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FRATTINI SUBALGEBRAS OF FINITELY GENERATED SOLUBLE LIE ALGEBRAS

BY

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ABSTRACT. This paper is motivated by a recent one of Stewart and Towers [8] investigating Lie algebras with "good Frattini structure" (definition below). One consequence of our investigations is to prove that any finitely generated metanilpotent Lie algebra has good Frattini structure, thereby answering a question of Stewart and Towers and providing a complete Lie theoretic analogue of the corresponding group theoretic result of Phillip Hall. It will also be shown that in prime characteristic, finitely generated nilpotent-by-finite-dimensional Lie algebras have good Frattini structure.

1. **Preliminaries.** We employ the notation of Amayo and Stewart [3]. For a fixed ground field $\mathfrak{k}$, $\mathfrak{A}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{N}$ denote the classes of abelian, finite-dimensional, finitely generated, and nilpotent Lie, algebras respectively. If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of Lie algebras, then $\mathfrak{X}\mathfrak{Y}$ is the class of all Lie algebras L having an ideal $I \in \mathfrak{X}$ with $L/I \in \mathfrak{Y}$. We write $\mathfrak{X}^2$ for the class $\mathfrak{X}\mathfrak{X}$, and, in general, $\mathfrak{X}^{n+1} = \mathfrak{X}^n\mathfrak{X}$. We also refer to $\mathfrak{X}\mathfrak{Y}$ as the class of $\mathfrak{X}$-by-$\mathfrak{Y}$ Lie algebras, and $\mathfrak{X}^2$ is the class of meta-$\mathfrak{X}$ algebras. Thus $\mathfrak{N}^2$ is the class of metanilpotent Lie algebras.

The symbol L will denote a Lie algebra of arbitrary dimension defined over the field $\mathfrak{k}$. The notation $A \subseteq L$, $A \leq L$, $A \lhd L$, A si L means that A is a subset, subalgebra, ideal, and subideal of L, respectively. By $A <\cdot L$ we mean that A is a maximal subalgebra of L. If $A, B \subseteq L$, then $[A, B]$ is the subspace of L spanned by all $[a, b]$ with $a \in A$ and $b \in B$, $[A,_{n+1}B] = [[A,_nB], B]$ and $[A,_0B] = A$; $[a,_0b] = a$ and $[a,_{n+1}b] = [[a,_nb], b]$.

The Frattini subalgebra $F(L)$ is the intersection of the maximal subalgebras of L or is L if there are no maximal subalgebras. The Frattini ideal $\Phi(L)$ is the largest ideal of L contained in $F(L)$. In general, $F(L) \neq \Phi(L)$.

A chief factor of L is a pair (H, K) of ideals of L such that $H > K$ and no ideal of L lies properly between H and K. We also refer to the corresponding factor ideal H/K of L/K as the chief factor.

If $A \lhd B \leq L$, then

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$$C_L(B/A) = \{x \in L: [B, x] \subseteq A\}.$$

If A and B are ideals of L, then $C_L(B/A)$ is also an ideal of L.

$$\psi(L) = \bigcap \{C_L(H/K): H/K \text{ is a chief factor of } L\}.$$

The Hirsch-Plotkin radical $\rho(L)$ is the unique maximal locally nilpotent ideal of L. The Fitting radical $\nu(L)$ is the sum of the nilpotent ideals of L. We always have $\nu(L) \leq \rho(L)$. We set

$$\tilde{\nu}(L)/\Phi(L) = \nu(L/\Phi(L)) \quad \text{and} \quad \tilde{\psi}(L)/\Phi(L) = \psi(L/\Phi(L)).$$

We say that L has good Frattini structure if $\nu(L)$ is nilpotent and $\nu(L) = \rho(L) = \psi(L) = \tilde{\nu}(L)$.

Let $U = U(L)$ be the universal enveloping algebra of L and let A be an L-module (and hence U-module). For a two-sided ideal I of U we let

$$Z(L : I) = Z^*(L; I)/I$$

be the center of U/I.

We say that B/C is a chief factor submodule of A in case B and C are submodules of A, $B > C$ and no submodule lies strictly between B and C. If $N \subseteq M$ are submodules of A, then

$$\mathrm{Ann}_U(M/N) = \{u \in U: Mu \subseteq N\}.$$

We define

$$\psi(A; U) = \bigcap \{\mathrm{Ann}_U(B/C): B/C \text{ is a chief factor submodule}\}.$$

Clearly $\psi(A; U)$ is a two-sided ideal of $U(L)$, and if we consider L as a module over itself under the adjoint action, then

$$L \cap \psi(L; U) = \psi(L).$$

We shall prove

THEOREM A. *If $L \in \mathfrak{F}$ and A is a finitely generated L-module, then there is an integer n such that $A(\psi(A; U) \cap Z^*(L; \mathrm{Ann}_U(A)))^n = 0$.*

Combining this with the fact that the universal enveloping algebras of nilpotent Lie algebras have centralizing sets of generators will yield

THEOREM B. *If $L \in \mathfrak{F} \cap \mathfrak{N}$ and A is a finitely generated Lie algebra, then there is an integer n such that $A(\psi(A; U))^n = 0$.*

When $\mathfrak{k}$ has prime characteristic, we can prove more, namely:

THEOREM C. *If $L \in \mathfrak{F}$ over a field of prime characteristic and if A is a finitely generated L-module, then there is an integer n such that $A(\psi(A; U))^n = 0$.*

Applications of these results yield:

THEOREM D. *Any finitely generated metanilpotent Lie algebra has good Frattini structure.*

THEOREM E. *Any finitely generated nilpotent-by-finite-dimensional Lie algebra over a field of prime characteristic has good Frattini structure.*

As is remarked in Stewart and Towers [8] we have

COROLLARY F. *The natural representation of L on $\nu(L)$ induces a faithful representation of $L/\nu(L)$ on $\nu(L)/\Phi(L)$ whenever $L \in \mathfrak{G} \cap \mathfrak{N}^2$.*

2. The Frattini ideal. In Stewart and Towers [8] it is proposed (though their proof is incorrect) that if $L \in \mathfrak{G} \cap \mathfrak{N}^2$, then $\nu(L)^2 \subseteq \Phi(L)$. This result is, in fact, true for any Lie algebra as we now show.

PROPOSITION 2.1. *Let L be any Lie algebra. Then*:
(a) $\nu(L) \leq \psi(L)$.
(b) $[\nu(L), \psi(L)] \subseteq \Phi(L)$.
(c) $[\tilde{\nu}(L), \tilde{\psi}(L)] \subseteq \Phi(L)$.
(d) *If $\tilde{\psi}(L)/\Phi(L)$ is a sum of solvable ideals of $L/\Phi(L)$, then $\tilde{\nu}(L) = \tilde{\psi}(L)$. In particular, $\tilde{\nu}(L)/\Phi(L)$ is abelian and $\nu(L)^2 \subseteq \Phi(L)$.*

PROOF. (a) Suppose that I is a nilpotent ideal and H/K a chief factor of L. Then $[H, I] + K$ is an ideal of L between H and K. If $H = [H, I] + K$, then $H = [H,_2 I] + K = \cdots = [H,_n I] + K \subseteq I^n + K$ for any positive integer n. Since $I^n = 0$ for some n this would imply that $H = K$, a contradiction. Thus $[H, I] + K = K$ and $[H, I] \subseteq K$ and $I \subseteq \psi(L)$.

(b) Suppose that $M <\cdot L$ and $[\nu(L), \psi(L)] \not\subseteq M$. Then there is a nilpotent ideal I such that $[I, \psi(L)] \not\subseteq M$. If $I^2 \not\subseteq M$, then $L = I^2 + M$, whence $I = I^2 + I \cap M = I^r + I \cap M$ for all r and so $I = I \cap M$, a contradiction. Thus $I^2 \subseteq M$ and $I^2 \subseteq I \cap M \neq I$, $L = I + M$ and $I \cap M$ is an ideal of L. As $M <\cdot L$, $I/I \cap M$ is a chief factor of L and so $[I, \psi(L)] \subseteq I \cap M \subseteq M$, a contradiction. So $[\nu(L), \psi(L)] \subseteq F(L)$, and since $[\nu(L), \psi(L)]$ is also an ideal of L, we have $[\nu(L), \psi(L)] \subseteq \Phi(L)$.

(c) follows from (b) and the definitions of $\tilde{\nu}(L)$ and $\tilde{\psi}(L)$.

(d) Suppose that $\tilde{\psi}(L) \neq \tilde{\nu}(L)$ and $\tilde{\psi}(L)/\Phi(L)$ is a sum of solvable ideals of $L/\Phi(L)$. Then there is an ideal I of L contained in $\tilde{\psi}(L)$ such that the derived length of $I/\Phi(L)$ is minimal with respect to $I^2 \not\subseteq \Phi(L)$. Then there is $M <\cdot L$ such that $L = I^2 + M$. Now $((I^2) + \Phi(L)) \neq I$ (else as $\Phi(L) \lhd L$ we would have $I = I^{(r)} + \Phi(L)$ for all r and so $I = \Phi(L)$), and hence $(I^2 + \Phi(L))^2 \subseteq \Phi(L)$. In particular, if $J = I^2 + \Phi(L)$, then $J \cap M \lhd J$ and so $J \lhd L$. Now $J/J \cap M$ is isomorphic to the chief factor $J/\Phi(L)/(J \cap M/\Phi(L))$ of $L/\Phi(L)$ and, hence, $[J, \tilde{\psi}(L)] \subseteq J \cap M$. This implies that $(\tilde{\psi}(L))^2 \subseteq M$, since $\tilde{\psi}(L) = J + \tilde{\psi}(L) \cap M$ and $J^2 \subseteq M$. Thus $L = J + M \subseteq (\tilde{\psi}(L))^2 +$

$M \subsetneq M$, a contradiction. This proves (d). The rest follows from (a)–(d). $\square$

LEMMA 2.2. *If $L \in \mathfrak{F}$ and A is an irreducible L-module then $Z(L; \mathrm{Ann}_U(A))$ is finite dimensional over $\mathfrak{k}$.*

PROOF. If $\mathfrak{k}$ has prime characteristic, then, by a result of Curtis [5, p. 952], A and, hence, $U/\mathrm{Ann}_U(A)$ is finite dimensional over $\mathfrak{k}$.

If $\mathfrak{k}$ has characteristic zero, then by Proposition 4.1.7 of Dixmier [7, p. 131], $Z(L; \mathrm{Ann}_U(A))$ is finite dimensional.

THEOREM 2.3. *Let $L \in \mathfrak{F}$ and A be a finitely generated L-module. Suppose that θ is an L-module endomorphism of A such that $B\theta \subsetneq C$ for any chief factor submodule B/C of A. Then there is an n such that $A\theta^n = 0$.*

PROOF. Let $X = L \oplus \mathfrak{k}\theta$ so that A is a finitely generated X-module. Set $V = U(X) = U(L) \otimes_{\mathfrak{k}} \mathfrak{k}[\theta]$. Then if $I = \mathrm{Ann}_V(A)$, $\theta + I \in Z(X; I)$. Evidently every L-submodule of A is an X-submodule. Suppose it is false that $A\theta^n = 0$ for some n. Now X is finite dimensional and so A is a noetherian X-module, whence there is a submodule N of A maximal with respect to $A\theta^n \not\subseteq N$ for any n.

By replacing A by A/N we may assume that if $0 \neq B \subseteq_X A$ then $A\theta^n \subseteq B$ for some n. Thus if $a \in A$ and $a\theta = 0$, then $aV\theta = a\theta V = 0$ and so $a = 0$. Thus θ is a V-module monomorphism of A.

By the proof of Theorem 3.3 of Stewart and Towers [8, p. 214] we can embed A in an X-module M such that $M = A\mathfrak{k}[T] = AU(X \oplus \mathfrak{k}T)$, is a finitely generated $(X \oplus \mathfrak{k}T)$-module, θ is an $(X \oplus \mathfrak{k}T)$-module automorphism of M and $T\theta - 1 = \theta T - 1 \in \mathrm{Ann}_W(M)$, where $W = U(X \oplus \mathfrak{k}T)$.

Let N be a nonzero W-submodule of M. Suppose, if possible, that $N \cap A = 0$. Let $a_0, \ldots, a_k \in A$ and $b = a_0 + a_1T + \cdots + a_kT^k \in N \setminus 0$ be such that k is minimal. Then $k \neq 0$ and $a_k \notin A\theta$. But $b\theta^k = a_0\theta^k + a_1\theta^{k-1} + \cdots + a_1\theta + a_k \in N \cap A = 0$ and, hence, $a_k \in A\theta$, a contradiction. Thus $N \cap A \neq 0$, whence $A\theta^n \subseteq N \cap A$ for some n and so $A = A\theta^nT^n \subseteq NT^n \subseteq N$, so that $A\mathfrak{k}[T] \subseteq N$ and $N = M$. Thus M is an irreducible W-module. By Lemma 2.2, $Z(M; \mathrm{Ann}_W(M))$ is finite dimensional, so there is a polynomial f of minimal degree with $Mf(\theta) = 0$. Let $f(t) = \lambda_0 t^n + \cdots + \lambda_n$. As θ is an automorphism of M we have $\lambda_n \neq 0$. If $a \in A$, then

$$a = \lambda_n^{-1}(\lambda_n a) = -\lambda_n^{-1}\big((f(\theta) - \lambda_n)a\big) \in A\theta.$$

So $A = A\theta$ and, therefore, A is an irreducible X-module, whence A is an irreducible L-module and so $A = A\theta = 0$, a contradiction. This proves the result. $\square$

The proof of Theorem 2.3 also yields the following useful corollary:

COROLLARY 2.4. *Let $L \in \mathfrak{F}$, A a finitely generated L-module and $z + I \in$*

$Z(L; \mathrm{Ann}_U(A))$. *If B is any submodule of A maximal with respect to $Az^n \nsubseteq B$ for any n, then B is a maximal submodule of A.* □

Let U be an arbitrary associative $\mathfrak{k}$-algebra and A a U-module. Then we may define $\psi(A; U)$ as before and Z as the center of U. Then U is said to have the *chief annihilator property* if $z \in \psi(A; U) \cap Z$ and A a finitely generated U-module implies $Az^n = 0$ for some n.

We denote by CAP the class of Lie algebras L such that $U(L)$ has the chief annihilator property. Evidently Theorem 2.3 states a stronger property, namely that if $z \in \psi(A; U) \cap Z^*(L; \mathrm{Ann}(U))$ then also $Az^n = 0$ for some n. We refer to the class of Lie algebras with this property as SCAP (strong chief annihilator property). Then we have

COROLLARY 2.5. $\mathfrak{F} <$ SCAP.

We leave open the question of whether or not the inclusion is strict. We note also that SCAP is closed under homomorphic images.

3. Proofs of the main results.

PROOF OF THEOREM A. Let $L \in \mathfrak{F}$ and A be a finitely generated L-module and set $N = \psi(A; U) \cap Z^*(L; \mathrm{Ann}_U(A))$. Clearly each element of $Z^*(L; \mathrm{Ann}_U(A))$ induces an L-module endomorphism of A and so if $z \in N$, then by Theorem 2.3 there is an $n = n(z)$ such that $Az^n = 0$. Using standard arguments and the fact that N mod $\mathrm{Ann}_U(A)$ is commutative, it follows that given any finite-dimensional subspace S of N there exists $n = n(S)$ such that $AS^n = 0$. Now $U(L)$ has the maximal condition on right ideals and so $UN = NU = SU = US$ (mod $\mathrm{Ann}_U(A)$) for some finite-dimensional subspace S of U, whence $A(NU)^n = A(SU)^n \subseteq AS^nU = 0$ for some n and Theorem A is proved. □

If L is a Lie algebra and I is an ideal of $U(L)$, then a subspace S of $U(L)$ is said to be L-invariant (mod I) if $[s, x] = sx - xs \in S + I$ for any $x \in L$ and $s \in S$. Now if L is nilpotent then for any u in U there is a c minimal with respect to $[u,_c L] = 0$. From this it follows easily that

(*) if L is nilpotent and S is L-invariant mod I, then either $S \subseteq I$ or else $(S + I) \cap Z^*(L; I) \nsubseteq I$.

PROOF OF THEOREM B. Let $L \in \mathfrak{F} \cap \mathfrak{N}$, $U = U(L)$, and A be a finitely generated submodule and $M = \psi(A; U)$. Clearly if E is a submodule or quotient module of A, then $M \subseteq \psi(E; U)$. Suppose that $AM^n \neq 0$ for any n. As A is a noetherian U-module we may replace A by a suitable quotient of A and assume then without loss of generality that if B is a nonzero submodule of A then $AM^n \subseteq B$ for some n depending on B. Consider the set of ideals of U which are annihilators of nonzero submodules of A and let I be a maximal element (U is noetherian) and B a nonzero submodule of A with $I = \mathrm{Ann}_U(B)$. If $M \subseteq I$, then $AM^n \subseteq B$ and so $AM^{n+1} = 0$ for some n, a contradiction. Assume that $M \not\subseteq I$. Then $M + I \neq I$ and $M + I$ is an L-invariant (mod I) subspace of $\psi(B; U)$ (M is an ideal of U), and, hence, by (*), $N_1 = (M + I) \cap Z^*(L; I) \not\subseteq I$, and $N_1 \subseteq N = \psi(B; U) \cap Z^*(L; I)$. By Theorem A there exists r minimal with respect to $BN^r = 0$, whence $I < NU + I = UN + I \subseteq \mathrm{Ann}_U(BN^{r-1})$ and BN^{r-1} is a nonzero submodule of A, and this contradicts the maximality of I. This proves Theorem B. □

Theorem C will follow from the more general Theorem 3.1 below. From Amayo and Stewart [3, pp. 225–232] we have the definition of the class Max-cu as consisting of all Lie algebras L with the property that if $U = U(L)$, then there exists a noetherian subring $R = \mathfrak{k}[z_1, z_2, \ldots, z_m]$ of the center of U such that $U = Ru_1 + \cdots + Ru_n$ is a finitely generated R-module. By the results of Curtis [5] and Amayo and Stewart [3, Chapter 11] we have the following facts:

(1) $\mathfrak{F} \cap \mathfrak{A} <$ Max-cu.

(2) In prime characteristic, $\mathfrak{F} <$ Max-cu.

(3) For $L \in$ Max-cu, $U = U(L)$ and $R = \mathfrak{k}[z_1, \ldots, z_s] \subseteq$ center of U, such that $U = Ru_1 + \cdots + Ru_n$;

(a) every irreducible L-module is finite dimensional;

(b) U is a noetherian R-module and so satisfies the maximal conditions on left and right ideals;

(c) there exists $n = n(L)$ such that to any $u \in U$ there correspond $r_1, r_2, \ldots, r_n \in R$ for which

$$(**) \qquad u^n + r_1 u^{n-1} + \cdots + r_{n-1} u + r_n = 0;$$

(d) every finitely generated L-module is noetherian.

Let A be an L-module and $U = U(L)$. Clearly for any submodules B, C of A with $C \subseteq B$, we have $\psi(A; U) \subseteq \psi(B/C; U)$. We refer to the factor module B/C and any L-module isomorphic to it (as L-modules) as an L-module section of A.

THEOREM 3.1. *If $L \in$ Max-cu and A is a finitely generated L-module, then there is an integer m such that $A(\psi(A; U))^m = 0$.*

PROOF. Let $U = U(L) = Ru_1 + \cdots + Ru_n$, where $R = \mathfrak{k}[z_1, \ldots, z_s] \subseteq$ center of U as above and let A be a finitely generated L-module for which the conclusion fails. Define

$$\mathfrak{z} = \{\psi(X; U) : X \text{ is an } L\text{-module section of } A \text{ and } X(\psi(X; U))^m \neq 0 \text{ for all } m\}.$$

As U is noetherian there is a section Y of A for which $N = \psi(Y; U)$ is a maximal element of ζ. Pick a submodule E of Y maximal with respect to the property that $YN^m \not\subseteq E$ for any m. Then if $A_1 = Y/E$ we have $N \subseteq \psi(A_1; U)$ and so $A_1\psi(A_1; U)^m \neq 0$ for any m, whence

$$N = \psi(A_1; U) = \psi(B; U) \quad \text{and} \quad A_1N^k \subseteq B \quad \text{for some } k,$$

for any nonzero L-submodule B of A_1. Let $\zeta_1 = \{\mathrm{Ann}_U(B);\ 0 \neq B \subseteq_L A_1\}$ and pick A_2 such that $J = \mathrm{Ann}_U(A_2)$ is a maximal element of ζ_1. Finally let

$$\zeta_2 = \{\mathrm{Ann}_R(a)\colon 0 \neq a \in A_2\}$$

and let $A_3 = aU$ be such that $P = \mathrm{Ann}_R(a)$ is a maximal element of ζ_2. By replacing A by A_3 we now have:

(i) $N = \psi(A; U) = \psi(B; U)$ for any nonzero submodule B of A;
(ii) if $0 \neq C \subseteq_L B \subseteq_L A$, then $B(\psi(A; U))^k \subseteq C$ for some $k = k(C)$;
(iii) $J = \mathrm{Ann}_U(A) = \mathrm{Ann}_U(B)$ if $0 \neq B \subseteq_L A$;
(iv) $P = \mathrm{Ann}_R(A) = \mathrm{Ann}_R(b)$ for all $0 \neq b \in A$.

Now let $u \in N \setminus 0$. Then there exist by (**), $r_1, r_2, \ldots, r_n \in R$ such that
(v) $u^n + r_1u^{n-1} + \cdots + r_n = 0$.

Suppose if possible that for some i, $r_i \notin P$ and let i be maximal such and set $r = r_i$. Thus $r_{i+1}, \ldots, r_n \in P$ in case $i < n$. Then r defines an L-module monomorphism of A by (iv). As in the proof of Theorem 2.3 we can embed A in an $(L \oplus \mathfrak{f}T)$-module $M = A\mathfrak{f}[T]$ such that $rT = Tr =$ identity map on M. By (v),

$$\begin{aligned} ru^{n-i} + r_{i-1}u^{n-i+1} + \cdots + r_1u^{n-1} + u^n \\ = \text{element of } PU \subseteq J \subseteq \mathrm{Ann}(M), \end{aligned}$$

and hence

$$Mu^{n-i} = (Mr)u^{n-i} \subseteq Mu^{n-i+1} \subseteq Mu^{n-i}.$$

Thus

$$Mu^{n-i} = \bigcap_{k=1}^{\infty} Mu^k = Mu^n.$$

Let $V = U(L \oplus \mathfrak{f}T) = U(L) \otimes_{\mathfrak{f}} \mathfrak{f}[T]$. Then $V = R_1u_1 + R_1u_2 + \cdots + R_1u_n$, where $R_1 = R \otimes \mathfrak{f}[T] \cong \mathfrak{f}[z_1, \ldots, z_s, T] \subseteq$ center of V and so $L_1 = L \oplus \mathfrak{f}T \in$ Max-cu. Suppose B is a nonzero submodule of M. Then it is easy to check that $B \cap A \neq 0$, and so $Au^k \subseteq B \cap A$ for some $k(u \in N)$ by (ii) and hence $Mu^k \subseteq B$. Thus if $\mu(M) = \bigcap\{B\colon 0 \neq B \text{ is a submodule of } M\}$ then $Mu^n \subseteq \mu(M)$. Now if $\mu(M) \neq 0$, then $\mu(M)$ is an irreducible L_1-module and so is finite dimensional, whence $\mu(M) \cap A = D$ is a finite-dimensional nonzero submodule of A, so $DN^l = 0$ for some l and, hence, $AN^{k+l} \subseteq DN^l = 0$ for some k, a contradiction. So $Mu^n \subseteq \mu(M) = 0$, and $Au^n = 0$.

Thus given $u \in N \setminus 0$, we have that u satisfies (v) for some r_j in R. If all r_j are in P, then $Au^n = 0$. If not, then as above, $Au^n = 0$ anyway. Therefore N/J is a nil ideal of U/J and U/J satisfies the left and right ascending chain conditions and so, by a well-known result (see, for example, Divinsky [6, Theorem 16, p. 51]), N/J is nilpotent, so $N^m \subseteq J$ for some m or $AN^m = 0$ for some m, a contradiction.

This proves Theorem 3.1. $\square$

4. Applications.

THEOREM D. *Every finitely generated metanilpotent Lie algebra has good Frattini structure.*

PROOF. Let $L \in \mathfrak{G} \cap \mathfrak{N}^2$. To show that L has good Frattini structure it is enough, as in Stewart and Towers [8], to assume that $L \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$ and to show that $\psi(L) = \nu(L)$. Let A be an abelian ideal of L for which $L/A \in \mathfrak{G} \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$, and let $N = \psi(L)$ and $X = N/A < Y + L/A$. Consider A as an L-module under the adjoint action. Then A is a finitely generated L-module (Amayo and Stewart [2]) and so a finitely generated Y-module. Now $X \subseteq \psi(A; U)$, where $U = U(Y)$ and so, by Theorem D, there exists m such that $0 = AX^m = [A,_m N]$. But $N^c \subseteq A$ for some c, and hence $N^{c+m} = 0$ whence $N = \psi(L) \subseteq \nu(L) \subseteq \psi(L)$ and the proof is complete. $\square$

For a Lie algebra L and $b \in L$ we define

$$E_L(b) = \{x \in L: [x,_n b] = 0 \text{ for some } n = n(x, b)\}.$$

The Lie algebra L is said to be an Engel algebra in case $L = E_L(b)$ for any b in L.

The class Max is the class of all Lie algebras satisfying the maximal condition on subalgebras. By Amayo and Stewart [2], Max-cu < Max. Furthermore by Amayo [1], $\mathfrak{E} \cap$ Max $< \mathfrak{F} \cap \mathfrak{N}$, where $\mathfrak{E}$ denotes the class of Engel Lie algebras.

Let $L \in \mathfrak{G} \cap (\mathfrak{N}\text{ Max-cu}) \cup \mathfrak{G} \cap (\mathfrak{F}\text{ Max-cu})$. Evidently $N = \nu(L) \in \mathfrak{N}$ and L has good Frattini structure if and only if L/N^2 has good Frattini structure (see Amayo and Stewart [3, p. 133] and Proposition 2.1). Thus to show that L has good Frattini structure we may assume that $\nu(L)$ is abelian.

Suppose if possible that L does not have good Frattini structure. As L satisfies the maximal condition for ideals (trivially if $L \in \mathfrak{G} \cap (\mathfrak{F}\text{ Max-cu})$ and, by Amayo and Stewart [3, pp. 225–240], if $L \in \mathfrak{G} \cap \mathfrak{N}$ Max-cu and $\nu(L) \in \mathfrak{A}$), we may factor out an ideal I of L maximal with respect to L/I not having good Frattini structure. Replacing L by this quotient we may now assume that

(i) If $0 \neq J \lhd L$, then L/J has good Frattini structure.

(ii) $A = \nu(L)$ is abelian.

(iii) $L \in \mathfrak{G}$ and has an ideal I such that $L/I \in$ Max-cu and either I is abelian or else I is finite dimensional.

Let $N = \psi(L)$, $M = \tilde{\nu}(L)$ and $P = \rho(L)$. Then A may or may not be zero. If $L \in \mathfrak{F}$ then by Barnes and Newell [4], L has good Frattini structure, a contradiction. So L is infinite dimensional.

Claim 1. $M < N$.

Let B/C be a chief factor with $[B, M] \nsubseteq C$. If $C \neq 0$ then L/C has good Frattini structure and $(M + C)/C \subseteq \tilde{\nu}(L/C) = \psi(L/C)$, hence $[B, M] \subseteq C$. Suppose then that $C = 0$. If $J \lhd L$ and $J \neq 0$, then $(B + J)/J$ is zero or a chief factor of L/J and so $0 \neq [B, M] \subseteq B \cap J < B$. Thus

$$B \subseteq \mu(L) = \bigcap\{J: 0 \neq J \lhd L\}.$$

If $I = 0$, then $L \in$ Max-cu, B is an irreducible L-module and so B is finite dimensional. If $I \neq 0$ and I is finite dimensional, then $B \subseteq I$, so B is finite dimensional. If $I \neq 0$ and I is abelian, then $0 \neq B \subseteq I$, I is a finitely generated L/I-module and, by Amayo and Stewart [3, pp. 225–240], this implies that I, and so B, is finite dimensional. So in all cases B is finite dimensional and $L \in$ Max. Now let $x \in M$. Then for some r,

$$[B,_r x] = [B,_{r+1} x] = \cdots = \bigcap_{i=1}^{\infty} [B,_i x]$$

(B is finite dimensional) and, since also $M \lhd L$ and $M^c \subseteq B$ for some c (L/B has good Frattini structure), we have

$$[L,_{c+1+r} x] \subseteq [B,_r x] = \bigcap_{i=1}^{\infty} [B,_i x] \subseteq [L,_{c+2+r} x].$$

Thus with $s = c + r + 1$, we have

$$[L,_s x] = [L,_{s+1} x], \quad \text{and, hence,} \quad L = E_L(x) + [L,_s x] = E_L(x) + B.$$

Let $D = C_L(B) \lhd L$. Then $L/D \in \mathfrak{F}$ and $L \notin \mathfrak{F}$, so $D \neq 0$ and $B \subseteq D$ (since $B \subseteq \mu(L)$). Then $D \neq B$, as otherwise, $L \in \mathfrak{F}$ and $D = D \cap (E_L(x) + B) = (D \cap E_L(x)) + B$. Evidently, $E_L(x) \cap D \lhd E_L(x)$ and $[E_L(x) \cap D, B] = 0$, and so $0 \neq E_L(x) \cap D \lhd L$, so that $B \subseteq E_L(x) \cap D$. Therefore, $L = E_L(x)$ and $M = E_M(x)$, whence $M \in \mathfrak{E} \cap \text{Max} < \mathfrak{F} \cap \mathfrak{N}$. Thus $M = \nu(L) < \psi(L)$, a contradiction which proves Claim 1.

The proof of Claim 1 also shows that we may assume that:

(iv) L has no minimal ideals (in particular, if $I \neq 0$ then I is not finite dimensional).

(v) If J_1, J_2 are nonzero ideals of L, then $J_1 \cap J_2 \neq 0$.

(vi) $P = \rho(L) < \psi(L)$ (proved in the same way as Claim 1).

If now $I \neq 0$, then from above, I is abelian and a finitely generated L/I module under the action $u(x + I) = [u, x]$ for $u \in I$ and $x \in L$ and, evidently, $(N + I)/I \subseteq \psi(I; U(L/I))$ and $L/I \in$ Max-cu and, hence, by Theorem 3.1, $[I,_r N] = 0$ for some r. But $N^c \subseteq I$ for some c (L/I has good Frattini structure) and, hence, $N^{c+r} \subseteq [I,_r N] = 0$, a contradiction. Similarly if $A = \nu(L) \neq 0$ we obtain a contradiction.

So we may finally assume that $I = 0$ and $L \in$ Max-cu $\cap\ \mathfrak{G}$, $\nu(L) = 0$, $N = \psi(L) \supseteq \tilde{\nu}(L) + \rho(L)$. But then L is finitely generated as a module over itself under the adjoint action and $N \subseteq \psi(L; U(L))$, whence by Theorem 3.1, $0 = L(N)^r = [L,_r N]$ for some r. Thus $N^{r+1} = 0$ and $N < \nu(L)$, a contradiction.

So we have proved

THEOREM 4.1. *If* $L \in \mathfrak{G} \cap (\mathfrak{N}$ Max-cu) *or* $L \in \mathfrak{G} \cap (\mathfrak{F}$ Max-cu), *then L has good Frattini structure.*

Since we do have $\mathfrak{F} <$ Max-cu in prime characteristic, Theorem 4.1 implies Theorem E. Further, Corollary F follows from Theorem D and the remarks in Stewart and Towers [8].

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