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LEAF PRESCRIPTIONS FOR CLOSED 3-MANIFOLDS

BY

JOHN CANTWELL AND LAWRENCE CONLON[1]

ABSTRACT. Our basic question is: What open, orientable surfaces of finite type occur as leaves with polynomial growth in what closed 3-manifolds? This question is motivated by other work of the authors. It is proven that every such surface so occurs for suitable C^∞ foliations of suitable closed 3-manifolds and for suitable C^1 foliations of all closed 3-manifolds. If the surface has no isolated nonplanar ends it also occurs for suitable C^∞ foliations of all closed 3-manifolds. Finally, a large class of surfaces with isolated nonplanar ends occurs in suitable C^∞ foliations of all closed, orientable 3-manifolds that are not rational homology spheres.

Introduction. It is natural to ask whether a given $(n-1)$-manifold can be a leaf of a foliation in a given closed n-manifold, or even in some closed n-manifold. This problem has been considered by J. Sondow [14] and S. Goodman [4].

We show in [18] that, for C^2 foliations of codimension one on closed n-manifolds, the condition that a leaf have polynomial growth [11] implies that it is a manifold of "finite type" in the sense defined in §1. In fact, polynomial growth of degree r implies that the type is at most r. Thus, it seems reasonable to narrow the scope of the general problem posed above by asking which $(n-1)$-manifolds of finite type occur as leaves with polynomial growth in which closed n-manifolds. For $n = 3$, this question assumes manageable proportions and we will obtain some strong results.

It is a pleasure to acknowledge helpful conversations with Paul Schweitzer and Sue Goodman.

1. **Statement of results.** Throughout this paper M will denote a closed, connected 3-manifold and N will denote a connected, open, orientable surface. All manifolds are understood to be paracompact.

In §2 we will review the notion of the *endset* $\mathfrak{E}(N)$, a compact, totally disconnected, metrizable space of ideal points at infinity. The concept makes sense, in fact, for open manifolds of arbitrary dimension.

Let E be a compact, totally disconnected, metrizable space and define $E^{(0)} = E$, and $E^{(r+1)}$ to be the set of accumulation points of $E^{(r)}$, $r \geq 0$.

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DEFINITION. The set E as above is of (finite) type r if $E^{(r)} \neq \emptyset = E^{(r+1)}$. The surface N is of (finite) type r if $\mathfrak{E}(N)$ is of type r.

Our constructions and proofs will proceed by induction on the type r. In §2 we will designate certain special type r surfaces $N_{r,i}$, $0 \leq i \leq q(r) < \infty$, such that every surface of finite type is a connected sum $N_* \# N_{r_1,i_1} \# \cdots \# N_{r_k,i_k}$ where N_* is compact. In particular, $N_{r,0}$ plays a slightly distinguished role in our results and is described as follows. Set $N_{0,0} = \mathbf{R}^2$ and, for $r \geq 1$, suppose $N_{r-1,0}$ has been defined. From $\mathbf{R}^2$ delete a sequence of disjoint open disks diverging to ∞ and to each boundary circle sew a copy of $N_{r-1,0}$ − (open disk), denoting the result by $N_{r,0}$. The cases $r = 1, 2$ are pictured in Figure 1.

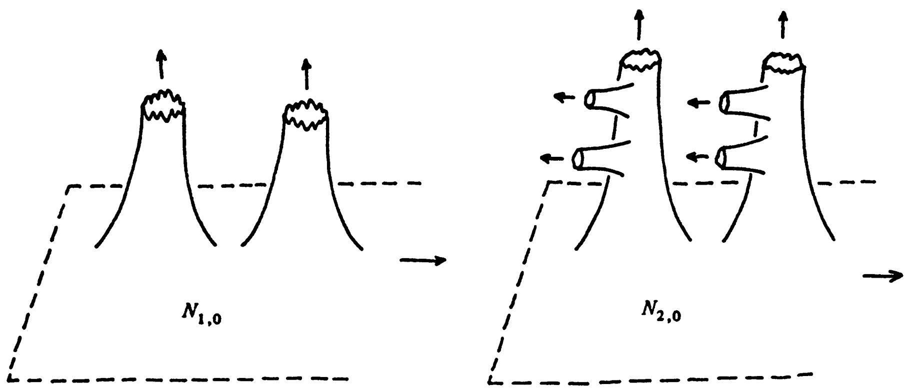

FIGURE 1

Recall from [11] the definition of "growth" of a leaf. The extreme types are *polynomial growth* and *exponential growth*, the latter being associated with such intuitively obscure asymptotic phenomena as exceptional minimal sets, etc. It will be convenient to say that a transversely orientable foliation is *tame* if it contains no exceptional minimal sets. Finally, recall that a leaf is *proper* if its relative topology and its manifold topology coincide.

THEOREM 1. *If N is of finite type, there is a closed, orientable,* 3-*manifold M with a tame C^∞ foliation having a leaf with polynomial growth homeomorphic to N. If $N \neq N_{r,0}$, $r \geq 1$, this leaf can be chosen to be proper.*

The difficulty with $N_{r,0}$, $r \geq 1$, is an essential one for foliations of class C^2 [2, Theorem 3].

THEOREM 2. *If N is of finite type, then every closed* 3-*manifold admits a tame C^1 foliation having a proper leaf with polynomial growth homeomorphic to N.*

Even relaxing the requirement that the desired leaf be proper will not allow greater smoothness in Theorem 2. We indicate briefly the main difficulty. In §2 we will identify the closed subset $\mathfrak{E}^*(N) \subset \mathfrak{E}(N)$ of *nonplanar* ends. An *isolated* nonplanar end in $\mathfrak{E}(N)$ corresponds to a connected sum decomposition $N \cong N_0 \# T_\infty$ where T_∞ is the surface pictured in Figure 2.

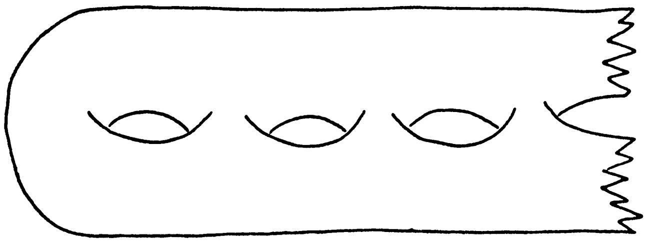

FIGURE 2

In C^2 foliations of S^3, no surface with an isolated nonplanar end, except T_∞ itself, can occur as a leaf with nonexponential growth [2, Theorem 1]. Furthermore, if there is a C^2 foliation of S^3 having T_∞ as a *proper* leaf, then this leaf must have exponential growth and be asymptotic to an exceptional minimal set [2, (4.3)]. These remarks explain the restrictive hypotheses in our remaining theorems.

By work of Goodman [4], each open, orientable surface with only finitely many ends and finite genus appears as a leaf with polynomial growth in suitable C^∞ foliations of a large number of closed 3-manifolds. In Theorem 3, we enlarge considerably this class of surfaces and remove all restrictions on the closed 3-manifold.

THEOREM 3. *Let N be of finite type without isolated nonplanar ends. Then every closed* 3-*manifold admits a tame* C^∞ *foliation having a leaf with polynomial growth homeomorphic to N* (*a proper leaf if* $N \neq N_{r,0}$, $r > 1$).

THEOREM 4. *Let N be of finite type and suppose either that all ends are nonplanar or that N is the connected sum of such an open surface with one as in Theorem* 3. *If M is orientable and is not a rational homology sphere, then M admits a tame* C^∞ *foliation having a proper leaf with polynomial growth homeomorphic to N.*

We do not know whether the condition on M in Theorem 4 is sufficient to allow every open, orientable surface of finite type to appear as a leaf with polynomial growth in some C^∞ foliation of M.

Our constructions depend on four basic foliations described in §4. Accordingly, that section is the core of the paper. The proof of Theorem 2 is completed in §5 and the proofs of the other theorems are completed in §6.

2. **The endset of** N. The one-point compactification of N is defined by introducing one ideal point at infinity and endowing it with a fundamental neighborhood system $V_1 \supset V_2 \supset \cdots$ where $V_i = N - K_i$, K_i is compact, $K_i \subset K_{i+1}$, and $\bigcup_{i=1}^{\infty} K_i = N$. A system $U_1 \supset U_2 \supset \cdots$ such that each U_i is a connected component of the corresponding V_i is said to be a fundamental neighborhood system of an *end* e of N, such distinct descending chains of components defining distinct ends. In this way we distinguish a whole set $\mathcal{E}(N)$ of ideal points at infinity. Another choice $V_1' \supset V_2' \supset \cdots$ as above gives systems $U_1' \supset U_2' \supset \cdots$ and the corresponding end e' is identified with e if and only if each U_i contains some U_j' and each U_i' contains some U_j. Implicit in this is a topology on $N \cup \mathcal{E}(N)$ making this set into a compact metrizable space with N an open subspace in its manifold topology. The subspace $\mathcal{E}(N)$ is compact and totally disconnected.

An end e is said to be *nonplanar* if none of its fundamental neighborhoods U_i is homeomorphic to an open subset of $\mathbf{R}^2$. The set of nonplanar ends is a closed subspace $\mathcal{E}^*(N)$ of $\mathcal{E}(N)$.

References for this material include [1] and [3].

Since N is orientable, the classification theory of B. Kerékjártó [7] and I. Richards [12] can be stated as follows.

(2.1) Proposition. *If* $\mathcal{E}^*(N) \neq \varnothing$, *then the homeomorphism type of the pair* $(\mathcal{E}(N), \mathcal{E}^*(N))$ *determines* N *up to diffeomorphism. If* $\mathcal{E}^*(N) = \varnothing$, *then the finite integer* genus(N) *together with the homeomorphism type of* $\mathcal{E}(N)$ *determine* N *up to diffeomorphism. Finally, every topological pair* (E, E^*) *such that* E *is a compact, totally disconnected, metrizable space and* E^* *a closed subspace, occurs as* $(\mathcal{E}(N), \mathcal{E}^*(N))$ *for suitable* N. *If* $\mathcal{E}^*(N) = \varnothing$, *any preassigned integer can be realized as* genus(N).

Surfaces of *finite type* as defined in §1 can be broken down into connected sums of certain canonical surfaces. In order to describe this process, we briefly discuss the point set topology of compact, totally disconnected, metrizable spaces E of finite type.

Let $\mathfrak{M}$ denote the class of all pairs (E, E^*) where E, as above, is of finite type and $E^* \subset E$ is a closed subspace.

(2.2) Proposition. *There is a function* $q: \mathbf{Z}^+ \to \mathbf{Z}^+$ *and a set* $\{(E_{n,i}, E_{n,i}^*) \in \mathfrak{M} \mid n \geq 0, 0 \leq i \leq q(n)\}$ *such that*:

(a) $E_{n,i}^{(n)} = \{e_{n,i}\}$ *is a singleton*;

(b) $E_{n,i} = E_n$ *is independent of* i, *and for* $k < n$ *the disjoint union* $E_k \cup E_n$ *is homeomorphic to* E_n;

(c) $E_{n,0}^* = \varnothing$ *and* $E_{n,q(n)}^* = E_{n,q(n)}$;

(d) *if* $(E, E^*) \in \mathfrak{M}$ *and* E *is of type* n, *then* (E, E^*) *is homeomorphic to a finite disjoint union of pairs* $(E_{n_\alpha, i_\alpha}, E_{n_\alpha, i_\alpha}^*)$ *where* $n_\alpha \leq n$.

PROOF. We proceed by induction on n. If $n = 0$, set $q(0) = 1$, $E_{0,0} = \{e_{0,0}\}$, $E^*_{0,0} = \emptyset$, $E_{0,1} = E^*_{0,1} = \{e_{0,1}\}$. If E is of type 0, then E and E^* are finite and all assertions are clear.

For the inductive step, assume the proposition true up through n. Suppose that $(E, E^*) \in \mathfrak{M}$ and that E has type $n + 1$. Let $E^{(n+1)} = \{e_1, \ldots, e_s\}$. Let U_i be an open compact neighborhood of e_i such that $U_i \cap U_j = \emptyset$, $i \neq j$, and let $X = E - \bigcup_{i=1}^{s} U_i$. Let $X^* = X \cap E^*$ and $U_i^* = U_i \cap E^*$. Then (X, X^*) and each (U_i, U_i^*) are elements of $\mathfrak{M}$ and (E, E^*) is the disjoint union of these pairs. Furthermore, X is of type at most n and each $U_i^{(n+1)} = \{e_i\}$, so by the inductive hypothesis we lose no generality in assuming $E^{(n+1)} = \{e\}$, a singleton.

Consider a fundamental system of open compact neighborhoods $V_1 \supset V_2 \supset \cdots \supset V_i \supset \cdots$ of e. By the inductive hypothesis we can find a finite set $A = \{(E_{n_\alpha,i_\alpha}, E^*_{n_\alpha,i_\alpha}) | n_\alpha \leq n,\ 1 \leq \alpha \leq m\}$ such that each $(V_i - V_{i+1}, (V_i \cap E^*) - V_{i+1})$, $i \geq 1$, is homeomorphic to a finite disjoint union of some elements of A. By choosing V_1 small enough we can guarantee that each element of A so appears for infinitely many values of i. Thus define $(E_A, E_A^*) \in \mathfrak{M}$ to be the one-point compactification of the disjoint union of countably many copies of all the elements of A. It is clear that $(V_1, V_1 \cap E^*)$ is homeomorphic to (E_A, E_A^*) while $(E - V_1, E^* - V_1) \in \mathfrak{M}$ is (by the inductive hypothesis) a disjoint union of finitely many pairs $(E_{p,i}, E^*_{p,k})$ where $p \leq n$.

Note that $V_i - V_{i+1}$ is a disjoint union $E_{k_1} \cup E_{k_2} \cup \cdots \cup E_{k_r}$ where all $k_j \leq n$ and, for infinitely many i, $k_r = n$. By the induction hypothesis, therefore, E_A is the one-point compactification of a countable disjoint union of copies of E_n, so $E_A = E_{n+1}$ depends only on $n + 1$.

There are only finitely many choices of A, so let $q(n + 1)$ be the number of topologically distinct pairs (E_A, E_A^*) so obtained and we number these pairs so that $E^*_{n+1,0} = \emptyset$ and $E^*_{n+1,q(n+1)} = E_{n+1,q(n+1)}$. All assertions are now clear. □

The function $q(n)$ is rather amusing. The first three values are $q(0) = 1$, $q(1) = 3$, and $q(2) = 19$, but then $q(3) > 10^6$ and $q(4) > 10^{300,000}$.

Also remark that, if $k < n$, the disjoint union $(E_{k,0}, E^*_{k,0}) \cup (E_{n,0}, E^*_{n,0}) \cong (E_{n,0}, E^*_{n,0})$ and $(E_{k,q(k)}, E^*_{k,q(k)}) \cup (E_{n,q(n)}, E^*_{n,q(n)}) \cong (E_{n,q(n)}, E^*_{n,q(n)})$.

By (2.1) and (2.2) we construct model endtypes $N_{n,i}$, these being open orientable surfaces such that $\mathcal{E}(N_{n,i}) \cong E_{n,i}$ and $\mathcal{E}^*(N_{n,i}) \cong E^*_{n,i}$, and with genus$(N_{n,0}) = 0$, $n > 0$. Observe that $N_{n,0}$ is homeomorphic to the surface denoted by that symbol and described in §1.

(2.3) COROLLARY. *Every N of finite type is homeomorphic to a connected sum $N_* \# N_{n_1,i_1} \# \cdots \# N_{n_r,i_r}$ where N_* is closed. If $\mathcal{E}^*(N) = \emptyset$, this can be chosen to involve only N_* and copies of some $N_{n,0}$. If $\mathcal{E}^*(N) = \mathcal{E}(N)$, the* *connected sum can be chosen to involve only copies of some* $N_{n,q(n)}$.

DEFINITION. If $\varnothing \neq E \subset \mathcal{E}(N)$ is a closed subset and if N_* is an orientable surface (open or closed), then $N \#_E N_*$ denotes the surface obtained by deleting from N a countable discrete family of open disks converging exactly to E, and then sewing to each boundary circle a copy of N_* − (open disk).

For example, $N_{n,q(n)} \cong N_{n,0} \#_{\mathcal{E}(N_{n,0})} T^2$.

(2.4) COROLLARY. *Every* $N_{n+1,i}$ *has the form*

$$N_{0,j} \#_{\{e_{0,j}\}} (N_{n_1,i_1} \# \cdots \# N_{n_m,i_m})$$

where $n_\alpha \leq n$ *for all* α *and* $j = 0, 1$.

Here we remark that the choice $j = 0$ is only required if $i = 0$ and that $j = 1$ is only required if $\mathcal{E}^*(N_{n+1,i}) = \{e_{n+1,i}\}$. The proof of (2.4) is easy from (2.1) and the proof of (2.2).

3. **Basic pasting principles.** Here we assemble a number of principles for pasting foliations together. All foliations will be assumed transversely orientable of class C^r, $1 \leq r \leq \infty$, the ambient manifold always being of class C^∞.

We will often be constructing leaves that are *proper* in a stronger sense than that term ordinarily denotes. Accordingly, let us agree to call a noncompact leaf *properly proper* if it intersects some closed transversal to the foliation exactly once.

Let W be a compact manifold with boundary. A foliation of W that is tangent to the boundary (i.e., the boundary components are leaves) will always be assumed to be *trivial at the boundary*. This means that the foliation extends to a C^r foliation of $W \cup_h (\partial W \times [0, 1))$ where $h: \partial W \times \{0\} \to \partial W$ is defined by $h(x, 0) = x$, with the collar having the product foliation. If W_1 and W_2 are so foliated and h is a C^r diffeomorphism of a boundary component of W_2 onto one of W_1, then $W_1 \cup_h W_2$ supports a C^∞ structure and is C^r foliated by the union of the two foliations.

If W_1 and W_2 have foliations transverse to the boundary and if the diffeomorphism h in the previous paragraph matches up the induced foliations of the boundary components, then standard techniques for smoothing corners give a C^r foliation on the C^∞ manifold $W_1 \cup_h W_2$.

Some special remarks will be needed about growth properties of leaves in an important case of the above construction. Consider the case in which $\partial W_1 \cong S^1 \times S^1 \cong \partial W_2$ and the induced foliations on the boundaries are by circles. Suppose a leaf L_0 in W_1 meets $\partial W_1 = S^1 \times S^1$ in a set of circles $\{\theta_n\} \times S^1$, where $0 < \alpha \leq \theta_n \leq \beta < 2\pi$ for fixed α and β and for all n. Suppose each leaf of W_2 meeting $\partial W_2 = S^1 \times S^1$ in the annulus $[\alpha, \beta] \times S^1$

intersects ∂W_2 in only one circle $\{\theta\} \times S^1$, and let L_θ be this leaf, $\alpha < \theta < \beta$. Here, of course, we have suitably coordinatized the boundaries and h = identity. Finally, suppose that L_0 has polynomial growth of degree p and that every L_θ, $\alpha \leq \theta \leq \beta$, has growth function dominated by a fixed polynomial Q of degree q. Then the leaf L of $W_1 \cup_h W_2$ containing L_0 has polynomial growth of degree at most $p + q$. We indicate how to establish this using the definition of growth in terms of a finitely generated holonomy pseudogroup Γ associated to a regular distinguished covering $\{U_i\}_{i=0}^m$ [11, §4]. In [11] each U_i is assumed homeomorphic to a product of open disks, in our case a 2-disk D and an interval J, but we can allow some $U_i \cong A \times J$ where A is an open annulus. In this way, we can arrange that U_0 is a normal neighborhood in $W_1 \cup_h W_2$ of the annulus $(\alpha, \beta) \times S^1$ on the interface. Choosing the remaining U_i with reasonable care, we easily arrange that the only words in the transition functions $\gamma_{ij} \in \Gamma$, $0 \leq i, j \leq m$, that must be considered in computing the growth function of L will be of the form

$$\gamma_{i_1 j_1} \circ \cdots \circ \gamma_{i_s 0} \circ \gamma_{0 j_{s+1}} \circ \cdots \circ \gamma_{i_t j_t}$$

where $\gamma_{i_k j_k}$ pertains to $\Gamma | W_1$ for $1 \leq k \leq s$ and to $\Gamma | W_2$ for $s + 1 \leq k \leq t$. For this, of course, it is crucial that L_θ meets ∂W_2 only in the one circle, $\alpha < \theta < \beta$. If the growth function $g_0(n)$ for L_0 is dominated by $P(n)$ and the growth functions $g_\theta(n)$ for L_θ (based at plaques in U_0) are dominated by $Q(n)$, then the growth function $g(n)$ for L is dominated by $P(n)Q(n)$.

We also remark on the method of producing tame foliations. Given any transversely orientable C^∞ foliation of W trivial at the boundary, it has only finitely many exceptional minimal sets [9], so a finite family of mutually disjoint transversals meets all of these minimal sets. The standard modification along each of these transversals (called "turbulization" in [5, §1.10] and carefully described in [17, §§2–3]) inserts a Reeb component along the transversal and spins all nearby leaves asymptotically along the resulting toral leaf. This destroys all of the exceptional minimal sets and introduces no other. If $\partial W = \emptyset$ and there is a compact saturated set $X \cong T \times [0, 1]$, $T \times \{0\}$ and $T \times \{1\}$ being leaves and X containing no exceptional minimal sets, then the above transversals can be chosen to miss X. Indeed, just work in the components of $M - T \times (0, 1)$.

After these long preliminaries we are ready to prove the promised pasting principles.

(3.1) Proposition. *If N is a surface that occurs in a tame foliation of $S^1 \times D^2$ trivial at the boundary, then N occurs as a leaf of a tame foliation in every closed 3-manifold. If N so occurs as a (properly) proper leaf in $S^1 \times D^2$, then it occurs as a (properly) proper leaf in every closed 3-manifold. Likewise, if*

N occurs with polynomial growth in $S^1 \times D^2$, it so occurs in every closed 3-manifold.

PROOF. Every closed 3-manifold admits a transversely orientable C^∞ foliation [16], [17], hence a tame C^∞ foliation. There is always a closed orientation preserving transversal σ and standard modification along σ produces a Reeb component R. Replacing R with the foliated $S^1 \times D^2$ in question gives the desired foliation. □

Suppose that $S^1 \times D^2$ has a tame foliation transverse to the boundary that induces the foliation of $\partial(S^1 \times D^2)$ by meridian circles $\{\theta\} \times \partial D^2$, and let N_1 be a surface such that $N_1' = N_1 -$ (open disk) is a leaf of this foliation. Equivalently, suppose that $S^1 \times S^2$ is tamely foliated so that N_1 is homeomorphic to a leaf met once by a closed transversal $\sigma = S^1 \times \{x_0\}$. Given any foliation, trivial at the boundary, of a compact orientable W and a properly proper leaf N_2 in W, let τ be a closed transverse circle in int(W) meeting N_2 once. A suitable normal neighborhood of τ can be replaced by the above foliated $S^1 \times D^2$ so that the foliations match smoothly and the ambient 3-manifold is unchanged. This proves the following.

(3.2) PROPOSITION. *Let N_1 (respectively N_1') occur as a leaf in $S^1 \times S^2$ (respectively $S^1 \times D^2$) as above. Let N_2 be a properly proper leaf in a tame foliation (trivial at the boundary) of a compact orientable 3-manifold W. Then the connected sum $N_1 \# N_2$ also occurs as a properly proper leaf in a tame foliation of W (trivial at the boundary). If N_1 and N_2 have polynomial growth, so does $N_1 \# N_2$.*

DEFINITION. The orientable surface N_* (closed or open) is *available* if, for some tame foliation of $S_1 \times D^2$ that is transverse to $\partial(S^1 \times D^2)$ and induces there the foliation by meridians, there is a nontrivial closed subarc $J \subset S^1$ such that each meridian circle $\{\theta\} \times S^1$, $\theta \in J$, is the border of a leaf L_θ homeomorphic to $N_* -$ (open disk), and such that the growth functions for all L_θ, $\theta \in J$, are dominated by a common polynomial Q.

In particular, of course, an available N_* satisfies the hypothesis on N_1 in (3.2).

Let N and N_* be orientable surfaces and let E be a nonempty closed subset of $\mathfrak{E}(N)$. Recall from §2 the definition of the infinite connected sum $N \#_E N_*$. If N_*^k denotes the k-fold connected sum $N_* \# \cdots \# N_*$, then $N \#_E N_*^k \cong N \#_E N_*$. The following is now rather obvious.

(3.3) PROPOSITION. *Let N be an open leaf of a tame foliation, trivial at the boundary, of a compact orientable 3-manifold W, σ a closed transverse circle such that $\sigma \cap N$ is a set of points converging exactly to each end in a certain nonempty closed subset $E \subset \mathfrak{E}(N)$, and suppose that $\sigma \cap N$ is contained in a* *proper closed subarc $J \subset \sigma$. Let N_*^k be available, some $k > 1$. Then W has a tame foliation, trivial at the boundary, in which the surface $N \#_E N_*$ occurs as a leaf. This leaf will be (properly) proper if N is (properly) proper, and will have polynomial growth if N has polynomial growth.*

There is another way to produce a leaf $N \#_E N_*$ that is frequently useful. Suppose that N is a properly proper leaf with polynomial growth of a tame foliation of $S^1 \times S^2$. Let $\sigma = S^1 \times \{x_0\}$ be a transversal, $J \subset \sigma$ a proper closed subarc, $\sigma \cap N \subset J$, and suppose that $\sigma \cap N$ has limit set exactly E in $N \cup \mathfrak{S}(N)$.

DEFINITION. In the above circumstances we will say that the pair (N, E) is *available*.

(3.4) PROPOSITION. *Let (N, E) be available and let W be a compact orientable 3-manifold and N_* an orientable surface. Suppose there is a tame foliation of W, trivial at the boundary, and a closed imbedded transverse circle σ' such that every point of a nontrivial closed subarc $J' \subset \sigma'$ belongs to a leaf L_θ, $\theta \in J'$, diffeomorphic to N_*^k, some fixed $k > 1$, that meets σ' in just the one point. Suppose the growth functions of all L_θ are dominated by a common polynomial Q. Then there is a tame foliation of W, trivial at the boundary, having a properly proper leaf with polynomial growth diffeomorphic to $N \#_E N_*$.*

PROOF. Excise a small open tubular neighorhood of σ in $S^1 \times S^2$ obtaining a foliation of $S^1 \times D^2$ meeting the boundary transversely in the meridian foliation. One of the leaves is the bordered surface obtained by deleting from N a discrete family of open disks approaching exactly the points of E. The boundary circles are meridians on $\partial(S^1 \times D^2)$ all lying in $J \times S^1$. Excise a small open normal neighborhood of σ' in W and sew in this foliated $S^1 \times D^2$ so as to match the foliations smoothly and so as to identify $J \times S^1$ with a subset of $J' \times S^1$. This produces $N \#_E N_*$ as desired. □

(3.5) COROLLARY. *If (N, E) is available and if the closed orientable 3-manifold M is not a rational homology sphere, then M admits $N \#_E T^2$ as a properly proper leaf with polynomial growth in a tame foliation.*

PROOF. Since $H_2(M) \neq 0$ [15, Théorème II. 27] implies the existence of a closed orientable surface T smoothly imbedded in M and not separating that manifold. It can be arranged that genus$(T) > 2$. Let $T \times [-1, 1] \subset M$ be the imbedding defined by a normal neighborhood of T. Using the relative version of Thurston's theorem [16], as does Goodman [4], we take a C^∞ foliation of $M - (T \times (-1, 1))$, trivial at the boundary, and complete it to a foliation of M by taking the product foliation in $T \times [-1, 1]$. By the remarks on exceptional minimal sets preceding (3.1), we assume this foliation to be tame.

By [4, Theorem 2.2] we produce a closed imbedded transverse circle σ' in M that meets $T \times [-1, 1]$ in a subarc coinciding with some $\{x_0\} \times [-1, 1]$. By (3.4) we obtain the desired leaf $N \#_E T \cong N \#_E T^2$. □

4. **Certain fundamental foliations.** Here we produce the four basic and elementary foliations promised in §1, together with some variations.

(A) *A foliation of class* C^1. We call attention to the following well-known result [8].

(4.1) THEOREM (N. KOPELL). *Let f and g be commuting C^2 diffeomorphisms of* $(0, 1]$ *onto subintervals containing* 1. *If $f(x) > x$ for all $x \neq 1$, then either g is the identity or g has no fixed point other than* 1.

There is a construction by D. Pixton [10] showing that this result fails if f and g are only required to be of class C^1. Minor modifications of his method give the following lemma.

(4.2) LEMMA (PIXTON). *There are C^1-diffeomorphisms f and g of $I = [0, 1]$ onto itself and a decomposition* $(0, 1) = \bigcup_{k=-\infty}^{\infty}[a_k, a_{k+1}]$ *into nontrivial subintervals such that*:

(a) $f(0) = g(0) = 0$, *hence* $f(1) = g(1) = 1$;

(b) $f'(0) = g'(0) = 1 = f'(1) = g'(1)$;

(c) *for all k, $g(x) = x$ if $a_{2k} \leq x \leq a_{2k+1}$ and $g(x) > x$ if $a_{2k-1} < x < a_{2k}$*;

(d) *for all k, $f[a_{2k}, a_{2k+2}] = [a_{2k+2}, a_{2k+4}]$, hence, in particular, $f(x) > x$ if* $0 < x < 1$;

(e) $f \circ g = g \circ f$.

The analogous lemma for *homeomorphisms* is trivial (suppressing, of course, property (b)). It is even easy to demand that f be C^∞ and that g be C^∞ on $(0, 1)$, differentiability of class C^1 failing for g only at the endpoints. As a consequence, the reader who does not want to be troubled with the somewhat delicate differentiability question can simply change all C^1 assertions to C^0 assertions, likewise weakening the statement of Theorem 1 in §1. In this case, the C^0 foliations so obtained have C^∞ leaves and, in fact, will be C^∞ except at a finite number of toral leaves.

Take f and g as in (4.2) and foliate $T^2 \times I$ as follows. On $I \times I$ take the product foliation with leaves $I \times \{t\}$ and make the identifications $(0, t) \sim (1, f(t))$, obtaining thereby a C^1 foliation of the annulus $A = S^1 \times I$ by spirals approaching the two boundary leaves. Remark that this foliation is invariant under the C^1 diffeomorphism $\bar{g}: A \to A$, $\bar{g}(\theta, t) = (\theta, g(t))$, and that the spiral leaves fall into two continuous classes: those left pointwise fixed by $\bar{g}$ and those that are carried to a distinct leaf by $\bar{g}$. Finally, foliate $I \times A$ by leaves $I \times L$, L ranging over the leaves of the above foliation, and make the identifications $(0, x) \sim (1, \bar{g}(x))$. This provides a C^1 foliation of $T^2 \times I$, trivial at the boundary, with two continuous classes of interior leaves, one class consisting of leaves diffeomorphic to $\mathbf{R}^2$, the other of leaves diffeomorphic to $S^1 \times \mathbf{R}$. All interior leaves wind in on both boundary components.

For a shorter definition, foliate $\mathbf{R}^2 \times I$ by planes $\mathbf{R}^2 \times \{t\}$ and divide out the C^1 action of $\mathbf{Z} \times \mathbf{Z}$ generated by

$$(x, y, t) \mapsto (x + 1, y, g(t)), \qquad (x, y, t) \mapsto (x, y + 1, f(t)).$$

It is fairly easy to see that the planar leaves have growth dominated by a common quadratic polynomial $P(n)$.

There are two particularly important closed transversals. The circle $\{0\} \times S^1 \times \{\frac{1}{2}\}$ in $I \times A$ defines a closed transversal σ in $T^2 \times I$ that crosses each interior leaf and meets each planar leaf infinitely often. The path $I \times \{\theta_0\} \times \{t_0\}$, $t_0 \in (a_1, a_2)$ and θ_0 arbitrary, passes to an arc in $T^2 \times I$ lying in a planar leaf, with endpoints that can be joined by a transverse arc cutting each remaining planar leaf exactly once. A standard construction [6, Lemme 1] allows us to produce a closed transversal τ disjoint from σ, cutting each planar leaf exactly once, and meeting no other leaf. By suitably coordinatizing $T^2 \times I$, we make τ isotopic to a longitude. We summarize all of this in a formal proposition.

(4.3) PROPOSITION. *The manifold $T^2 \times I$ admits a tame C^1 foliation, trivial at the boundary, admitting closed transversals τ and σ with the following properties. Each leaf met by τ is so met just once and is diffeomorphic to $\mathbf{R}^2$, and there is a (quadratic) polynomial dominating the growth of all of these leaves. There is a proper closed subarc $J \subset \sigma$ such that each leaf L that meets τ has $L \cap \sigma \subset J$, $L \cap \sigma$ being a discrete subset of $L \cong \mathbf{R}^2$ approaching the one end of that leaf. Finally, a suitable coordinatization of T^2 makes τ isotopic to a longitude on $T^2 \times \{0\}$.*

(4.4) COROLLARY. *The solid torus $S^1 \times D^2$ and the manifold $S^1 \times S^2$ admit tame C^1 foliations with closed transversals τ and σ haivng all of the above properties, τ being isotopic to a circle $S^1 \times \{x_0\}$.*

PROOF. Suitably glue a Reeb component to the above foliated $T^2 \times I$ along the boundary component $T^2 \times \{0\}$ to obtain the desired foliated solid torus. Another Reeb component suitably glued along the remaining boundary component gives $S^1 \times S^2$ appropriately foliated. □

(B) *A C^∞ foliation of $S^1 \times S^2$.* We begin by describing a useful realization of $S^1 \times S^2$ as a quotient space. Let W_0 be obtained from $S^1 \times D^2$ by removing an open tubular neighborhood T of an imbedded circle $\tau \subset \operatorname{int}(S^1$

$\times D^2$) that bounds a disk in that manifold. Thus ∂W_0 is the disjoint union of two tori. Since $S^1 \times D^2$ is diffeomorphic to $D^3 \cup h$, the solid ball with one solid handle attached, we see that W_0 can be viewed as $D^3 \cup h - T$. Suitably blowing up the toral worm hole T, we see that W_0 is diffeomorphic to $S^2 \times I \cup h_0 \cup h_1$, the thickened 2-sphere with solid handles h_i attached to $S^2 \times \{i\}$, $i = 0, 1$. For an intuition of this metamorphosis, refer to Figure 3.

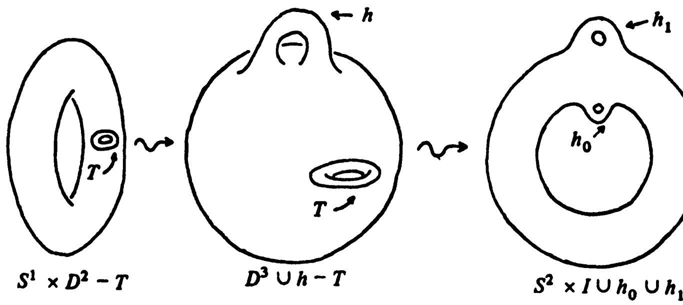

FIGURE 3

In $S^1 \times S^2$ fix a factor $\{\theta_0\} \times S^2$ and modify this to an imbedded copy of T^2 by adjoining a handle within a Euclidean neighborhood in $S^1 \times S^2$. Cutting the manifold open along this copy of T^2 produces a manifold diffeomorphic to $S^2 \times I \cup h_0 \cup h_1 \cong W_0$. Here one uses the elementary diffeomorphism between the two solids obtained from $\mathbf{R}^2 \times [0, \infty)$ by respectively boring out or attaching a solid handle at the boundary.

As a consequence of these observations, there is a diffeomorphism between the two boundary components of W_0 that can be used to glue together these components so as to produce $S^1 \times S^2$. The arc ζ from the one boundary component to the other as pictured in Figure 4 can be assumed to become a circle σ isotopic to an S^1-factor under this identification.

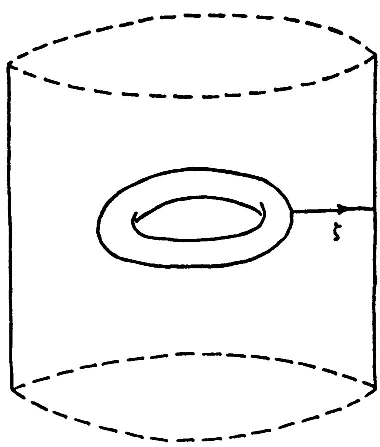

FIGURE 4

The next move is to produce a C^∞ foliation of W_0, trivial at the boundary, with ζ as a transverse arc, hence a C^∞ foliation of $S^1 \times S^2$ having a toral leaf and a closed transversal σ isotopic to an S^1-factor and meeting the toral leaf exactly once.

There is a standard C^∞ foliation of $T^2 \times I$ by cylinders winding asymptotically in on the two boundary components. By suitably coordinatizing T^2, we specify the longitudes on each boundary component to be loops with trivial holonomy and the meridians to be loops with contracting holonomy. We glue a Reeb component to $T^2 \times \{0\}$ by a diffeomorphism preserving meridians and longitudes and obtain a foliated $S^1 \times D^2$ with a transverse imbedded circle $\tau \subset \operatorname{int}(S^1 \times D^2)$ that is unknotted (τ is isotopic to a meridian on $\partial(S^1 \times D^2)$). We perform the standard modification [5, §1.10] along τ and remove the interior of the resulting Reeb component R. The foliation of W_0 so produced has leaves that are spun asymptotically along the toral boundary leaves. If the direction of spin along ∂R is correctly chosen, the arc ζ in Figure 4 is transverse to the foliation.

As one added wrinkle, thicken the leaf diffeomorphic to T^2 so as to obtain $T^2 \times I$, each $T^2 \times \{t\}$, $0 < t < 1$, being a leaf. We summarize the salient properties of this foliation as follows.

(4.5) PROPOSITION. *There is a tame C^∞ foliation of $S^1 \times S^2$ having a closed transversal $\sigma = S^1 \times \{x_0\}$ and a nontrivial closed subarc $J \subset \sigma$, each leaf meeting J being diffeomorphic to T^2 and meeting σ in only the one point.*

(4.6) COROLLARY. *The surface T^2 is available (in the sense of* §3).

(C) *A related foliation of $S^1 \times S^2$.* Let Σ_k denote $S^2 - \{k \text{ points}\}$. The foliation we now construct will show that $(\Sigma_{k+1}, \mathfrak{S}(\Sigma_{k+1}))$ is available (in the sense of §3) if $k > 1$.

Consider the Reeb foliation of $S^1 \times D^2$ and let γ be a closed transversal that pierces each interior leaf exactly k times. Perform the standard modification along γ and remove the interior of the resulting Reeb component R, obtaining a foliated manifold with two toral boundary leaves and all interior leaves diffeomorphic to Σ_{k+1}, the "ends" of these leaves each spinning asymptotically along one or another of the boundary tori. For the case $k = 2$ this is indicated in Figure 5, where a transverse arc ξ joining the boundary components and a closed transversal τ meeting ξ in one point and each interior leaf in one point are also pictured.

In place of R glue in a copy of W_0 foliated as in (B) so that the resulting foliated manifold is again diffeomorphic to W_0. Also make sure that the transverse arc ζ of (B) joins with the transverse arc ξ to produce a transverse arc ζ' in the new W_0, again having the property that a suitable identification

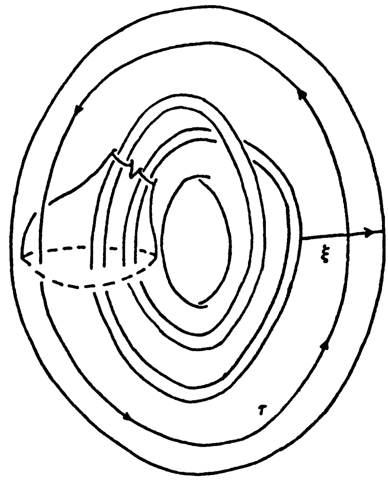

FIGURE 5

of the boundary components of W_0 carries ζ' to a closed transversal σ in a foliation of $S^1 \times S^2$. For each leaf $L \cong \Sigma_{k+1}$, $\sigma \cap L$ is a discrete subset converging exactly to the $k + 1$ ends of L. It is an elementary observation that the foliation is tame and that L has polynomial growth, so we have proven the following.

(4.7) PROPOSITION. *For each $k \geq 1$, $(\Sigma_{k+1}, \mathfrak{S}(\Sigma_{k+1}))$ is available.*

In the above foliation, observe that τ is (isotopic to) a circle on a factor $\{\theta_0\} \times S^2$, and that σ meets τ. Also observe that τ meets each leaf $L \cong \Sigma_{k+1}$ exactly once and meets no other leaves. These observations lead to a useful trick. By slight perturbations of σ we produce disjoint copies σ_1 and σ_2 meeting $\{\theta_0\} \times S^2$ on opposite sides of τ. Performing the standard modifications along σ_1 and σ_2 and removing the interiors of the resulting Reeb components, we obtain a foliation of $T^2 \times I$, trivial at the boundary, a closed transversal τ, and each leaf met by τ is so met only once and is diffeomorphic to $N_{1,0}^{k+1}$ (the connected sum of $k + 1$ copies of $N_{1,0}$). There is also a polynomial dominating the growth of all these leaves. Sew the Reeb components back in by suitable diffeomorphisms on the boundaries interchanging longitudes and meridians. This produces a foliated $S^1 \times S^2$ in which the above τ is isotopic to an S^1-factor.

(4.8) PROPOSITION. *For $r \geq 1$ and $k \geq 1$, the surface $N_{r,0}^{k+1}$ is available.*

PROOF. We proceed by induction on r. The above remarks have proved the case $r = 1$, so assume the assertion for some $r \geq 1$. Considering the foliation of $S^1 \times S^2$ that shows $(\Sigma_{k+1}, \mathfrak{S}(\Sigma_{k+1}))$ to be available, let each leaf $L \cong$

Σ_{k+1} play the role of N in (3.3), let $\mathcal{E}(L)$ play the role of E, let $S^1 \times S^2 = W$, let σ be the same in both contexts, and for N_*^k take the surface $N_{r,0}^2$, this being available by the inductive hypothesis. By (3.3) obtain a foliation of $S^1 \times S^2$ with an array of leaves diffeomorphic to $\Sigma_{k+1} \#_{\mathcal{E}(\Sigma_{k+1})} N_{r,0} \cong N_{r+1,0}^{k+1}$. The slightly displaced copies σ_1 and σ_2 of σ as above survive in this new foliation as does (a slightly displaced) τ. The modifications and surgeries along σ_1 and σ_2, carried out as before, only alter the leaves $N_{r+1,0}^{k+1}$ by punching out an extra set of isolated planar ends approaching the elements of $\mathcal{E}^{(r+1)}(N_{r+1,0}^{k+1})$, hence the diffeomorphism classes of these leaves are unaltered, while τ becomes isotopic to an S^1-factor in the new $S^1 \times S^2$. □

(D) *Another related foliation of* $S^1 \times S^2$. Let T' denote the surface obtained by attaching one handle to $\mathbf{R}^2$ (equivalently, by removing one point from T^2). The foliation developed here will show that $(T', \mathcal{E}(T'))$ is available.

In the foliation of W_0 in (B), there was one Reeb component R. Let τ be a closed transversal to the foliation isotopic to a meridian circle of ∂R as pictured in Figure 6.

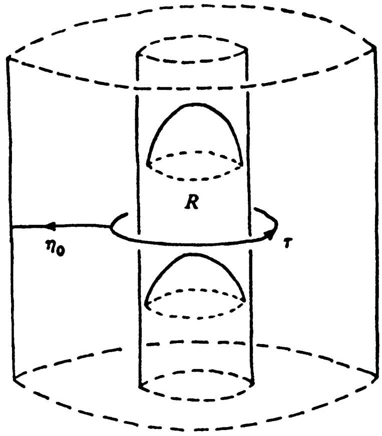

FIGURE 6

We have also pictured a transverse arc η_0 joining τ to the outer boundary component of W_0. Passing to the foliation of $S^1 \times S^2$ as in (B) and removing an open tubular neighborhood of σ, we produce $S^1 \times D^2$ foliated transversely to the boundary (and inducing the meridian foliation there) with a whole interval of leaves $L_t \cong T^2 -$ (open disk), $0 < t < 1$, meeting the boundary in meridian circles. In Figure 7, we picture L_0, τ, and η_0 in this solid torus.

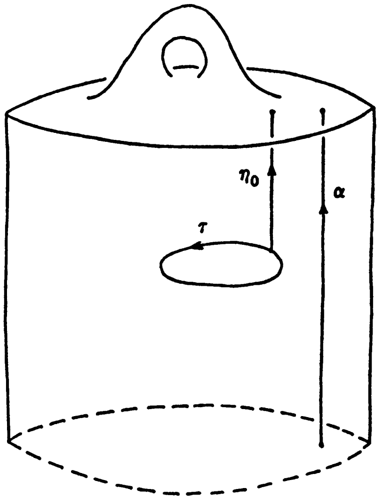

FIGURE 7

Here we also picture a segment of a closed transversal α that coincides with an S^1-factor.

Perform the standard modification along τ and remove the interior of the resulting Reeb component. If the direction of spin along τ is correctly chosen, η_0 (or rather what is left of it) remains a transverse arc. If the foliation is also spun in the proper direction along the outer toral boundary component, η_0 extends to a transverse arc η from the one boundary component to the other as indicated in Figure 8.

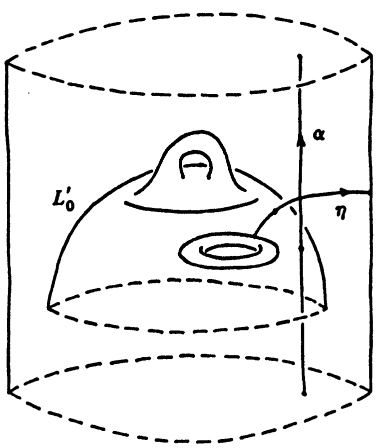

FIGURE 8

Here we arrange that α and η meet in one point. We have again foliated W_0, the transverse arc η is isotopic to the arc ζ of (B), and we have a continuum of leaves $L'_t \cong T'$, $0 < t < 1$, each of which is met infinitely often by η. The closed transversal α meets each L'_t just once, the points of intersection filling out a proper nontrivial closed subarc $J \subset \alpha$. As in (B), identify the boundary components of W_0 so that η becomes a closed transversal β isotopic to an S^1-factor in a foliation of $S^1 \times S^2$.

(4.9) PROPOSITION. *Let $N = T'$ or $N_{0,1}$ (defined as in §2). Then $(N, \mathfrak{S}(N))$ is available. Indeed, there is a tame C^∞ foliation of $S^1 \times S^2$ together with two closed transversals α and β that intersect in a single point and satisfy the following conditions.*

(a) *There is a nontrivial closed subarc $J \subset \alpha$ such that each leaf meeting J meets α in just one point, is diffeomorphic to N, and there is a (linear) polynomial dominating the growth of all these leaves;*

(b) *there is a proper closed subarc $I \subset \beta$ such that each leaf L meeting J meets β in a subset of I approaching the end of L;*

(c) *β is isotopic to an S^1-factor and α is isotopic to a circle on an S^2-factor.*

PROOF. For $N = T'$, this is entirely contained in the above construction. Noting that $N_{0,1} \cong T' \#_{\mathfrak{S}(T')} T^2$ and combining (4.6) and (3.3) with the above, we obtain the same conclusion for $N = N_{0,1}$. $\square$

(4.10) COROLLARY. *For each $n > 0$, the pair $(N_{n,q(n)}, \mathfrak{S}(N_{n,q(n)}))$ is available. Indeed, there is a tame C^∞ foliation $\mathfrak{F}_n$ of $S^1 \times S^2$ with disjoint closed transversals α_n and β_n and nontrivial proper closed subintervals $J_n \subset \alpha_n$ and $I_n \subset \beta_n$, all satisfying the following.*

(a) *Each leaf of $\mathfrak{F}_n$ that meets J_n meets α_n just once, is diffeomorphic to $N_{n,q(n)}$, and there is a polynomial dominating the growth of all these leaves.*

(b) *Each of these leaves is crossed by β_n in a set of points approaching all of its ends.*

(c) *β_n is isotopic to the S^1-factor and meets all of the above leaves in a subset of I_n.*

PROOF. We proceed by induction on n. For $n = 0$, use (4.9) for the case $N = N_{0,1}$, taking $\beta_0 = \beta$ and slightly displacing α to obtain α_0 disjoint from β_0. Let J_0 and I_0 correspond to J and I, respectively. If the assertion holds for some n, consider the foliation of $S^1 \times D^2$ obtained from the foliation in (4.9) for the case $N = T'$ by deleting a small open tubular neighborhood of β. By perturbing α slightly we assure that it perdures as a closed transversal in $S^1 \times D^2$. In $\mathfrak{F}_n$ replace an open tubular neighborhood of α_n with this foliated $S^1 \times D^2$ so as to match the foliations smoothly and so as to match $I \times S^1$ to the leaves crossing α_n in the subarc J_n. Exactly as in (3.4) this produces leaves diffeomorphic to $T' \#_{\mathfrak{S}(T')} N_{n,q(n)} \cong N_{n+1,q(n+1)}$. Furthermore, since α was contained in the foliated $S^1 \times D^2$, it determines a closed transversal α_{n+1} in the new foliation, and $J \subset \alpha$ becomes $J_{n+1} \subset \alpha_{n+1}$. Likewise β_n becomes the desired β_{n+1} and I_n the desired I_{n+1}, and the assertions all follow. □

For the next application of this foliation, and for later use, the following definition is convenient.

DEFINITION. For an open orientable surface N, let $N^{(0)}$ coincide with N and, for $r \geq 1$, obtain $N^{(r)}$ by deleting from $N^{(r-1)}$ a closed discrete subset approaching all of the points of $\mathcal{E}(N^{(r-1)})$.

Thus, $\mathcal{E}^{(r)}(N^{(r)}) = \mathcal{E}(N)$. As an example, remark that $(\mathbf{R}^2)^{(r)} \cong N_{r,0}$. Also, $(T')^{(r)} \cong N_{r,0} \# T^2$.

(4.11) PROPOSITION. *The surfaces $(T')^{(r)}$ and $(N_{0,1})^{(r)}$ are available, $r \geq 1$.*

PROOF. Using the information in (4.9), we are able to proceed much as in the proof of (4.8). View α as a circle on $\{\theta_0\} \times S^2$ and take slightly displaced disjoint copies $\beta', \beta'', \beta'''$ of β, arranging that β' and β'' meet $\{\theta_0\} \supset S^2$ on opposite sides of α. Standard modification along β''' produces a continuum of leaves $(T')^{(1)}$ (respectively, $(N_{0,1})^{(1)}$), while if $r \geq 2$, application of (4.8) and (3.3) along β''' produces a continuum of leaves $(T')^{(r)} \cong T' \#_{\mathcal{E}(T')} N^2_{r-1,0}$ (respectively, $(N_{0,1})^{(r)} \cong N_{0,1} \#_{\{e_{0,1}\}} N^2_{r-1,0}$). Then modifications and surgeries along β' and β'' convert α to an S^1-factor of $S^1 \times S^2$ without changing the homeomorphism class of these leaves. Since α meets this continuum of leaves precisely in the subarc $J \subset \alpha$ in the required way, we have shown $(T')^{(r)}$ (respectively, $(N_{0,1})^{(r)}$) to be available. □

We now have all of the raw material needed for the proofs of the theorems stated in §1. These proofs will be carried out in the next two sections.

5. **Leaves of finite type, the C^1 case.** Here we prove Theorem 2 of §1. All foliations will be tame of class C^1 on closed C^∞ manifolds M^3, and we must show that every N of finite type occurs as a proper leaf in every such M.

(5.1) PROPOSITION. *Let $N_{n,i}$ be a model endtype. There is a tame C^1 foliation $\mathfrak{F}_{n,i}$ of $S^1 \times D^2$, trivial at the boundary, and a tame C^1 foliation $\mathfrak{F}'_{n,i}$ of $S^1 \times S^2$, each with a closed transversal $\tau = S^1 \times \{x_0\}$, such that each leaf met by τ is so met exactly once and is diffeomorphic to $N_{n,i}$, and there is a polynomial P dominating the growth of every such leaf. As a consequence, in the class of C^1 foliations the model endtype $N_{n,i}$ is available.*

PROOF. We proceed by induction on n. If $n = 0$ there are two endtypes, $N_{0,0}$ and $N_{0,1}$. Since $N_{0,0} = \mathbf{R}^2$, we obtain $\mathfrak{F}_{0,0}$ and $\mathfrak{F}'_{0,0}$ by (4.4). The closed transversal σ of (4.4) meets each leaf of type $N_{0,0}$ in a countable discrete set contained in a proper closed subinterval $J \subset \sigma$. By (4.6) and (3.3) we produce the desired foliations $\mathfrak{F}_{0,1}$ and $\mathfrak{F}'_{0,1}$. A slightly displaced copy of σ is still transverse to the foliation and meets each leaf of type $N_{0,1}$ in a countable discrete set as above.

If $\mathfrak{F}'_{n,i}$ has been obtained as desired, excising a suitable tubular neighborhood of τ shows that $N_{n,i}$ is available.

Assume inductively that $\mathfrak{F}_{r,i}$ and $\mathfrak{F}'_{r,i}$ have been constructed for all $r \leq n$, $0 \leq i \leq q(r)$. Consider any $N_{n+1,i}$. By (2.4) we can write

$$N_{n+1,i} = N_{0,j} \#_{\{e_{0,j}\}} \{N_{n_1,i_1} \# \cdots \# N_{n_m,i_m}\}$$

where $j = 0, 1$ and each $n_\alpha \leq n$. In $\mathfrak{F}_{0,j}$ and $\mathfrak{F}'_{0,j}$ take m slightly displaced disjoint copies of σ. By the inductive hypothesis, N_{n_α,i_α} is available, $1 \leq \alpha \leq m$, so m applications of (3.3) along the disjoint copies of σ produce the desired foliations $\mathfrak{F}_{n+1,i}$ and $\mathfrak{F}'_{n+1,i}$. □

It is now easy to prove Theorem 2. Let N be of finite type and, by (2.3), write

$$N = N_* \# N_{n_1,i_1} \# \cdots \# N_{n_r,i_r}$$

where N_* is a closed orientable surface of genus g. The surface T^2 satisfies the hypothesis on N_1 in (3.2) because of (4.5), and the surfaces N_{n_α,i_α}, $2 \leq \alpha \leq 4$, satisfy the same hypothesis because of the existence of $\mathfrak{F}'_{n_\alpha,i_\alpha}$. The surface N_{n_1,i_1} satisfies the hypothesis on N_2 in (3.2), where we take $W = S^1 \times D^2$. Thus, $g + r - 1$ applications of (3.2) produces a properly proper leaf (with polynomial growth)

$$N \cong N_{n_1,i_1} \# T^2 \# \cdots \# T^2 \# N_{n_2,i_2} \# \cdots \# N_{n_r,i_r}$$

in $S^1 \times D^2$, the foliation being trivial at the boundary. By (3.1), N occurs as a properly proper leaf with polynomial growth in a tame C^1 foliation in every closed 3-manifold.

6. **Leaves of finite type, the C^∞ cases.** We prove the remaining three theorems of §1, beginning with Theorem 3 and ending with Theorem 1. In all cases, even should we fail to keep saying so, we are producing a leaf with polynomial growth in a tame foliation.

(6.1) LEMMA. *For each $r \geq 0$, the surface $N_{r,0}$ occurs as a leaf with polynomial growth of a tame C^∞ foliation in every closed 3-manifold.*

PROOF. The case $r = 0$ is known, so assume $r \geq 1$. In the foliation (4.5) of $S^1 \times S^2$, a set of toral leaves forms a saturated set X diffeomorphic to $J \times T^2$, J a closed bounded nontrivial interval. Replace X by a copy of $J \times T^2$ foliated so that the interior leaves are planes with quadratic growth and are transverse to the J-factors. Such a foliation is standard and is described, for instance, in [13, p. 2], but it should be noted that these planar leaves are dense in $J \times T^2$. This is why we fail to produce a *proper* leaf. Nonetheless, the closed transversal $\sigma = S^1 \times \{x_0\}$ crosses this foliated $J \times T^2$ in the factor $J \times \{x_0\}$, hence σ meets each planar leaf L in a discrete subset of L converging to the end. If $r = 1$, make the standard modification along σ and delete the resulting Reeb component, thereby producing $N_{1,0}$ as a leaf with polynomial growth in a transversely orientable foliation of $S^1 \times D^2$ trivial at the boundary. If $r > 2$, use (4.8) and (3.3) to produce in $S^1 \times S^2$ a leaf $\mathbf{R}^2 \#_{\{\infty\}} N_{r-1,0} \cong N_{r,0}$. Another closed transversal σ', isotopic to the S^1-factor, crosses this leaf in a countable discrete set approaching the original end ∞, hence the standard modification along σ' and deletion of the Reeb component produce a similar foliation of $S^1 \times D^2$ with $N_{r,0}$ as a leaf. The lemma follows by (3.1). □

(6.2) PROPOSITION. *Let N be of finite type with* genus$(N) < \infty$. *Then N occurs as a leaf with polynomial growth of a tame C^∞ foliation in every closed 3-manifold, this leaf being properly proper (as defined in* §3) *if $N \neq N_{r,0}$, $r > 1$.*

PROOF. Because of (6.1), we assume $N \neq N_{r,0}$, $r > 0$. Also, if N is obtained by deleting a finite set of points from $\mathbf{R}^2$, it occurs as a properly proper leaf in every closed 3-manifold (just produce a tame foliation with a Reeb component R and modify along finitely many core transversals of R). So if genus$(N) = 0$, we can assume $N = N_{r,0}^{k+1}$ where $r > 1$ and $k > 1$. By (4.8), (3.2), and the fact that every M admits a tame foliation with a Reeb component, we see that $\mathbf{R}^2 \# N_{r,0}^{k+1} \cong N_{r,0}^{k+1}$ occurs as a properly proper leaf in all closed 3-manifolds as desired.

For the remaining cases, first remark that if N_1 occurs as a properly proper leaf in M, so does $N_1 \# T^2 \# \cdots \# T^2$. This is by (4.6), (3.2), and induction. If genus $(N) = g \neq 0$, we write $N = N_1 \# T^2 \# \cdots \# T^2$ where genus$(N_1) = 0$ and there are g copies of T^2. If $N_1 \neq N_{r,0}$, $r > 1$, the previous paragraph and the above remark show that N occurs as a properly proper leaf in every closed 3-manifold. But if $N_1 = N_{r,0}$, $r > 1$, write

$$N = (N_{r,0} \# T^2) \# \cdots \# T^2 = (T')^{(r)} \# T^2 \# \cdots \# T^2$$

(where there are $g - 1$ copies of T^2). By (4.11) and (3.2), $\mathbf{R}^2 \# (T')^{(r)} \cong (T')^{(r)}$ occurs as a properly proper leaf in every closed 3-manifold; hence so does N. □

(6.3) LEMMA. *Let $N_{n,i}$ have no isolated nonplanar end and suppose $i > 0$. Then $N_{n,i}$ is available.*

PROOF. We use induction on n. If $n = 0$, there is no such surface and, if $n = 1$, the only one is diffeomorphic to $(N_{0,1})^{(1)}$, so the assertion for $n = 1$ is contained in (4.11). Generally, we will have

$$N_{n+1,i} = N_{0,1} \#_{\{e_{0,1}\}} (N_{n_1,i_1} \# \cdots \# N_{n_m,i_m})$$

where each $n_\alpha < n$ and no N_{n_α,i_α} has an isolated nonplanar end. By assumption there must be at least one sequence of planar ends converging to $e_{0,1} = e_{n+1,i} \in \mathcal{E}(N_{n+1,i})$, hence we have

$$N_{n+1,i} \cong (N_{0,1})^{(1)} \#_{\{e_{0,1}\}}(N_{n_1,i_1} \# \cdots \# N_{n_m,i_m}).$$

But we can modify the proof of (4.11) by taking $m+2$ disjoint closed transversals $\beta', \beta'', \ldots, \beta^{(m+2)}$ isotopic to β and missing α, β' and β'' being as in that proof, and the surgeries along β' and β'' produce a continuum of leaves diffeomorphic to $(N_{0,1})^{(1)}$ meeting α in a nontrivial closed subinterval as usual. The transversals $\beta^{(3)}, \ldots, \beta^{(m+2)}$ meet each of these leaves in a discrete set approaching the end $e_{0,1}$. By the inductive hypothesis together with (4.8) (in case some of the $i_\alpha = 0$) and m applications of (3.3), the assertion follows. □

We can now prove Theorem 3. Let N be of finite type without isolated nonplanar ends. That is, N is a finite connected sum $N_0 \# N_1 \# \cdots \# N_p$ where N_0 is of finite genus, and N_i is one of the surfaces of (6.3), $1 \leq i \leq p$. If $p = 0$, we are reduced to (6.2), and otherwise we are allowed to assume genus$(N_0) \neq 0$. By (6.2), N_0 occurs as a properly proper leaf in every closed 3-manifold, and we use (6.3) and (3.2) to produce N itself as such a leaf. This proves Theorem 3.

In order to prove Theorem 4, we need the following.

(6.4) Lemma. *If N is of finite type and has only nonplanar ends, then $(N, \mathcal{E}(N))$ is available.*

Proof. We have $N = N_{n,q(n)} \# \cdots \# N_{n,q(n)}$ for some $n \geq 0$. If there is just one summand, then (4.10) says that $(N, \mathcal{E}(N))$ is available. Otherwise, suppose the number of summands is $k+1$, $k \geq 1$. If $n = 0$, (4.7) and (4.6) readily give the result. If $n \geq 1$, consider the foliation $\mathcal{F}_{n-1}$ of (4.10). Let α_{n-1} in that foliation play the role of σ' and J_{n-1} the role of J' in (3.4), and let $(\Sigma_{k+1}, \mathcal{E}(\Sigma_{k+1}))$ as in (4.7) play the role of (N, E) in (3.4). Then (3.4) and (4.10) give a foliation of $S^1 \times S^2$ with a leaf diffeomorphic to $\Sigma_{k+1} \#_{\mathcal{E}(\Sigma_{k+1})} N_{n-1,q(n-1)} \cong N$ and meeting the closed transversal $\beta_{n-1} = S^1 \times \{x_0\}$ in a set of points approaching all of the ends of N. Thus $(N, \mathcal{E}(N))$ is available. □

We prove Theorem 4. Let M be a closed orientable 3-manifold that is not a rational homology sphere. If all ends of N are nonplanar, then by (6.4) and (3.5), the surface $N \#_{\mathcal{E}(N)} T^2 \cong N$ occurs in the desired way as a properly proper leaf in M. If $N = N_0 \# N_*$ where all ends of N_0 are nonplanar and N_* has no isolated nonplanar end, we can assume genus$(N_*) \neq 0$, and write $N = N_0 \# N_1 \# N_2 \# \cdots \# N_p$ where N_i is a surface as in (6.3), $2 \leq i \leq p$, and $N_1 = N_{r,0}^k \# T^2$ for $k \geq 1$. If $k = 1$, N_1 is available by (4.11), and if $k \geq 2$ availability follows from (4.6) and (4.8). By (6.3) each N_i, $2 \leq i \leq p$, is also available. By the above and (3.2), N occurs as a properly proper leaf with polynomial growth in a tame C^∞ foliation of M. The proof of Theorem 4 is complete.

The main step in the proof of Theorem 1 is the following.

(6.5) LEMMA. *Let $n \geqslant 0$ and $1 \leqslant i \leqslant q(n)$. There is a compact manifold $W_{n,i}$ with $\partial W_{n,i} \cong S^1 \times S^1$ and a nontrivial closed subarc $J \subset S^1$ and a C^∞ foliation of $W_{n,i}$, transverse to the boundary and inducing there the foliation by circles $\{\theta\} \times S^1$, such that each leaf meeting $J \times S^1$ is diffeomorphic to $N_{n,i}$ − (open disk), and such that the growth of the leaves is dominated by a common polynomial P.*

PROOF. Proceed by induction on n. If $n = 0$, the only endtype to be considered is $N_{0,1}$. In the foliation $\mathfrak{F}_0$ of (4.10) delete an open tubular neighborhood of α_0 obtaining a foliated manifold $W_{0,1}$ as desired. If the assertion has been established for all nonnegative integers less than $n + 1$, consider any $N_{n+1,i}$, $1 \leqslant i \leqslant q(n + 1)$. Since $i \neq 0$, (2.4) and the associated remark show that

$$N_{n+1,i} \cong N_{0,1} \#_{\{e_{0,1}\}} (N_{n_1,i_1} \# \cdots \# N_{n_m,i_m})$$

where each $n_\alpha \leqslant n$. Let $\beta_1, \ldots, \beta_m$ be disjoint closed transversals to the $\mathfrak{F}_0$ of (4.10) all isotopic to β_0. If $1 \leqslant \alpha \leqslant m$ and $i_\alpha \neq 0$, we have W_{n_α,i_α} foliated as desired and we glue it in suitably to replace a tubular neighborhood of β_α. If $i_\alpha = 0$ and $n_\alpha \neq 0$, remark that $N_{n_\alpha,0}$ can be replaced in the above expression by $T^2 \# N_{n_\alpha,0}$, and by (4.11) this surface is available. In this case, foliate $S^1 \times D^2$ so as to exhibit this availability and suitably replace a tubular neighborhood of β_α with this foliated solid torus. If $(n_\alpha, i_\alpha) = (0, 0)$, perform the standard modification that inserts a Reeb component along β_α and punches out a sequence of planar ends in $N_{0,1}$ approaching $e_{0,1}$. Finally, remove an open tubular neighborhood of the closed transversal α_0, thereby producing $W_{n+1,i}$ suitably foliated. □

We prove Theorem 1. If N is of finite type and has only planar ends, the desired conclusion is contained in Theorem 3. Otherwise write

$$N = N_* \# N_{n_1,i_1} \# \cdots \# N_{n_m,i_m}$$

where $i_\alpha \neq 0$, $1 \leqslant \alpha \leqslant m$, and N_* is either S^2 or a surface of finite type with only planar ends and nonzero genus. In any case, N_* occurs as a properly proper leaf of a tame C^∞ foliation in $S^1 \times S^2$. Let $\sigma_1, \ldots, \sigma_m$ be disjoint closed transversals, each meeting this leaf just once. Replace an open tubular neighborhood of σ_α with W_{n_α,i_α}, $1 \leqslant \alpha \leqslant m$, suitably matching foliations so as to obtain N as a proper leaf with polynomial growth in the resulting manifold. The proof is complete.

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Department of Mathematics, St. Louis University, St. Louis, Missouri 63103

Department of Mathematics, Washington University, St. Louis, Missouri 63130