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APPROXIMATION THEOREMS FOR UNIFORMLY CONTINUOUS FUNCTIONS

BY

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ABSTRACT. Let X be a set, A a family of real-valued functions on X which contains the constants, μ_A the weak uniformity generated by A, and $U(\mu_A X)$ the collection of uniformly continuous functions to the real line R. The problem is how to construct $U(\mu_A X)$ from A. The main result here is: *For A a vector lattice, the collection of suprema of countable, finitely A-equiuniform, order-one subsets of A^+ is uniformly dense in $U(\mu_A X)$*. Two less technical corollaries: *If A is a vector lattice (resp., vector space), then the collection of functions which are finitely A-uniform and uniformly locally-A (resp., uniformly locally piecewise-A) is uniformly dense in $U(\mu_A X)$. Further, for any A, a finitely A-uniform function is just a composition $F \circ (a_1, \ldots, a_p)$ for some $a_1, \ldots, a_p \in A$ and F uniformly continuous on the range of $(a_1, \ldots, a_p)$ in R^p*. Thus, such compositions are dense in $U(\mu_A X)$. For $BU(\mu_A X)$, the compositions with $F \in BU(R^p)$ are dense (B denoting bounded functions). So, in a sense, to know $U(\mu_A X)$ it suffices to know A and subspaces of the spaces R^p, and to know $BU(\mu_A X)$, A and the spaces R^p suffice.

In case A is a vector lattice and $A = BA$ (i.e., A consists of bounded functions), the problem of describing $U(\mu_A X)$ in terms of A has an elegant solution: A is uniformly *dense* in $U(\mu_A X)$ (i.e., if $f \in U(\mu_A X)$ and $\varepsilon > 0$ there is $a \in A$ with $|f(x) - a(x)| < \varepsilon$ for all $x \in X$). This can be seen to be essentially equivalent to the usual Stone-Weierstrass Theorem. The result appears to have been first published in 1955 by Maak [M] and Nöbeling and Bauer [NB]; we give a short proof in the course of proving our main theorem.

If A contains unbounded functions, the situation is more complicated: $A = U(R)|N$ is a "uniformly closed" vector lattice, while μ_A is (uniformly) discrete. With $X = R$, let A be the piecewise linear functions (finitely many pieces); μ_A is the usual uniformity on R and the closure of A consists only of continuous eventually linear functions.

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Preliminary versions of the main result were presented at the Pittsburgh Symposium, December, 1972 (see [H]) and to the Prague seminar on uniform spaces, May, 1975.

The theorem that we prove shortly derives from the usual Stone-Weierstrass Theorem (or the one mentioned above) and some results of Fenstad [F] and Császár [C] giving conditions sufficient for density and characterizations of the structures $U(\mu X)$. See 1.5 below.

1. **The main theorem.** This is 1.3 below, which explicitly, if rather technically, constructs $U(\mu_A X)$ from A. We require some preliminaries.

We make the standing assumption that A is a subset of R^X which contains the constants. Saying that A is a vector space or vector lattice refers to the pointwise operations.

μ_A denotes the weak uniformity generated by A, that is, the smallest uniformity such that each function in A is uniformly continuous. If $\mathfrak{S}(\varepsilon)$ denotes the cover of R consisting of ε-balls, then $\{\mathfrak{S}(\varepsilon)|\varepsilon > 0\}$ is a base for the usual uniformity of R (using the covering description of uniformities [I(2)]). Thus, a subbase for μ_A is $\{f^{-1}\mathfrak{S}(\varepsilon)|f \in A, \varepsilon > 0\}$.

Thus, a family $\mathfrak{F} \subset R^X$ is *A-equiuniform* (i.e., equiuniform for μ_A) if given $\varepsilon > 0$, there is μ_A-basic cover $\mathfrak{B}$ such that whenever $B \in \mathfrak{B}$ and $x, y \in B$, then $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathfrak{F}$. This means that given $\varepsilon > 0$ there are $\{a_1, \ldots, a_p\} \subset A$ and $\delta > 0$ such that whenever $|a_i(x) - a_i(y)| < \delta$ for $i = 1, \ldots, p$, then $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathfrak{F}$–since basic $\mathfrak{B}$ is of the form $\bigwedge_{i=1}^p a_i^{-1}\mathfrak{S}(\delta)$.

In this definition, if the family $\{a_1, \ldots, a_p\}$ may be chosen independently of ε, we shall call $\mathfrak{F}$ *finitely-A-equiuniform*. Evidently, this means that there is finite $F \subset A$ such that $\mathfrak{F}$ is F-equiuniform.

As is well known, if $\mathfrak{F} \subset R^X$ is μ-equiuniform (for a uniformity μ on X), and if the pointwise supremum $\bigvee \mathfrak{F}$ is finite at each point, then $\bigvee \mathfrak{F} \in U(\mu X)$. We shall need a generalization:

1.1 LEMMA. *If $\mathfrak{F}$ is finitely-A-equiuniform, using finite $F \subset A$, with $\bigvee \mathfrak{F}$ finite at each point, and if $\mathfrak{S}$ is any collection of subsets of $\mathfrak{F}$, then $\{\bigvee S | S \in \mathfrak{S}\}$ is finitely-A-equiuniform, and using F.*

This is an immediate consequence of the inequality [F(1)]: for $\mathfrak{B}, \mathfrak{C} \subset R$, $|\bigvee \mathfrak{B} - \bigvee \mathfrak{C}| \leqslant \bigvee\{|b - c| \,|b \in \mathfrak{B}, c \in \mathfrak{C}\}$.

Finally, a family of sets will be said to be *of order one* if each three members have empty intersection. We shall deal with families $\mathfrak{F} \subset R^X$ for which coz $\mathfrak{F}$ is a cover of order one; here, $\operatorname{coz} \mathfrak{F} = \{\operatorname{coz} f | f \in \mathfrak{F}\}$ and $\operatorname{coz} f = \{x \in X | f(x) \neq 0\}$.

1.2 *Notation.* A_0 stands for the collection of functions of the form $\bigvee \mathfrak{F}$, where

(a) $\mathfrak{F} \subset A^+$,
(b) $\mathfrak{F}$ is countable,
(c) $\mathfrak{F}$ is finitely-A-equiuniform,

(d) coz $\mathfrak{F}$ is a μ_A-uniform cover of order one.

1.3 THEOREM. *If A is a vector lattice, then A_0 is uniformly dense in $U(\mu_A X)^+$.*

The proof of 1.3 is a bit involved. We first sketch it for an almost prototypical special case, and give a corollary.

1.4 Let $X = [0, +\infty)$, with A the vector lattice of continuous piecewise linear functions (with finitely many pieces). Here, μ_A is the usual uniformity.

Given $f \in U(\mu_A X)^+$, and $\varepsilon > 0$, choose $\delta > 0$ and uniformly continuous g which is linear on each interval $[n\delta, (n+1)\delta] \equiv I_n$ and which approximates f within ε. By uniform continuity, a number $s > 0$ can be chosen so that $g_n \leq g$ for each n, where g_n is defined like this: let $L_n(x)$ be linear of slope s with $L_n(n\delta) = g(n\delta)$, $M_n(x)$ linear of slope $-s$ with $M_n((n+1)\delta) = g((n+1)\delta)$; then $L_n|I_n = g|I_n$; for $x \leq n\delta$, $g_n = 0 \vee L_n$; for $x \geq (n+1)\delta$, $g_n = M_n \vee 0$.

Clearly, $\bigvee_n g_n = g$, and $\{g_n\}$ satisfies 1.2(a), (b), and (c) using $\{a_1, \ldots, a_p\} =$ the singleton $\{a_1(x) = sx\}$. But (d) does not yet hold. To achieve (d), define inductively $K_1 < K_2 < \ldots$ in such a way that (d) holds for

$$f_1 = \bigvee \{ g_n | K_i + 1 \leq n \leq K_{i+1} \}.$$

This is done by making the differences $|K_{i+1} - K_i|$ grow rapidly. Then $\bigvee_i f_i = \bigvee_n g_n$, and $\{f_i\}$ is finitely A-equiuniform by 1.1.

1.5 COROLLARY. *Let A be a vector lattice.*

(a) *If $A_0 = A^+$, then A is uniformly dense in $U(\mu_A X)$.*

(b) *$A = U(\mu_A X)$ iff $A_0 = A^+$ and A is "uniformly closed".*

PROOF. (a) Given $f \in U(\mu_A X)$, write $f = f^+ - f^-$. By 1.3, approximate f^+ and f^- by g and h in $A_0 = A^+$. Then $g - h \in A$, and approximates f.

(b) Any $U(\mu X)$ is closed under taking uniform limits and taking sups of families $\mathfrak{F}$ satisfying 1.2(c). The converse follows from (a).

(a) can be viewed as an improvement of a combination of theorems of Fenstad and Császár. [F] A is dense in $U(\mu_A X)$ if A is closed under the taking of suprema of countable, equiuniform, *star-finite* families. (1.2(d) implies $\mathfrak{F}$ is star-two. The result here is a combination of 4.1 of [F(1)] and 4.3 of [F(2)]; see also [H].) [C] A is dense in $U(\mu_A X)$ if A is closed under the taking of pointwise limits of finitely equiuniform sequences. (This is approximately Satz 3 of [C].) Here note that: If $\{f_n\}$ is a finitely equiuniform sequence, then $g_n \equiv \bigvee_{k \leq n} f_k$ defines a sequence $\{g_n\}$ which converges pointwise to $\bigvee_n f_n$, and which is finitely equiuniform by 1.1; thus Császár's condition implies that $A_0 = A^+$.

Fenstad and Császár also derive corollaries like (b), which similarly (b) improves.

PROOF OF 1.3. If $\mathfrak{F}$ satisfies 1.2, then $\mathfrak{F}$ is equiuniform for μ_A, and $\bigvee \mathfrak{F} \in U(\mu_A X)$. So $A_0 \subset U(\mu_A X)$.

For the density: Let $\mathcal{S}(c, r)$ be the open interval in R with center c and radius r. Let $\mathcal{L}(r) = \{S(nr, r) | n \in Z\}$. Evidently, $\{\mathcal{L}(r) | r > 0\}$ is a base for the (usual) uniformity of R, and $\{a^{-1}\mathcal{L}(r) | a \in A, r > 0\}$ is a subbase for μ_A.

Given $f \in U(\mu_A X)^+$ and $\varepsilon > 0$, choose a basic cover $\mathcal{Q} = \bigwedge_{i=1}^{p} a_i^{-1}\mathcal{L}(\delta) < f^{-1}\mathcal{L}(\varepsilon)$. Members of $\mathcal{Q}$ will be denoted $\alpha, \beta, \gamma, \ldots$.

Given $\alpha = \bigcap_{i=1}^{p} a_i^{-1} S(c_i, \delta)$,

(1) define $n(\alpha) \equiv$ least integer n with $\alpha \subset f^{-1}S(n\varepsilon, \varepsilon)$, and

$$e(\alpha)(x) \equiv 2 - 2 \wedge \frac{1}{\delta}\left[\bigvee_i |a_i(x) - c_i|\right] \vee 1.$$

Then

$$\left|\bigvee_\alpha \varepsilon n(\alpha)e(\alpha)(x) - f(x)\right| \leq 2\varepsilon \quad \text{for all } x \in X.$$

This will follow from

$$e(\alpha)(x) = 1 \quad \text{iff } |a_i(x) - c_i| \leq \delta/2 \text{ for each } i;$$

(2) $$e(\alpha)(x) = 0 \quad \text{iff } |a_i(x) - c_i| \geq \delta \text{ for some } i;$$

$$0 \leq e(\alpha) \leq 1, \quad \text{and} \quad \operatorname{coz} e(\alpha) = \alpha.$$

(3) If $e(\alpha)(x) \neq 0$, then $e(\beta)(x) = 1$ for some β with $\beta \cap \alpha \neq \varnothing$.

(4) If $\beta \cap \alpha \neq \varnothing$ then $n(\beta) \leq n(\alpha) + 1$.

PROOFS. (2) is computed straightforwardly. (4) follows from the facts that $\mathcal{Q} < f^{-1}\mathcal{L}(\varepsilon)$ and the latter is star-two.

For (3), if $e(\alpha)(x) = 1$ then $\beta = \alpha$ works, so suppose that $0 < e(\alpha)(x) < 1$. Then both inequalities in (2) are violated: $|a_j(x) - c_j| > \delta/2$ for some j, and $|a_i(x) - c_i| < \delta$ for each i. Then let

$$K_j' = K_j \pm \delta, \text{ in such a way that } |a_j(x) - c_j| \leq \delta/2,$$

$$K_i' = K_i \text{ or } K_i \pm \delta, \text{ so that } |a_i(x) - c_i| \leq \delta/2.$$

Set $\beta = \bigcap_i a_i^{-1} S(K_i', \delta)$; then $\beta \cap \alpha \neq \varnothing$ and $e(\beta)(x) = 1$.

PROOF OF (1). Given x, $x \in f^{-1}S(n\varepsilon, \varepsilon)$ for at most two consecutive n's. Let n be the first, so that $n\varepsilon - \varepsilon < f(x) < (n + 1)\varepsilon + \varepsilon$.

If $e(\alpha)(x) \neq 0$ then $x \in \alpha$ and $n \leq n(\alpha) \leq n + 1$. So first: $\bigvee_\alpha \varepsilon n(\alpha)e(\alpha)(x) \leq \varepsilon(n + 1) \cdot 1$. And second: there is α with $x \in \alpha$ so (by (4)) there is β with $e(\beta)(x) = 1$. Since $x \in \beta$, $n \leq n(\beta)$, and

$$\varepsilon n \leq \varepsilon n(\beta)e(\beta)(x) \leq \bigvee_\alpha \varepsilon n(\alpha)e(\alpha)(x).$$

DIGRESSION. (a) If $A = BA$, then A is dense in $U(\mu_A X)$. We have essentially shown this: By writing arbitrary $f \in U(\mu_A X)$ as $f = f^+ - f^-$, we see that it suffices to approximate functions in $U(\mu_A X)^+$. Given such f and ε, proceed as above. Because each a_i is bounded, $\mathcal{Q} = \{\alpha\}$ is finite. Thus $\bigvee_\alpha \varepsilon n(\alpha)e(\alpha) \in A$ and (1) applies.

(b) In the general case, the approximation in (1) is only preliminary. Clearly, $\{e(\alpha)|\alpha \in \mathcal{Q}\}$ is finitely equiuniform (using $\{a_1, \ldots, a_p\}$), but not so for $\{\varepsilon n(\alpha)e(\alpha)|\alpha \in \mathcal{Q}\}$ which, roughly speaking, grow from 0 to $\varepsilon n(\alpha)$ nonuniformly in α; also, $e(\beta) \wedge e(\alpha) \neq 0$ for possibly $3^p - 1$ other β's because coz $e(\alpha) = \alpha$ (from (2)), and $\mathcal{Q}$ has the same "starring" properties as the cover $\bigwedge_{i=1}^p \pi_i^{-1}\mathcal{L}(\alpha)$ of R^p.

So we shall, first, spread out the support of each $\varepsilon n(\alpha)e(\alpha)$ so as to achieve equiuniformity, and then do some collecting together as in 1.4 to make the supports of order one.

Given α, let $\alpha_0 = \alpha$, $\alpha_1 = \bigcup\{\beta \in \mathcal{Q}|\beta \cap \alpha \neq \varnothing\}$, and ... $\alpha_i = \bigcup\{\beta \in \mathcal{Q}|\beta \cap \alpha_{i-1} \neq \varnothing\}$. As noted earlier, the "starring" properties of $\mathcal{Q}$ are those of $\bigwedge_{i=1}^p \pi_i^{-1}\mathcal{L}(\delta)$ in R^p: so each α meets $< 3^p$ members of $\mathcal{Q}$; note that 3^p is the volume of a cube in R^p of side 3. $|\{\beta|\beta \cap \alpha_1 \neq \varnothing\}| <$ the volume of the cube in R^p of side 5, and, in general, $|\{\beta|\beta \cap \alpha_{i-1} \neq \varnothing\}| < (i+2)^p$, which is the volume of the cube of side $(i+2)$.

Let $e(\alpha_i) \equiv \bigvee\{e(\beta)|\beta \subset \alpha_i\}$. There are $< (i+2)^p$ such β's, so $e(\alpha_i) \in A$. Moreover, $\{e(\alpha_i)|\alpha \in \mathcal{Q}, i = 0, 1, 2, \ldots\}$ is finitely A-equiuniform by 1.1.

(5) Let $f(\alpha) \equiv 2\varepsilon\Sigma\{e(\alpha_i)|0 < i < n(\alpha)/2\}$; then $\{f(\alpha)|\alpha \in \mathcal{Q}\}$ is finitely A-equiuniform, and $|\bigvee_\alpha f(\alpha)(x) - f(x)| < 3\varepsilon$ for each $x \in X$.

Note that $f(\alpha)$ is approximately $\varepsilon n(\alpha)e(\alpha)$ on the set α, but the jump in $\varepsilon n(\alpha)e(\alpha)$ has been spread over

$$\operatorname{coz} f(\alpha) = \bigcup\left\{\operatorname{coz} e(\alpha_i)|0 < i < \frac{n(\alpha)}{2}\right\} = \bigcup\left\{\alpha_i|0 < i < \frac{n(\alpha)}{2}\right\}.$$

To prove (5), observe that

$$(6) \qquad e(\alpha_{i+1})|\alpha_i \equiv 1; \quad \beta \subset \alpha_i \text{ implies } n(\beta) < n(\alpha) + i.$$

PROOF. The first is implied by (3), and the second follows by iterating (4).

PROOF OF (5). Given $\varepsilon_0 > 0$, choose $\delta > 0$ so that whenever $|a_i(x) - a_i(y)| < \delta$ for $i = 1, \ldots, p$, then (i) whenever $x \in \alpha$ then $y \in \alpha_1$, and if $y \in \alpha$ then $x \in \alpha_1$, and (ii) $|e(\alpha_i)(x) - e(\alpha_i)(y)| < \varepsilon_0$ for every α_i. Suppose x and y are "this close".

Given α: if $f(\alpha)(x) = f(\alpha)(y) = 0$, there is nothing to prove. Suppose, say, that $f(\alpha)(x) \neq 0$, and let i be the least integer $< n(\alpha)/2$ with $e(\alpha_i)(x) \neq 0$. Then $e(\alpha_{i+1})(x) = 1$. We then have: If $j < i$, $e(\alpha_j)(x) = 0$; so if $j < i - 1$, $e(\alpha_j)(y) = 0$. If $j > i$, $e(\alpha_j)(x) = 1$; so if $j > i + 1$, $e(\alpha_j)(y) = 1$. Thus,

$$|f(\alpha)(x) - f(\alpha)(y)| = 2\varepsilon\left|\sum_j \{e(\alpha_j)(x) - e(\alpha_j)(y)\}\right|$$
$$= 2\varepsilon\left|\sum \{e(\alpha_j)(x) - e(\alpha_j)(y)|j = i-1, i, i+1\}\right|$$
$$< 2\varepsilon \cdot 3 \cdot \varepsilon_0.$$

For present purposes, ε is fixed; so $\{f(\alpha)|\alpha \in A\}$ is finitely A-equiuniform.

Next, we shall show that $|\bigvee_\alpha f(\alpha)(x) - \bigvee_\alpha \varepsilon n(\alpha)e(\alpha)(x)| < \varepsilon$ for all $x \in X$. Then (1) will apply.

We always have $e(\alpha) < e(\alpha_i)$, and the number of terms in $f(\alpha)$ is at least $(n(\alpha) - 1)/2$. So

$$\varepsilon(n(\alpha) - 1)e(\alpha) < 2\varepsilon\left(\frac{n(\alpha) - 1}{2}\right)e(\alpha) < 2\varepsilon\sum e(\alpha_i) = f(\alpha).$$

Thus for any α, $\varepsilon n(\alpha)e(\alpha) \leqslant f(\alpha) + \varepsilon$, and thus $\bigvee_\alpha \varepsilon n(\alpha)e(\alpha) \leqslant \bigvee_\alpha f(\alpha) + \varepsilon$.

Now, given α and $x \in X$, we show there is β with $f(\alpha)(x) < \varepsilon n(\beta)e(\beta) + \varepsilon$. Then $\bigvee_\alpha f(\alpha) < \bigvee_\alpha \varepsilon n(\alpha)e(\alpha) + \varepsilon$ follows. If $f(\alpha)(x) = 0$, then take $\beta = \alpha$. Otherwise, there is least $i < n(\alpha)/2$ with $e(\alpha_i)(x) \neq 0$, and

$$f(\alpha)(x) = 2\varepsilon\sum \{e(\alpha_j)(x)|i < j \leqslant n(\alpha)/2\}$$
$$< 2\varepsilon(n(\alpha)/2 - i) = \varepsilon(n(\alpha) - 2i).$$

Choose $\beta \subset \alpha_{i+1}$ with $e(\beta)(x) = 1$, so $\alpha \subset \beta_{i+1}$ as well, and $n(\alpha) < n(\beta) + (i + 1)$. Then

$$n(\alpha) - 2i < n(\alpha) - i \leqslant n(\beta) + 1.$$

Thus $f(\alpha)(x) < \varepsilon(n(\beta) + 1) = \varepsilon n(\beta)e(\beta)(x) + \varepsilon$.

We now complete the construction by making the supports of order one.

$\mathfrak{A}$ breaks into equivalence classes of the relation $\beta \sim \alpha$ if $\beta \subset \alpha_i$ for some positive integer i. Let $\mathfrak{R}$ consist of one representative from each class.

Given $\alpha \in \mathfrak{R}$ and $\beta \sim \alpha$, let $d(\beta) =$ the least i with $\beta \subset \alpha_i$. We shall determine a sequence $K_1 < K_2 < \ldots$ of positive integers (perhaps finite, and depending on α) so that with

$$(7)\qquad f_i(\alpha) \equiv \bigvee\{f(\beta)|K_i + 1 < d(\beta) < K_{i+1}\},$$

the family $\{\operatorname{coz} f_i(\alpha)|i = 1, 2, \ldots\}$ is of order one. Note that according to the remarks preceding (5),

$$|\{\beta|\beta \subset \alpha_{K_{i+1}}\}| < (K_{i+1} + 2)^p, \text{ so that each } f_i(\alpha) \in A.$$

To define $\{K_i\}$ precisely, we use

(8) If $f(\beta) \wedge f(\alpha) \neq 0$, then $\beta \subset \alpha_{2n(\alpha)+3}$.

PROOF. If $f(\beta) \wedge f(\alpha) \neq 0$, then there are γ and $i < n(\alpha)/2, j < n(\beta)/2$

with $\gamma \subset \alpha_{i+1} \cap \beta_{j+1}$. Then $\beta \subset \gamma_{j+1} \subset (\alpha_{i+1})_{j+1} = \alpha_{i+j+2}$, and thus

$$n(\beta) \leq n(\alpha) + i + j + 2 \leq n(\alpha) + n(\alpha)/2 + n(\beta)/2 + 2$$

(using (6)). Solving for $n(\beta)$, $n(\beta) \leq 3n(\alpha) + 1$. Thus

$$i + j + 2 \leq n(\alpha)/2 + \left(3n(\alpha)/2 + \tfrac{1}{2}\right) + 2 \leq 2n(\alpha) + 3,$$

and $\beta \subset \alpha_{2n(\alpha)+3}$.

Now, what is required of $\{K_i\}$ is that $f(\beta) \wedge f(\gamma) = 0$ whenever $d(\beta) > K_{i+1}$ and $d(\gamma) \leq K_i$. $d(\gamma) \leq K_i$ implies $n(\gamma) \leq n(\alpha) + K_i$ (by (6)), so if $f(\beta) \wedge f(\gamma) \neq 0$ then (using (8)),

$$\beta \subset \gamma_{2n(\gamma)+3} \subset (\alpha_{K_i})_{2(n(\alpha)+K_i)+3} = \alpha_{2n(\alpha)+2K_i+3}.$$

So take $K_1 = 1$, and inductively, $K_{i+1} = 2n(\alpha) + 2K_i + 3$. Then $\{\operatorname{coz} f_i(\alpha) | i = 1, 2, \ldots\}$ is of order one.

Let $\mathcal{F} \equiv \{f_i(\alpha) | \alpha \in \mathcal{R};\ i = 1, 2, \ldots\}$. If $\alpha \neq \beta$ in $\mathcal{R}$, then $f_i(\alpha) \wedge f_j(\beta) = 0$; so coz $\mathcal{F}$ is also of order one. To see that it is a μ_A-uniform cover: Fix α. Then

$$\{\operatorname{coz} f_i(\alpha)\}_i > \{\operatorname{coz} f(\beta) | \beta \sim \alpha\} > \{\operatorname{coz} e(\beta) | \beta \sim \alpha\} > \{\beta | \beta \sim \alpha\}.$$

Thus coz $\mathcal{F}$ is refined by the μ_A-uniform cover $\mathcal{A}$ consisting of all α's.

By (5) and 1.1, $\mathcal{F}$ is finitely A-equiuniform. For each $\beta \in \mathcal{A}$, $f(\beta)$ is a term in some $f_i(\alpha)$, so that $\bigvee \mathcal{F} = \bigvee \{f(\beta) | \beta \in \mathcal{A}\}$; by (5) $\bigvee \mathcal{F}$ approximates f.

2. **Locally-*A* functions.** We present a less technical, perhaps more memorable, corollary of 1.3, isolating a class of functions dense in $U(\mu_A X)$ with a considerably simpler description.

2.1 DEFINITION. A_1 stands for the collection of functions $g \in R^X$ which are

(a) finitely A-uniform: $g \in U(\mu_F X)$ for some finite $F \subset A$; and

(b) uniformly locally-A: there is a μ_A-uniform cover on each member of which g agrees with some function in A.

There are other ways to put (b). For example, each of the following is equivalent to "g is uniformly locally-A": (b′) There are $a_1, \ldots, a_p \in A$ and $\delta > 0$ such that if S is a subset of X for which each $\operatorname{osc}_S a_i < \delta$, then there is $a_S \in A$ with $g|S = a_S|S$. (b″) There is a sequence $\{a_n\} \subset A$ such that $\{x \in X | g(x) = a_n(x)\}_n$ is a μ_A-uniform cover.

§4 will say more about the finitely A-uniform functions.

2.2 THEOREM. *If A is a vector lattice, then A_1 is uniformly dense in $U(\mu_A X)$.*

PROOF. First, $A_0 \subset A_1$: Let $\mathcal{F}$ be as in 1.2. By 1.1, $\bigvee \mathcal{F}$ is finitely A-uniform. By 1.2(d), coz $\mathcal{F}$ is μ_A-uniform, and on the member coz g, $\bigvee \mathcal{F} = \bigvee \{h \in \mathcal{F} | \operatorname{coz} h \cap \operatorname{coz} g \neq \varnothing\}$; the latter function is in A because of the "order one" condition.

By 1.3, $A_0 - A_0$ is dense in $U(\mu_A X)$; of course, $A_0 - A_0 \subset A_1 - A_1$. But

$A_1 - A_1 = A_1$, as is readily checked. (In fact, A_1 is a vector lattice, because A is.)

2.3 REMARK. If in 2.2 (or 1.3) we assume that A is only a vector space, then the conclusion fails: Consider the vector space A of linear functions on R, for which μ_A is the usual uniformity. Any uniformly piecewise-A function is locally linear, and an easy chaining argument shows that a locally linear function is in fact linear. Thus, $A_1 = A_0 = A$, and is not dense in $U(\mu_A R) = U(R)$.

The next two sections present theorems in which the hypotheses on A are weakened.

3. **Piecewise-A functions.** We prove, again as a corollary of 1.3, a theorem like 2.2 but applied to a vector space; what is needed is a condition (3.1(b)) more permissive than 2.1(b).

3.1 DEFINITION. A function $g \in R^X$ is piecewise-A if there is finite $F \subset A$ such that at each point of X, g agrees with some function in F.

g is uniformly locally piecewise-A if there is a μ_A-uniform cover $\mathfrak{U}$ such that g agrees with a piecewise-A function on each member of $\mathfrak{U}$.

A_2 (respectively, A_3) stands for the collection of functions in R^X which are

(a) finitely A-uniform, and

(b) piecewise-A (respectively, uniformly locally piecewise-A).

3.2 THEOREM. *If A is a vector space, then A_3 is uniformly dense in $U(\mu_A X)$.*

The following prepares for application of 1.3:

3.3 PROPOSITION. *Let A be a vector space. Then*

(a) *A_2 and A_3 are vector lattices;*

(b) $\mu_{A_2} = \mu_{A_3} = \mu_A$;

(c) *a family $\mathfrak{F}$ is finitely-A-equiuniform if (and only if) $\mathfrak{F}$ is finitely-A_2-equiuniform;*

(d) $(A_2)_0 \subset A_3$.

PROOF OF 3.3. (a) It is easy to see that A_2 and A_3 are vector spaces, because A is. They are lattices because at each point, $a \vee b$ and $a \wedge b$ agree with either a or b.

(b) is implied by 3.1(a).

(c) The "only if" part is obvious. Conversely, let $\mathfrak{F}$ be F-equiuniform with finite $F \subset A_2$. For $f \in F$, there is finite $G_f \subset A$ such that at each point of X, f agrees with some function in G_f. With $H = \bigcup\{G_f | f \in F\}$, $\mathfrak{F}$ is H-equiuniform, hence finitely-A-equiuniform.

(d) Let $\mathfrak{F}$ be a countable subset of A_2^+ which is finitely-A_2-equiuniform, with coz $\mathfrak{F}$ a μ_{A_2}-uniform cover of order one. We show $\bigvee \mathfrak{F} \in A_3$: By (c) and 1.1, $\bigvee \mathfrak{F}$ is finitely-A-uniform. By (b), coz $\mathfrak{F}$ is μ_A-uniform. Let $g \in \mathfrak{F}$, and let $F = \{h \in \mathfrak{F} | \operatorname{coz} h \cap \operatorname{coz} g \neq \varnothing\}$. By "order-one", F has at most three elements, so $\bigvee F \in A_2$ by (a). But on the set $\operatorname{coz} g$, $\bigvee \mathfrak{F} = \bigvee F$. So $\bigvee \mathfrak{F} \in A_3$.

PROOF OF 3.2. Using 3.3(a) and (b), then 1.3, $(A_2)_0$ is dense in $U(\mu_A X)^+$. Using 3.3(a), (b) and (d), A_3 is dense in $U(\mu_A X)$.

4. **Compositions.** We derive some corollaries concerning functions of the form $G \circ (a_1, \ldots, a_p)$, where $(a_1, \ldots, a_p)$: $X \to R^p$ is the evaluation (or parametric) map defined by $(a_1, \ldots, a_p)(x) = (a_1(x), \ldots, a_p(x))$, and G is a function defined at least on the range $(a_1, \ldots, a_p)(X)$. The essential observation is this:

4.1. PROPOSITION. *Let $A \subset R^X$, and $g \in R^X$. Then g is finitely-A-uniform (2.1) iff $g = G \circ (a_1, \ldots, a_p)$ for some $a_1, \ldots, a_p \in A$ and $G \in U((a_1, \ldots, a_p)(X))$.*

PROOF. If $g = G \circ (a_1, \ldots, a_p)$, with G uniformly continuous, then certainly $g \in U(\mu_F X)$ for $F = \{a_1, \ldots, a_p\}$.

Let $g \in U(\mu_F X)$, for $F = \{a_1, \ldots, a_p\} \subset A$. Clearly, if $a_i(x) = a_i(y)$ for $i = 1, \ldots, p$, then $g(x) = g(y)$. Thus, defining G: $(a_1, \ldots, a_p)(X) \to R$ by $G((a_1(x), \ldots, a_p(x))) = g(x)$ makes sense. Let $\varepsilon > 0$. Since $g \in U(\mu_F X)$, there is $\delta > 0$ with $g^{-1}\mathbb{S}(\varepsilon) > \bigwedge a_i^{-1}\mathbb{S}(\delta)$. Now,

$$G^{-1}\mathbb{S}(\varepsilon) = (a_1, \ldots, a_p) g^{-1}\mathbb{S}(\varepsilon) > (a_1, \ldots, a_p) \bigwedge a_i^{-1}\mathbb{S}(\delta)$$
$$= \left(\bigwedge \pi_i^{-1}\mathbb{S}(\delta)\right) \cap (a_1, \ldots, a_p)(X).$$

This last is a uniform cover, so is $G^{-1}\mathbb{S}(\varepsilon)$, and G is uniformly continuous.

Given $A \subset R^X$, let $\operatorname{comp} A$ denote the class of functions g described in 3.1. Note that, here, we are not assuming A to have any algebraic properties.

4.2 THEOREM. *For any $A \subset R^X$*, comp *A is uniformly dense in $U(\mu_A X)$.*

PROOF. $\operatorname{comp} A$ is itself a vector lattice: for example, $a_1 + a_2 = G \circ (a_1, a_2)$, where $G(x, y) = x + y$; the other operations go similarly.

It is clear that a finitely $\operatorname{comp} A$-uniform function is finitely-A-uniform, hence by 4.1, in $\operatorname{comp} A$. This shows that $(\operatorname{comp} A)_1 = \operatorname{comp} A$, so by 2.2, $\operatorname{comp} A$ is dense in $U(\mu_{\operatorname{comp} A} X)$. But, of course, $\mu_{\operatorname{comp} A} = \mu_A$.

In a sense, 4.2 reduces the problem of describing $U(\mu_A X)$ to the problem for subsets of R^p $(p = 1, 2, \ldots)$. One feels that one knows more about the functions in $U(R^p)$ than about those in $U(S)$ for $S \subset R^p$. So cases in which one can reduce to this may be worth considering.

4.3 COROLLARY. *$\{\{G \circ (a_1, \ldots, a_p) | p \in N;\ a_1, \ldots, a_p \in A;\ G \in BU(R^p)\}$ is dense in $BU(\mu_A X)\}$.*

PROOF. If $f \in U(\mu_A X)$, then by 4.2, f is approximable by a composition $G \circ (a_1, \ldots, a_p)$ with $G \in U((a_1, \ldots, a_p)(X))$. If f is bounded, G is also bounded and thus extends over all of R^p by the Katětov Theorem [K].

4.4 COROLLARY. *$U(\mu_A X) = \{G \circ (a_n) | \{a_n\} \subset A;\ G \in U((a_n)(X))\}$ and $BU(\mu_A X) = \{G \circ (a_n) | \{a_n\} \subset A;\ G \in BU(R^{\aleph_0})\}$ (where the sets $\{a_n\}$ are countable).*

PROOF. The inclusions $\supset$ are automatic.

If $f \in U(\mu_A X)$, then by 4.2, f is the uniform limit of a sequence $\{f_n\}$, where $f_n = F_n \circ (a_1^n, \ldots, a_{p_n}^n)$. It is easily arranged inductively that if $m < n$, then $p_m < p_n$ and for $i < p_m$, $a_i^m = a_i^n$. We thus write $f_n = F_n \circ (a_1, \ldots, a_{p_n})$. Now let $G_n \in U((a_n)(X))$ be defined by $G_n((a_n(x))) = F_n((a_1(x), \ldots, a_{p_n}(x)))$. Since $g_n \to f$ uniformly, G_n converges uniformly to some $G \in U((a_n)(X))$. That $f = G \circ (a_n)$ follows.

The proof for $BU(\mu_A X)$ then uses the Katětov Theorem, as in 4.3.

Finally, we derive a theorem of Isbell [I(1)].

To say that A has *continuous composition* is to say that if $p \in N$, $a_1, \ldots, a_p \in A$ and $G \in C(R^p)$, then $G \circ (a_1, \ldots, a_p) \in A$. Assuming only this about A, it follows easily that A is a vector lattice and ring, as in the proof of 4.2.

4.5 COROLLARY. *If A has continuous composition, then A is dense in $U(\mu_A X)$.*

PROOF. Given $f \in U(\mu_A X)$, approximate within ε by $G \circ (a_1, \ldots, a_p)$, using 4.2. Extend G to a uniformly continuous function G_1 on the closure of $(a_1, \ldots, a_p)(X)$, and then continuously over R^p, to G_2, by the Tietze Extension Theorem. Then $G \circ (a_1, \ldots, a_p) = G_2 \circ (a_1, \ldots, a_p) \in A$.

This proof uses 1.3, of course, Isbell's proof is rather simple.

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