TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 236, February 1978

THE IMMERSION CONJECTURE FOR $\mathbf{R}P^{8l+7}$ IS FALSE[1]

BY

DONALD M. DAVIS AND MARK MAHOWALD

ABSTRACT. Let $\alpha(n)$ denote the number of 1's in the binary expansion of n. It is proved that if $n \equiv 7\ (8)$, $\alpha(n) = 6$, and $n \neq 63$, then $\mathbf{R}P^n$ can be immersed in $\mathbf{R}^{2n-14}$. This provides the first counterexample to the well-known conjecture that the best immersion is in $\mathbf{R}^{2n-2\alpha(n)+1}$ (when $\alpha(n) \equiv 1$ or 2 mod 4). The method of proof is obstruction theory.

1. Introduction. In [6] we announced that we had proved the nonimmersion part of the well-known conjecture [7], [10]:

Let $\alpha(n)$ denote the number of 1's in the binary expansion of n. Let

$$\beta(n) = \begin{cases} 2\alpha(n) & \text{if } \alpha(n) \equiv 1, 2\ (4), \\ 2\alpha(n) + 1 & \text{if } \alpha(n) \equiv 0\ (4), \\ 2\alpha(n) + 2 & \text{if } \alpha(n) \equiv 3\ (4). \end{cases}$$

If $n \equiv 7\ (8)$, then $\mathbf{R}P^n \subseteq \mathbf{R}P^{2n-\beta(n)+1}$ but $\mathbf{R}P^n \not\subseteq \mathbf{R}^{2n-\beta(n)}$.

Our proof was wrong [25] and, moreover, the result is wrong for $\alpha(n) = 6$ (and probably also for most values of $\alpha(n)$). Indeed, our main result shows that the conjectured best immersion dimension is at least three too large if $\alpha(n) = 6$.

THEOREM 1.1 *If* $n \equiv 7\ (8)$, $\alpha(n) = 6$, *and* $n \neq 63$, $\mathbf{R}P^n \subseteq \mathbf{R}^{2n-14}$ *but* $\mathbf{R}P^n \not\subseteq \mathbf{R}^{2n-18}$.

The best immersion dimension for $\mathbf{R}P^n$ when $n + 1$ is a 2-power was established in [9] and [13]. The lesson of our paper is that the immersion problem for $\mathbf{R}P^n$ is much harder than was previously expected: the only nonimmersions detected by the Adams operation $\psi^3 - 1$ are those of [13].

In [6] we also announced that we had obtained the conjectured immersions when $\alpha(n) = 5$, 6, 8, or 9. That proof is valid and is sketched in this paper, although we no longer conjecture these immersions to be best possible.

Received by the editors September 13, 1976.

AMS (MOS) subject classifications (1970). Primary 57A35; Secondary 55G45, 55F20, 55F45, 55H15.

Key words and phrases. Immersions of projective spaces, obstruction theory, Adams spectral sequence.

[1]This work was partially supported by NSF Grants GP 25335 and MPS 75-04299.

THEOREM 1.2. *If* $n \equiv 7\ (8)$, $\mathbf{R}P^n \subseteq \mathbf{R}^{2n-D}$; *if* $n \equiv 3\ (8)$, $\mathbf{R}P^n \subseteq \mathbf{R}^{2n-E}$, *where* D *and* E *are given by the table*:

$\alpha(n)$	5	6	7	8	9	$\geqslant 10$
D	9	11	14	16	17	18
E	9	10	15	17	17*	17*

The numbers indicated with * *may be increased to* 18 *if* $n \equiv 11\ (16)$.

It is well known [21] that $\mathbf{R}P^n$ immerses in $\mathbf{R}^{2n-e}$ if and only if the map $\mathbf{R}P^n \to BO$ which classifies the stable normal bundle $(2^L - n - 1)\xi_n$ lifts to BO_{n-e} (where L is any sufficiently large integer). When $n \equiv 7\ (8)$, this bundle is trivial on $\mathbf{R}P^7$ (since $\widetilde{KO}(\mathbf{R}P^7) \approx \mathbf{Z}_8$) and so its classifying map lifts to the 7-connected covering $BO[8]$. The nonimmersion of Theorem 1.1 is proved by showing that the map $\mathbf{R}P^n \xrightarrow{g} BO[8] \to BO[8]/BO_{n-18}[8]$ is essential, by studying the Adams spectral sequence which converges to $[\mathbf{R}P^n, BO[8]/BO_{n-18}[8]]$.

The immersion of Theorem 1.1 is proved by considering the diagram

$$\begin{tikzcd}
 & BO_{n-14}[8] \arrow[d] & \\
\Omega(BO[8]/BO_{n-14}[8]) \times \mathcal{E} \arrow[r, "\mu"] & \mathcal{E} \arrow[r, "k_1"] \arrow[d] & \mathcal{E}/BO_{n-14}[8] \\
\mathbf{R}P^n \arrow[r, "g"] \arrow[ur, "l"] & BO[8] \arrow[r, "k_0"] & BO[8]/BO_{n-14}[8]
\end{tikzcd}$$

where ξ is the fiber of k_0. The immersion follows from

LEMMA 1.4. (a) *There is a lifting* l *of* g.

(b) *There is a map* f: $\mathbf{R}P^n \to \Omega(BO[8]/BO_{n-14}[8])$ *such that* $k_1\mu(f \times l)$ *is null homotopic*.

(c) *The fibre of* k_1 *has the same* n-*type as* $BO_{n-14}[8]$.

2. Preliminaries and discussion of Theorem 1.2. Let bo denote the connected Ω-spectrum whose $8k$th space is the $(8k-1)$-connected space $BO[8k]$ localized at 2 [10], [4]. Let $\widetilde{BSp}_N$ denote the classifying space for symplectic vector bundles of real geometric dimension N, and let $B_N^0 = \widetilde{BSp}_N \wedge_{BSp} bo$ denote the space which was called E_N^0 in [5]. Let h: $\mathbf{R}P^{4k+3} \to QP^k$ denote the canonical map. If g': $QP^k \to BSp$ classifies pH_k, then $g = g' \cdot h$ classifies $4p\xi_{4k+3}$.

Let E_i and E_i^0 (or sometimes $E_i(N)$ and $E_i^0(N)$) denote the ith space in an n-MPT [8] for the fibrations $V_N \to \widetilde{BSp}_N \to BSp$ and $V_N \wedge bo \to B_N^0 \to BSp$, respectively. By the techniques of [15] there is a map $E_i \to E_i^0$ which respects the projections.

Let $P_N^M = \mathbf{R}P^M/\mathbf{R}P^{N-1}$ and $P_N = P_N^\infty$. P_N and V_N have the same $2N$-type [12], and since our resolutions are through degree n with $n < 2N$, P_N and V_N are interchangeable. A k-invariant in $H^m(E_i)$ (all coefficients are $\mathbf{Z}_2$ unless otherwise indicated) corresponds to an element in $\mathrm{Ext}_{\mathcal{A}}^{i,i+m-1}(P_N)$, which is tabulated in [17]. Here, and throughout the paper, $\mathcal{A}$ denotes the mod 2 Steenrod algebra, and we have abbreviated $\mathrm{Ext}(\tilde{H}^*(X), \mathbf{Z}_2)$ to $\mathrm{Ext}(X)$. The tables of [17] will be used extensively without always explicitly saying so. We will denote by k_m^i a k-invariant in $H^m(E_i)$, where E_i is the ith space in some MPT.

Let $X\langle i\rangle$ denote the space formed from X by killing classes in $\mathrm{Ext}_{\mathcal{A}}^{s,t}(X)$ for $0 \leq s < i$. More precisely,

DEFINITION 2.1. Let $X\langle 0\rangle = X$. Let $X\langle i\rangle$ denote the fibre of the map $X\langle i-1\rangle \to K(\mathbf{Z}_2, \mathrm{Ext}_{\mathcal{A}}^{0,*}(X\langle i-1\rangle))$. Then $\mathrm{Ext}_{\mathcal{A}}^{s,t}(X\langle i\rangle) \approx \mathrm{Ext}_{\mathcal{A}}^{s-i,t-i}(X)$.

DEFINITION 2.2. Let $X\langle 0, i\rangle$ denote the mapping cone of the map $X\langle i+1\rangle \to X$.

In the stable range (which is the only range in which these constructions will be considered) the 2-primary homotopy groups of $X\langle 0, i\rangle$ are those obtained by considering the elements of the Adams spectral sequence [1] of X of filtration $\leq i$.

If $F \to E \to^p B$ is a fibration which is totally transgressive [8] in the range of interest and E_i is the ith stage of a MPT for p, then $\mathrm{fibre}(E_i \to B) = F\langle 0, i-1\rangle$ (in the stable range). This follows easily from the map of fibrations

$$\begin{tikzcd}
F\langle i\rangle \arrow[r] \arrow[d] & F \arrow[d] \\
E \arrow[r] \arrow[d] & E \arrow[d] \\
E_i \arrow[r] & B
\end{tikzcd}$$

The proof of Theorem 1.2 is quite similar to that of [5, 1.4, 1.3(b)]. It involves a detailed study of the MPT's for the lifting question

$$\begin{tikzcd}
 & & \widetilde{BSp}_N \arrow[r] \arrow[d] & B^0_{N=4k+3-D} \arrow[d] \\
 & & E_i \arrow[r] \arrow[d] & E_i^0 \arrow[d] \\
RP^{4k+3} \arrow[r, "h"] & QP^k \arrow[r, "g'"] & BSp \arrow[r, "="] & BSp
\end{tikzcd}$$

We use [5, 1.8] and maps of MPT's for

$$\begin{tikzcd} B_N^0 \arrow[r] \arrow[d] & B_{N+\Delta}^0 \arrow[d] \\ BSp \arrow[r, "="] & BSp \end{tikzcd}$$

to get QP^k to lift to E_i^0 for largest possible i. The possible obstructions to pulling this lifting back to E_i are the elements of $\mathrm{Ext}_{\mathscr{Q}}^{s,s+4j-1}(P_N)$ for $s < i$ and $j < k$ which are not bo-primary, i.e. which are in the kernel of the homomorphism $\mathrm{Ext}_{\mathscr{Q}}(P_N) \to \mathrm{Ext}_{\mathscr{Q}}(P_N \wedge bo)$. These are just the dots in $t - s \equiv 3$ (4) in the tables of [17] which are not part of the regular towers. It is the existence of such possible obstructions which cause our inability to assert liftings of $\mathbf{R}P^n$ to BO_{n-e} for $e > 18$. However, we can sometimes show these obstructions are zero by naturality using the maps of MPT's for

$$\begin{tikzcd} \widetilde{BSp}_{N-\Delta} \arrow[r] \arrow[d] & \widetilde{BSp}_N \arrow[d] \\ BSp \arrow[r, "="] & BSp \end{tikzcd}$$

Having lifted QP^k to E_i for some i we can often lift $\mathbf{R}P^{4k+3}$ to $E_{i+\varepsilon}$ for some small positive integer ε by indeterminancy computations. This gives us a lifting to $\widetilde{BSp}_N$ if $\mathrm{Ext}_{\mathscr{Q}}^{s,t}(P_N) = 0$ whenever $s > i + \varepsilon$ and $t - s < n$, for then $\widetilde{BSp}_N \to E_{i+\varepsilon}$ is an n-equivalence.

We exemplify with the case $\alpha(n) = 7$, $n = 8l + 7$. Then $N = 8l - 7$, $\alpha(l) = 4$, and g' classifies $(2^{L-2} - 2l - 2)H_{2l+1}$. As in [5, §4] we compute

$$\nu\binom{2^{L-2} - 2l - 2}{i} = \begin{cases} 3, & i = 2l - 2, \\ 5 + \nu(l), & i = 2l - 1, \\ 4, & i = 2l, \\ 5, & i = 2l + 1, \end{cases}$$

where $\nu(2^a(2b + 1)) = a$. Thus by [5, 1.8] g' lifts to B_{8l-5}^0 and $g'|QP^{2l-1}$ lifts to B_{8l-11}^0. (We use here that if $0 < \varepsilon < 3$ and $i > m$,

$$\nu(\pi_{4i-1}(P_{4m+\varepsilon} \wedge bo)) = 2(i - m) + \begin{cases} 1 & \text{if } i - m \text{ odd}, \varepsilon = 0, 1, \\ -1 & \text{if } i - m \text{ odd}, \varepsilon = 3, \\ 0 & \text{otherwise}, \end{cases}$$

so that

$$\nu(\pi_{4i-1}(P_{8l-5} \wedge bo)) = \left.\begin{cases} 0, & i < 2l - 2, \\ 1, & i = 2l - 1, \\ 4, & i = 2l, \\ 5, & i = 2l + 1. \end{cases}\right).$$

Since

$$\begin{aligned} &\pi_{4i-1}\big(\mathrm{fibre}\big(E_4(8l - 11) \to E_4^0(8l - 11)\big)\big) \\ &\quad = \pi_{4i-1}(\mathrm{fibre}(P_{8l-11}\langle 0, 3\rangle \to P_{8l-11} \wedge bo\langle 0, 3\rangle)) = 0 \end{aligned}$$

for $i < 2l$, $g'|QP^{2l-1}$ lifts to $E_4(8l-11)$. (The last equality holds because in [17, Tables 8.6 and 8.14] all elements in columns 2, 6, and 10 occur in the regular tower, which is also present after applying $\wedge bo$). Since

$$\pi_{8l-1}\Big(\text{fibre}\big(E_4(8l-11) \to E_4^0(8l-5)\big)\Big) = 0$$

$g'|QP^{2l}$ lifts to $E_4(8l-11)$ and, hence, to $E_4(8l-7) = E_5(8l-7)$. (The use of $8l-11$ was to get past the non-bo-primary element labeled ${}_0h_2^2$ in [17, Tables 8.2 and 8.10].) Since $\pi_{8l+3}(\text{fibre}(E_5(8l-7) \to E_5^0(8l-5))) = 0$, g' lifts to $E_5(8l-7)$, and, hence, there is a lifting l of $g'h$ to $E_5(8l-7)$ which sends the k-invariants k_{8l+3}^5 (corresponding to the top element in column 9 of [17, Tables 8.2 and 8.10]) and k_{8l+7}^5 to zero (because the lifting factors through QP^{2l+1}). If $l^*(k_{8l+4}^5) \neq 0$ (which it will since

$$\nu\binom{2^{L-2}-2l-2}{2l+1} = 5\Big),$$

we form a new lifting $l' = \mu(x_{8l+3} \times l)$, where μ: $(K_{8l+1} \times K_{8l+3} \times K_{8l+6}) \times E_5^0 \to E_5^0$ is the action of the fibre on the total space of the principal fibration $E_5^0 \to E_4^0$, $K_i = K(\mathbf{Z}_2, i)$, and x_{8l+3} is the map $\mathbf{R}P^{8l+7} \to K_{8l+1} \times K_{8l+3} \times K_{8l+6}$ which is nontrivial in H^{8l+3}. $\mu^*(k_{8l+4}^5) = 1 \times k_{8l+4}^5 + \mathrm{Sq}^1\iota_{8l+3} \times 1 +$ other terms. This can be shown as in [8] by computing the relations in the MPT. (Alternatively, the existence of $\mathrm{Sq}^1\iota$ in $\mu^*(k)$ corresponds to the action of h_0 in Ext, so this information can be read directly from the tables of [17].) Then $l'^*(k_{8l+4}^5) = l^*(k_{8l+4}^5) + \mathrm{Sq}^1x_{8l+3}^*(\iota_{8l+3}) = 0$. We must also check that $l'^*(k_{8l+7}^5) = 0$, but this is so since $\mathrm{Sq}^4x_{8l+3} = 0$, $\mathrm{Sq}^3\mathrm{Sq}^1x_{8l+3} = 0$, and $w_4((2^L - 8l - 8)\xi) = 0$, so that if $\mathrm{Sq}^4\iota_{8l+3} \times 1$ and $\iota_{8l+3} \times w_4$ are present in $\mu^*(k_{8l+7}^5)$ (which in fact they are), they do not contribute to $l'^*(k_{8l+7}^5)$. Thus l' lifts to l_6: $\mathbf{R}P^{8l+7} \to E_6(8l-7)$. If $l_6^*(k_{8l+4}^6) \neq 0$, it can be varied as above. Thus there is a lifting to $E_7 = \widetilde{BSp}_{8l-7}$.

3. The spaces $BO[8]/BO_N[8]$. Throughout the remainder of the paper we shall let $C_N = BO[8]/BO[8]$. In this section we compute $\pi_i(C_N)$ for $i < N + 14$ and N odd. We also compute $[P^n, \Omega C_N]$ and $[P^n, C_N]$ for certain values of N and n.

THEOREM 3.1. *There is an isomorphism of $\mathcal{Q}$-modules* $H^*(BO[8], BO_N[8]) \approx \tilde{H}^*(\Sigma P_n) \otimes \mathcal{Q}//\mathcal{Q}_2$ *through degree* $N + 16$, *where $\mathcal{Q}_2$ is the subalgebra of $\mathcal{Q}$ generated by* Sq^1, Sq^2, *and* Sq^4.

PROOF. Let k: $BO[8] \to BO$ and $\bar{k} = BO[8]/BO_N[8] \to BO/BO_N$. Let i: $\Sigma V_N = CV_N/V_N \to BO[8]/BO_N[8]$ be induced from the fibration $V_N \to BO_N[8] \to BO[8]$. The Serre spectral sequence [20] of the relative fibration $(CV_N, V_N) \to (BO[8], BO_N[8]) \to BO[8]$ collapses through degree $2N$ because it is mapped onto by that of $(BO, BO_N) \to BO$. Thus there is a vector space isomorphism

$$H^*(BO[8], BO_N[8]) \approx \langle\{\bar{k}^* w_m\colon m > N\}\rangle \otimes H^*(BO[8]),$$

where $\langle S\rangle$ denotes the vector space spanned by S. Here we use the external cup product and the fact that $i^*\bar{k}^* w_m = s\alpha^{m-1}$, the nonzero element in $H^m(\Sigma V_N)$.

By Stong [23],

$$H^i(BO[8]) \approx \begin{cases} \mathbf{Z}_2 & i = 0, 8, 12, 14, 15, \\ 0 & \text{other } i < 16, \end{cases} \qquad \text{with } \mathrm{Sq}^i u_8 = u_{i+8}$$

for $i = 4, 6, 7$, and $k^* w_i = u_i$ for $i = 8, 12, 14, 15$. By the Wu formula, for $i < 15$,

$$\mathrm{Sq}^i(\bar{k}^* w_m) = \sum_{\substack{j<i \\ j=0,8,12,14,15}} \binom{m-1-j}{i-j} \bar{k}^* w_{m+i-j} \cup k^* w_j.$$

For $i < 15$,

$$(\mathcal{A}//\mathcal{A}_2)_i = \begin{cases} \mathbf{Z}_2, & i = 0, 8, 12, 14, 15, \\ 0, & \text{other } i < 16, \end{cases} \qquad \text{generated by } \mathrm{Sq}^i.$$

$\psi\colon \tilde{H}^*(\Sigma P_N) \otimes \mathcal{A}//\mathcal{A}_2 \to H^*(BO[8], BO_N[8])$ defined by $\psi(s\alpha^m \otimes \mathrm{Sq}^i) = \bar{k}^* w_{m+1} \cup k^* w_i$ is an $\mathcal{A}$-module isomorphism in the desired range. For example,

$$\begin{aligned}
&\psi(\mathrm{Sq}^{13}(s\alpha^m \otimes \mathrm{Sq}^0)) \\
&\quad = \psi\left(\binom{m}{13} s\alpha^{m+13} \otimes \mathrm{Sq}^0 + \binom{m}{5} s\alpha^{m+5} \otimes \mathrm{Sq}^8 + \binom{m}{1} s\alpha^{m+1} \otimes \mathrm{Sq}^{12}\right) \\
&\quad = \binom{m}{13}\bar{k}^* w_{m+14} + \binom{m}{5}\bar{k}^* w_{m+6} \cup k^* w_8 + \binom{m}{1}\bar{k}^* w_{m+2} \cup k^* w_{12} \\
&\quad = \binom{m}{13}\bar{k}^* w_{m+14} + \binom{m-8}{5}\bar{k}^* w_{m+6} \cup k^* w_8 \\
&\qquad + \binom{m-12}{1}\bar{k}^* w_{m+2} \cup k^* w_{12} \\
&\quad = \mathrm{Sq}^{13}(\bar{k}^* w_{m+1}) = \mathrm{Sq}^{13}\psi(s\alpha^m \otimes \mathrm{Sq}^0). \qquad \square
\end{aligned}$$

Thus by the change-of-rings theorem [3], [4, Lemma 3.1], $\mathrm{Ext}_{\mathcal{A}}^{s,t}(C_N) \approx \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\Sigma P_N)$, and this is as in Table 3.2. As usual, vertical and diagonal (/) lines indicate multiplication by h_0 and h_1, respectively. Diagonal (\) lines are

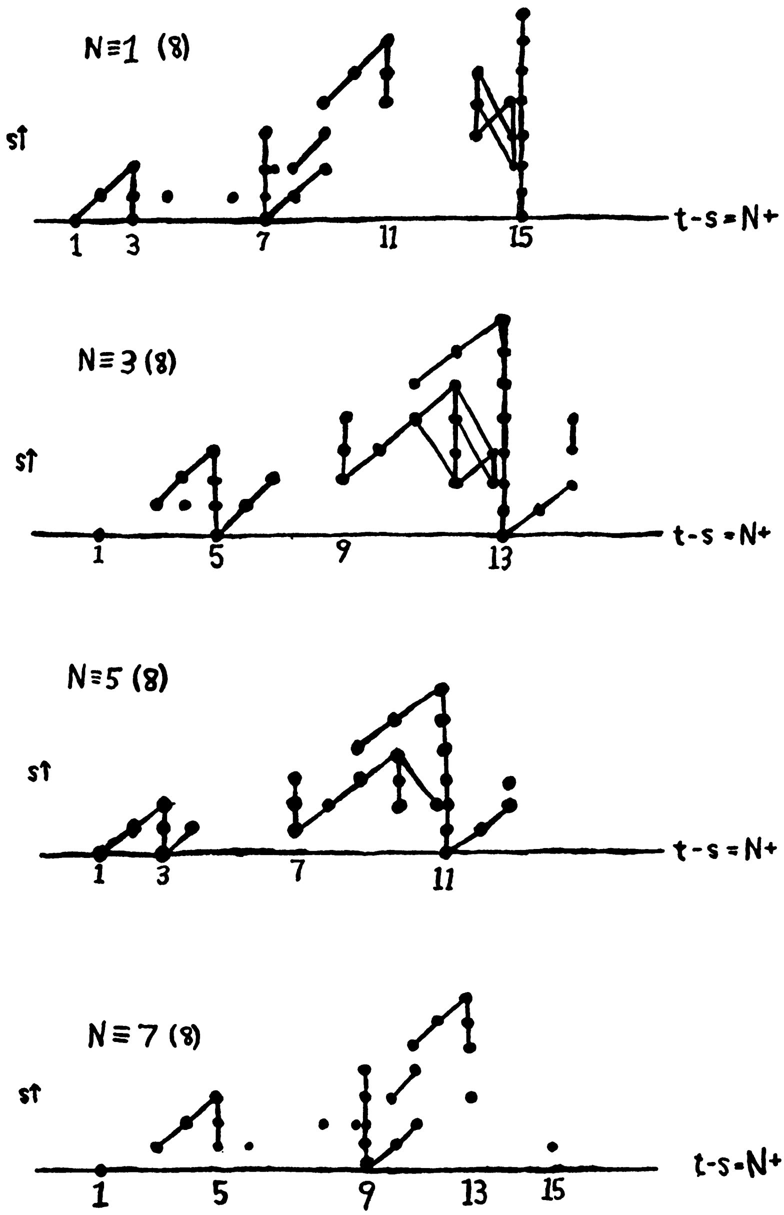


TABLE 3.2

d_2-differentials in the Adams spectral sequences, which will be established in Theorems 3.3 and 3.4.

$\mathrm{Ext}_{\mathcal{a}_2}(P_N)$ can be computed by minimal resolution and this is listed in §6 when $N \equiv 1, 5, 7$ (8). A second way of computing it for $t - s < N + 15$ is by the exact sequence

$$\to \operatorname{Ext}^{s,t}_{\mathcal{Q}}(P_N) \to \operatorname{Ext}^{s,t}_{\mathcal{Q}_2}(P_N) \to \operatorname{Ext}^{s,t}_{\mathcal{Q}_1}(\Sigma^8 P_N) \to \operatorname{Ext}^{s+1,t}_{\mathcal{Q}}(P_N) \to$$

using the tables of $\operatorname{Ext}_{\mathcal{Q}}(P_N)$ in [17]. This exact sequence is obtained by applying $\operatorname{Ext}_{\mathcal{Q}}(H^* P_N \otimes -, \mathbf{Z}_2)$ to the short exact sequence through degree 15

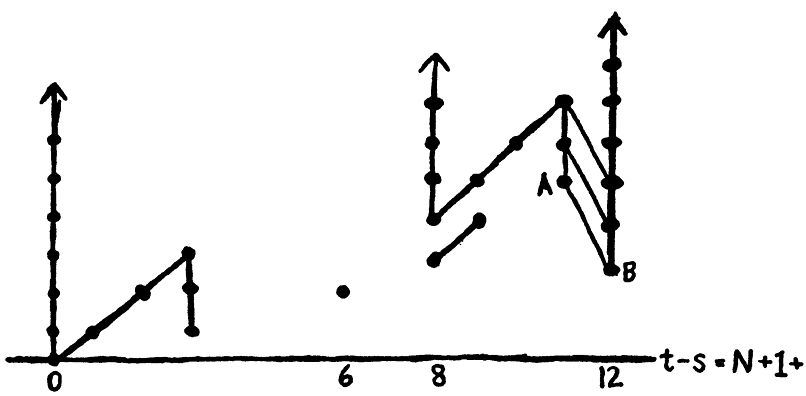
$$0 \to \Sigma^8 \mathcal{Q}//\mathcal{Q}_1 \to \mathcal{Q}//\mathcal{Q}_2 \to \mathbf{Z}_2 \to 0.$$

Theorem 3.3. *There are d_2-differentials in the Adams spectral sequences for C_N as indicated in Table* 3.2.

Proof. As in Theorem 3.1, $H^*(BO_{N+1}[8], BO_N[8]) \approx s^{N+1}\mathcal{Q}//\mathcal{Q}_2$, and so

$$\operatorname{Ext}^{s,t}_{\mathcal{Q}}(BO_{N+1}[8]/BO_N[8]) \approx \operatorname{Ext}^{s,t-N-1}_{\mathcal{Q}_2}(\mathbf{Z}_2, \mathbf{Z}_2),$$

which begins (by [11])



The d_2-differential is present because

$$BO_{N+1}[8]/BO_N[8]^{(N+13)} = S^{N+1} \cup {}_{\sigma}e^{N+9} \cup {}_{\nu}e^{N+13}$$

and, since A comes from $\operatorname{Ext}(S^{N+1})$, by [26, 9.4] A represents the composite

$$S^{N+12} \xrightarrow{\langle 8,\nu,\sigma\rangle} S^{N+1} \xrightarrow{i} S^{N+1} \cup {}_{\sigma}e^{N+9} \cup {}_{\nu}e^{N+13},$$

which is null homotopic because $\langle 8, \nu, \sigma\rangle$ is, by definition, the composite

$$S^{N+12} \xrightarrow{8} S^{N+8} \cup {}_{\nu}e^{N+12} \xrightarrow{\sigma} S^{N+1}$$

and $i\sigma$ is null homotopic.

If $N \equiv 3$ (8), the inclusion $BO_{N+1}[8]/BO_N[8] \to BO[8]/BO_N[8]$ sends the Ext-classes A and B nontrivially. Thus the indicated d_2: $E_2^{3,N+16}(C_N) \to E_2^{5,N+17}(C_N)$ when $N \equiv 3$ (8) is present, and the other d_2's for $N \equiv 3$ (8) follow by h_0- and h_1-naturality. The d_2's for $N \equiv 1$ and $N \equiv 5$ follow by naturality from the maps $C_N \to C_{N+2}$. $\square$

Theorem 3.4. *In the Adams spectral sequences pictured in Table* 3.2 *there are no nonzero differentials except those indicated.*

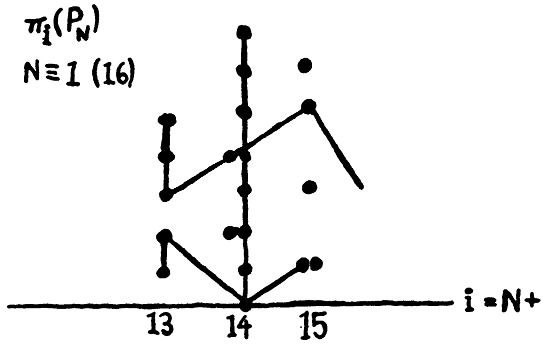
PROOF. We first show d_3 annihilates the element g_{15} of $\mathrm{Ext}_{\mathcal{A}}^{0,N+15}(C_N)$ when $N \equiv 1\ (16)$. We consider the exact sequence

$$\pi_{N+15}(C_N/\Sigma P_N) \xrightarrow{\sigma^*} \pi_{N+14}(\Sigma P_N) \xrightarrow{i^*} \pi_{N+14}(C_N).$$

By Theorem 3.1,

$$\tilde{H}^*(C_N/\Sigma P_N) \approx \tilde{H}^*(\Sigma P_N) \otimes \overline{\mathcal{A}//\mathcal{A}_2} \approx \tilde{H}^*(\Sigma P_N) \otimes \Sigma^8 \mathcal{A}//\mathcal{A}_1$$

in our range. Thus $\mathrm{Ext}_{\mathcal{A}}(C_N/\Sigma P_N) \approx \mathrm{Ext}_{\mathcal{A}_1}(\Sigma^9 P_N)$, and so $\pi_{N+15}(C_N/\Sigma P_N) \approx \mathbf{Z}_{16}$ by [4] or [10]. If $d_3(g_{15}) \neq 0$, then $\pi_{N+14}(C_N) = 0$, so that σ_* is surjective by exactness. Hence $\pi_{N+14}(\Sigma P_N)$ is cyclic. But this is not true (by [17, Table 8.2]), contradicting the hypothesis $d_3(g_{15}) \neq 0$. (To see that $\pi_{N+14}(\Sigma P_N)$ is not cyclic, we reproduce the relevant portion of [17, Table 8.2]

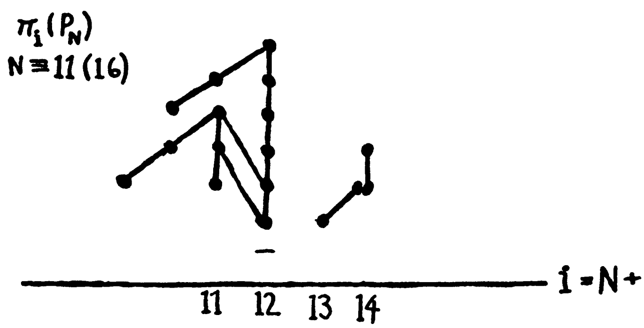


d_3: $\mathrm{Ext}_{\mathcal{A}}^{1,N+16}(P_N) \to \mathrm{Ext}_{\mathcal{A}}^{4,N+18}(P_N)$ is zero, because one element comes from P_{N-2}, where d_3 is clearly 0, while the other is ${}_0h_4$, which survives the Adams spectral sequence of P_N^{N+1} (because of the Adams decomposition of Sq^{16}) and, hence, survives the Adams spectral sequence of P_N. Thus η times a filtration 3 element of $\pi_{N+13}(P_N)$ is nonzero and so no class of filtration 3 can be divisible by 2.)

We next show d_3 annihilates the element g_{13} of $\mathrm{Ext}_{\mathcal{A}}^{0,N+13}(C_N)$ when $N \equiv 11\ (16)$. We consider the exact sequence

$$\pi_{N+13}(C_N/\Sigma P_N) \xrightarrow{\sigma_*} \pi_{N+12}(\Sigma P_N) \xrightarrow{i_*} \pi_{N+12}(C_N).$$

From [17, Table 8.12] we have



As before, if $d_3(g_{13}) \neq 0$, $\sigma_*(G)$ is nonzero in filtration 3, where G generates $\pi_{N+13}(C_N/\Sigma P_N)$. But this cannot happen, for $G \circ \nu \in \pi_{N+16}(C_N/\Sigma P_N) = 0$, whereas if x is any element of $\pi_{N+11}(P_N)$ of filtration 3, $x \circ \nu \neq 0$ (because h_2: $\mathrm{Ext}_{\mathcal{A}}^{3,N+14}(P_N) \to \mathrm{Ext}_{\mathcal{A}}^{4,N+18}(P_N)$ is nonzero, and by [17, Table 8.12] the element in $\mathrm{Ext}_{\mathcal{A}}^{4,N+18}(P_N)$ survives the Adams spectral sequence).

All other differentials are zero by naturality. □

THEOREM 3.5. *If* $N \equiv 7\ (8)$, $\pi_{N+13}(C_N) \approx \mathbf{Z}_{16}$.

REMARK. The content of this theorem is the nontrivial extension in the Adams spectral sequence.

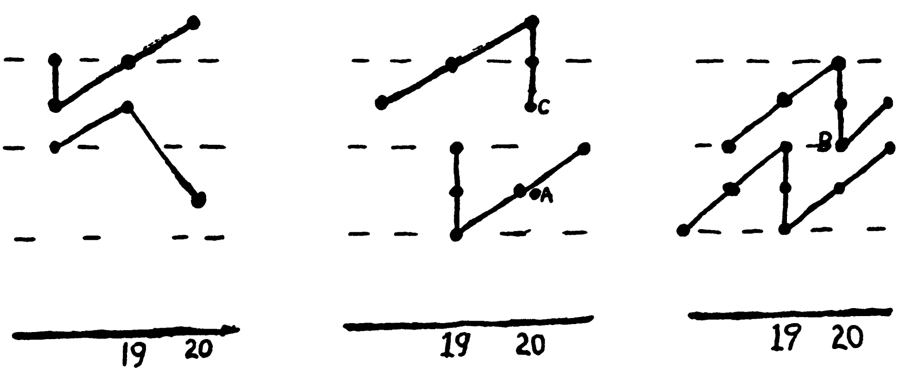
PROOF. We consider the maps of Adams spectral sequences induced by the maps

$$BO_{8l+9}[8]/BO_{8l+7}[8] \to BO_{8l+13}[8]/BO_{8l+7}[8] \xrightarrow{k} BO_{8l+13}[8]/BO_{8l+9}[8].$$

As in Theorem 3.1 the E_2-terms are

$$\mathrm{Ext}_{\mathcal{A}_2}(\Sigma P_{8l+7}^{8l+8}) \to \mathrm{Ext}_{\mathcal{A}_2}(\Sigma P_{8l+7}^{8l+12}) \to \mathrm{Ext}_{\mathcal{A}_2}(\Sigma P_{8l+9}^{8l+12}).$$

These are in $t - s = 8l + 19, 20$ as below.



The d_2-differential in $BO_{8l+9}[8]/BO_{8l+7}[8]$ follows as in the proof of Theorem 3.3. Thus a homotopy class $\hat{A}$ corresponding to the Ext-element A is mapped nontrivially by k_* to a homotopy class corresponding to the Ext-element B. Thus $k_*(2\hat{A}) = 2\hat{B}$ and hence $2\hat{A} = \hat{C}$. By naturality this implies the extension in $BO[8]/BO_{8l+7}[8]$. □

THEOREM 3.6. *Any map* $P^{8l+6} \to C_{8l-7}$ *of filtration* > 4 *is null homotopic.*

PROOF. One first computes

$$\mathrm{Ext}_{\mathcal{A}}^{s,s}(H^*(C_{8l-7}), H^*(P^{8l+6})) = \begin{cases} \mathbf{Z}_2, & s = 4, \\ 0, & s > 4. \end{cases}$$

For example,

$$\operatorname{Ext}_{\mathscr{Q}}^{4,4}(H^*C_{8l-7}, H^*P^{8l+6}) \approx \operatorname{Ext}_{\mathscr{Q}_2}^{4,4}(H^*(\Sigma P_{8l-7}), H^*P^{8l+6})$$
$$\approx \operatorname{Ext}_{\mathscr{Q}_2}^{4,3}(H^*P_1, H^*P^{14})$$
$$= \frac{\ker\left(d_5^*: \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-3}C_4, H^*P^{14}) \to \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-3}C_5, H^*P^{14})\right)}{\operatorname{im}\left(d_4^*: \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-3}C_3, H^*P^{14}) \to \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-3}C_4, H^*P^{14})\right)},$$

where C_i are the $\mathscr{Q}$-modules in the minimal resolution of P_1 given in §6. If l_i denotes a generator of C_4, denote by $\hat{l}_i$ the $\mathscr{Q}$-homomorphism $\Sigma^{-3}C_4 \to H^*P^{14}$ dual to $\Sigma^{-3}l_i$. Then $d_5^*(\hat{l}_9) = \hat{m}_{10}$ because

$$d_5^*(\hat{l}_9)(\Sigma^{-3}m_{10}) = \hat{l}_9\left(d_5(\Sigma^{-3}m_{10})\right) = \hat{l}_9\left(\mathrm{Sq}^2(\Sigma^{-3}l_9)\right) = \mathrm{Sq}^2(\alpha^{10}) \neq 0.$$

Similarly, $d_5^*(\hat{l}_{11}) = 0$ and $\operatorname{im}(d_4^*) = 0$.

Thus the only possible map of filtration > 4 is the unique extension over P^{8l+6} of $P^{8l+4} \to^c S^{8l+4} \to^u C_{8l-7}$, where c is the collapsing map and u a filtration 4 generator of $\pi_{8l+4}(C_{8l-7})$. By Theorem 3.5, u factors as $S^{8l+4} \to^{u'} C_{8l-9} \to^i C_{8l-7}$. The composite $u'c$ is trivial because the corresponding element of $\operatorname{Ext}_{\mathscr{Q}}^{3,3}(H^*C_{8l-9}, H^*P^{8l+4}) \approx \operatorname{Ext}_{\mathscr{Q}_2}^{3,2}(H^*P_7, H^*P^{20})$ is zero since $d_2^*\hat{h}'_{15} = \hat{k}_{19}$ (using the notation of the previous paragraph and the resolution of H^*P_7 given in §6). □

The above proof may be rephrased as: a d_1-differential in the Adams spectral sequence (ASS) converging to $[P^{8l+6}, C_{8l-9}]$ gives rise to a d_2-differential in that of $[P^{8l+6}, C_{8l-7}]$.

PROPOSITION 3.7. *A filtration* 2 *map* $P^{8l} \to C_{8l-11}$ *which sends only* k_{8l-8} *nontrivially is essential.*

PROOF. As in the previous proof,

$$\operatorname{Ext}_{\mathscr{Q}}^{2,2}(H^*C_{8l-11}, H^*P^{8l})$$
$$\approx \frac{\ker\left(d_3^*: \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-1}C_2, H^*P^{16}) \to \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-1}C_3, H^*P^{16})\right)}{\operatorname{im}\left(d_2^*: \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-1}C_1, H^*P^{16}) \to \operatorname{Hom}_{\mathscr{Q}}(\Sigma^{-1}C_2, H^*P^{16})\right)}$$
$$\approx \langle \hat{h}_7, \hat{h}_{15}, \hat{h}_{12} + \hat{h}_{14}\rangle / \langle \hat{h}_7 + \hat{h}_{15}\rangle,$$

where C_i are part of the minimal resolution of P_5. $\hat{h}_7$ corresponds to $\hat{k}_{8l-8}$ under the isomorphism

$$\operatorname{Hom}(\Sigma^{-1}C_2, H^*P^{16}) \approx \operatorname{Hom}\left(H^*(C_{8l-11}\langle 2\rangle), H^*P^{8l}\right)$$

and so gives a nontrivial element of Ext. It cannot be hit by a differential in the ASS converging to $[P^{8l}, C_{8l-11}]$ because $\operatorname{Ext}_{\mathscr{Q}}^{0,1}(H^*C_{8l-11}, H^*P^{8l}) = \mathbf{Z}_2$ and the nonzero class survives to give the map $P^{8l} \to V_{8l-11} \to \Omega C_{8l-11}$. □

Proposition 3.8. *The cokernel of* $i_*\colon [P^{8l+7}, \Omega C_{8l-7}] \to [P^{8l+7}, \Omega C_{8l-3}]$ *is generated by an element* $[f_2]$ *of Adams filtration* 2 *whose restriction to* P^{8l+5} *is trivial.*

Proof. $\pi_{8l+7}(\Omega C_{8l-7}) \to \pi_{8l+7}(\Omega C_{8l-3})$ is surjective by Table 3.2. $[P^{8l+6}, \Omega C_{8l-3}]$ can be computed from $\operatorname{Ext}_{\mathcal{A}_2}^{s,s+2^L}(P_{8l-3} \wedge P_{2^L-8l-7}^{2^L-2})$; this was done in an earlier version of this paper, but the Ext computations were quite complicated. It is much easier to use the method of 3.6 and the second resolution of §6 to obtain

$$\operatorname{Ext}_{\mathcal{A}}^{s,s}\left(H^*(\Omega C_{8l-3}), H^*(P^{8l+6})\right) = \begin{cases} \mathbf{Z}_2, & s < 5, s \neq 2, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2, & s = 2, \\ 0, & s > 5. \end{cases}$$

Thus $[P^{8l+6}, \Omega C_{8l-3}]$ contains the cyclic summand of order 2^6 generated by $P^{8l+6} \to V_{8l-3} \to \Omega C_{8l-3}$, which is in the image of i_*, plus $\mathbf{Z}_2$ generated by $[f_2]$ which corresponds to Ext class $\hat{h}_{14}$. Since $\operatorname{Ext}_{\mathcal{A}}^{s,s}(H^*(C_{8l-3}), H^*(P^{8l+6})) = 0$ for $s > 4$, there are no possible differentials on this class in the ASS converging to $[P^{8l+6}, \Omega C_{8l-3}]$. □

4. Proof of the immersions of Theorem 1.1. The fibrations $B_N^0 \to BSp$ and $\mathcal{E} \to BO[8]$ are not directly comparable. We compare them by the following maps of fibrations.

$$\begin{tikzcd}
\Omega(BO[8]/BO_N[8]) \arrow[d] & V_n \arrow[l] \arrow[r, "="] \arrow[d] & V_N \arrow[d] & V_N \arrow[l, "="'] \arrow[r] \arrow[d] & V_N \wedge bo \arrow[d] \\
\mathcal{E} \arrow[d] & BO_N[8] \arrow[l] \arrow[r] \arrow[d] & B\mathrm{Spin}_N \arrow[d] & BSp_N \arrow[l] \arrow[r] \arrow[d] & B_N^0 \arrow[d] \\
BO[8] & BO[8] \arrow[l, "="'] \arrow[r] & B\mathrm{Spin} & BSp \arrow[l] \arrow[r, "="] & BSp
\end{tikzcd} \tag{4.1}$$

Here we have $n = 8l + 7$, $N = 8l - 7$ and a commutative diagram

$$\begin{tikzcd}
BO[8] \arrow[r] & B\mathrm{Spin} & BSp \arrow[l] \\
\mathbf{R}P^{8l+7} \arrow[u, "g"] \arrow[r, "h"] & QP^{2l+1} \arrow[ur, "g'"'] &
\end{tikzcd} \tag{4.1'}$$

where g classifies $(2^L - 8l - 8)\xi$ and g' classifies $(2^{L-2} - 2l - 2)H$. Let

$$E_i' \leftarrow E_i''[8] \to E_i'' \leftarrow E_i \to E_i^0 \tag{4.1''}$$

denote the ith spaces in the n-MPT's for the 5 fibrations of (4.1).

The first step to proving Lemma 1.4(a) is to prove

Proposition 4.2. *If $\alpha(n) = 6$ (and $N = 8l - 7$), g' lifts to E_3 and g lifts to E_4'.*

Proof. The method is that outlined in §2. As in [5, §4] we compute

$$\nu\binom{2^{L-2} - 2l - 2}{i} = \begin{cases} 2, & i = 2l - 2, \\ 4 + \nu(l), & i = 2l - 1, \quad \text{if } \alpha(l) = 3. \\ 3, & i = 2l, \\ 4, & i = 2l + 1, \end{cases}$$

Thus by [5, 1.8] $g'|QP^{2l-1}$ lifts to B^0_{8l-10} and g' lifts to B^0_{8l-3}. Since

$$\pi_{4i-1}\Big(\text{fibre}\big(E_2(8l-11) \to E_2^0(8l-10)\big)\Big)$$
$$\approx \pi_{4i-1}\big(\text{fibre}(P_{8l-11}\langle 0,1\rangle \to P_{8l-10} \wedge bo\langle 0,1\rangle)\big) = 0$$

for $i < 2l - 1$, QP^{2l-1} lifts to $E_2(8l - 11)$. Since

$$\pi_{4i-1}\Big(\text{fibre}\big(E_2(8l-11) \to E_2^0(8l-3)\big)\Big) = 0$$

for $i = 2l$ and $2l + 1$, g' lifts to E_2. For this lifting the k-invariants in degrees $8l - 4$ and $8l$ map to zero, because they are the images of the corresponding k-invariants for B^0_{8l-7} and B^0_{8l-3}, respectively. Similarly, when this lifting g_2' is followed into $E_2(8l - 7)$ all k-invariants except possibly the one corresponding to the element of $\text{Ext}^2_{\mathcal{Q}}(P_{8l-7})$ labeled ${}_0h_2^2$ in [17, Tables 8.2 and 8.10] map to zero. If l is even, the image of this k-invariant k^2 in $H^*(E_2(8l - 11))$ can be computed to be

$$\begin{aligned} &\text{Sq}^1k_{8l-1} + \text{Sq}^1k'_{8l-1} + \text{Sq}^2k_{8l-2} + (\text{Sq}^4 + \text{Sq}^3\text{Sq}^1)k_{8l-4} \\ &\qquad + (\text{Sq}^8 + \text{Sq}^6\text{Sq}^2 + w_8 + w_6\text{Sq}^2 + w_7\text{Sq}^1 + w_4\text{Sq}^3\text{Sq}^1 + w_4^2)k_{8l-8}. \end{aligned}$$

Thus

$$\begin{aligned} (ig_2')^*(k^2) &= \big(\text{Sq}^8 + \text{Sq}^6\text{Sq}^2 + w_8 + w_6\text{Sq}^2 \\ &\qquad\qquad + w_7\text{Sq}^1 + w_4\text{Sq}^3\text{Sq}^1 + w_4^2\big)\, g_2'^*(k_{8l-8}) \\ &= \text{Sq}^8x_{8l-8} + w_8\big((2^{L-2} - 2l - 2)H\big)\cdot x_{8l-8} = 2x_{8l-8} = 0. \end{aligned}$$

If l is odd, $\text{Sq}^8x_{8l-8} = 0$ and $w_8 = 0$, so k^2 certainly maps to zero. Thus all k-invariants in $H^*(E_2(8l - 7))$ map to zero, so there is a lifting to E_3. Since $E_3''[8]$ is the pullback of E_3'', there is a lifting of g into $E_3''[8]$ and, hence, into E_3'. This lifting, l_3, sends k^3_{8l} nontrivially and k^3_{8l+2} and k^3_{8l+7} trivially. This is because they correspond to the k-invariants in E_3, where k^3_{8l} maps nontrivially (to QP^{2l+1}) since by [5, 1.8] QP^{2l} does not lift to B^0_{8l-7}, and k_{8l+2} and k_{8l+7} map trivially since $H^{8l+2}(QP^{2l+1}) = 0 = H^{8l+7}(QP^{2l+1})$.

The relations for the k^3-invariants in the MPT for $\mathcal{E} \to BO[8]$ are

$$k_{8l}^3\colon \mathrm{Sq}^1 k_{8l}^2 + \mathrm{Sq}^2\mathrm{Sq}^3 k_{8l-4}^2 = 0,$$
$$k_{8l+2}^3\colon \mathrm{Sq}^2 k_{8l+1}^2 + (\mathrm{Sq}^7 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)k_{8l+4}^2 = 0,$$
$$k_{8l+7}^3\colon \mathrm{Sq}^5\mathrm{Sq}^1 k_{8l+2}^2 + (\mathrm{Sq}^7 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)k_{8l+1}^2 + \mathrm{Sq}^7\mathrm{Sq}^1 k_{8l}^{2'} + (\mathrm{Sq}^{10}\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^3\mathrm{Sq}^1)k_{8l-4}^2 = 0.$$

Thus (by [8]) under the action

$$\mu\colon K_{8l-5} \times K_{8l-1} \times K'_{8l-1} \times K_{8l} \times K_{8l+1} \times E'_3 \to E'_3,$$
$$\mu^*(k_{8l}^3) = \mathrm{Sq}^1\iota_{8l-1}^2 + \mathrm{Sq}^2\mathrm{Sq}^3\iota_{8l-5}^2 + 1 \otimes k_{8l}^3,$$

etc. If $f\colon \mathbf{R}P^{8l+7} \to K_{8l-1}$ is nontrivial, then $l'_3 = \mu(f \times l_3)$ satisfies

$$l_3'^*(k_{8l}^3) = (f^* \otimes l_3^*)(\mathrm{Sq}^1\iota_{8l-1}^2 + \mathrm{Sq}^2\mathrm{Sq}^3\iota_{8l-5}^2 + 1 \otimes k_{8l}^3)$$
$$= \mathrm{Sq}^1 x_{8l-1} + l_3^* k_{8l}^3 = 0,$$

and, similarly, $l_3'^*(k_{8l+2}^3) = 0 = l_3'^*(k_{8l+7}^3)$. Thus l'_3 lifts to E'_4. $\square$

By Table 3.2, $\pi_{8l+7}(C_{8l-7})$ contains no nontrivial elements of filtration > 4. Thus, by Theorem 3.6, since g lifts to E'_4, $k_0 \circ g$ is null homotopic, proving Lemma 1.4(a).

We now begin to prove Lemma 1.4(b) and (c).

PROPOSITION 4.3. *Through degree $N + 16$, there is an isomorphism of $\mathcal{A}$-modules*

$$H^*(\mathcal{E}, BO_N[8]) \approx \tilde{H}^*(\Sigma^8 P_N) \otimes \mathcal{A}//\mathcal{A}_1,$$

where $\mathcal{A}_1$ is the subalgebra of $\mathcal{A}$ generated by Sq^1 *and* Sq^2.

PROOF. We use the relative Serre spectral sequence [20] for the fibration $(CF, F) \to (\mathcal{E}, BO_N[8]) \to \mathcal{E}$, where $F = \mathrm{fibre}(BO_N[8] \to \mathcal{E}) \approx \mathrm{fibre}(V_N \to \Omega C_N)$. By Theorem 3.1 in this range

$$\tilde{H}^*(\Sigma F) \approx \tilde{H}^*(P_N) \otimes \overline{\mathcal{A}//\mathcal{A}_2} \approx \tilde{H}^*(\Sigma^8 P_N) \otimes \mathcal{A}//\mathcal{A}_1.$$

The proposition follows, since $\tilde{H}^i(\mathcal{E}) = 0$ for $i < 8$. $\square$

Lemma 1.4(c) now follows immediately from a standard property of the stable range [19, Chapter 15]-If $A \subset X$, then $A \to \mathrm{fibre}(X \to X/A)$ is an $(a + b + 1)$-equivalence if A is a-connected and X/A is b-connected.

Because of the decomposition $\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$, it is convenient to consider the suspensions of the maps in (1.3). Thus if $f\colon \mathbf{R}P^n \to \Omega C_N$ is any map, $[\Sigma k_1 \mu(f \times l)]$ is the homotopy sum of three maps

(4.4) $$\Sigma P^n \xrightarrow{\Sigma l} \Sigma\mathcal{E} \xrightarrow{\Sigma k_1} \Sigma(\mathcal{E}/BO_N[8]),$$

(4.5) $$\Sigma P^n \xrightarrow{\Sigma f} \Sigma\Omega C_N \xrightarrow{\Sigma(k_1 i)} \Sigma(\mathcal{E}/BO_N[8]),$$

$$(4.6)\qquad \Sigma P^n \xrightarrow{\Sigma(f\wedge l)} \Sigma(\Omega C_N \wedge \mathfrak{S}) \xrightarrow{H(\mu)} \Sigma\mathfrak{S} \xrightarrow{\Sigma k_1} \Sigma(\mathfrak{S}/BO_N[8]),$$

where $H(\mu)$ denotes the Hopf construction on μ. We will consider two candidates for f: f_0 is the composite

$$P^n \xrightarrow{\text{coll}} P^n_N \hookrightarrow V_N \to \Omega\Sigma V_N \xrightarrow{\Omega i} \Omega C_N,$$

and f_1 is the composite

$$P^n \xrightarrow{\text{coll}} S^n \xrightarrow{u} \Omega C_N,$$

where u_* is nonzero in $\mathbf{Z}_2$-cohomology so that $[u]$ is represented by the element in $\operatorname{Ext}^{0,N+15}_{\mathcal{Q}}(C_N)$.

PROPOSITION 4.7. (4.5) *with $f = f_0$ is null homotopic.*

PROOF. This follows from the general fact: If $F \to E \xrightarrow{p} B$ is a fibration and $\mathfrak{S} = \text{fibre}(B \to B/E)$, then the composite $F \to \Omega\Sigma F \to \Omega(B/E) \to \mathfrak{S} \to \mathfrak{S}/E$ is null homotopic. To prove this let $F = p^{-1}(b_0)$ and view B/E as the reduced mapping cone of p. Then $\mathfrak{S}$ is the set of paths in B/E beginning at b_0 and ending in B. The map l: $E \to \mathfrak{S}$ can be chosen to send a point e to the path $\sigma_e(t) = [t, e]$. Then l restricted to F is homotopic to $F \to \Omega\Sigma F \to \Omega(B/E) \to \mathfrak{S}$, but $E \to \mathfrak{S} \to \mathfrak{S}/E$ is null homotopic. □

PROPOSITION 4.8. (4.6) *with $f = f_1$ is null homotopic.*

PROOF. Since $f_1|P^{n-1}$ is null homotopic and $l|P^7$ is null homotopic, $f_1 \wedge l$: $P^n \wedge P^n \to \Omega C_N \wedge \mathfrak{S}$ is null homotopic on the $(n + 7)$-skeleton. Preceding this by a skeletal version of the diagonal map shows $P^n \to \Omega C_n \wedge \mathfrak{S}$ to be null homotopic. □

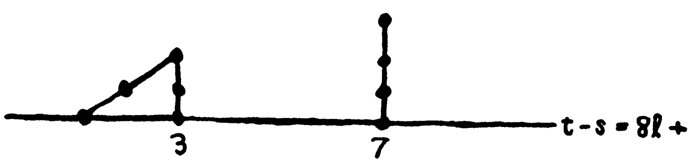
PROPOSITION 4.9.

$$[P^{8l+7}, \mathfrak{S}/BO_{8l-7}[8]] \approx [P^{8l+6}_{8l+1}, \mathfrak{S}/BO_{8l-7}[8]] \oplus \pi_{8l+7}(\mathfrak{S}/BO_{8l-7}[8])$$
$$\approx \mathbf{Z}_8 \oplus \mathbf{Z}_{16}.$$

PROOF. The splitting follows immediately from the observations

$$[P^{8l+7}, \mathfrak{S}/BO_{8l-7}[8]] \approx [P^{8l+7}_{8l+1}, \mathfrak{S}/BO_{8l-7}[8]] \quad \text{and}$$
$$P^{8l+7}_{8l+1} \simeq P^{8l+6}_{8l+1} \vee S^{8l+7}.$$

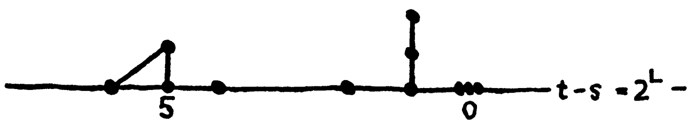
The Adams spectral sequence converging to $\pi_*(\mathfrak{S}/BO_{8l-7}[8])$ has E_2-term $\operatorname{Ext}_{\mathcal{Q}_1}(\Sigma^8 P_{8l-7})$ by Proposition 4.3 and the change-of-rings theorem. This was computed in [10] to begin



and so the 2-component of $\pi_{8l+7}(\mathfrak{S}/BO_{8l-7}[8]) \approx \mathbf{Z}_{16}$. For odd primes p, $\tilde{H}^*(\mathfrak{S}/BO_{8l-7}[8]: \mathbf{Z}_p) = 0$ and so the odd component of $\pi_{8l+7}(\mathfrak{S}/BO_{8l-7}[8])$ is zero.

$$[P_{8l+1}^{8l+6}, \mathfrak{S}/BO_{8l-7}[8]] \approx \pi_{2^L-1}\left(D(P_{8l+1}^{8l+6}) \wedge \mathfrak{S}/BO_{8l-7}[8]\right),$$

where $D(P^{8l+6})$ is a Spanier-Whitehead (2^L-1)-dual of P^{8l+6} [22], which has the homotopy type of $P_{2^L-8l-7}^{2^L-8l-2}$. The Adams spectral sequence converging to this has E_2-term $\mathrm{Ext}_{\mathcal{Q}_1}^{s,t}(P_{2^L-8l-7} \wedge \Sigma^8 P_{8l-7})$, which is easily computed by the methods of [4, §3] to be



The d_2-differential on the classes in $s = 0$, $t-s = 2^L$ is zero because the classes are present in $\mathrm{Ext}_{\mathcal{Q}_1}(P_{2^L-8l-7} \wedge \Sigma^8 P_{8l-9})$, where d_2 is zero by h_0-naturality. Thus the 2-primary component of $[P^{8l+6}, \mathfrak{S}/BO_{8l-7}[8]]$ is $\mathbf{Z}_8$, and the odd component is zero because $\tilde{H}^*(P_{2^L-8l-7} \wedge \mathfrak{S}/BO_{8l-7}[8]; \mathbf{Z}_p) = 0$ for p odd. $\square$

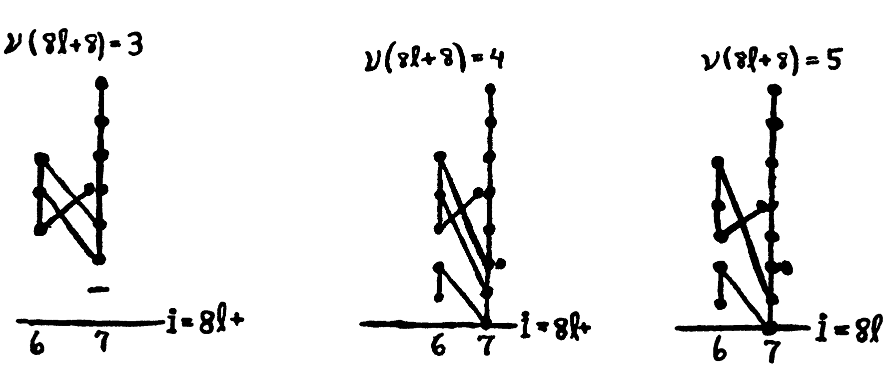
We shall denote by G_1 and G_2 the generators of the $\mathbf{Z}_8$ and $\mathbf{Z}_{16}$, respectively.

PROPOSITION 4.10. (4.5) *for* $f = f_1$ *is* (*up to odd multiples*) $2^{\nu(l+1)}G_2$.

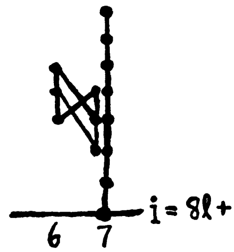
PROOF. We must determine $(k_1 i)_*(u)$ in the exact sequence

$$\pi_{8l+7}(P_{8l-7}) \to \pi_{8l+7}(\Omega C_{8l-7}) \xrightarrow{(k_1 i)_*} \pi_{8l+7}(\mathfrak{S}/BO_{8l-7}[8]) \to \pi_{8l+6}(P_{8l-7})$$

by considering the induced map of Adams spectral sequences. For $\pi_i(P_{8l-7})$ we have from [17, Tables 8.2 and 8.10, Theorem 7.4]



Note that for $\alpha(8l+7)=6$, $l \neq 7$, we must have $\nu(8l+8) < 5$. $\pi_i(\Omega C_{8l-7})$ is, by Table 3.2,



Thus the classes of $\pi_{8l+7}(\Omega C_{8l-7})$ of filtration 0 and 1 map nontrivially, while for $\nu(8l+8) < 4$ a class of filtration 2 also maps nontrivially, and for $\nu(8l+8) = 3$ so also does the class of filtration 3. That is, the image of $(k_1 i)_*$ in $\mathbf{Z}_{16}$ is a subgroup of order $2^{4-\nu(l+1)}$, which must be generated by $2^{\nu(l+1)}G_2$. $\square$

$[P^{8l+7}, \Omega C_{8l-7} \wedge \mathcal{E}] \overset{(1\wedge p)^*}{\approx} [P^{8l+7}, \Omega C_{8l-7} \wedge BO[8]]$, and $[P^{8l+7}, BO[8]]$ is a cyclic 2-group such that the map "$8i\xi$" classifying the vector bundle $8i\xi$ is i times the generator.

Proposition 4.11. *Suppose q: $P^{8l+7} \to \Omega C_{8l-7} \wedge \mathcal{E}$ is such that $(1 \wedge p)q = f_0 \wedge$"$8i\xi$" for some odd integer i. Then $\Sigma k_1 \cdot H(\mu) \cdot \Sigma q$ is nontrivial in H^{8l+2} and hence is $a_1 G_1 + b_2 G_2$, where a_1 is an odd integer and b_2 is some integer.*

Proof. There is a commutative diagram of exact sequences in the stable range defined using [8, 2.9].

$$\begin{array}{ccccccc}
\to H^*(\mathcal{E}, BO_N[8]) & \xrightarrow{\delta} & H^{*+1}(BO[8], \mathcal{E}) & \to & H^{*+1}(BO[8], BO_N[8]) & \to \\
\mu^* k_1^* \downarrow & & \approx \downarrow \mu'^* & & \downarrow & \\
\to H^*(\Omega C_N \times \mathcal{E}) & \to & H^{*+1}(\mathcal{E}, \Omega C_N \times \mathcal{E}) & \to & H^{*+1}(\mathcal{E}) & \longrightarrow
\end{array}$$

where as usual $N = 8l - 7$. $\tilde{H}^*(\Omega C_N)$ has classes $(\Omega j)^* w_{N+i}$ for $i > 0$, where j: $BO[8]/BO_N[8] \to BO/BO_N$, and these classes are also present in $H^*(BO[8], \mathcal{E})$. If z is the first nonzero class in $H^*(\mathcal{E}, BO_N[8])$, then

$$\delta z = (\mathrm{Sq}^8 + w_8)w_{N+1} + [(N+1)/8]\mathrm{Sq}^4 w_{N+5},$$

and hence

$$\begin{aligned}\mu^* k_1^* z = {}& \mathrm{Sq}^8 w_{N+1} \otimes 1 + w_{N+1} \otimes w_8 \\ & + [(N+1)/8]\mathrm{Sq}^4 w_{N+5} \otimes 1 + 1 \otimes \text{something}.\end{aligned}$$

Thus the $\wedge$-component of $\mu^* k_1^* z$ is $w_{N+1} \otimes w_8$, which is the first nonzero class in $H^*(\Omega C_N \wedge \mathcal{E})$.

$$q^*(w_{N+1}\otimes w_8)=f_0^*(w_{N+1})\cdot\text{“}8i\xi\text{”}^*(w_8)=x^N\cdot w_8(8i\xi)\neq 0$$

since i is odd. $\square$

COROLLARY 4.12. *If $l\colon P^{8l+7}\to \mathfrak{S}$ is a lifting of $8i\xi$, then* (4.6) *with $f=f_0$ is $ia_1G_1+ib_2G_2$, where a_1 is an odd integer and b_2 is an integer.*

PROOF. Under the homomorphisms

$$\begin{aligned}[][P^n,\Omega C_N]\otimes[P^n,BO[8]]&\to[\Sigma P^n,\Sigma\Omega C_N\wedge BO[8]]\\ &\overset{(1\wedge p^*)^{-1}}{\to}[\Sigma P^n,\Sigma\Omega C_N\wedge\mathfrak{S}]\\ &\overset{\Sigma k_1\circ H(\mu)}{\to}[\Sigma P^n,\Sigma(\mathfrak{S}/BO_N[8])],\end{aligned}$$

$[\Sigma k_1\circ H(\mu)\circ\Sigma(f_0\wedge l)]$ is the image of $f_0\wedge$“$8i\xi$”, which is i times the image of $f_0\wedge$“8ξ”.

In order to show that this pairing is a homomorphism we note that since $n<N+15$,

$$[P^n,\Omega C_N\wedge BO[8]]\approx\left[P^n,\Omega C_N\wedge\left(BO[8]^{(15)}\right)\right],$$

where $BO[8]^{(15)}$ denotes the 15-skeleton. Hence $[P^n, BO[8]]$ may be replaced by $[P^n, BO[8]^{(15)}]$. But $BO[8]^{(15)}$ is a stable space and so the Whitney sum and stable ($=\Omega^\infty S^\infty$) additions agree. It is well known that if stable addition is used in all spaces, the smash product gives a homomorphism. $\square$

In order to show that f can be chosen so that the sum of the maps (4.4)–(4.6) is zero, we must get some hold on what (4.4) can be. To do this, we use the fact [18] that there is a lifting of P^n to $BO_{n-10}[8]$. There is a map of (1.3) into the corresponding diagram with $n-14$ replaced by $n-10$.

$$\begin{tikzcd}
\Omega C_{n-14}\times\mathfrak{S} \arrow[r, "\mu"] \arrow[d, "j_1\times j_2"] & \mathfrak{S} \arrow[r, "k_1"] \arrow[d, "j_2"] & \mathfrak{S}/BO_{n-14}[8] \arrow[d, "j_3"] \\
\Omega C_{n-10}\times\mathfrak{S}' \arrow[r, "\mu'"] & \mathfrak{S}' \arrow[r, "k_1'"] & \mathfrak{S}'/BO_{n-10}[8]
\end{tikzcd}$$

$[P^n,\mathfrak{S}/BO_{n-14}[8]]\xrightarrow{j_{3*}}[P^n,\mathfrak{S}'/BO_{n-10}[8]]$ is a surjection $\mathbf{Z}_8\oplus\mathbf{Z}_{16}\to\mathbf{Z}_4\oplus\mathbf{Z}_8$ sending $G_1\mapsto G_1'$ and $G_2\mapsto G_2'$. By Proposition 3.8 the cokernel of $[P^n,\Omega C_{n-14}]\xrightarrow{j_{1*}}[P^n,\Omega C_{n-10}]$ is generated by an element f_2 of Adams filtration 2 whose restriction to P^{8l+5} is trivial. The analogue (with $N=n-10$ and $\mathfrak{S}$ replaced by $\mathfrak{S}'$) of (4.6) with $f=f_2$ is null homotopic by the argument of Proposition 4.8, and the analogue of (4.5) with $f=f_2$ has filtration >2 so that it is a multiple of $4G_2'$.

Let $l\colon P^n\to\mathfrak{S}$ be some lifting of g. Since P^n lifts to $BO_{n-10}[8]$, there exists some $f'\colon P^n \to \Omega C_{n-10}$ such that $k_1' \mu'(f' \times j_2 l) = 0$. Either f' or $f' - f_2$ factors as $P^n \xrightarrow{f} \Omega C_{n-14} \xrightarrow{j_1} \Omega C_{n-10}$. Thus $k_1' \mu'(j_1 \times j_2)(f \times l)$ is trivial or $4G_2'$, and hence $k_1 \mu(f \times l)$ is $4aG_1 + 4bG_2$ for some $a \in \mathbf{Z}_2$, $b \in \mathbf{Z}_4$. If $a = 0$, then

$$k_1 \mu\big((f - b \cdot 2^{2-\nu(l+1)} f_1) \times l\big) = 0$$

by Propositions 4.8 and 4.10. If $a = 1$, then by Propositions 4.7 and 4.12,

$$k_1 \mu\big((f + 2^{2-\nu(l+1)} f_0) \times l\big) = 4cG_2$$

for some $c \in \mathbf{Z}_4$ and

$$k_1 \mu\big((f + 2^{2-\nu(l+1)} f_0 - c \cdot 2^{2-\nu(l+1)} f_1) \times l\big) = 0,$$

proving Lemma 1.4(b).

5. Proof of the nonimmersion of Theorem 1.1. It suffices to prove $\alpha(l) = 3$ implies

$$P^{8l} \xrightarrow{\text{“}(2^L - 8l - 8)\xi\text{”}} BO[8] \to C_{8l-11}$$

is essential. We shall again use diagrams (4.1), (4.1′), and (4.1″), with $N = 8l - 11$. The coefficients

$$\nu\binom{2^{L-2} - 2l - 2}{i}$$

are as in Proposition 4.2, and by the method of Proposition 4.2, $g' | QP^{2l}$ lifts to E_2 and $g | RP^{8l}$ lifts to E_2'.

PROPOSITION 5.1. *There is a lifting of* $\mathbf{R}P^{8l}$ *to* E_2' *sending only* k_{8l-8} *nontrivially.*

PROOF. We first show that there is a lifting of QP^{2l} to E_2^0 sending only k_{8l+8} nontrivially. The underlying reason is that

$$\nu\binom{2^{L-2} - 2l - 2}{2l - 2} = 2 \quad \text{while } \nu\binom{2^{L-2} - 2l - 2}{i} > 2$$

for $i = 2l - 1, 2l$; this becomes a proof by using naturality of k-invariants and the fact that [5, 1.8] implies QP^{2l-2} does not lift to B^0_{8l-11}, QP^{2l-1} lifts to B^0_{8l-10}, and QP^{2l} lifts to B^0_{8l-3}.

Since $\operatorname{Ext}_{\mathcal{Q}}^{2,4i+1}(P_{8l-11}) \to \operatorname{Ext}_{\mathcal{Q}}^{2,4i+1}(P_{8l-11} \wedge bo)$ is an isomorphism for $i < 2l$, there is a lifting of QP^{2l} to E_2 sending only k_{8l-8} nontrivially, and hence there is a lifting l_2 of RP^{8l} to $E_2''[8]$ sending only k_{8l-8} nontrivially. By computing the induced map of minimal $\mathcal{Q}(BO[8])$-resolutions, we see that $RP^{8l} \xrightarrow{l_2} E_2''[8] \xrightarrow{i} E_2'$ also sends only k_{8l-8} nontrivially. For example, there is a k-invariant $k_{8l}' \in H^{8l}(E_2')$ due to the element in Table 3.2, $N \equiv 5$ (8), $s = 2$, $t - s = N + 11$, which is annihilated by h_0.

$$i^*k'_{8l} = \mathrm{Sq}^1 k_{8l-1} + \mathrm{Sq}^2 k_{8l-2} + \mathrm{Sq}^2\mathrm{Sq}^1 k_{8l-3} + (\mathrm{Sq}^8 + \mathrm{Sq}^6\mathrm{Sq}^2 + w_8)k_{8l-8}.$$

Then $l_2^* i^* k'_{8l} = (\mathrm{Sq}^8 + \mathrm{Sq}^6\mathrm{Sq}^2 + w_8)x^{8l-8} = 0$, since $\mathrm{Sq}^8 x^{8l-8} \neq 0$ if and only if $w_8((2^L - 8l - 8)\xi) \neq 0$. □

There is a map from this MPT into an Adams resolution of C_{8l-11}, and so 5.1 and 3.7 imply that our map $P^{8l} - C_{8l-11}$ is essential. □

6. Minimal resolutions. In this section we tabulate the minimal $\mathcal{Q}_2$-resolutions of P_1, P_5, and P_7 which were used in §3. See [19, Chapter 18] or [2]. We form an exact sequence

$$0 \longleftarrow H^*P_N \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \longleftarrow \cdots$$

of $\mathcal{Q}_2$-modules, where C_i is free, such that $\ker(d_i) \subset \bar{\mathcal{Q}}_2 \cdot C_i$. Then $\mathrm{Ext}^{s,t}_{\mathcal{Q}_2}(P_N) \approx \mathrm{Hom}_{\mathcal{Q}_2}(\Sigma^{-t}C_s, \mathbf{Z}_2)$.

We shall denote generators of C_s by g, h, k, l, m, n, p for $s = 1, 2, 3, 4, 5, 6, 7$, respectively, with a subscript to denote $t - s$, and primes to denote a second generator for the same s and t. We omit Sq for Steenrod squares; thus $62g$ denotes $\mathrm{Sq}^6\mathrm{Sq}^2 g$. $\tilde{H}^*(P_{N+8}) \approx \tilde{H}^*(\Sigma^8 P_N)$ as $\mathcal{Q}_2$-modules so all $\mathrm{Ext}_{\mathcal{Q}_2}(P_N)$ for $N \equiv 1, 5, 7$ (8) follows from these tables.

Resolution of H^*P_1 for $t - s < 15$.

generators $\alpha_1, \alpha_3, \alpha_7, \alpha_{15}$,
g_2: $2\alpha_1$ (this means $d_0(g_2) = \mathrm{Sq}^2\alpha_1$),
g_3: $1\alpha_3 + 21\alpha_1$,
g_4: $4\alpha_1$,
g_6: $4\alpha_3$,
g_7: $1\alpha_7 + 41\alpha_3$,
g_8: $2\alpha_7 + 42\alpha_3$,
g_{15}: $1\alpha_{15} + 27\alpha_7$,
h_3: $1g_3 + 2g_2$,
h_7: $1g_7 + 23g_3$,
h'_7: $(4 + 31)g_4 + 6g_2$,
h_8: $21g_6 + 6g_3 + (7 + 421)g_2 + 23g_4$,
h_9: $2g_8 + 3g_7 + 4g_6$,
h_{15}: $1g_{15} + 27g_7$,
h'_{15}: $(46 + 73)g_6 + 651g_4 + (67 + 463)g_3$,
k_7: $1h_7 + 23h_3$,
k_9: $2h_8 + (7 + 421)h_3$,
k_{14}: $51h_9 + (7 + 421)h_8 + 71h'_7 + (\widehat{10}, 2 + 831)h_3$,
k_{15}: $1h_{15} + 27h_7$,
k'_{15}: $1h'_{15} + 52h_9 + (27 + 72)h_7 + 27h'_7 + (\widehat{13} + \widehat{12}, 1 + 94 + 841)h_3$,

l_9: $21k_7$,
l_{11}: $3k_9 + 41k_7$,
l_{14}: $1k_{14} + 42k_9 + 71k_7$,
l_{15}: $1k_{15} + 27k_7$,
l'_{15}: $1k'_{15} + 2k_{14} + (7 + 421)k_9 + 27k_7$,
m_{10}: $2l_9$,
m_{11}: $1l_{11} + 21l_9$,
m_{14}: $1l_{14} + 4l_{11} + 6l_9$,
m_{15}: $1l_{15} + 23l_{11}$,
n_{11}: $1m_{11} + 2m_{10}$,
n_{15}: $1m_{15} + 23m_{11}$,
p_{15}: $1n_{15} + 23n_{11}$.
Resolution of H^*P_5 for $t - s < 15$.
generators $\alpha_5, \alpha_7, \alpha_{15}$,
g_6: $2\alpha_5$,
g_7: $1\alpha_7 + 21\alpha_5$,
g_8: $2\alpha_7 + 4\alpha_5$,
g_{11}: $41\alpha_7$,
g_{15}: $1\alpha_{15} + 4241\alpha_5$,
h_7: $1g_7 + 2g_6$,
h_{11}: $1g_{11} + 212g_7$,
h_{12}: $2g_{11} + 6g_7 + 23g_8 + (7 + 421)g_6$,
h_{14}: $4g_{11} + 62g_7$,
h_{15}: $1g_{15} + 521g_8 + (27 + 72 + 621)g_7$,
h'_{15}: $(62 + 521)g_8 + (27 + 72 + 63 + 621)g_7 + (\widehat{10} + 82 + 91)g_6$,
k_{11}: $1h_{11} + 23h_7$,
k_{13}: $2h_{12} + 3h_{11} + (7 + 421)h_7$,
k_{14}: $1h_{14} + 21h_{12} + 4h_{11} + 62h_7$,
k_{15}: $1h_{15} + 41h_{11}$,
l_{13}: $21k_{11}$,
l_{14}: $1k_{14} + 2k_{13} + 4k_{11}$,
l_{15}: $1k_{15} + 23k_{11}$,
m_{14}: $2l_{13}$,
m_{15}: $1l_{15} + 21l_{13}$,
n_{15}: $1m_{15} + 2m_{14}$.
Resolution of H^*P_7 for $t - s < 19$.
generators α_7, α_{15},
g_9: $21\alpha_7$,
g_{11}: $41\alpha_7$,
g_{12}: $42\alpha_7$,
g_{15}: $1\alpha_{15} + 45\alpha_7$,

g_{16}: $2\alpha_{15} + 424\alpha_7$,
h_{10}: $2g_9$,
h_{11}: $1g_{11} + 21g_9$,
h_{14}: $4g_{11} + 6g_9$,
h_{15}: $1g_{15} + 7g_9 + 421g_9$,
h'_{15}: $(4 + 31)g_{12} + 61g_9$,
h_{17}: $2g_{16} + 3g_{15} + 42g_{12} + 52g_{11}$,
k_{11}: $1h_{11} + 2h_{10}$,
k_{15}: $1h_{15} + 41h_{11}$,
k_{16}: $21h_{14} + 6h_{11} + (7 + 421)h_{10}$,
k_{19}: $3h_{17} + 5h_{15} + 5h'_{15} + 6h_{14} + (631 + 82 + \widehat{10} + 91)h_{10}$,
l_{15}: $1k_{15} + 23k_{11}$,
l_{17}: $2k_{16} + (7 + 421)k_{11}$,
m_{17}: $21l_{15}$,
m_{19}: $3l_{17} + 41l_{15}$,
n_{18}: $2m_{17}$,
n_{19}: $1m_{19} + 21m_{17}$,
p_{19}: $1n_{19} + 2n_{18}$.

REFERENCES

1. J. F. Adams, *Stable homotopy theory*, Lecture Notes in Math., vol. 3, Springer-Verlag, Berlin and New York, 1969.

2. ______, *The nonexistence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104.

3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956.

4. D. M. Davis, *Generalized homology and the generalized vector field problem*, Quart. J. Math. Oxford Ser. **25** (1974), 169–193 .

5. D. M. Davis and M. Mahowald, *The geometric dimension of some vector bundles over projective spaces*, Trans. Amer. Math. Soc. **205** (1975), 295–316.

6. ______, *A strong nonimmersion theorem for* RP^{8l+7}, Bull. Amer. Math. Soc. **81** (1975), 155–156.

7. S. Gitler, *Immersion and embedding of manifolds*, Proc. Sympos. Pure. Math., vol. 22, Amer. Math. Soc., Providence, R.I., 1971, pp. 87–96.

8. S. Gitler and M. Mahowald, *The geometric dimension of real stable vector bundles*, Bol. Soc. Mat. Mexicana **11** (1966), 85–107.

9. ______, *The immersion of manifolds*, Bull. Amer. Math. Soc. **73** (1967), 696–700.

10. S. Gitler, M. Mahowald and R. J. Milgram, *The nonimmersion problem for* RP^n *and higher order cohomology operations*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 432–437.

11. A. Iwai and N. Shimada, *On the cohomology of some Hopf algebras*, Nagoya Math. J. **30** (1967), 103–111.

12. I. M. James, *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. **9** (1959), 115–140.

13. ______, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc. **69** (1963), 231–238.

14. ______, *Euclidean models of projective spaces*, Bull. London Math. Soc. **3** (1971), 257–276.

15. D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc. **107** (1963), 432–450.

16. R. J. Milgram, *The Steenrod algebra and its dual for connective K-theory*, Proc. Homotopy Theory Conf., Mono. Mexican Math. Soc. **1** (1975), 127–159.

17. M. Mahowald, *The metastable homotopy of* S^n, Mem. Amer. Math. Soc. No. 72 (1967).

18. R. J. Milgram, *Immersing projective space*, Ann. of Math. (2) **85** (1967), 473–482.

19. R. Mosher and M. Tangora, *Cohomology operations and applications to homotopy theory*, Harper and Row, New York, 1968.

20. F. Nussbaum, *Semi-principal bundles and stable non-orientable obstruction theory*, Loyola Univ., Chicago, Ill. (mimeograph), 1973.

21. B. Sanderson, *Immersions and embeddings of projective spaces*, Proc. London Math. Soc. **53** (1964), 137–153.

22. E. H. Spanier, *Function spaces and duality*, Ann. of Math. (2) **70** (1959), 338–378.

23. R. Stong, *Determination of* $H^*(BO(k, \ldots, \infty); Z_2)$ *and* $H^*(BU(k, \ldots, \infty); Z_2)$, Trans. Amer. Math. Soc. **107** (1963), 526–544.

24. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N.J., 1962.

25. D. M. Davis and M. Mahowald, *Obstruction theory and ko-theory*, (to appear).

Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015

Department of Mathematics, Northwestern University, Evanston, Illinois 60201