonempty interior since X is minimal, and so $(x/S(\varphi), w)t \in (U_i/S(\varphi)) \times V_i$ for some t in T since $X/S(\varphi) \perp Y$. So $(x', w)t \in U_i \times V_i$ for some x' in $S(\varphi)(x)$. So $N \cap (U_i \times V_i) \neq \emptyset$ and therefore there exist s in T and w' in W with $(x, w')s \in U_i \times V_i$. Now since W was arbitrary, the set $A_i = \{a \in Y: (x, a)t \in U_i \times V_i$ for some t in $T\}$ is an open dense subset of B. Thus $D(x) = \bigcap_1^\infty A_i$ is a dense G_δ subset of Y.

Now suppose $\lambda(A_i) \neq 1$. Then $\lambda(Y \setminus A_i) \neq 0$. Let $N = \text{cls}([\{x\} \times (Y \setminus A_i)]T)$ and $C = \bigcap\{N_{x'}: x' \in S(\varphi)(x)\}$, then $\lambda(C) \neq 0$ and $C \neq \emptyset$. Now $S(\varphi)(x) \times C \subseteq N$ and for c in C, $(x/S(\varphi), c)t \in U_i/S(\varphi) \times V_i$ for some t in T, since $X/S(\varphi) \perp Y$ and $U_i/S(\varphi)$ has nonempty interior. So there exists x' in $S(\varphi)(x)$ with $(x', c)t \in U_i \times V_i$. So $N \cap (U_i \times V_i) \neq \emptyset$ and there exist s in T and y in $Y \setminus A_i$ with $(x, y)s \in U_i \times V_i$. But then $y \in A_i$ by the definition of A_i, a contradiction. Thus $\lambda(A_i) = 1$ and $\lambda(D(x)) = 1$.

(1.12) Remark. If X is minimal and metric, $(\theta, \lambda) \in \mathbb{S}$ $(X = Y, \varphi = \theta)$, $S(\varphi) = R(\varphi)$, and φ is open, then for x in X, $D(x) = \{x' \in B(x): (x, x')$ has dense orbit in $X \circ^Z X\}$ is a dense G_δ subset of $B(x)$ and $\lambda_{\varphi(x)}(D(x)) = 1$. Moreover the transformation group $W_n = X \circ^Z X \circ^Z X \circ^Z \cdots \circ^Z X$ (n-times) is point-transitive for each integer n, by 1.8. As a consequence $P_m(\varphi_n)$ is dense in $R(\varphi_n)$ where φ_n: $W_n \to Z$, since $R(\varphi_n) = W_{2n}$ and $P_m(\varphi_n) = \{(w, w') \in W_n \circ^Z W_n = W_{2n}$: w, w' are transitive points in W_n and there exists a net t_i in T with $(w, w')t_i \to (w, w)\}$ contains the transitive points in W_{2n}.

(1.13) Proposition. *Suppose Z is a singleton, X is minimal and has an invariant Borel probability measure λ, X is metric, $S_X = X \times X$, and $W_n = X \times \cdots \times X$ (n-times). Then for each w in $(W_n)_m$ (the set of transitive points in W_n), the set $D(w) = \{x \in X: (w, x) \in (W_{n+1})_m\}$ is a dense G_δ subset of X.*

Proof. $S_m(\varphi_n) = (W_n)_m \times (W_n)_m$ since $P_m(\varphi_n)$ is dense in $R(\varphi_n) = (W_n) \times (W_n)$. The proposition follows from Lemma 1.10 upon simultaneously replacing Y by X and X by W_n and noting that then $Y_m = Y$.

2.

(2.1) Proposition. *Suppose φ: $X \to Z$ has $D(\varphi)$ dense in $R(\varphi)$, X is minimal, Z' is minimal, and ψ: $Z' \to Z$ is a proximal homomorphism. Let X' be the unique minimal set in $X \circ^Z Z'$ and let φ' and ψ' be the projections of X' onto Z' and X respectively then*

(1) *If $(x, x^*) \in Q(\varphi)$, then there exists z in Z' with $((x, z), (x^*, z)) \in Q(\varphi')$.*

(2) *$S(\varphi) = \{(x, x^*) \in R(\varphi)$: $(xu, x^*u) \in Q(\varphi)\}$ for some and thus every idempotent u in J.*

(3) *$S(\varphi') = \{(x', x_1') \in R(\varphi')$: $(x'u, x_1'u) \in Q(\varphi')\}$ for some and thus every idempotent u in J.*

(4) *If $(x, x^*) \in S(\varphi)$, then $((x, z'), (x^*, z')) \in S(\varphi')$ for some z' in Z'.*

(5) *Suppose $u \in J$ and $x_0 \in X$, with $x_0u = x_0$. Then there exists a compact*

nonempty subset B of $R(\varphi)(x_0)$ such that if $x \in R(\varphi)(x_0)$, $xu = x$, $x' \in B$, and $(x, x') \in S(\varphi)$, then $(x, x') \in Q(\varphi)$ and moreover there exist nets t_n in T and x_n in B with $x_n \to x$, $x_n t_n \to x$, $x' t_n \to x$. So for x' in X, $B \cap S(\varphi)x' = B \cap Q(\varphi)(x') = \bigcap \{\mathrm{cls}(\alpha T(x') \cap B): \alpha = V \times V$ where V is a neighborhood of x_1 in $X\}$ for any fixed x_1 in X.

(6) *If $S(\varphi) = R(\varphi)$ and $x_i u \in R(\varphi)(x_0 u)$ for $i = 1, 2, 3, \ldots$, $u \in J$, then there exist a nonempty compact subset B of $R(\varphi)(x_0 u)$ and a dense G_δ subset F of B such that $(x_i, f) \in P$ for f in F and $i = 0, 1, 2, \ldots$.*

Proof of (1). Suppose $(x, x^*) \in Q(\varphi)$. Then there exist nets t_n in T and $(x_n, x_n^*) \in R(\varphi)$ such that $(x_n, x_n^*) \to (x, x^*)$ and $(x_n, x_n^*)t_n \to (x, x)$. Now since $D(\varphi)$ is dense in $R(\varphi)$, there exists a net z_n' in Z' with $((x_n, z_n'), (x_n^*, z_n')) \in R(\varphi')$. We may assume z_n' and $z_n' t_n$ converge. Thus (1) follows with $z' = \lim z_n'$.

Proof of (2). See 2.4 of [MW] or 2.6.2 of [V_2].

Proof of (3). If $((x, z^\sim), (x^*, z^\sim)) \in S(\varphi')$, then

$$((xu, z^\sim u), (x^*u, z^\sim u)) \in S(\varphi') \subseteq S(\psi \circ \varphi') = S(\varphi \circ \psi').$$

So $(xu, x^*u) \in S(\varphi)$. Then $(xu, x^*u) \in Q(\varphi)$ by (2) and $((xu, z^*), (x^*u, z^*)) \in Q(\varphi')$ for some $z^* \in Z'$. Then $((xu, z^*u), (x^*u, z^*u)) \in Q(\varphi')$ and (3) follows since $z^*u = z^\sim u$.

Proof of (4). If $(x, x^*) \in S(\varphi)$ and $u \in J$, then $(xu, x^*u) \in Q(\varphi)$ and so there exists z in Z' with $((xu, z), (x^*u, z)) \in Q(\varphi')$. So $((xu, zu), (x^*u, zu)) \in Q(\varphi')$. Also there exists z' in Z' with $((x, z'), (x^*, z')) \in R(\varphi')$. Then $((xu, z'u), (x^*u, z'u)) \in Q(\varphi')$ since Z' is a proximal extension of Z implies $z'u = zu$. Thus $((x, z'), (x^*, z')) \in S(\varphi)$ by (3).

Proof of (5). (5) is a special case of 2.10 of [MW]. The proof here is completely different. Let Z' be $Z^\sim$ as in 4.1 of [G]. So φ' has a section. Given $u \in J$ and $x_0 \in X$ with $x_0 u = x_0$, take $z \in Z'$ with $(x_0, z) \in X'$ and $zu = z$. Then take $B(x_0, z)$ as in 1.5. Note $B(x_0, z) \subseteq R(\varphi)(x_0) \times \{z\}$. Clearly (5) holds with $B = \{x \in X: (x, z) \in B(x_0, z)\}$.

Proof of (6). Follows easily from the proof of (5).

In 3.1 of [V], it is shown that for $\varphi: X \to Y$ with X metric there exists a residual set A of points y in Y such that φ is open at each point of $\varphi^{-1}(y)$. In 3.3 of [G], it is shown that if, in addition to X metric, $(\varphi, \lambda) \in \mathfrak{S}$, then there exists a residual set B of points y in Y such that $\varphi^{-1}(y)$ equals the support B_y of λ_y. The next proposition shows that $B \subseteq A$; Example 3.2.3 shows $B \neq A$, in general.

(2.2) Proposition. *Suppose $\varphi: X \to Z$, $(\varphi, \lambda) \in \mathfrak{S}$, and x is in the support of $\lambda_{\varphi(x)}$. Then φ is open at x.*

Proof. Suppose φ is not open at x and $x \in B(x) =$ support of $\lambda_{\varphi(x)}$. Then there exists an open neighborhood V of x such that $\varphi(x) \notin \operatorname{int}(\varphi(V))$. Then there exists t in T with $\lambda_{yt}(V) > 0$ and $\varphi^{-1}(y)t \cap V = \varnothing$. This is a contradiction.

3.

(3.1) *Note*. Example 2.2 of [M] is an example of an action on the torus X that has the almost periodic points in $X \times X$ dense in $X \times X$ but does not have an invariant measure.

(3.2) EXAMPLES. The following examples are based on 2.1 of [M]. Note the examples will be nonmetric; however, metric examples could be obtained by letting (Y, S) in 2.1 of [M] be a countable dense subgroup S of the Cantor group provided with the discrete topology acting on the Cantor set by right multiplication. The verifications of the properties of the following examples are straightforward.

(1) Define g' by

$$g'(y, w) = \begin{cases} 0 & \text{if } h(y) > h(w), \\ 1 & \text{if } h(y) < h(w) \text{ and } h(y) \text{ is even}, \\ 2 & \text{if } h(y) < h(w) \text{ and } h(y) \text{ is odd}. \end{cases}$$

Then $\varphi_{g'}: X_{g'} \to Y$ has a RIM, $S(\varphi_{g'}) = R(\varphi_{g'})$, $\varphi_{g'}$ is not open, and $R(\varphi_{g'})$ is not point-transitive.

(2) Define g^* by

$$g^*(y, w) = \begin{cases} 0 & \text{if } h(y) > h(w) \text{ or } h(y) \text{ is odd}, \\ 1 & \text{if } h(y) < h(w) \text{ and } h(y) \text{ is even}. \end{cases}$$

Then $R(\varphi_{g^*})$ is point-transitive and φ_{g^*} is not open.

(3) Define f by

$$f(y, w) = \begin{cases} 0 & \text{if } h(y) > h(w), \\ 1 & \text{if } h(y) < h(w). \end{cases}$$

Then φ_f has a section λ, φ_f is open, and the support of λ_{y_0} does not equal $\varphi^{-1}(y_0)$, where (y_0, w_0) is the point of discontinuity of f.

4. For this section we refer the reader to III.3 of [G_b] for a sketch of a construction of a universal point-transitive transformation group $|L|$ with transitive point x_e for a topological group T with identity e. Note in [G_b] the group acts on the left, whereas it acts on the right in this paper. Because of this the definition of left uniformly continuous functions should be changed by defining f^t to be $f^t(s) = f(st)$. Such a function is sometimes called right uniformly continuous, [K]; such functions are exactly the functions that are uniformly continuous with respect to the left uniform structure as defined in

[HR]. Using 8.2 of [HR], we note that the mapping $t \to x_e t$ is a homeomorphism of T onto $x_e T$; identifying t with $x_e t$, we note that for any subset S of T and any neighborhood W of the identity, int(cls(SW)) is an open set in $|L|$ containing cls S.

Let M be a minimal right ideal in $|L|$. As sketched in 3.1 of [G_b], T is amenable iff M has an invariant measure.

(4.1) PROPOSITION. *Let T be an amenable group, M be the universal minimal set for T, and u be an idempotent in M. Suppose B is a syndetic subset of T and W is an open neighborhood of e in T. Then there exists a neighborhood V of u/Q in the compact Hausdorff group M/Q such that $t \in W^{-1}B^{-1}BUB^{-1}BW$ for all neighborhoods U of e in T and for all t in T with ut/Q in V. If B is discretely syndetic, then there exists a neighborhood V of u/Q in M/Q such that $t \in B^{-1}BUB^{-1}B$ for all neighborhoods U of e in T and all t in T with ut/Q in V.*

PROOF. By definition $T = BK$ for some compact subset K of T and so $|L| = \text{cls}(BK)$. Then note that $M \cap \text{cls}(B) \neq \emptyset$. So the set $A = M \cap \text{int}(\text{cls}(BW))$ is a nonempty open (in M) subset of M. (When B is discretely syndetic, $BF = T$, where F is finite, let $C = M \cap \text{cls}\, B$ and note $CF = M$, so $A = \text{int}_M(C) \subseteq \text{cls}\, B$ is a nonempty open subset of M.) Now A/Q is a subset of M/Q with nonempty interior, int(A/Q), since M is minimal. Let $m \in A$ with M/Q in int(A/Q) and let u be an idempotent in M with $mu = m$. Then $m^{-1}A$ is a neighborhood of u and $V = \text{int}(m^{-1}A/Q)$ is a neighborhood of u/Q (since M/Q is a regular minimal set). Now suppose $t \in T$ with $ut/Q \in V$. Then $(ut, p) \in Q$ for some p in $m^{-1}A$. So $(mt, mp) \in Q$ and by 1.5 there exist q in M and s in T with q in A and qs, mts in cls(BU) for any neighborhood U of e in T. Also $m \in A$. We note that q and m can be approximated by b_1w_1 and b_2w_2 so that b_1w_1s and b_2w_2ts are in cls(BU). So $b_1w_1sU \cap BU \neq \emptyset$ and $b_2w_2tsU \cap BU \neq \emptyset$. So $t \in W^{-1}B^{-1}BUU^{-1}UU^{-1}B^{-1}BW$. To get the stated result simply note that we could have used a U_0 with $U_0U_0^{-1}U_0U_0^{-1} \subseteq U$ in the proof. When B is discretely syndetic, $A \subseteq \text{cls}\, B$ and so for any neighborhood U of e in T, cls(BU) is a neighborhood of A in M and q and m may be approximated by b_1 and b_2. Then $t \in B^{-1}BUU^{-1}UU^{-1}B^{-1}B$.

REMARK. Motivated by 3.12 of [EK], we prove the following.

(4.3) COROLLARY. *Let T be abelian, B syndetic, and W a neighborhood of E in T. Then there exist b_0 in B and a neighborhood V of u/Q in M/Q such that $t \in WB^{-1}BBb_0^{-1}$ for all t with ut/Q in V.*

PROOF. We will merely sketch the portion of the proof that parallels that of 4.1. When $A = M \cap \text{int}(\text{cls}(BW))$ there is $at_0 = b_0w_0$ with $(u/Q)b_0w_0 \in \text{int}(A/Q)$. Take $V = \text{int}(A/Q)(b_0w_0)^{-1}$. If $(ut, p) \in Q$, then $(utt_0, pt_0) \in Q$

and $(vtt_0, pt_0) \in Q$ where v is an idempotent in M with $pv = p$. Take nets q_n in M and s_n in T with $q_n \to pt_0$, $q_n s_n \to pt_0$, $vtt_0 s_n \to pt_0$. Note pt_0 is in A. We may assume $q_n v = q_n$ since Mv is dense in M, and we may assume $r = \lim s_n \in M$ since vs_n has the same effect as s_n on q_n and vtt_0 (let $a_\lambda \to v$ and use the iterated limit theorem). Now $vtt_0 s_n \to rtt_0$, so $rtt_0 = pt_0$ and $s_n tt_0 \to rtt_0 = pt_0$. So we may fix an n such that $s_n tt_0 \in A$ and also $q_n \in A$ and $q_n s_n \in A$. Approximate q_n by $b_1 w_1$ so that $b_1 w_1 s_n \in \operatorname{cls}(BW)$. Then $s_n t b_0 w_0 = s_n tt_0 \in BWW^{-1}$ and $b_1 w_1 s_n \in BWW^{-1}$. So $t \in WW^{-1}B^{-1}BWW^{-1}w_0^{-1}b_0^{-1}$. The corollary clearly follows.

(4.4) EXAMPLE. The b_0^{-1} in the above cannot be dropped as the following example shows. Let T be the discrete group of integers and let T act on the unit circle K in the complex plane by irrational rotation, so (K, T) is minimal. Let V be a neighborhood of 1 in K and $k \in K$ such that $V \cap V^3 k = \varnothing$. Let $B = \{t \in T\colon 1t \in V_k\}$. Then $V \cap B^{-1}BB \subseteq V \cap V^3 k = \varnothing$.

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