FREE STATES OF THE GAUGE INVARIANT CANONICAL ANTICOMMUTATION RELATIONS

BY

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ABSTRACT. The gauge invariant subalgebra of the canonical anticommutation relations (henceforth GICAR) is viewed as an inductive limit of finite-dimensional C^*-algebras, and a study is made of a simple class of its representations. In particular, representations induced by restricting the well-known gauge invariant generalized free states from the entire canonical anticommutation relations (henceforth CAR) are considered. Denoting (a) a state of the CAR by ω and its restriction to the GICAR by ω°, (b) the unique gauge invariant generalized free state of the CAR such that $\omega(a(f)^* a(g))$ $= (f, Ag)$ by ω_A, it is shown that (1) ω_A° induces (an impure) factor representation of the GICAR if and only if $\operatorname{Tr} A(I - A) = \infty$, (2) two (impure) GICAR factor representations ω_A° and ω_B° are quasi-equivalent if and only if $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are Hilbert-Schmidt class operators.

1. **Introduction.** In this work we study the gauge invariant[1] subalgebra of the canonical anticommutation relations via a simple class of its representations. Following [13] and [4] we view the canonical anticommutation relations (henceforth CAR) and its $U(1)$ or gauge invariant subalgebra (henceforth GICAR) as C^*-algebras which are in fact limits of ascending sequences of finite-dimensional matrix algebras; in the terminology of [6] and [4] they are examples of uniformly hyperfinite (UHF) and approximately finite (AF) algebras respectively. We then proceed to examine the representations induced by restricting the gauge invariant generalized free states of the CAR (see §2) to the GICAR. Previously, many authors have studied UHF algebras in general (e.g. [6], [11]) and the generalized free states of the CAR in particular, as in [1], [2], [5], [12], and [13]. AF algebras (including the GICAR) are introduced and studied in [4].[2]

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[1] In field theory, automorphisms induced by the action of $U(1)$ on the field algebra are called gauge transformations of the first kind.

[2] In [4] the GICAR is referred to as the fermion current algebra, following [9].

The present work is motivated by [7]–[9] where C^*-algebras of observables are studied in general, and that of [1], [2], [5], and [12] in which techniques of field theory are used to analyze the generalized free states of the CAR. Throughout, the generalizations of the work of [6] and [11] on UHF algebras to the AF case accomplished in [4] are taken as the starting point. The main result of §3, a necessary and sufficient condition for a gauge invariant generalized free state of the CAR to induce a factor representation of the GICAR by restriction, naturally extends the results of [14]. The main result of §4, a necessary and sufficient condition for two factor states of the GICAR to induce quasi-equivalent representations, is identical in conclusion to Theorem 5.1 of [12], whose techniques and results are relied upon heavily.

I would like to thank my thesis advisor, Robert T. Powers. Without his many helpful suggestions this work could not have been done. I also thank Herbert Wilf for his statement and proof of Proposition 3.19.

2. Definitions, notation. Let $\mathcal{K}$ be a separable Hilbert space with orthonormal basis $\{f_n\}$, $n = 1, 2, \ldots$, and let $\mathfrak{M}_n = \operatorname{span}\{f_1, \ldots, f_n\}$. We denote by $\mathcal{C}(\mathfrak{M}_n)$ the CAR algebra over $\mathfrak{M}_n$. As sketched below, this is constructed via a linear mapping $f \to a(f)$ satisfying the relations

$$a(f)a(g) + a(g)a(f) = 0,$$

$$a(f)a(g)^* + a(g)^*a(f) = (g, f)I,$$

for all f, g in $\mathfrak{M}_n$. We denote by $\mathcal{C}(\mathcal{K})$ the CAR algebra over $\mathcal{K}$, the completion of $\bigcup_n \mathcal{C}(\mathfrak{M}_n)$, written $\mathcal{C}(\mathcal{K}) = \overline{\bigcup_n \mathcal{C}(\mathfrak{M}_n)}$. This is a UHF algebra in the sense of [6].

Setting

$$\mathfrak{B}_1 = \mathcal{C}(\mathfrak{M}_1), \quad \mathfrak{B}_k = \mathcal{C}(\mathfrak{M}_k) \cap \mathcal{C}(\mathfrak{M}_{k-1})^c, \quad k = 2, 3, \ldots,$$

we may choose matrix units for these algebras as follows:

$$e_{11}^{(k)} = a(f_k)a(f_k)^*, \qquad e_{12}^{(k)} = a(f_k)V_k,$$

$$e_{21}^{(k)} = a(f_k)^*V_k, \qquad e_{22}^{(k)} = a(f_k)^*a(f_k),$$

where $V_1 = I$, $V_k = \prod_{j=1}^{k-1}(I - 2a(f_j)^*a(f_j))$, $k = 2, 3, \ldots$. It follows straightforwardly that (1) each $\mathfrak{B}_k$ is isomorphic to a 2×2 matrix algebra, (2) the $\{\mathfrak{B}_k\}$, $k = 1, 2, \ldots$, commute pairwise for differing k and (3) $\mathcal{C}(\mathfrak{M}_n)$ is isomorphic to $\otimes_{k=1}^n \mathfrak{B}_k$, written $\mathcal{C}(\mathfrak{M}_n) \approx \otimes_{k=1}^n \mathfrak{B}_k$. From (3) we conclude $\mathcal{C}(\mathfrak{M}_n)$ isomorphic to a $2^n \times 2^n$ matrix algebra with matrix units

$$e_{i_1j_1}^{(1)} \cdot e_{i_2j_2}^{(2)} \cdot \cdots \cdot e_{i_nj_n}^{(n)}, \quad 1 \leqslant i_k, j_k \leqslant n, k = 1, \ldots, n.$$

We denote by $\mathscr{Q}^\circ(\mathscr{K})$ the $U(1)$ or gauge invariant subalgebra of $\mathscr{Q}(\mathscr{K})$. More explicitly, the mapping $a(f) \to e^{-it}a(f)$ is extended to an automorphism of $\mathscr{Q}(\mathscr{K})$, which we denote χ_t, and we define $\mathscr{Q}^\circ(\mathfrak{M}_n) = \{x \text{ in } \mathscr{Q}(\mathfrak{M}_n): \chi_t(x) = x, 0 \leqslant t < 2\pi\}$. It follows that $\mathscr{Q}^\circ(\mathfrak{M}_n) \approx \oplus_{k=0}^n M_{(nk)}$ with $M_{(nk)}$ an $\binom{n}{k} \times \binom{n}{k}$ matrix algebra and

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

(see [4]). Finally, we define $\mathscr{Q}^\circ(\mathscr{K}) = \overline{\cup_n \mathscr{Q}^\circ(\mathfrak{M}_n)}$, an AF algebra in the sense of [4].

In the sequel we will investigate a simple class of representations of $\mathscr{Q}^\circ(\mathscr{K})$ obtained by restricting the gauge invariant generalized free states of $\mathscr{Q}(\mathscr{K})$ to $\mathscr{Q}^\circ(\mathscr{K}) \subset \mathscr{Q}(\mathscr{K})$. In general, a state of $\mathscr{Q}(\mathscr{K})$ is determined by its values on monomials of the form $a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)$ since (1) a state is linear and continuous, (2) polynomials in the $a(f)^*$, $a(g)$ with f, g in $\mathscr{K}$ are norm dense in $\mathscr{Q}(\mathscr{K})$, (3) polynomials may be written as a combination of monomials in the indicated form via the anticommutation relations. In particular, a state ω of $\mathscr{Q}(\mathscr{K})$ is called a generalized free state if it vanishes on monomials odd in the $a(f)$, $a(g)^*$ and its truncated N-point functions (see, e.g., [10]) vanish for $N > 2$; it follows from the latter property that ω is completely determined by its 2-point functions $\omega(a(f)^* a(g))$ and $\omega(a(f)a(g))$. Here we consider gauge invariant generalized free states; these are determined by the single 2-point function $\omega(a(f)^* a(g))$ which by linearity and boundedness properties can be written as (f, Ag) for some linear operator A, $0 \leqslant A \leqslant I$. More precisely we define the gauge invariant generalized free state ω_A of $\mathscr{Q}(\mathscr{K})$ by the formula

$$\omega_A(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det((f_i, Ag_j)).$$

We denote by ω_A° the restriction of ω_A to $\mathscr{Q}^\circ(\mathscr{K})$.

A state ω of $\mathscr{Q}(\mathscr{K})$ is said to factorize with respect to the $\{\mathscr{B}_k\}$, $k = 1, 2, \ldots$, or $\{\mathfrak{M}_k\}$, $k = 1, 2, \ldots$, defined above, if $\omega(xy) = \omega(x)\omega(y)$ whenever x is in $\mathscr{B}_l$ and y is in $\mathscr{B}_m$, $m \neq l$. Two states ω_1 and ω_2 are said to be unitarily equivalent, denoted $\omega_1 \underset{u}{\sim} \omega_2$, if they induce unitarily equivalent representations. Two states are said to be quasi-equivalent, denoted $\omega_1 \underset{q}{\sim} \omega_2$, if they induce quasi-equivalent representations; letting Π_1 (Π_2) be the representation induced by ω_1 (ω_2) this means there is an isomorphism h between the von Neumann algebras Π_1'' and Π_2'' such that $h(\Pi_1(x)) = \Pi_2(x)$ for all x in $\mathscr{Q}(\mathscr{K})$.

For convenience, we summarize some results on gauge invariant generalized free states, giving an appropriate reference for each.

STATEMENT	EQUIVALENT CONDITION	REFERENCE
CAR		
ω_A is a pure state	A is a projection	[13]
$\omega_E \sim_u \omega_F$ (E, F are projections)	$E\text{-}F$ is Hilbert-Schmidt class	[12]
ω_A is a factor state	automatic	[13]
$\omega_A \sim_q \omega_B$	$A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are Hilbert-Schmidt class	[12]
GICAR		
ω_A° is a pure state	A is a projection	[14]
$\omega_E^\circ \sim_u \omega_F^\circ$ (E, F are projections)	$\mathrm{Tr}((I - E)F(I - E))$ $= \mathrm{Tr}((I - F)E(I - F)) < \infty$	[14]
ω_A° is a factor state	$\mathrm{Tr}\, A(I - A) = \infty$	§3
$\omega_A^\circ \sim_q \omega_B^\circ$	$A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are Hilbert-Schmidt class	§4

We note that nongauge invariant generalized free states are studied in [1], [3], and [5]. For further elaboration on notation, definitions, or results we ask the reader to consult the references, especially [10].

3. Factor condition. In this section, we wish to prove

THEOREM. *Let A be an operator on a Hilbert space $\mathcal{K}$, with $0 < A < I$. Suppose A is not a projection. Let ω_A° be the associated generalized free state of $\mathcal{C}^\circ(\mathcal{K})$. Then ω_A° is a factor state if and only if $\mathrm{Tr}\, A(I - A) = \infty$.*

We note that when A is a projection, it can be shown that ω_A° is a pure state, as in [14, III, 1, Proposition]. Our method of attack will be to eventually apply the following theorem.

THEOREM 3.1 [4, THEOREM 4.4]. *Let $\mathcal{C} = \overline{\bigcup_n \mathcal{C}_n}$ and suppose that ω is a state of $\mathcal{C}$ and Π_ω the representation associated to ω by the Gel'fand-Segal construction. Then the following conditions are equivalent:*

(i) *ω is a factor state.*

(ii) *For all x in $\mathcal{C}$ there exists an integer $r > 0$ such that $|\omega(xy) - \omega(x)\omega(y)| \leqslant \|\Pi_\omega(y)\|$ for all y in $\mathcal{C}_r^c$.*

(iii) *For all x in $\mathcal{C}$ there exists a finite-dimensional *-algebra $\mathcal{B} \subseteq \mathcal{C}$ containing e such that $|\omega(xy) - \omega(x)\omega(y)| \leqslant \|\Pi_\omega(y)\|$ for all y in $\mathcal{B}^c$.*

In order to use condition (ii) above, for the case $\mathcal{C} = \mathcal{C}^\circ(\mathcal{K}) = \overline{\bigcup_r \mathcal{C}^\circ(\mathfrak{M}_r)}$, we will examine the relative commutant of $\mathcal{C}^\circ(\mathfrak{M}_r)$ in $\mathcal{C}^\circ(\mathcal{K})$. (Here it will be necessary to distinguish between the relative commutant in $\mathcal{C}^\circ(\mathcal{K})$ and the relative commutant in $\mathcal{C}(\mathcal{K})$. Thus we will use the superscripts c_0 for the

former and c for the latter.) The Lemmas 3.2–3.10 below, together with Proposition 3.11, provide us with a useful form for $\mathcal{C}^\circ(\mathfrak{M}_r)^{c_0}$.

LEMMA 3.2. *Let $\mathfrak{M}$ be a finite-dimensional Hilbert space with orthonormal basis $\{f_1, \ldots, f_n\}$, and*

$$N(\mathfrak{M}) = \sum_{i=1}^{n} a(f_i)^* a(f_i) = \sum_{i=1}^{n} e_{22}^{(i)}.$$

Then x in $\mathcal{C}(\mathfrak{M})$ commutes with $N(\mathfrak{M})$ if and only if x is in $\mathcal{C}^\circ(\mathfrak{M})$.

PROOF. First recall that $\mathcal{C}(\mathfrak{M})$ is a $p \times p$ matrix algebra with $p = 2^n$. Letting Φ, ψ be index functions such that Φ, ψ: $\{1, \ldots, n\} \to \{1, 2\}$ we define

$$f_{\Phi\psi} = e_{\Phi(1)\psi(1)}^{(1)} e_{\Phi(2)\psi(2)}^{(2)} \cdot \cdots \cdot e_{\Phi(n)\psi(n)}^{(n)}$$

which is a matrix unit for $\mathcal{C}^\circ(\mathfrak{M})$ if and only if Φ and ψ take on the value 2 (and hence 1) the same number of times, i.e., $\sum_{j=1}^{n} \psi(i) = \sum_{i=1}^{n} \Phi(i)$. Now note

$$\left[\sum_{\Phi, \psi} c_{\Phi, \psi} f_{\Phi, \psi}, N(\mathfrak{M}) \right]_{-} = \sum_{\Phi\psi} c_{\Phi\psi} \sum_{i} [f_{\Phi\psi}, e_{22}^{(i)}]$$

$$= \sum_{\Phi\psi} c_{\Phi\psi} \left(\sum_{i} (\psi(i) - \Phi(i)) \right) f_{\Phi\psi} = 0 \Leftrightarrow c_{\Phi\psi} = 0$$

for all Φ, ψ such that $\sum_i \Phi(i) \neq \sum_i \psi(i)$. Since any x in $\mathcal{C}^\circ(\mathfrak{M})$ may be written as such a sum, we have x in $\mathcal{C}^\circ(\mathfrak{M}) \Leftrightarrow [x, N(\mathfrak{M})]_{-} = 0$. Done.

LEMMA 3.3. *Let $\mathcal{K}$ be a separable Hilbert space and $\mathfrak{M} \subset \mathcal{K}$ a finite-dimensional subspace. Let $N(\mathfrak{M})$ be as above, and c denote the commutant relative to $\mathcal{C}(\mathcal{K})$. Then*

$$\mathcal{C}^\circ(\mathfrak{M}) = \mathcal{C}(\mathfrak{M}) \wedge N(\mathfrak{M})^c.$$

PROOF. Immediate, from Lemma 3.2.

LEMMA 3.4. *Let $z(\mathcal{C}^\circ(\mathfrak{M}))$ denote the center of $\mathcal{C}^\circ(\mathfrak{M})$ and $\mathcal{C}(N(\mathfrak{M}))$ the C^*-algebra generated by the single element $N(\mathfrak{M})$. Then $z(\mathcal{C}^\circ(\mathfrak{M})) = \mathcal{C}(N(\mathfrak{M}))$.*

PROOF. From Lemma 3.3 we have $\mathcal{C}^\circ(\mathfrak{M}) = \mathcal{C}(\mathfrak{M}) \wedge N(\mathfrak{M})^c$. Setting $\mathcal{K} = \mathfrak{M}$, we have c denoting the commutant relative to $\mathcal{C}(\mathfrak{M})$. Taking the commutant we obtain $\mathcal{C}^\circ(\mathfrak{M})^c = \mathcal{C}(\mathfrak{M})^c \vee N(\mathfrak{M})^{cc}$ recalling that for von Neumann algebras R_1 and R_2, $(R_1 \wedge R_2)' = R_1' \vee R_2'$. By the double commutant theorem we have $N(\mathfrak{M})^{cc} = \mathcal{C}(N(\mathfrak{M}))$, and since $\mathcal{C}(\mathfrak{M})$ is a factor we conclude

$$\mathcal{C}^\circ(\mathfrak{M})^c = \{\lambda I\} \vee \mathcal{C}(N(\mathfrak{M})) = \mathcal{C}(N(\mathfrak{M})).$$

Now $N(\mathfrak{M})$ is in $\mathcal{C}^{\circ}(\mathfrak{M})$ so $\mathcal{C}(N(\mathfrak{M})) \subset \mathcal{C}^{\circ}(\mathfrak{M})$; hence $z(\mathcal{C}^{\circ}(\mathfrak{M})) = \mathcal{C}^{\circ}(\mathfrak{M})^{c} \wedge \mathcal{C}^{\circ}(\mathfrak{M}) = \mathcal{C}(N(\mathfrak{M})) \wedge \mathcal{C}^{\circ}(\mathfrak{M}) = \mathcal{C}(N(\mathfrak{M}))$. Done.

LEMMA 3.5 [12, LEMMA 2.2]. *Let $\mathcal{K}$ be a Hilbert space, and $\mathfrak{M} \subset \mathcal{K}$ a finite-dimensional subspace with orthonormal basis $\{f_1, \ldots, f_n\}$. Let*

$$V = \prod_{i=1}^{n} (I - 2a(f_i)^{*} a(f_i)),$$

and let c denote the commutant relative to $\mathcal{C}(\mathcal{K})$. Then $\mathcal{C}(\mathfrak{M})^{c}$ is generated by the elements $a(f)V$ for all f in $\mathfrak{M}^{\perp}$.

LEMMA 3.6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be finite-dimensional Hilbert spaces with $\mathfrak{M} \subset \mathfrak{N}$. Let c denote the commutant relative to $\mathcal{C}(\mathfrak{N})$. Then $\mathcal{C}^{\circ}(\mathfrak{M})^{c} = \mathcal{C}(\mathfrak{M}^{\perp}) \vee N(\mathfrak{M})$.*

PROOF. From Lemma 3.3 we have $\mathcal{C}^{\circ}(\mathfrak{M}) = \mathcal{C}(\mathfrak{M}) \wedge N(\mathfrak{M})^{c}$, and taking the commutant we get $\mathcal{C}^{\circ}(\mathfrak{M})^{c} = \mathcal{C}(\mathfrak{M})^{c} \vee N(\mathfrak{M})^{cc} = \mathcal{C}(\mathfrak{M})^{c} \vee N(\mathfrak{M})$. By the above lemma, we have

$$\mathcal{C}^{\circ}(\mathfrak{M})^{c} = \mathcal{C}(\{a(f)V, f \text{ in } \mathfrak{M}^{\perp}\}) \vee N(\mathfrak{M}) = \mathcal{C}(\mathfrak{M}^{\perp}) \vee N(\mathfrak{M})$$

since $V = \exp(i\pi N(\mathfrak{M}))$ is in $\mathcal{C}(N(\mathfrak{M}))$. Done.

LEMMA 3.7. *Let $\mathcal{K}$ be a Hilbert space and $\mathfrak{M} \subset \mathcal{K}$ a finite-dimensional subspace. Let c denote the commutant relative to $\mathcal{C}(\mathcal{K})$. Then $\mathcal{C}^{\circ}(\mathfrak{M})^{c} = \mathcal{C}(\mathfrak{M}^{\perp}) \vee N(\mathfrak{M})$.*

PROOF. ($\supset$) As in Lemma 3.5, for f in $\mathfrak{M}^{\perp}$ we have $a(f)V$ an element of $\mathcal{C}(\mathfrak{M})^{c} \subset \mathcal{C}^{\circ}(\mathfrak{M})^{c}$. Further $N(\mathfrak{M})$ and $V = \exp(i\pi N(\mathfrak{M}))$ are in $\mathcal{C}^{\circ}(\mathfrak{M})^{c}$ by Lemma 3.2. Since $V^{2} = I$ we conclude such $a(f)$ to be elements of $\mathcal{C}^{\circ}(\mathfrak{M})^{c}$. Hence $\mathcal{C}(\mathfrak{M}^{\perp}) \vee N(\mathfrak{M}) \subset \mathcal{C}^{\circ}(\mathfrak{M})^{c}$. Done.

($\subset$) Let x be in $\mathcal{C}^{\circ}(\mathfrak{M})^{c}$, and let $\{\mathfrak{N}_k\}$, $k = 1, 2, \ldots$, be an increasing sequence of subspaces of $\mathcal{K}$ such that $\mathfrak{M} = \mathfrak{N}_1$ and $\overline{\bigcup_n \mathfrak{N}_n} = \mathcal{K}$. Certainly, x is in $\mathcal{C}(\mathcal{K}) = \overline{\bigcup_n \mathcal{C}(\mathfrak{N}_n)}$; hence there exists a sequence $\{x_n\}$, $n = 1, 2, \ldots$, such that x_n is in $\mathcal{C}(\mathfrak{N}_n)$ and $x_n \to x$. Further, for all y in $\mathcal{C}^{\circ}(\mathfrak{M})$ and $\delta_1 > 0$, there exists N_1 such that $n > N_1$ implies $\|x_n y - y x_n\| < \delta_1$, since multiplication is continuous. We now construct a sequence x_n', $n = 1, 2, \ldots$, such that x_n' is in $\mathcal{C}(\mathfrak{M}^{\perp}) \vee N(\mathfrak{M})$ and $x_n' \to x$ as $n \to \infty$. Let $e_{ij}^{(k)}$ be a set of matrix units for $\mathcal{C}^{\circ}(\mathfrak{M})$. Define $x_n' = \sum_i \sum_k e_{i1}^{(k)} x_n e_{1i}^{(k)}$, $1 \leqslant i < q < \infty$, $1 \leqslant k \leqslant p < \infty$. By construction x_n' is in $\mathcal{C}(\mathfrak{N}_n)$ and by straightforward computation, x_n' is in $\mathcal{C}^{\circ}(\mathfrak{M})^{c}$. Further,

$$\|x_n - x'_n\| = \left\| x_n - \sum_i \sum_k e_{i1}^{(k)} x_n e_{1i}^{(k)} \right\| = \left\| e x_n - \sum_i \sum_k e_{i1}^{(k)} x_n e_{1i}^{(k)} \right\|$$

$$= \left\| \sum_i \sum_k e_{ii}^{(k)} x_n - \sum_i \sum_k e_{i1}^{(k)} x_n e_{1i}^{(k)} \right\| \leqslant \sum_i \sum_k \| e_{ii}^{(k)} x_n - e_{i1}^{(k)} x_n e_{1i}^{(k)} \|$$

$$= \sum_i \sum_k \| e_{i1}^{(k)} (e_{1i}^{(k)} x_n - x_n e_{1i}^{(k)}) \| \leqslant \sum_i \sum_k \| e_{1i}^{(k)} x_n - x_n e_{1i}^{(k)} \|.$$

Thus for $n > N_1$, we have $\|x_n - x'_n\| \leqslant pq\delta_1$. Also, since $x_n \to x$, for any $\delta_2 > 0$ there is an N_2 such that $n > N_2$ implies $\|x - x_n\| < \delta_2$. We now pick $\delta_1 = \varepsilon/2pq$ and $\delta_2 = \varepsilon/2$, and $N = \max(N_1, N_2)$. Combining, we obtain $\|x - x'_n\| < \varepsilon$. Thus $x'_n \to x$ as $n \to \infty$, x'_n is in $\mathcal{C}(\mathfrak{N}_n)$, and denoting the commutant relative to $\mathcal{C}(\mathfrak{N}_n)$ by c, we have x'_n in $\mathcal{C}^\circ(\mathfrak{M})^c = \mathcal{C}(\mathfrak{M}^\perp) \vee N(\mathfrak{M})$ by Lemma 3.6, since $\mathfrak{M} \subset \mathfrak{N}_n$ and both are finite-dimensional. Finally, since $x'_n \to x$ we conclude x in $\mathcal{C}(\mathfrak{M}^\perp) \vee N(\mathfrak{M})$. Done.

LEMMA 3.8. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be finite-dimensional Hilbert spaces such that $\mathfrak{M} \subset \mathfrak{N}$. Let c_0 denote the commutant relative to $\mathcal{C}^\circ(\mathfrak{N})$. Then $\mathcal{C}^\circ(\mathfrak{M})^{c_0} = \mathcal{C}^\circ(\mathfrak{M}^\perp) \vee z(\mathcal{C}^\circ(\mathfrak{M})) = \mathcal{C}^\circ(\mathfrak{M}^\perp) \vee N(\mathfrak{M})$.*

PROOF. Since $z(\mathcal{C}^\circ(\mathfrak{M})) = \mathcal{C}(N(\mathfrak{M}))$ by Lemma 3.4, the second equality is immediate. Now we show $\mathcal{C}^\circ(\mathfrak{M})^{c_0} = \mathcal{C}^\circ(\mathfrak{M}^\perp) \vee N(\mathfrak{M})$. Consider x in $\mathcal{C}^\circ(\mathfrak{M})^{c_0} \subset \mathcal{C}^\circ(\mathfrak{M})^c$. By Lemma 3.6 and the fact that $z(\mathcal{C}^\circ(\mathfrak{M})) = \mathcal{C}(N(\mathfrak{M}))$, we can write $x = \sum_i y_i z_i$ with y_i in $\mathcal{C}(\mathfrak{M}^\perp)$ and z_i in $\mathcal{C}(N(\mathfrak{M}))$. Letting χ_t be the gauge automorphism and

$$\alpha = \frac{1}{2\pi} \int_0^{2\pi} \chi_t \, dt$$

we note that $\alpha: \mathcal{C}(\mathfrak{M}) \to \mathcal{C}^\circ(\mathfrak{M})$ and leaves $\mathcal{C}^\circ(\mathfrak{M})$ fixed. Hence for x in $\mathcal{C}^\circ(\mathfrak{M})^{c_0} \subset \mathcal{C}^\circ(\mathfrak{M})$ we have $\alpha(x) = x = \sum_i \alpha(y_i z_i) = \sum_i \alpha(y_i) z_i$, using the fact that $N(\mathfrak{M})$ and thus z_i is gauge invariant. Further, since y_i is in $\mathcal{C}^\circ(\mathfrak{M}^\perp)$, $\alpha(y_i)$ is in $\mathcal{C}^\circ(\mathfrak{M}^\perp)$, and we conclude $\mathcal{C}^\circ(\mathfrak{M})^{c_0} = \mathcal{C}^\circ(\mathfrak{M}^\perp) \vee N(\mathfrak{M})$. Done.

LEMMA 3.9. *Let $\mathcal{K}$ be a separable Hilbert space, and $\mathfrak{M} \subset \mathcal{K}$ a finite-dimensional subspace. Let c_0 denote the commutant relative to $\mathcal{C}^\circ(\mathcal{K})$. Then $\mathcal{C}^\circ(\mathfrak{M})^{c_0} = \mathcal{C}^\circ(\mathfrak{M}^\perp) \vee z(\mathcal{C}^\circ(\mathfrak{M}))$.*

PROOF. $(\supset)$ Trivially, $z(\mathcal{C}^\circ(\mathfrak{M})) \subset \mathcal{C}^\circ(\mathfrak{M})^{c_0}$, so we need only show $\mathcal{C}^\circ(\mathfrak{M}^\perp) \subset \mathcal{C}^\circ(\mathfrak{M})^{c_0}$. Certainly $\mathcal{C}^\circ(\mathfrak{M}^\perp) \subset \mathcal{C}^\circ(\mathcal{K})$, and $\mathcal{C}^\circ(\mathfrak{M}^\perp) \subset \mathcal{C}(\mathfrak{M}^\perp) \subset \mathcal{C}^\circ(\mathfrak{M})^c$. Hence $\mathcal{C}^\circ(\mathfrak{M}^\perp) \subset \mathcal{C}^\circ(\mathfrak{M})^{c_0}$. Done.

$(\subset)$ Let x be in $\mathcal{C}^\circ(\mathfrak{M})^{c_0}$, and let $\{\mathfrak{N}_k\}$, $k = 1, 2, \ldots$, be an increasing sequence of subspaces of $\mathcal{K}$ such that $\mathfrak{M} = \mathfrak{N}_1$ and $\overline{\bigcup_n \mathfrak{N}_n} = \mathcal{K}$. Certainly, x is in $\mathcal{C}^\circ(\mathcal{K}) = \overline{\bigcup_n \mathcal{C}^\circ(\mathfrak{N}_n)}$, hence there exists a sequence $\{x_n\}$, $n = 1, 2, \ldots$,

such that x_n is in $\mathcal{C}^\circ(\mathfrak{M}_n)$ and $x_n \to x$. Arguing exactly as in the same inclusion of Lemma 3.7, we may construct a sequence $\{x_n'\}$, $n = 1, 2, \ldots$, such that x_n' is in $\mathcal{C}^\circ(\mathfrak{M}^\perp) \vee z(\mathcal{C}^\circ(\mathfrak{M}))$ and $x_n' \to x$ as $n \to \infty$, using Lemma 3.8 to conclude x_n' is in $\mathcal{C}^\circ(\mathfrak{M}^\perp) \vee z(\mathcal{C}^\circ(\mathfrak{M}))$ for each $n = 1, 2, \ldots$. Done.

LEMMA 3.10. *Let $\mathcal{C}^\circ(\mathfrak{K}) = \overline{\cup_r \mathcal{C}^\circ(\mathfrak{M}_r)}$, with $\{\mathfrak{M}_r\}$, $r = 1, 2, \ldots$, an increasing sequence of r-dimensional subspaces of $\mathfrak{K}$ such that $\overline{\cup_r \mathfrak{M}_r} = \mathfrak{K}$. For convenience, let $N_r = N(\mathfrak{M}_r)$. Then $z(\mathcal{C}^\circ(\mathfrak{M}_r)) = \mathcal{C}(N_r)$ is spanned by $r + 1$ orthogonal projections $E_0, \ldots, E_r$.*

PROOF. By a straightforward analysis of the matrix units [4, Lemma 5.2] we have $\mathcal{C}^\circ(\mathfrak{M}_r)$ isomorphic to a direct sum of $r + 1$ finite-dimensional factors. Defining E_k to be the projection on the kth such factor, $0 \leqslant k \leqslant r$, we have $r + 1$ orthogonal projections which clearly span $z(\mathcal{C}^\circ(\mathfrak{M}_r))$. Done.

PROPOSITION 3.11. *Let $\mathcal{C}^\circ(\mathfrak{K}) = \overline{\cup_r \mathcal{C}^\circ(\mathfrak{M}_r)}$ with $\{\mathfrak{M}_r\}$, $r = 1, 2, \ldots$, an increasing sequence of r-dimensional subspaces of $\mathfrak{K}$ such that $\overline{\cup_r \mathfrak{M}_r} = \mathfrak{K}$, and $N_r = N(\mathfrak{M}_r)$ as above. Let c_0 denote the commutant relative to $\mathcal{C}^\circ(\mathfrak{K})$. Then for arbitrary y in $\mathcal{C}^\circ(\mathfrak{M}_r)^{c_0}$ we can write*

$$y = \int_0^{2\pi} y(t) e^{iN_r t} \, dt$$

with $y(t)$ in $\mathcal{C}^\circ(\mathfrak{M}_r^\perp)$.

PROOF. Let y be in $\mathcal{C}^\circ(\mathfrak{M}_r)^{c_0}$. By Lemmas 3.9 and 3.4 we have $\mathcal{C}^\circ(\mathfrak{M}_r)^{c_0} = \mathcal{C}^\circ(\mathfrak{M}_r^\perp) \vee N_r$; thus N_r in $\mathcal{C}^\circ(\mathfrak{M}_r)$ commutes with $\mathcal{C}^\circ(\mathfrak{M}_r^\perp)$. Using Lemma 3.10, then, we can write $y = \sum_{k=0}^r y_k E_k$ with y_k in $\mathcal{C}^\circ(\mathfrak{M}_r^\perp)$. Now letting

$$y(t) = \frac{1}{2\pi} \sum_{k=0}^r y_k e^{-ikt}$$

we have $y(t)$ in $\mathcal{C}^\circ(\mathfrak{M}_r^\perp)$ since the y_k are. Now

$$e^{iN_r t} = \exp\left(i \sum_{j=0}^r a(f_j)^* a(f_j) t \right) = \prod_{j=0}^r \exp(i e_{22}^{(j)} t)$$

$$= \prod_{j=0}^r (e_{11}^{(j)} + e_{22}^{(j)} e^{it})$$

and we obtain

$$\int_0^{2\pi} y(t) e^{iN_r t} \, dt = \frac{1}{2\pi} \sum_{k=0}^r y_k \int_0^{2\pi} e^{-ikt} \times \prod_{j=0}^r (e_{11}^{(j)} + e_{22}^{(j)} e^{it}) \, dt.$$

Clearly, this integral picks out of the product the coefficient of the e^{ikt} term. An inspection of the matrix units for $\mathcal{C}^\circ(\mathfrak{M}_r)$ shows that these coefficients are

the central projections E_k of $\mathcal{C}^\circ(\mathfrak{M}_r)$ above. Hence we have

$$\int_0^{2\pi} y(t)e^{iN_r t}\,dt = \sum_{k=0}^r y_k E_k = y.$$

Done.

In the following section, we will relate the factor condition for a given generalized free state to properties of a particular series. We begin with some definitions.

DEFINITION 3.12. Given a function $f(t) = \sum_{r=0}^\infty \gamma_r e^{irt}$, we define its Fourier-$s$ norm, denoted $\|\ \|_s$, by the formula

$$\|f\|_s = \left\{ \sum_{r=0}^\infty |\gamma_r|^s \right\}^{1/s}.$$

DEFINITION 3.13. Let ω be a state of $\mathcal{C}^\circ(\mathfrak{K}) = \overline{\bigcup_r \mathcal{C}^\circ(\mathfrak{M}_r)}$ with $\{\mathfrak{M}_r\}$, $r = 1$, $2, \ldots$, an increasing sequence of finite-dimensional subspaces of $\mathfrak{K}$ such that $\overline{\bigcup_r \mathfrak{M}_r} = \mathfrak{K}$. Let $N_r = N(\mathfrak{M}_r)$ as above. We define a function $P_{mn}(t)$ associated with ω by the formula $P_{mn}(t) = \omega(e^{i(N_m - N_n)t})$.

PROPOSITION 3.14. *Let ω be a generalized free state of $\mathcal{C}(\mathfrak{K}) = \overline{\bigcup_r \mathcal{C}(\mathfrak{M}_r)}$, which factorizes with respect to the increasing sequence of finite-dimensional subspaces $\{\mathfrak{M}_r\}$, $r = 1, 2, \ldots$. Let x be in $\mathcal{C}^\circ(\mathfrak{M}_n)$, and y be in $\mathcal{C}^\circ(\mathfrak{M}_m)^{c_0}$, $m > n$. Then $|\omega(xy) - \omega(x)\omega(y)| \to 0$ as $m \to \infty$ if $\|(1 - e^{it})P_{mn}(t)\|_1 \to 0$ as $m \to \infty$.*

PROOF. By Lemma 3.11 and the continuity of ω, we have

$$\omega(xy) - \omega(x)\omega(y) = \int_0^{2\pi} (\omega(xy(t)e^{iN_m t}) - \omega(x)\omega(y(t)e^{iN_m t}))\,dt.$$

Also,

$$\omega(xy(t)e^{iN_m t}) = \omega(xe^{iN_m t}y(t)) = \omega(xe^{iN_m t})\omega(y(t))$$

and

$$\omega(y(t)e^{iN_m t}) = \omega(y(t))\omega(e^{iN_m t})$$

using the factorization properties of ω. Now noting $e^{iN_n t} = e^{i(N_m - N_n)t}e^{iN_n t}$ and combining, we obtain

$$\omega(xy) - \omega(x)\omega(y) = \int_0^{2\pi} \omega(y(t))P_{mn}(t)[\omega(xe^{iN_n t}) - \omega(x)\omega(e^{iN_n t})]\,dt$$

$$= \int_0^{2\pi} \omega(y(t))P_{mn}(t)q_x(t)\,dt$$

defining $q_x(t) = \omega(xe^{iN_n t}) - \omega(x)\omega(e^{iN_n t})$. Now $q_x(0) = 0$ and, recalling E_k, $k = 0, 1, \ldots, n$, to be the central projections of $\mathcal{C}^\circ(\mathfrak{M}_n)$ we have $e^{iN_n t} = \sum_{k=0}^{n} E_k e^{ikt}$ by straightforward comparison. Substituting, we see that $q_x(t)$ is a finite Fourier sum with highest power n. Hence, there are coefficients a_k, $k = 1, 2, \ldots, n$, such that

$$q_x(t) = \sum_{k=1}^{n} a_k(1 - e^{ikt}).$$

Further, $y(t) = (1/2\pi) \sum_{p=0}^{m} y_p e^{-ipt}$, and defining $\omega(y_p) = b_p$, we get $\omega(y(t)) = (1/2\pi) \sum_{p=0}^{m} b_p e^{-ipt}$. Substituting these into our last expression for $\omega(x)\omega(y)$, we obtain

$$\omega(xy) - \omega(x)\omega(y) = \int_0^{2\pi} \left(\sum_{p=0}^{m} b_p e^{-ipt} \right) \left(\sum_{k=1}^{n} a_k(1 - e^{ikt}) \right) P_{mn}(t) \, dt$$

$$= \sum_{k=1}^{n} \sum_{p=0}^{m} a_k b_p \int_0^{2\pi} e^{-ipt}(1 - e^{ikt}) P_{mn}(t) \, dt.$$

Now define $\sum_{r=0}^{m-n+k} c_r e^{irt} = (1 - e^{ikt}) P_{mn}(t)$; the sum limit follows from the equation

$$e^{i(N_m - N_n)t} = \prod_{l=n}^{m} (e_{11}^{(l)} + e_{22}^{(l)} e^{it}).$$

Substituting and taking absolute values, we get

$$|\omega(xy) - \omega(x)\omega(y)| = \left| \sum_{k=1}^{n} \sum_{p=0}^{m} \sum_{r=0}^{m-n+k} a_k b_p c_r \int_0^{2\pi} e^{-ipt} e^{irt} \, dt \right|$$

$$= \left| 2\pi \sum_{k=1}^{n} \sum_{r=0}^{m-n+k} a_k b_r c_r \right|$$

$$\leqslant 2\pi \sum_{k=1}^{n} |a_k| \sum_{r=0}^{m-n+k} |b_r||c_r| \leqslant 2\pi \sum_{k=1}^{n} |a_k| \sum_{r=0}^{m-n+k} |c_r|,$$

since we may assume $\|y\| = 1$, and it follows that $|b_r| \leqslant 1$. Now note n is fixed and $k \leqslant n$, so $m \to \infty$ implies $m - n + k \to \infty$. Hence, for arbitrary x in $\mathcal{C}^\circ(\mathfrak{M}_n)$ and y in $\mathcal{C}^\circ(\mathfrak{M}_m)^{c_0}$ we have $|\omega(xy) - \omega(x)\omega(y)| \to 0$ as $m \to \infty$ if $\sum_{r=0}^{m-n+k} |c_r| \to 0$ as $m \to \infty$. By definition, $\sum_{r=0}^{m-n+k} |c_r| = \|(1 - e^{ikt}) P_{mn}(t)\|_1$, and observing $(1 - e^{ikt}) = \sum_{s=0}^{k-1} e^{ist}(1 - e^{it})$ we obtain

$$\|(1 - e^{ikt}) P_{mn}(t)\|_1 = \left\| \sum_{s=0}^{k-1} e^{ist}(1 - e^{it}) P_{mn}(t) \right\|_1$$

$$\leqslant (n - 1)\|(1 - e^{it}) P_{mn}(t)\|_1$$

since $k \leqslant n$. But n is fixed, so $\|(1 - e^{it})P_{mn}(t)\|_1 \to 0$ as $m \to \infty$ implies $|\omega(xy) - \omega(x)\omega(y)| \to 0$ as $m \to \infty$. Done.

LEMMA 3.15. *Let A be a positive operator on a Hilbert space $\mathfrak{K}$, with $0 < A < I$, pure point spectrum $\{\lambda_i\}$, $i = 1, 2, \ldots$, and associated eigenvectors $\{f_i\}$, $i = 1, 2, \ldots$. Let $\mathfrak{M}_r = \mathrm{span}\{f_1, \ldots, f_r\}$ and let ω_A be the gauge invariant generalized free state of $\mathcal{C}(\mathfrak{K}) = \overline{\bigcup_r \mathcal{C}(\mathfrak{M}_r)}$ associated with A. Define*

$$P_{mn}(t) = \omega_A(e^{i(N_m - N_n)t})$$

as above. Then

$$\|(1 - e^{it})P_{mn}(t)\|_1 = \left\|(1 - e^{it}) \prod_{j=n+1}^{m} \{(1 - \lambda_j) + \lambda_j e^{it}\}\right\|_1.$$

PROOF. Recall

$$P_{mn}(t) = \omega_A(e^{i(N_m - N_n t)}) = \omega_A\left(\prod_{j=n+1}^{m} e_{11}^{(j)} + e^{it} e_{22}^{(j)}\right).$$

By construction, ω_A factorizes with respect to the $\{\mathfrak{M}_r\}$, $r = 1, 2, \ldots$, and the $e_{ij}^{(k)}$ lie in $\mathfrak{B}_k$ with

$$\mathcal{C}(\mathfrak{K}) \approx \bigotimes_{k=1}^{\infty} \mathcal{C}(\mathfrak{M}_{k+1}) \cap \mathcal{C}(\mathfrak{M}_k)^c = \bigotimes_{k=1}^{\infty} \mathfrak{B}_k.$$

Hence

$$P_{mn}(t) = \prod_{j=n+1}^{m} \{\omega_A(e_{11}^{(j)}) + e^{it}\omega_A(e_{22}^{(j)})\}$$

$$= \prod_{j=n+1}^{m} \{(1 - \lambda_j) + \lambda_j e^{it}\}. \quad \text{Done.}$$

DEFINITION 3.16. We define the functions P_n and χ_n by the formulae

$$P_n(t) = \prod_{j=1}^{n} \{(1 - \lambda_j) + \lambda_j e^{it}\},$$

$$\chi_n(t) = (1 - e^{it})P_n(t).$$

DEFINITION 3.17. We define the Fourier coefficients α_k and γ_k by the formulae

$$P_n(t) = \sum_{k=0}^{n} \alpha_k e^{ikt}, \qquad \chi_n(t) = \sum_{k=0}^{n+1} \gamma_k e^{ikt}.$$

REMARK. Since $\chi_n(t) = (1 - e^{it})P_n(t)$, we observe

$$\gamma_k = \alpha_k - \alpha_{k-1} \text{ for } 0 \leqslant k \leqslant n + 1 \text{ taking } \alpha_{-1} = \alpha_{n+1} = 0.$$

LEMMA 3.18 (CHANGE OF SIGN). *Let* $\chi_n(t) = \sum_{k=0}^{n+1} \gamma_k e^{ikt}$ *as above. Then, for all* $n > 0$, *there is a* $p < n$ *such that*

$$\gamma_k \begin{cases} \geqslant 0, & k \leqslant p, \\ \leqslant 0, & k > p. \end{cases}$$

PROOF. By induction. For $n = 1$, we have

$$\chi_1 = (1 - e^{it})\{(1 - \lambda_1) + \lambda_1 e^{it}\}$$

whose coefficients are $1 - \lambda_1, 2\lambda_1 - 1, -\lambda_1$, in order, and clearly

$$p = \begin{cases} 1, & \lambda_1 < \frac{1}{2}, \\ 2, & \lambda_1 \geqslant \frac{1}{2}. \end{cases}$$

Now suppose true for χ_n. We can write the coefficients in order as $a_{n_+}, \ldots,$ $a_3, a_2, a_1, -b_1, -b_2, -b_3, \ldots, -b_{n_-}$ with $n_+ + n_- = n + 2$, $p = n_+$, and a_i, $b_i \geqslant 0$. Since $\chi_{n+1} = ((1 - \lambda_{n+1}) + \lambda_{n+1} e^{it})\chi_n$ we can generate the χ_{n+1} coefficients as indicated below:

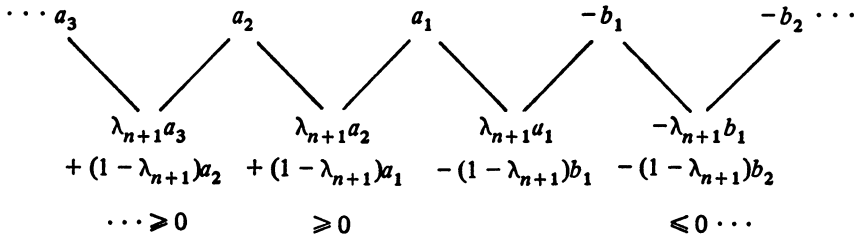FIGURE 1. Generation of coefficients

Hence we have the two cases

$$\lambda_{n+1} a_1 - (1 - \lambda_{n+1})b_1 \begin{cases} \geqslant 0, \\ < 0, \end{cases}$$

and we conclude

$$p = \begin{cases} n_+ + 1, \\ n_+, \end{cases}$$

respectively.

PROPOSITION 3.19. *Let* $\chi_n(t)$, $P_n(t)$ *and* α_k *be as in Definitions* 3.16 *and* 3.17. *Then*

$$\|\chi_n\|_1 = 2 \max_{0 < k \leqslant n} \{\alpha_k\}.$$

PROOF. Recall

$$P_n(t) = \sum_{k=0}^{n} \alpha_k e^{ikt}, \quad \chi_n(t) = \sum_{k=0}^{n+1} \gamma_k e^{ikt}, \, \gamma_k = \alpha_k - \alpha_{k-1},$$

with $\alpha_k = 0$ for $k < 0$ or $k > n$. By the change of sign lemma (Lemma 3.18) there is a p such that

$$\gamma_k \geqslant 0, \quad k = 0, 1, \ldots, p - 1,$$

$$\gamma_k \leqslant 0, \quad k = p, p + 1, \ldots, n + 1.$$

Thus,

$$\|\chi_n\|_1 = \sum_{k=0}^{n+1} |\gamma_k| = \sum_{k=0}^{p-1} \gamma_k - \sum_{k=p}^{n+1} \gamma_k$$

$$= \sum_{k=0}^{p-1} (\alpha_k - \alpha_{k-1}) - \sum_{k=p}^{n+1} (\alpha_k - \alpha_{k-1})$$

$$= \{\alpha_0 + (\alpha_1 - \alpha_0) + \cdots + (\alpha_{p-1} - \alpha_{p-2})\}$$

$$-\{(\alpha_p - \alpha_{p-1}) + (\alpha_{p+1} - \alpha_p) + \cdots + (\alpha_n - \alpha_{n-1}) - \alpha_n\}$$

$$= \alpha_{p-1} - (-\alpha_{p-1}) = 2\alpha_{p-1}.$$

But $\alpha_{p-1} = \max_{0 < k \leqslant n}\{\alpha_k\}$. Hence

$$\|\chi_n\|_1 = 2\alpha_{p-1} = 2 \max_{0 < k \leqslant n} \{\alpha_k\}.$$

THEOREM 3.20. *Let A be a positive operator on a separable Hilbert space $\mathcal{K}$ with $0 < A < I$ and pure point spectrum $\{\lambda_i\}$, $i = 1, 2, \ldots$. Then ω_A°, the associated generalized free state of $\mathcal{C}^\circ(\mathcal{K})$, is a factor state if $\sum_{i=1}^{\infty} \lambda_i(1 - \lambda_i) = \infty$.*

PROOF. By Proposition 3.14, given $\|(1 - e^{it})P_{mn}(t)\|_1 \to 0$ as $m \to \infty$, we have $|\omega_A^\circ(xy) - \omega_A^\circ(x)\omega_A^\circ(y)| \to 0$ as $m \to \infty$ for x in $\mathcal{C}^\circ(\mathfrak{M}_n)$, y in $\mathcal{C}^\circ(\mathfrak{M}_m)^{c_0}$. Since ω_A° is continuous and the $\cup_n \mathcal{C}^\circ(\mathfrak{M}_n)$ is dense in $\mathcal{C}^\circ(\mathcal{K})$ we may assume x is an arbitrary element of $\mathcal{C}^\circ(\mathcal{K})$. Now invoking Theorem 3.1, Lemma 3.15 and Definition 3.16 (relabeling the product lower limit $n + 1$) we conclude ω_A° is a factor state if $\|\chi_m\|_1 \to 0$ as $m \to \infty$. By Proposition 3.19 it is sufficient to show that $\max_{0 < k \leqslant m}\{\alpha_k\} \to 0$ as $m \to \infty$. Now note

$$|\alpha_k| = \left| \frac{1}{2\pi} \int_0^{2\pi} P_{mn}(t) e^{-ikt} \, dt \right| = \left| \int_0^{2\pi} \frac{e^{ikt}}{(2\pi)^{1/2}} \frac{P_{mn}}{(2\pi)^{1/2}}(t) \, dt \right|$$

$$\leqslant \frac{1}{2\pi} \int_0^{2\pi} |P_{mn}(t)|^2 \, dt$$

by the Cauchy-Schwarz inequality. Further,

$$|P_{mn}(t)|^2 = P_{mn}(t) P_{mn}^*(t) = \prod_{i=n+1}^m \{1 - 2(1 - \cos t)\lambda_i(1 - \lambda_i)\}.$$

But, for all t in $(0, 2\pi)$ and $n = 1, 2, \ldots,$

$$\sum_{i=n+1}^\infty 2(1 - \cos t)\lambda_i(1 - \lambda_i) = 2(1 - \cos t) \sum_{i=n+1}^\infty \lambda_i(1 - \lambda_i) = \infty$$

if and only if $\sum_{i=1}^\infty \lambda_i(1 - \lambda_i) = \infty$. Hence $|P_{mn}(t)|^2 \to 0$ as $n \to \infty$ by the residual theorem for infinite products, for all t in $(0, 2\pi)$. Finally, $|P_{mn}(t)|^2 \leqslant 1$ and $|P_{mn}(t)| \to 0$ as $n \to \infty$ almost everywhere on $(0, 2\pi)$ so $\int_0^{2\pi} |P_{mn}(t)|^2 \, dt \to 0$ as $n \to \infty$. Thus $|\alpha_k| \to 0$ as $n \to \infty$, independent of k, by our first inequality. Done.

The following lemma, as modified in Proposition 3.22, will be useful in proving the main result (Theorem 3.24).

LEMMA 3.21 [LEMMA 5.3,(i)]. *Let ω_A be a gauge invariant generalized free state of the CAR algebra $\mathfrak{C}(\mathcal{K})$. Then ω_A is type I if and only if there exists a spectral projection E of A and a trace class operator T such that $A = E + T$.*

PROPOSITION 3.22. *Let A be a positive operator on a separable Hilbert space $\mathcal{K}$ with $0 < A < I$, and let ω_A be the associated generalized free state of $\mathfrak{C}(\mathcal{K})$. Then ω_A induces a type I factor representation of $\mathfrak{C}(\mathcal{K})$ if and only if $\mathrm{Tr}\, A(I - A) < \infty$.*

PROOF. *Sufficiency.* Suppose ω_A induces a type I factor representation. By the above lemma, $A = E + T$, with E a projection and T trace class. Hence

$$A(I - A) = (E + T)((I - E) - T) = -ET + T(I - E) - T^2$$

and

$$\mathrm{Tr}\, A(I - A) = -\mathrm{Tr}(ET) + \mathrm{Tr}(T(I - E)) - \mathrm{Tr}(T^2) < \infty$$

since the trace class operators form an ideal in $\mathfrak{B}(\mathcal{K})$.

Necessity. Suppose $\mathrm{Tr}\, A(I - A) < \infty$. Then A clearly has pure point spectrum, and there is an orthonormal basis $\{f_i\}$, $i = 1, 2, \ldots,$ for $\mathcal{K}$ such that $Af_i = \lambda_i f_i$ and $0 \leqslant \lambda_i \leqslant 1$. Let E be the spectral projection of A projecting onto $\overline{\mathrm{span}}\,(\{f_i : \lambda_i \geqslant \frac{1}{2}\})$. Then $AE \geqslant \frac{1}{2}E$ and

$$\text{Tr}(|A - E|) = \sum_i (f_i, (|A - E|)f_i) = \sum_{i \in I - E} \lambda_i + \sum_{i \in E} (1 - \lambda_i)$$

where $i \in E$ means $Af_i = \lambda_i f_i$ and $\lambda_i \geqslant \frac{1}{2}$, $i \in (I - E)$ means $Af_i = \lambda_i f_i$ and $\lambda_i < \frac{1}{2}$. Note that

$$\lambda_i(1 - \lambda_i) \leqslant \min(\lambda_i, 1 - \lambda_i) \leqslant 2\lambda_i(1 - \lambda_i)$$

follows straightforwardly from $0 < \lambda_i < 1$. Hence

$$\sum_{i=1}^{\infty} \lambda_i(1 - \lambda_i) \leqslant \sum_{i \in I - E} \lambda_i + \sum_{i \in E} (1 - \lambda_i) \leqslant 2 \sum_{i=1}^{\infty} \lambda_i(1 - \lambda_i)$$

or $\text{Tr}(A - E) < \infty$ if and only if $\sum_{i=1}^{\infty} \lambda_i(1 - \lambda_i) < \infty$. Thus $A - E$ is trace class or $A = E + T$ with T trace class, which implies ω_A induces a type I factor representation by the above lemma. Done.

In proving the main result of this section (Theorem 3.24) we will (a) make use of [14, Theorem 1] where sufficiency (i.e., ω_A° is a factor state implies $\text{Tr} A(I - A) = \infty$) is already proved and (b) invoke Theorem 4.12 and Lemmas 3.23, 4.10, 4.16 for the necessity argument. We now state

LEMMA 3.23 [12, **LEMMA 2.1**]. *Let $\mathcal{K}$ be a Hilbert space and ω_1, ω_2 two factor states of $\mathcal{C}(\mathcal{K})$. Let $\mathfrak{M}_1 \subset \mathfrak{M}_2 \cdots$ be an increasing sequence of finite-dimensional subspaces of $\mathcal{K}$, the closure of whose union is $\mathcal{K}$. Then the following statements are equivalent:*

(i) $\omega_1 \underset{q}{\sim} \omega_2$.

(ii) *For every $\varepsilon > 0$ there is an integer n such that*

$$\|\omega_1 | \mathcal{C}(\mathfrak{M}_n)^c - \omega_2 | \mathcal{C}(\mathfrak{M}_n)^c\| < \varepsilon.$$

(iii) *There is a finite-dimensional subspace $\mathfrak{N} \subset \mathcal{K}$ such that*

$$\|\omega_1 | \mathcal{C}(\mathfrak{N})^c - \omega_2 | \mathcal{C}(\mathfrak{N})^c\| < 2.$$

Finally, we have

THEOREM 3.24. *Let A be a positive operator on a separable Hilbert space $\mathcal{K}$, with $0 < A < I$. Suppose A is not a projection, and let ω_A° be the associated generalized free state of $\mathcal{C}^\circ(\mathcal{K})$. Then ω_A° is a factor state if and only if $\text{Tr} A(I - A) = \infty$.*

PROOF. *Sufficiency.* This is [14, Theorem 1]. The proof proceeds by assuming $\text{Tr} A(I - A) < \infty$, and constructing nontrivial central elements of the von Neumann algebra associated with the representation. These are given by $\exp(i\Pi(V_n)t)$ where $V_n = N_n - \omega_A(N_n)I$, $0 < t < 1$. We note that this construction is repeated, with minor modification, in [3] to which we refer the reader for further detail.

Necessity. Suppose $\operatorname{Tr} A(I - A) = \infty$. By Lemma 4.16 we can get a sequence of operators $\{A_p\}$, $p = 1, 2, \ldots$, such that

(i) $0 < A_p < I$,

(ii) A_p has pure point spectrum,

(iii) $\|A^{1/2} - A_p^{1/2}\|_{\text{H.S.}} \to 0$ as $p \to \infty$,

(iv) $\|(I - A)^{1/2} - (I - A_p)^{1/2}\|_{\text{H.S.}} \to 0$ as $p \to \infty$.

Further, by Lemma 4.10 we have

(v) $\|\omega_A - \omega_{A_p}\| \to 0$ as $p \to \infty$,

with ω_A, ω_{A_p} the gauge invariant generalized free states of $\mathcal{C}(\mathcal{K})$ associated with A, A_p. Now, by Lemma 3.23 we have $\omega_A \sim_q \omega_{A_p}$, $p = 1, 2, \ldots$. But $\operatorname{Tr} A(I - A) = \infty$ implies ω_A induces a factor representation which is not type I; since $\omega_A \sim_q \omega_{A_p}$ we have ω_{A_p} inducing a factor representation not of type I. (Recall quasi-equivalent states induce representations whose associated von Neumann algebras are *-isomorphic, and type is an isomorphism invariant.) Since ω_{A_p} is not type I, we have $\operatorname{Tr} A_p(I - A_p) = \infty$ by Proposition 3.22. Since the A_p have pure point spectrum, we conclude the ω_A° are factor states of $\mathcal{C}^\circ(\mathcal{K})$ by Theorem 3.20. But $\|\omega_A^\circ - \omega_{A_p}^\circ\| \leqslant \|\omega_A - \omega_{A_p}\| \to 0$ as $p \to \infty$ and the uniform limit of factor states is a factor state; hence ω_A° is a factor state. Done.

We conclude this section by proving that the representations induced by such factor states of $\mathcal{C}^\circ(\mathcal{K})$ are faithful, i.e., they have trivial kernel–hence no nontrivial two-sided ideals are mapped to zero. To do so, we use the characterization of all two-sided ideals of $\mathcal{C}^\circ(\mathcal{K})$ given in [4, 5.5]. We summarize this information with a remark.

REMARK. The ideals of $\mathcal{C}^\circ(\mathcal{K})$, denoted $_nI_m$:

(i) are generated by the factors $M_{(n+m,n)}$,

(ii) contain as a dense set all gauge invariant polynomials in $a(f)$, $a(g)^*$ each of whose addends has:

(a) at least m creators when ordered so that all creators are standing to the left of all annihilators and

(b) at least n annihilators when ordered so that all annihilators are standing to the left of all creators.

We are now ready for

PROPOSITION 3.25. *Let A be a positive operator on a separable Hilbert space $\mathcal{K}$ such that $0 < A < I$ and $\operatorname{Tr} A(I - A) = \infty$. Let ω_A° be the associated generalized free factor state of $\mathcal{C}^\circ(\mathcal{K})$, and Π_A the representation of $\mathcal{C}^\circ(\mathcal{K})$ induced by ω_A° via the Gel'fand-Naĭmark-Segal construction. Then Π_A is faithful.*

PROOF. First define $\mathcal{K}_{0,1} = \{f: Af = 0 \text{ or } Af = f\}$. Clearly, $\mathcal{K} \ominus \mathcal{K}_{0,1}$ is infinite dimensional, since if not, $\operatorname{Tr} A(I - A) < \infty$. Now assume Π_A is not faithful; if so it must map one of the $_nI_m$ to zero. We observe that, by (ii)

above, $_nI_m \subset {}_pI_p$ with $p = \max(m, n)$ and by (i), $M_{(2p,p)}$ generates $_pI_p$. Further, the element $y = e_{11}^{(1)} \cdots e_{11}^{(p)} \cdot e_{22}^{(p+1)} \cdots e_{22}^{(2p)}$ is in $M_{(2p,p)}$ by its definition; hence it is in $_pI_p$. We now construct an element x in $_nI_m$ such that $\Pi_A(x) \neq 0$. To do so, we observe that: (a) we can find $2p$ orthonormal vectors $\{g_1, \ldots, g_{2p}\}$ in $\mathcal{K} \ominus \mathcal{K}_{0,1}$ since it is infinite dimensional; (b) A is selfadjoint when restricted to $\mathfrak{N} = \mathrm{span}(\{g_1, \ldots, g_{2p}\})$; (c) therefore we can find $\{f_1, \ldots, f_{2p}\}$ which are an orthonormal basis of $\mathfrak{N}$ and eigenvectors of A, and (d) A and $I - A$ are strictly positive on $\mathfrak{N}$. Now we define $n(f_k) = a(f_k)^* a(f_k)$, $k = 1, \ldots, 2p$ and let

$$x = n(f_1) \cdot \cdots \cdot n(f_p) \cdot (I - n(f_{p+1})) \cdot \cdots \cdot (I - n(f_{2p})).$$

By (ii) above, we have that x is in $_pI_p$ since y is, i.e., the left (right) ordering procedures will result in polynomials of minimal degree p in creators (destructors) respectively.

Finally, letting $\lambda_1, \ldots, \lambda_{2p}$ be the eigenvalues corresponding to $f_1, \ldots, f_{2p}$ we have

$$\omega_A(x) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_p \cdot (1 - \lambda_{p+1}) \cdot (1 - \lambda_{p+2}) \cdot \cdots \cdot (1 - \lambda_{2p}) > 0.$$

Thus $\Pi_A(x) \neq 0$ or $\Pi_A(_nI_m) \neq 0$, giving a contradiction. Done.

In this section, we have characterized factor states of $\mathcal{C}^\circ(\mathcal{K})$ which were obtained from gauge invariant generalized free states of $\mathcal{C}(\mathcal{K})$. In the following section, we turn our attention to the question of quasi-equivalence for two such states of $\mathcal{C}^\circ(\mathcal{K})$.

4. Quasi-equivalence of representations. In this section, we wish to prove

THEOREM. *Let ω_A° and ω_B° be factor states of $\mathcal{C}^\circ(\mathcal{K})$, that is, $\mathrm{Tr}\, A(I - A) = \mathrm{Tr}\, B(I - B) = \infty$. Then $\omega_A^\circ \sim_q \omega_B^\circ$ if and only if $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are of Hilbert-Schmidt class.*

To do so, we will make use of the fact that $\|\omega_A^\circ - \omega_B^\circ\| < 2$ implies $\omega_A^\circ \sim_q \omega_B^\circ$, by the theorem below and Proposition 3.25. (See (i) if and only if (iv).)

THEOREM 4.1 [4, THEOREM 4.5]. *Let $\mathcal{C} = \overline{\bigcup_n \mathcal{C}_n}$ and let Π_1 and Π_2 be two factor representations of $\mathcal{C}$ such that $\ker \Pi_1 = \ker \Pi_2$. Let ω_1 and ω_2 be vector spaces of Π_1 and Π_2 respectively. Then the following statements are equivalent:*

(i) *Π_1 and Π_2 are quasi-equivalent.*

(ii) *For all $\varepsilon > 0$ there is an integer r such that*

$$|\omega_1(x) - \omega_2(x)| < \varepsilon \|\Pi_1(x)\| \quad \text{for all } x \text{ in } \mathcal{C}_r^c.$$

(iii) *For all $\varepsilon > 0$ there is a finite-dimensional *-algebra $\mathcal{B} \subset \mathcal{C}$ containing e such that*

$$|\omega_1(x) - \omega_2(x)| < \epsilon\|\Pi_1(x)\| \quad \text{for all } x \text{ in } \mathfrak{B}^c.$$

(iv) *There exists a finite-dimensional *-algebra $\mathcal{C} \subset \mathcal{C}$ containing e such that* $\text{Sup}\{|\omega_1(x) - \omega_2(x)|: x \text{ in } \mathcal{C}^c, \|\Pi_1(x)\| \leqslant 1\} < 2.$

In previous work ([12], [14]) projection states of the CAR and GICAR algebras were successfully analysed by the device of "doubling" the Hilbert space (from $\mathcal{K}$ to $\mathcal{K} \oplus \mathcal{K}$). Here, we find

$$E_A = \begin{pmatrix} A & A^{1/2}(I-A)^{1/2} \\ A^{1/2}(I-A)^{1/2} & I-A \end{pmatrix}$$

is a projection, i.e., $E_A = E_A^* E_A$, and the restriction of ω_{E_A} (a state of $\mathcal{C}(\mathcal{K} \oplus \mathcal{K})$) to $\mathcal{C}(\mathcal{K} \oplus 0)$ is simply ω_A. Our ability to analyse projection states will suggest a method of proof.

THEOREM 4.2 [12, THEOREM 2.8 AND LEMMA 4.5]. *Let E and F be projections on a Hilbert space $\mathcal{K}$. Let ω_E and ω_F be the associated gauge invariant generalized free states of $\mathcal{C}(\mathcal{K})$. Then $\omega_E \sim \omega_F$ if and only if $E - F$ is Hilbert-Schmidt class. Further, for E_A and E_B as above, $E_A - E_B$ is Hilbert-Schmidt class if and only if $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are Hilbert-Schmidt class.*

The above theorem indicates that projections nearby in Hilbert-Schmidt norm produce equivalent states. Hence we will attempt to construct a path from E_A to E_B with an appropriate Hilbert-Schmidt continuity property; this will give a path from ω_A° to ω_B° as the following diagram indicates:
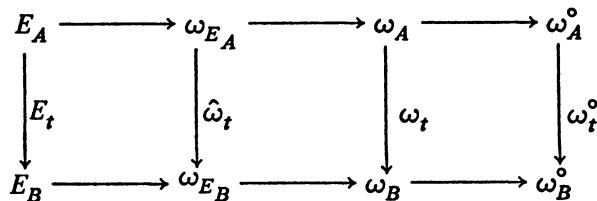
$$
\begin{array}{ccccccc}
E_A & \longrightarrow & \omega_{E_A} & \longrightarrow & \omega_A & \longrightarrow & \omega_A^\circ \\
\downarrow{\scriptstyle E_t} & & \downarrow{\scriptstyle \hat{\omega}_t} & & \downarrow{\scriptstyle \omega_t} & & \downarrow{\scriptstyle \omega_t^\circ} \\
E_B & \longrightarrow & \omega_{E_B} & \longrightarrow & \omega_B & \longrightarrow & \omega_B^\circ
\end{array}
$$

FIGURE 2. Deformation diagram

where $\hat{\omega}_t$ is defined to be ω_{E_t}, the state obtained from E_t in the usual way, $\omega_t = \hat{\omega}_t|\mathcal{C}(\mathcal{K} \oplus 0)$, and $\omega_t^\circ = \omega_t|\mathcal{C}^\circ(\mathcal{K} \oplus 0)$. Finally, the continuity properties of the original path will carry over to the restriction paths, and the quasi-equivalence of ω_A° and ω_B° will follow from these. We begin, then, by studying pairs of projections, in order to construct the desired path E_t.

LEMMA 4.3. *Let E and F be projections on a Hilbert space. Then*
(i) $E - F$ *is selfadjoint.*

(ii) *If $E - F$ is Hilbert-Schmidt, then it is compact with discrete spectrum contained in $[-1, 1]$.*

PROOF. $E - F$ is selfadjoint since E and F are, and $E - F$ is Hilbert-Schmidt if and only if $\text{Tr}(E - F)^2 < \infty$ which implies $E - F$ is compact, and hence has pure point spectrum. Further

$$(f, (E - F)f) = (f, Ef) - (f, Ff) = \|Ef\|^2 - \|Ff\|^2$$

which implies $-1 \leqslant (f, (E - F)f) \leqslant 1$. Done.

LEMMA 4.4. *Let E and F be projections on a Hilbert space $\mathcal{K}$, such that $E - F$ is Hilbert-Schmidt class. Then we can write*

$$E = E^{(0)} + E^{(1)}, \qquad F = F^{(0)} + F^{(1)},$$

so that

(i) $E^{(0)}, E^{(1)}, F^{(0)}, F^{(1)}$ *are projections and $E^{(0)}E^{(1)} = E^{(1)}F^{(1)} = F^{(0)}F^{(1)} = 0$.*

(ii) *The dimensions of the ranges of $E^{(1)}$ and $F^{(1)}$ are finite.*

(iii) $\|E^{(0)} - F^{(0)}\| < 1$.

PROOF. Since $E - F$ is selfadjoint and compact, it follows that differing nonzero eigenvalues of $E - F$ correspond to mutually orthogonal finite-dimensional eigenspaces. That is, defining $\mathcal{K}_\alpha = \{f \in \mathcal{K}: (E - F)f = \alpha f\}$ and $n_\alpha = \dim(\mathcal{K}_\alpha)$, we have $\mathcal{K}_\alpha \perp \mathcal{K}_{\alpha'}$, $\alpha \neq \alpha'$, and $n_\alpha < \infty$ for all $\alpha \neq 0$ in $[-1, 1]$. Now we observe that, for all $f \in \mathcal{K}_{\pm 1}$ we have $(E - F)f = \pm f$ implies

$$\|Ef\| = 1 \quad \text{and} \quad \|Ff\| = 0,$$

$$\|Ff\| = 1 \quad \text{and} \quad \|Ef\| = 0,$$

respectively. That is,

$$f \in \begin{cases} \text{Ran } E \cap (\text{Ran } F)^\perp = \mathcal{K}_1, \\ \text{Ran } F \cap (\text{Ran } E)^\perp = \mathcal{K}_{-1}, \end{cases}$$

respectively. Hence we define $E^{(1)}, F^{(1)}$ as the projections on $\mathcal{K}_{+1}, \mathcal{K}_{-1}$, respectively, and $E^{(0)}, F^{(0)}$ as $E - E^{(1)}, F - F^{(1)}$, respectively. Since we have split off the spaces where $(E - F)f = \pm f$, and clearly $|(g, (E - F)g)| < 1$ for all $g \in \mathcal{K}$, we observe $\|E^{(0)} - F^{(0)}\| < 1$, giving (iii). Finally, by the definitions of $E^{(0)}, E^{(1)}, F^{(0)}, F^{(1)}$ and the first remark of the proof, we have (i) and (ii). Done.

LEMMA 4.5. *See, e.g., [14, p. 74]. Let $E^{(0)}$ and $F^{(0)}$ be projections on a Hilbert space $\mathcal{K}$ such that $\|E^{(0)} - F^{(0)}\| < 1$ and $E^{(0)} - F^{(0)}$ is Hilbert-Schmidt class. Then we can decompose $\mathcal{K}$ into orthogonal subspaces $\mathcal{K}_{u_i}$, $i = 1, 2, \ldots,$ such that*

(i)(a) *The $\mathcal{K}_{u_i}$ are one-dimensional (corresponding to independent vectors of the eigenspace $\mathcal{K}_0$ of $E^{(0)} - F^{(0)}$), or*

(b) *the $\mathcal{K}_{\pm u_i}$ are 2n-dimensional (corresponding to n pairs of nonzero eigenvalues $\pm u_i$ of $E^{(0)} - F^{(0)}$ with $0 < u_i < 1$ and n the finite multiplicity of u_i).*

(ii) *The projections $E^{(0)}$ and $F^{(0)}$ are reduced by the $\mathcal{K}_{u_i}$.*

(iii) *The projections $E_i = E^{(0)}|\mathcal{K}_{u_i}$ and $F_i = F^{(0)}|\mathcal{K}_{u_i}$ are*

(a) *rank one or zero for $u_i = 0$,*

(b) *rank one for $u_i \neq 0$.*

PROOF. For convenience, we reproduce the argument in [14]. For (i)(b), let $0 < |u| < 1$ and suppose $(E^{(0)} - F^{(0)})f = uf$, with $f \neq 0$. Let $g = -(E^{(0)} - F^{(0)} - uI)E^{(0)}f$. This is (1) an eigenvector of $E^{(0)} - F^{(0)}$ and (2) nonzero since

$$
(1) \quad
\begin{aligned}
(E^{(0)} - F^{(0)})g &= -(E^{(0)} - F^{(0)})^2 E^{(0)}f + u(E^{(0)} - F^{(0)})E^{(0)}f \\
&= -u(-E^{(0)} + F^{(0)} + uI)E^{(0)}f = -ug
\end{aligned}
$$

and

$$
(2) \quad
\begin{aligned}
(E^{(0)} - F^{(0)})f = uf \quad &\text{implies} \quad E^{(0)}f = F^{(0)}f + uf \\
&\text{implies} \quad g = -u(1 - u)f + 2uF^{(0)}f;
\end{aligned}
$$

hence if $g = 0$, either $u = 0$ or $F^{(0)}f = (1 - u)f/2$, in which case $u = 1$. Since either contradicts the assumption $0 < u < 1$, we have $g \neq 0$. Now note that

$$
\begin{aligned}
(f, g) &= -(f, (E^{(0)} - F^{(0)} - uI)E^{(0)}f) \\
&= -((E^{(0)} - F^{(0)} - uI)f, E^{(0)}f) = (uf - uf, E^{(0)}f) = 0;
\end{aligned}
$$

hence $f \perp g$. Choosing mutually orthogonal vectors in $\mathcal{K}_{u_i}$ until the space is exhausted, we obtain n pairs of eigenvectors, corresponding to eigenvalues $\pm u$, where n is the finite multiplicity of u_i. Hence $\mathcal{K}_{\pm u_i}$ is 2n-dimensional, completing (i)(b). For (i)(a), we note that an orthonormal basis for the zero eigenspace of $E^{(0)} - F^{(0)}$ gives the desired one-dimensional spaces.

To show (ii) for the case $0 < u_i < 1$, we solve $(E^{(0)} - F^{(0)})f_i = u_i f_i$ for $E^{(0)}f$ and substitute in $g_i = -(E^{(0)} - F^{(0)} - u_i I)E^{(0)}f_i$ to obtain $E^{(0)}f_i = \frac{1}{2}(1 + u_i)f_i + (1/2u_i)g_i$ and $F^{(0)}f_1 = \frac{1}{2}(1 - u_i)f_i + (1/2u_i)g_i$. If $u_i = 0$, we note $E^{(0)}f = F^{(0)}f$, and thus $[E^{(0)}|\mathcal{K}_0, F^{(0)}|\mathcal{K}_0] = 0$. Hence the orthonormal basis for $\mathcal{K}_0$ may be chosen to diagonalize simultaneously $E^{(0)}$ and $F^{(0)}$, completing (ii).

Finally, for (iii) we argue that for $0 < u_i < 1$ the E_i, F_i are rank one projections since

$$E_i = \begin{cases} 0|\mathcal{K}_{u_i} \\ I|\mathcal{K}_{u_i} \end{cases} \quad \text{or} \quad F_i = \begin{cases} 0|\mathcal{K}_{u_i} \\ I|\mathcal{K}_{u_i} \end{cases}$$

are precluded by the inequality $0 < u_i < 1$, together with (1) and (2). For $u_i = 0$, the basis chosen above consists of eigenvectors for both $E^{(0)}$ and $F^{(0)}$; since they are projections the eigenvalues are zero or one, hence the corresponding E_i, F_i are rank zero or one respectively–completing (iii) and the lemma.

We are now ready to construct a path from $E^{(0)}$ to $F^{(0)}$, by rotation.

LEMMA 4.6. *Let $E^{(0)}$, $F^{(0)}$ be as above. There is a path $E_t^{(0)}$, $t \in [-1, 1]$, such that $E_{-1}^{(0)} = E^{(0)}$, $E_1^{(0)} = F^{(0)}$, and $\|E_r^{(0)} - E_s^{(0)}\|_{\text{H.S.}} \to 0$ as $r \to s$.*

PROOF. For i such that $u_i \neq 0$, we have E_i, F_i rank one. Hence there are vectors h_i, k_i such that $\text{Ran}(E_i) = \text{span}\{h_i\}$, $\text{Ran}(F_i) = \text{span}\{k_i\}$. An easy computation shows that for such rank one operators, $\|E_i - F_i\|_{\text{H.S.}}^2 = 2 - 2|(h_i, k_i)|^2$. Since

$$\frac{1}{\|f_i\|^2}(f_i, (E_i - F_i)^2 f_i) = \frac{1}{\|g_i\|^2}(g_i, (E_i - F_i)^2 g_i) = u_i^2$$

we have $\|E_i - F_i\|_{\text{H.S.}}^2 = 2u_i^2 = 2 - 2|(h_i, k_i)|^2$, and letting $|(h_i, k_i)|^2 = \cos^2\theta_i$, we get $u_i = \sin\theta_i$. As shown in the figure, we rotate k_i into h_i by varying the angle from, e.g., $(t + 1)\theta_i/2 \to \theta_i$, $t \in [-1, 1]$. Defining $k_{i,t}$ in the obvious way, we then define $E_{i,t}$ to be the projection onto $k_{i,t}$. Then

$$\|E_{i,r} - E_{i,s}\|_{\text{H.S.}}^2 = 2 - 2\cos^2\frac{(r-s)}{2}\theta_i = \sin^2\frac{(r-s)}{2}\theta_i;$$

clearly, we have $E_{i,-1} = E_i$, $E_{i,1} = F_i$ and $\|E_{i,r} - E_{i,s}\|_{\text{H.S.}}^2 \to 0$ as $r \to s$. Now define

$$E_r^{(0)} = E^{(0)}|\mathcal{K}_0 + \sum_{\{i:u_i \neq 0\}} E_{i,r} = F^{(0)}|\mathcal{K}_0 + \sum_{\{i:u_i \neq 0\}} E_{i,r}.$$

Then $E_{-1}^{(0)} = E^{(0)}$, $E_1^{(0)} = F^{(0)}$ and

$$\|E_r^{(0)} - E_s^{(0)}\|_{\text{H.S.}}^2 = \sum_{i=1}^{\infty} 2\sin^2\frac{(r-s)}{2}\theta_i.$$

Since $E^{(0)} - F^{(0)}$ is Hilbert-Schmidt, $\sum_{i=1}^{\infty} u_i^2 = \sum_{i=1}^{\infty} \sin^2\theta_i < \infty$. Further, for $0 < x < \pi/2$, $2x/\pi < \sin x < x$ and $4x^2/\pi^2 < \sin^2 x < x^2$, so $\sum_{i=1}^{\infty} \sin^2\theta_i < \infty$ implies $\sum_{i=1}^{\infty} \theta_i^2 = M < \infty$.

Finally,

$$\|E_r^{(0)} - E_s^{(0)}\|_{\text{H.S.}}^2 = \sum_{i=1}^{\infty} 2 \sin^2 \frac{(r-s)}{2} \theta_i < \frac{(r-s)^2}{2} M \to 0$$

as $r \to s$. Done.

For convenience we state

LEMMA 4.7 [12, LEMMA 4.1]. *Let A and B be positive operators on a Hilbert space $\mathcal{K}$. Then*

$$\|A^{1/2} - B^{1/2}\|_{\text{H.S.}}^2 \leqslant \|A - B\|_{\text{TR.}}.$$

LEMMA 4.8. *Let E and F be projections on a Hilbert space $\mathcal{K}$, with $E - F$ Hilbert-Schmidt class. Define $E = E^{(0)} + E^{(1)}$, $F = F^{(0)} + F^{(1)}$ as in Lemma 4.4. Then there are operators E_t, $t \in [-2, 2]$, such that*
  (i) $E_{-2} = E$, $E_2 = F$.
  (ii) $0 < E_t < I$.
  (iii) $\|E_r - E_s\|_{\text{H.S.}} \to 0$ *as $r \to s$, for all r, s in $[-1, 1]$.*
  (iv) $\|E_r^{1/2} - E_s^{1/2}\|_{\text{H.S.}} \to 0$, $\|(I - E_r)^{1/2} - (I - E_s)^{1/2}\|_{\text{H.S.}} \to 0$ *as $r \to s$, for all r, s in $[-2, 1]$ or $[1, 2]$.*

PROOF. We construct paths, with the above continuity properties as follows:

$$E_t = \begin{cases} E^{(0)} - (t+1)E^{(1)}, & t \in [-2, -1], \\ E_t^{(0)}, & t \in [-1, 1], \\ F^{(0)} + (t-1)F^{(1)}, & t \in [1, 2], \end{cases}$$

with $E_t^{(0)}$ as in the previous lemma, whose conclusion gives (ii). First, we note the E_t are clearly positive by definition giving (i). Now, applying Lemma 4.7 we obtain, for $r, s \in [-2, -1]$,

$$\|E_r^{1/2} - E_s^{1/2}\|_{\text{H.S.}}^2 \leqslant \|E_r - E_s\|_{\text{TR.}}$$

$$= \|(s-r)E^{(1)}\|_{\text{TR.}} = |(s-r)|n_1 \to 0 \quad \text{as } r \to s,$$

recalling n_1 to be the (finite) dimension of the range of $E^{(1)}$. Likewise, we have

$$\|(I - E_r)^{1/2} - (I - E_s)^{1/2}\|_{\text{H.S.}}^2 \leqslant \|(I - E_r) - (I - E_s)\|_{\text{TR.}} = \|E_r - E_s\|_{\text{TR.}}$$

$$= |(s-r)|n_1 \to 0 \quad \text{as } r \to s.$$

The identical argument for the F_r yields, for $r, s \in [1, 2]$,

$$\|F_r^{1/2} - F_s^{1/2}\|_{\text{H.S.}} \to 0 \quad \text{as } r \to s,$$

$$\|(I - F_r)^{1/2} - (I - F_s)^{1/2}\|_{\text{H.S.}} \to 0 \quad \text{as } r \to s.$$

Hence, we have (iii). Done.

SUMMARY. We have a path from E to F as shown below.

$$E = E^{(0)} + E^{(1)} \longrightarrow E^{(0)}$$
$$\downarrow$$
$$F = F^{(0)} + F^{(1)} \longleftarrow F^{(0)}$$

FIGURE 3.   Projection deformation

We are now ready to show the inherited state norm continuity. We will need, however, the following results of [12].

THEOREM 4.9 [12, THEOREM 2.6]. *Suppose ω_E and ω_F are pure generalized free states of $\mathcal{C}(\mathcal{K})$. Let $\alpha_1 = \det(I - E(I - F)E)$, and $\alpha_2 = \det(I - (I - E)F(I - E))$, and $\alpha = \min(\alpha_1, \alpha_2)$. Then $\|\omega_E - \omega_F\| = 2(1 - \alpha)^{1/2}$.*

LEMMA 4.10 [12, LEMMA 4.7]. *Let ω_A and ω_B be gauge invariant generalized free states of $\mathcal{C}(\mathcal{K})$. Suppose $0 < \varepsilon < 2$. Then if $\|A^{1/2} - B^{1/2}\|_{\text{H.S.}} < \varepsilon/12$ and $\|(I - A)^{1/2} - (I - B)^{1/2}\|_{\text{H.S.}} < \varepsilon/12$, then $\|\omega_A - \omega_B\| < \varepsilon$.*

Now we show

LEMMA 4.11. *Let E_A, E_B be projections on $\mathcal{K} \oplus \mathcal{K}$, defined as above. Suppose $E_A - E_B$ is Hilbert-Schmidt class. We define E_t as in Lemma 4.8, setting $E_A = E$ and $E_B = F$, and a gauge invariant generalized free state of $\mathcal{C}(\mathcal{K} \oplus \mathcal{K})$, $\hat{\omega}_t$ by the formula $\hat{\omega}_t = \omega_{E_t}$. Also, we define*

$$\omega_t = \omega_t|\mathcal{C}(\mathcal{K} \oplus 0), \qquad \omega_t^\circ = \omega_t|\mathcal{C}^\circ(\mathcal{K} \oplus 0).$$

Then
  (i) *$\omega_{-2}^\circ = \omega_A^\circ$ and $\omega_2^\circ = \omega_B^\circ$.*
  (ii) *$\|\omega_r^\circ - \omega_s^\circ\| \to 0$ as $r \to s$, for all $r, s \in [-2, 2]$.*

PROOF. For (i), we simply observe that $\hat{\omega}_{-2} = \omega_{E_{-2}} = \omega_{E_A}$ and

$$\omega_{-2} = \hat{\omega}_{-2}|\mathcal{C}(\mathcal{K} \oplus 0) = \omega_A, \qquad \omega_{-2}^\circ = \hat{\omega}_{-2}|\mathcal{C}^\circ(\mathcal{K} \oplus 0) = \omega_A^\circ.$$

Likewise, $\omega_2^\circ = \omega_B^\circ$. For (ii) we first consider $r, s \in [-1, 1]$. Then by Theorem 4.9 above, we have

$$\|\hat{\omega}_r - \hat{\omega}_s\| = \|\omega_{E_r} - \omega_{E_s}\| = 2(1 - \alpha)^{1/2}$$

with

$$\alpha = \min(\det(I - (I - E_r)E_s(I - E_r)), \det(I - E_r(I - E_s)E_r)).$$

But since $\det(I - A) \geqslant 1 - \operatorname{Tr}(A)$ for $0 \leqslant A \leqslant I$ we obtain

$$\alpha \geqslant \det(I - (I - E_r)E_s(I - E_r)) \geqslant 1 - \operatorname{Tr}((I - E_r)E_s(I - E_r)).$$

By positivity of $E_r(I - E_s)E_r$ we have

$$\alpha \geqslant 1 - \operatorname{Tr}((I - E_r)E_s(I - E_r) + E_r(I - E_s)E_r)$$

$$= 1 - \operatorname{Tr}((E_r - E_s)^2) = 1 - \|E_r - E_s\|_{\text{H.S.}}^2.$$

Hence, for all $r, s \in [-1, 1]$, $\|\hat{\omega}_r - \hat{\omega}_s\| = 2(1 - \alpha)^{1/2} \leqslant 2\|E_r - E_s\|_{\text{H.S.}} \to 0$ as $r \to s$, by Lemma 4.8. Now consider $r, s \in [-2, -1]$ or $r, s \in [1, 2]$. Again $\|\hat{\omega}_r - \hat{\omega}_s\| = \|\omega_{E_r} - \omega_{E_s}\|$ and from Lemma 4.8(iii), we have $\|E_r^{1/2} - E_s^{1/2}\|_{\text{H.S.}} \to 0$ and $\|(I - E_r)^{1/2} - (I - E_s)^{1/2}\|_{\text{H.S.}} \to 0$. But by direct application of Lemma 4.10, these imply $\|\omega_{E_r} - \omega_{E_s}\| \to 0$ as $r \to s$. Finally, $\|\hat{\omega}_r - \hat{\omega}_s\| \geqslant \|\omega_r - \omega_s\| \geqslant \|\omega_r^\circ - \omega_s^\circ\|$ by the properties of restriction. Thus $\|\omega_r^\circ - \omega_s^\circ\| \to 0$ as $r \to s$, for all $r, s \in [-2, 2]$, completing (ii). Done.

Of use in proving our final results in this section will be

THEOREM 4.12 [12, THEOREM 5.1]. *Every gauge invariant generalized free state of $\mathcal{Q}(\mathcal{K})$ is a factor state. Two gauge invariant generalized free states ω_A and ω_B are quasi-equivalent if and only if the operators $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are of Hilbert-Schmidt class.*

We will also use the following two lemmas.

LEMMA 4.13 [12, LEMMA 2.3]. *Suppose ω_1 and ω_2 are even states of $\mathcal{Q}(\mathcal{K})$ and $\mathfrak{M}$ is a finite-dimensional subspace of $\mathcal{K}$. Then*

$$\|\omega_1|\mathcal{Q}(\mathfrak{M})^c - \omega_2|\mathcal{Q}(\mathfrak{M})^c\| = \|\omega_1|\mathcal{Q}(\mathfrak{M}^\perp) - \omega_2|\mathcal{Q}(\mathfrak{M}^\perp)\|.$$

REMARK. We note that generalized free states are defined to be even.

LEMMA 4.14 [12, LEMMA 4.5]. *Let A and B be operators on a Hilbert space $\mathcal{K}$ such that $0 \leqslant A \leqslant I$ and $0 \leqslant B \leqslant I$. Then $E_A - E_B$ is a Hilbert-Schmidt class operator if and only if the operators $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are of Hilbert-Schmidt class.*

We are now ready to prove

THEOREM 4.15. *Let A and B be operators with point spectrum such that $0 < A < I$, $0 < B < I$ and $\operatorname{Tr} A(I - A) = \operatorname{Tr} B(I - B) = \infty$. Let ω_A° and ω_B° be the corresponding factor series of $\mathcal{Q}^\circ(\mathcal{K})$. Then $\omega^\circ \sim_q \omega_B^\circ$ if and only if $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are of Hilbert-Schmidt class.*

PROOF. *Sufficiency.* Suppose $A^{1/2} - B^{1/2}$ and/or $(I - A)^{1/2} - (I - B)^{1/2}$ are not Hilbert-Schmidt. Then by Theorem 4.12 above, we have $\omega_A \nsim_q \omega_B$, where ω_A and ω_B are the gauge invariant generalized free states of the whole CAR, $\mathcal{C}(\mathcal{K})$. Hence, for all finite-dimensional $\mathfrak{M} \subset \mathcal{K}$, we have

(1)
$$\|(\omega_A - \omega_B)|\mathcal{C}(\mathfrak{M})^c\| = 2$$

by, e.g., Lemma 3.23 above. Now, by Lemma 3.9, $\mathcal{C}^\circ(\mathfrak{M})^{c_0} \subset \mathcal{C}^\circ(\mathfrak{M}^\perp)$, so we have

(2)
$$\|(\omega_A^\circ - \omega_B^\circ)|\mathcal{C}^\circ(\mathfrak{M})^{c_0}\| \geqslant \|(\omega_A^\circ - \omega_B^\circ)|\mathcal{C}^\circ(\mathfrak{M}^\perp)\|$$

and from the definitions, we conclude

(3) $\|(\omega_A^\circ - \omega_B^\circ)|\mathcal{C}^\circ(\mathfrak{M}^\perp)\| = \|(\omega_A - \omega_B)|\mathcal{C}^\circ(\mathfrak{M}^\perp)\| = \|(\omega_A - \omega_B)|\mathcal{C}(\mathfrak{M}^\perp)\|.$

Finally from Lemma 4.13 we have

(4)
$$\|(\omega_A - \omega_B)|\mathcal{C}(\mathfrak{M}^\perp)\| = \|(\omega_A - \omega_B)|\mathcal{C}(\mathfrak{M})^c\|,$$

since generalized free states are necessarily even. Combining (1) through (4) gives

$$\|(\omega_A^\circ - \omega_B^\circ)|\mathcal{C}^\circ(\mathfrak{M})^{c_0}\| \geqslant 2;$$

hence by Theorem 4.1 and Proposition 3.24 we conclude $\omega_A^\circ \nsim_q \omega_B^\circ$. Done with sufficiency.

Necessity. Suppose $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are Hilbert-Schmidt. It follows from above that $E_A - E_B$ is Hilbert-Schmidt. Now we invoke the path ω_t of Lemma 4.11. Recall $\|\omega_r - \omega_s\| \to 0$ as $r \to s$, for all $r, s \in [-2, 2]$, and $\omega_{-2} = \omega_A$, $\omega_2 = \omega_B$. It follows that we can choose a partition of $[-2, 2]$, say $\{t_k\}$, $1 \leqslant k \leqslant N$, such that $t_1 = -2$ and $t_N = 2$ and $\|\omega_{t_{k+1}} - \omega_{t_k}\| < 2$, $1 \leqslant k \leqslant N - 1$. Since the ω_t are restrictions of the gauge invariant generalized free states ω_{E_t}, they are likewise generalized free states, and hence factors, by Theorem 4.12 above. Now observe that since $\omega_{t_1} = \omega_A$ and $\mathrm{Tr}\, A(I - A) = \infty$, ω_{t_1} is not type I by Proposition 3.22. But by Lemma 3.23, since $\|\omega_{t_{k+1}} - \omega_{t_k}\| < 2$, we have $\omega_{t_{k+1}} \sim_q \omega_{t_k}$, $1 \leqslant k \leqslant N - 1$. Thus the ω_{t_k} are not type I, and the positive operators A_{t_k} defined by the two-point function of ω_{t_k} have the property that $\mathrm{Tr}\, A_{t_k}(I - A_{t_k}) = \infty$, again by Proposition 3.22. We can then conclude, by Theorem 3.24, that $\omega_{t_k}^\circ$ is a factor state of $\mathcal{C}^\circ(\mathcal{K})$, $1 \leqslant k \leqslant N$. Finally, we have $\omega_{-2}^\circ = \omega_A^\circ$, $\omega_2^\circ = \omega_B^\circ$ and $\|\omega_{t_{k+1}}^\circ - \omega_{t_k}^\circ\| < 2$, the latter by restriction, $1 \leqslant k \leqslant N - 1$. Thus $\omega_{t_k}^\circ \sim_q \omega_{t_{k+1}}^\circ$, $1 \leqslant k \leqslant N - 1$ and since quasi-equivalence is an equivalence relation, $\omega_{t_1}^\circ \sim_q \omega_{t_N}^\circ$ or $\omega_A^\circ \sim_q \omega_B^\circ$. Done.

To remove the point spectrum condition, we will make use of Lemma 4.10 and the following lemma.

LEMMA 4.16 [12, LEMMA 4.4]. *Let A be an operator such that $0 \leqslant A \leqslant I$, and let $\varepsilon > 0$. Then there exists an operator B, $0 \leqslant B \leqslant I$, with pure point spectrum such that $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are of Hilbert-Schmidt class with Hilbert-Schmidt norm less than ε. Furthermore, the eigenvalues of B are dense in the spectrum of A.*

THEOREM 4.17. *Let A and B be operators such that $0 < A < I$, $0 < B < I$ and $\mathrm{Tr}\, A(I - A) = \mathrm{Tr}\, B(I - B) = \infty$. Let ω_A° and ω_B° be the corresponding factor states of $\mathfrak{C}^\circ(\mathcal{K})$. Then $\omega_A^\circ \sim_q \omega_B^\circ$ if and only if $A^{1/2} - B^{1/2}$ and $(I - A)^{1/2} - (I - B)^{1/2}$ are of Hilbert-Schmidt class.*

PROOF. *Sufficiency.* Same proof as Theorem 4.15, sufficiency.

Necessity. By Lemma 4.16 there exist operators $\hat{A}$ and $\hat{B}$, each with pure point spectrum, such that

$$\|A^{1/2} - \hat{A}^{1/2}\|_{\text{H.S.}} < 1/6 \qquad\qquad \|B^{1/2} - \hat{B}^{1/2}\|_{\text{H.S.}} < 1/6,$$

$$\|(I - A)^{1/2} - (I - \hat{A})^{1/2}\|_{\text{H.S.}} < 1/6, \qquad \|(I - B)^{1/2} - (I - \hat{B})^{1/2}\|_{\text{H.S.}} < 1/6.$$

Further, by Lemma 4.10 the associated states of $\mathfrak{C}(\mathcal{K})$, $\omega_{\hat{A}}$ and $\omega_{\hat{B}}$, have the properties $\|\omega_A - \omega_{\hat{A}}\| < 2$, $\|\omega_B - \omega_{\hat{B}}\| < 2$; hence, by Lemma 3.23, $\omega_A \sim_q \omega_{\hat{A}}$ and $\omega_B \sim_q \omega_{\hat{B}}$. By hypothesis and the above Proposition 3.22, ω_A and ω_B are not type I, and therefore $\omega_{\hat{A}}$ and $\omega_{\hat{B}}$ are not type I by quasi-equivalence. Thus $\mathrm{Tr}\,\hat{A}(I - \hat{A}) = \mathrm{Tr}\,\hat{B}(I - \hat{B}) = \infty$, which implies $\omega_{\hat{A}}^\circ$ and $\omega_{\hat{B}}^\circ$ are factor states of $\mathfrak{C}^\circ(\mathcal{K})$ by Theorem 3.24. Further

$$\|\hat{A}^{1/2} - \hat{B}^{1/2}\|_{\text{H.S.}} = \|\hat{A}^{1/2} - A^{1/2} + A^{1/2} - B^{1/2} + B^{1/2} - \hat{B}^{1/2}\|_{\text{H.S.}}$$

$$\leqslant 1/6 + \|A^{1/2} - B^{1/2}\|_{\text{H.S.}} + 1/6 < \infty$$

by hypothesis. Likewise $\|(I - \hat{A})^{1/2} - (I - \hat{B})^{1/2}\|_{\text{H.S.}} < \infty$, and by the previous theorem, $\omega_{\hat{A}}^\circ \sim_q \omega_{\hat{B}}^\circ$. Hence we have $\omega_A^\circ \sim_q \omega_{\hat{A}}^\circ \sim_q \omega_{\hat{B}}^\circ \sim_q \omega_B^\circ$ or $\omega_A^\circ \sim_q \omega_B^\circ$. Done necessity, and theorem.

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