TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 236, February 1978

A GLOBAL THEOREM FOR SINGULARITIES OF MAPS BETWEEN ORIENTED 2-MANIFOLDS

BY

J. R. QUINE

ABSTRACT. Let M and N be smooth compact oriented connected 2-manifolds. Suppose $f: M \to N$ is smooth and every point $p \in M$ is either a fold point, cusp point, or regular point of f i.e., f is excellent in the sense of Whitney. Let M^+ be the closure of the set of regular points at which f preserves orientation and M the closure of the set of regular points at which f reverses orientation. Let $p_1, \ldots, p_n$ be the cusp points and $\mu(p_k)$ the local degree at the cusp point p_k. We prove the following:

$$\chi(M) - 2\chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N)$$

where χ is the Euler characteristic and deg is the topological degree. We show that it is a generalization of the Riemann-Hurwitz formula of complex analysis and give some examples.

Proof of the theorem. Let $C^\infty(M, N)$ be the set of smooth maps from M to N topologized in the usual fashion, where two functions are close if all of their derivatives are close. Let $f \in C^\infty(M, N)$. We say p is a regular point of f if we can find local coordinates (x, y) near p and (u, v) near $f(p)$ with p and $f(p)$ having coordinates (0, 0) such that f is given by $u = x$, $v = y$. Say p is a fold point if the equations can be put in the form $u = x$, $v = y^2$. Say p is a cusp point if the equations can be put in the form $u = x$, $v = y^3 - xy$. It is well known for these dimensions that these three types of local maps are the only stable types and that the stable types are dense. (See Whitney [17] or Callahan [2].) Say f is excellent if every point $p \in M$ is one of the above types for f. In the following discussion, we will assume that f is excellent and that the images of the cusp points are distinct.

We will find it convenient to use a different form for the equation at a cusp point. Let $x = 3t^2 + s, y = t$. This clearly defines a C^∞ homeomorphism and the cusp equation becomes $u = 3t^2 + s$, $v = -2t^3 - st$. The general fold is now along the t axis. If we want the ordered pairs of vectors $(\partial/\partial s, \partial/\partial t)$ and $(\partial/\partial u, \partial/\partial v)$ to be positively oriented in the tangent spaces TM and TN

Presented to the Society, November, 6, 1976; received by the editors May 18, 1976 and, in revised form, August 22, 1976.

AMS (MOS) subject classifications (1970). Primary 57D35, 57D45, 58C25; Secondary 57D25, 30A90.

Key words and phrases. Singularities, cusp points, folds, nanifold, Euler characteristic, excellent map, Riemann-Hurwitz formula.

respectively, we must distinguish between two equations for a cusp:

(1) $$u = 3t^2 + s, \qquad v = -2t^3 - st$$

and

(2) $$u = 3t^2 - s, \qquad v = -2t^3 + st.$$

In case (1), the local degree of f at p is $+1$, i.e, for a sufficiently small neighborhood U of $f(p)$ the map $f\colon V \to U$ has degree $+1$ where V is the component of $f^{-1}(U)$ containing p. In case (2), the local degree is -1. If p is a cusp point, we will define $\mu(p)$ to be the local degree of f at p.

Let $p_1, \ldots, p_n$ be the cusp points of f. Let (x_k, y_k) be local coordinate systems defined in disjoint neighborhoods U_k of p_k, $k = 1, \ldots, n$, and let (u_k, v_k) be local coordinate systems in disjoint neighborhoods V_k of $f(p_k)$, $k = 1, \ldots, n$. Let these coordinate systems be chosen so that f takes the form (1) or (2) depending on the local degree, and such that $(\partial/\partial x_k, \partial/\partial y_k)$ and $(\partial/\partial u_k, \partial/\partial v_k)$ are positively oriented. We define a geometry on U_k by

$$\left\langle a\frac{\partial}{\partial x_k} + b\frac{\partial}{\partial y_k}, c\frac{\partial}{\partial x_k} + d\frac{\partial}{\partial y_k} \right\rangle = ac + bd,$$

i.e., the matrix (g_{ij}) defining the geometry in this coordinate system is the identity matrix. Extending $\langle\ ,\ \rangle$ to all of TM we get a Riemannian geometry on M. The inner product $\langle\ ,\ \rangle$ is just the usual dot product in the given coordinate systems on U_k, $k = 1, \ldots, n$. Follow the same procedure on N. Let $\omega\colon M \to \Lambda^2 TM$ and $\sigma\colon N \to \Lambda^2 TN$ be the dual volume (area) forms in these geometries. Clearly $\omega = \partial/\partial x_k \wedge \partial/\partial y_k$ on U_k and $\sigma = \partial/\partial u_k \wedge \partial/\partial v_k$ on V_k, $k = 1, \ldots, n$. We have $f_*\colon \Lambda^2 TM \to \Lambda^2 TN$ and we may define $\phi\colon M \to R$ by $f_*\omega = \phi\sigma$. Here R is the real numbers. The following facts are clear:

(a) If $p \in M$ and (x, y) are local coordinates at p and (u, v) are local coordinates at $f(p)$ then

$$\phi(x, y) = \psi(x, y)\partial(u, v)/\partial(x, y)$$

where $\partial(u, v)/\partial(x, y)$ is the Jacobian and ψ is a nonzero real-valued function.

(b) On U_k,

$$\phi(x_k, y_k) = \partial(u_k, v_k)/\partial(x_k, y_k) \quad \text{for } k = 1, \ldots, n.$$

(c) $\phi(p) > 0$ iff p is a regular point of f and f preserves orientation at p.

(d) $\phi(p) < 0$ iff p is a regular point of f and f reverses orientation at p.

(e) $\phi(p) = 0$ iff p is a singular point of f.

Let $S = \phi^{-1}(0)$, then S is the set of singular points of f. We will refer to the components of S as general folds after Whitney [17]. We claim that 0 is a regular value of ϕ. To show this, we take $p \in S$. By (a) and (e), $\partial\phi/\partial x = \psi\partial J/\partial x$ and $\partial\phi/\partial y = \psi\partial J/\partial y$ at p, where J is the Jacobian. If (x, y) and (u, v) are chosen so that the equation of f is in normal form for a fold or cusp at p, then we can easily check that $\partial J/\partial x$ and $\partial J/\partial y$ are not both zero at p, and p is a regular point for ϕ. Since 0 is a regular value of ϕ, we know that $S = \phi^{-1}(0)$ has the structure of a smooth 1-submanifold of M.

We now define a vector field on M which will be normal to S. We define $X = \text{grad}\,\phi$ by $\langle \text{grad}\,\phi, W\rangle = W(\phi)$ for any vector $W \in TM$ (cf. Milnor [10]). The zeros of $X = \text{grad}\,\phi$ are at the critical points of ϕ. We wish the zeros of X on M to be isolated, therefore we modify ϕ outside of a neighborhood of $S \cup (\bigcup_k U_k)$ in such a way that ϕ has only isolated critical points. This is done in such a way as to create no new zeros of ϕ. (We may assume that ϕ is a Morse function although we will not need this.) Now we define the vector field Y on M to be normal to X, i.e., $\langle X, Y\rangle = 0$, and such that $\|X\| = \|Y\|$ on M and (X, Y) has positive orientation. Now $Y(\phi) = 0$ and therefore Y is tangent to the level curves of ϕ and provides a vector field tangent to S. The vector fields X and Y are not zero on S since 0 is a regular value of f. We let the tangent field $Y|_S$ define positive orientation on S. Now since we modified ϕ outside of U_k, $k = 1, \ldots, n$, we still have $\phi(x_k, y_k) = \partial(u_k, v_k)/\partial(x_k, y_k)$ in U_k, $k = 1, \ldots, n$. In these coordinate systems, f takes the form (1) or (2) and we check that $\phi(x_k, y_k) = -x_k$. Since the geometry is given by the usual dot product, we have $X = -\partial/\partial x_k$ and $Y = -\partial/\partial y_k$ in U_k for $k = 1, \ldots, n$.

We have $Y\colon M \to TM$ and $f_*\colon TM \to TN$. Thus $f_*Y\colon M \to TN$. The theorem will follow from an investigation of the zeros of f_*Y. If p is an isolated zero of f_*Y, we define $\text{ind}_p(f_*Y)$ as follows: Choose (x, y) and (u, v), coordinate systems in neighborhoods U of p and V of $f(p)$ respectively, such that $(\partial/\partial x, \partial/\partial y)$ and $(\partial/\partial u, \partial/\partial v)$ are positively oriented. Write $f_*Y = \alpha_1\partial/\partial u + \alpha_2\partial/\partial v$ where $\alpha = (\alpha_1, \alpha_2)\colon U \to R^2$. Now choose a positively oriented coordinate circle C about p such that α is not zero inside or on C except at p. Then we define

$$\text{ind}_p\big(f_*Y\big) = \deg \left.\frac{\alpha}{|\alpha|}\right|_C$$

where deg is the topological degree and the image of $\alpha/|\alpha|$ is $S^1 \subseteq R^2$, with the usual counterclockwise orientation. It is easily checked that the definition is independent of the positively oriented coordinate system (u, v) chosen. This definition is analogous to the usual definition of the index of a vector field at an isolated zero (see Guillemin and Pollack [5]). Here, however, the vectors are on a different manifold from the points. In Greub [4, p. 385], this is referred to as the local dashed degree.

It is clear from the definition of fold point and cusp point that the only zeros of f_*Y on S are at cusp points. Since f is nonsingular on $M - S$, the only other zeros of f_*Y are the zeros of Y. Let us look at f_*Y in the neighborhood U_k of the cusp point p_k. We have

$$f_*Y = -\frac{\partial u_k}{\partial y_k}\frac{\partial}{\partial u_k} - \frac{\partial v_k}{\partial y_k}\frac{\partial}{\partial v_k}$$

$$= \begin{cases} -6y_k\dfrac{\partial}{\partial u_k} + (6y_k^2 + x_k)\dfrac{\partial}{\partial v_k} & \text{if } \mu(p_k) = +1, \\ -6y_k\dfrac{\partial}{\partial u_k} + (6y_k^2 - x_k)\dfrac{\partial}{\partial v_k} & \text{if } \mu(p_k) = -1. \end{cases}$$

Let $\alpha = (-6y_k, 6y_k^2 \pm x_k)$, the sign depending on $\mu(p_k)$. We check that

$$\left.\frac{\partial\alpha}{\partial y_k}\right|_{p_k} = (-6, 0),$$

$$\left.\frac{\partial\alpha}{\partial x_k}\right|_{p_k} = (0, 1) \quad \text{if } \mu(p_k) = +1,$$

$$\left.\frac{\partial\alpha}{\partial x_k}\right|_{p_k} = (0, -1) \quad \text{if } \mu(p_k) = -1.$$

By checking the orientation of $((\partial\alpha/\partial x_k)|_{p_k}, (\partial\alpha/\partial y_k)|_{p_k})$, we see that

$$\operatorname{ind}_{p_k}(f_*Y) = \mu(p_k).$$

We now define $M^+ = \phi^{-1}([0, \infty))$ and $M^- = \phi^{-1}((-\infty, 0])$. Now M^+ and M^- are oriented 2-submanifolds of M with boundary S. The 1-manifold S with positive orientation given by the vector field Y provides a positively oriented boundary for M^- and a negatively oriented boundary for M^+. By remarks (c) and (d) above, f preserves orientation on the interior of M^+ and reverses orientation on the interior of M^-. We remark that M^+ and M^- are not usually connected and we will write

$$M^+ = M_1^+ \cup \cdots \cup M_l^+ \quad \text{and} \quad M^- = M_1^- \cup \cdots \cup M_m^-$$

where M_k^+, $k = 1, \ldots, l$, are the connected components of M^+ and M_k^-, $k = 1, \ldots, m$, are the connected components of M^-. We recall that Y is a nonzero vector field on S. Thus

$$\sum_{p \in M^-} \operatorname{ind}_p(Y) = \chi(M^-)$$

where χ is the Euler characteristic. The notion of Euler characteristic of an oriented manifold with boundary is an extension of the notion for unbounded manifolds (see Spivak [12]). If M^- is formed from a compact manifold of genus g by cutting out j discs, then $\chi(M^-) = 2 - 2g - j$. Also $\chi(M^-) = \Sigma_k\chi(M_k^-)$.

Now we have

$$\operatorname{ind}_p(f_*Y) = \begin{cases} \operatorname{ind}_p Y & \text{if } p \in M^+, \\ -\operatorname{ind}_p Y & \text{if } p \in M^-. \end{cases}$$

Also on S we have shown that $f_* Y$ is zero only at the cusps $p_1, \ldots, p_n$ where

$$\operatorname{ind}_{p_k}(f_* Y) = \mu(p_k), \qquad k = 1, \ldots, n.$$

Thus

$$\sum_{p \in M} \operatorname{ind}_p(f_* Y) = \chi(M^+) - \chi(M^-) + \sum_{k=1}^{n} \mu(p_k). \tag{3}$$

To complete the proof of the formula, we need the following (cf. Greub [4, p. 386, problem 17])

Lemma. *Let $\tilde{f}: M \to TN$ have only isolated zeros, and let $f = \pi \circ \tilde{f}$ where $\pi: TN \to N$ is the usual projection. Then*

$$\sum_{p \in M} \operatorname{ind}_p(\tilde{f}) = (\deg f)\chi(N).$$

Proof. Let q be a regular value of f such that $\tilde{f}$ is not zero on $f^{-1}(q)$. Let W_1 and W_2 be vector fields on N such that (W_1, W_2) is a positively oriented basis at each point of $N - \{q\}$ and W_1 and W_2 vanish at q with index $\chi(N)$. Write $\tilde{f} = \alpha_1 W_1 + \alpha_2 W_2$ on $M - f^{-1}(q)$ where

$$\alpha = (\alpha_1, \alpha_2): (M - f^{-1}(q)) \to R^2.$$

If $p \in f^{-1}(q)$ and S_p is a small enough coordinate circle about q, positively oriented, then

$$\deg \left.\frac{\alpha}{|\alpha|}\right|_{S_p} = \begin{cases} -\chi(N) & \text{if } f \text{ preserves orientation at } p, \\ \chi(N) & \text{if } f \text{ reverses orientation at } p. \end{cases}$$

Now we have

$$\sum_{p \in N - f^{-1}(q)} \operatorname{ind}_p \alpha = - \sum_{p \in f^{-1}(q)} \deg \left.\frac{\alpha}{|\alpha|}\right|_{S_p} = \chi(N)(\deg f).$$

This completes the proof of the lemma.

Now by the previous lemma

$$\sum_{p \in M} \operatorname{ind}_p(f_* Y) = (\deg f)\chi(N).$$

Combining this with (3) above, we get

$$\chi(M^+) - \chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N).$$

Since $\chi(M) = \chi(M^+) + \chi(M^-)$, we get

$$\chi(M) - 2\chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N)$$

and this is the desired formula. We mention that at this point we may, by continuity, drop the assumption that the images of the cusp points are distinct.

Summarizing, we have

THEOREM 1. *Let M and N be smooth, compact, oriented, connected 2-manifolds and let f: $M \to N$ be excellent. Let M^+ be the closure of the set of regular points where f preserves orientation and M^- the closure of the set of regular points at which f reverses orientation. Let $p_1, \ldots, p_n$ be the cusp points and let $\mu(p_k)$ be the local degree at the cusp point p_k, then*

$$\chi(M) - 2\chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N) \tag{4}$$

where χ is the Euler characteristic. The conclusion may also be stated

$$\chi(M^+) - \chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N).$$

The following corollary is a result of A. Haefliger [6, p. 53].

COROLLARY 1. *Let M be a smooth, compact, oriented, connected 2-manifold and let f: $M \to R^2$ be excellent. Let M^+, M^-, p_k be defined as in the theorem. Then the number of cusp points n is less than or equal to $|\chi(M^+) - \chi(M^-)|$.*

PROOF. We may consider R^2 as S^2 minus a point. Letting $N = S^2$ in the theorem, we have $\deg f = 0$ and $\Sigma\,\mu(p_k) = \chi(M^-) - \chi(M^+)$. Now $n < |\Sigma\,\mu(p_k)|$ and the result follows.

We also have the following corollary (cf. Levine [8], [9], and Thom [14]).

COROLLARY 2. *Under the hypotheses of the theorem, the number of cusp points is even.*

PROOF. We note that that $\chi(M)$ and $\chi(N)$ are even and $\chi(p_k)$ has the same parity as n.

For another formula related to (4), see Tucker [16]. Also, in combinatorially classifying stable maps from bordered surfaces to the sphere, Francis and Troyer [3], [18] have proved a formula related to (4).

We now show how the standard Riemann-Hurwitz formula (see Hille [7, p. 110] or Bliss [1, p. 89]) follows from (4). Suppose f: $M \to N$ is orientation-preserving except for branch points at $p_1, \ldots, p_n$ of orders $\nu_1, \ldots, \nu_n$, respectively. (Such a map is called sense preserving or SP by Titus [15].) Now f is not excellent but f may be perturbed slightly so that instead of a branch point of order ν_k at p_k, we have $\nu_k + 1$ cusps of positive type on a simple closed general fold. This general fold bounds a component of M^- in its interior (see Callahan [2] for a demonstration of this for $\nu_k = 2$). Thus $\chi(M^-) = n$ and $\Sigma\,\mu(p_k) = n + \Sigma\nu_k$. Thus (4) gives

$$\chi(M) + \sum (\nu_k - 1) = (\deg f)\chi(N). \tag{5}$$

If N is the sphere, $\chi(N) = 2$ and (5) is the Riemann-Hurwitz formula.

SOME EXAMPLES. We give two examples of formula (4).

1. Let M be a torus considered as a sphere N with one handle. Let f be obtained by projecting M onto N. The configuration of cusps and general folds is shown in Figure 1. The four cusps are of positive type. We have $\Sigma\, \mu(p_k) = 4$, $\chi(M) = 0$, $\chi(N) = 2$, $\chi(M^-) = 1$, and $\deg f = 1$.

2. Let M be a torus and N a sphere (see Figure 2). Construct f by sending S_k to T_k, $k = 1, \ldots, 4$, preserving the orientations indicated, and sending the indicated points on S_1 and S_3 to the cusps on T_1 and T_3. All cusps are of positive type and $\Sigma\, \mu(p_k) = 4$, $\chi(M^-) = 0$, $\chi(M) = 0$, $\chi(N) = 2$, and $\deg f = 2$. We remark that f is a perturbation of the usual branched covering of a torus over a sphere given by the algebraic function

$$w^2 - (1 - z^2)(1 - k^2z^2) = 0.$$

See Springer [13] introduction.

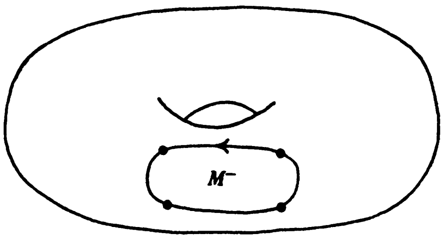

FIGURE 1

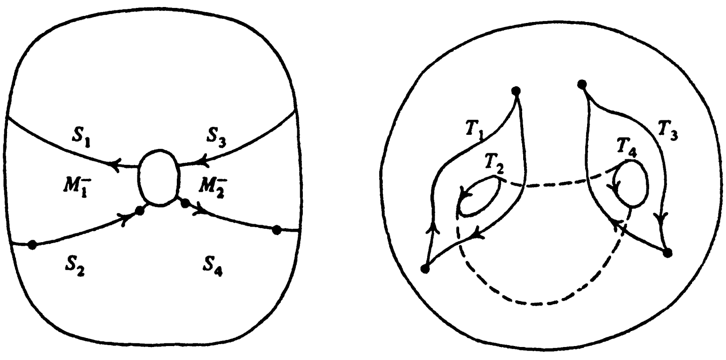

FIGURE 2

REFERENCES

1. G. A. Bliss, *Algebraic functions*, Dover, New York, 1966. MR 34 #2866.

2. J. Callahan, *Singularities and plane maps*, Amer. Math. Monthly 81 (1974), 211–240. MR 50 #5820.

3. G. K. Francis and S. F. Troyer, *Excellent maps with given folds and cusps* (preprint).

4. W. Greub, S. Halperin and R. Vanstone, *Connections, curvature, and cohomology*, Vol. I, Academic Press, New York, 1972. MR **49** #1423.

5. V. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall, Englewood Cliffs, N.J., 1974. MR **50** #1276.

6. A. Haefliger, *Quelques remarques sur les applications différentiables d'une surface dans le plan*, Ann. Inst. Fourier (Grenoble) **10** (1960), 47–60. MR **22** #7145.

7. E. Hille, *Analytic function theory*, Vol. II, Blaisdell, Waltham, Mass., 1962. MR **34** #1490.

8. H. I. Levine, *Elimination of cusps*, Topology 3 (1965) suppl. 2, 263–296. MR **31** #756.

9 ______, *Mappings of manifolds into the plane*, Amer. J. Math. **88** (1966), 357–365. MR **34** #8418.

10. J. Milnor, *Morse theory*, Ann. of Math. Studies, No. 51, Princeton Univ. Press, Princeton, N.J., 1963. MR **29** #634.

11. J. R. Quine, *Tangent winding numbers and obstructions to regular homotopy as intersection numbers* (to appear).

12. M. Spivak, *A comprehensive introduction to differential geometry*, Vol. 5, Publish or Perish, Boston, Mass., 1975.

13. G. Springer, *Introduction to Riemann surfaces*, Addison-Wesley, Reading, Mass., 1957. MR **19**, 1169.

14. R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier (Grenoble) **6** (1955–1956), 43–87. MR **19**, 310.

15. C. J. Titus, *Extensions through codimension one to sense preserving mappings*, Ann. Inst. Fourier (Grenoble) **23** (1973), 215–227. MR **50** #1265.

16. A. W. Tucker, *Branched and folded coverings*, Bull. Amer. Math. Soc. **42** (1936), 859–862.

17. H. Whitney, *On singularities of mappings of euclidean spaces*. I: *Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410. MR **17**, 518.

18. G. K. Francis, *Branched and folded parametrizations of the sphere*, Bull. Amer. Math. Soc. **80** (1974), 72–76. MR **50** #3245.

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306