TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 238, April 1978

BENDER GROUPS AS STANDARD SUBGROUPS

BY

ROBERT L. GRIESS, JR.[1] DAVID R. MASON AND GARY M. SEITZ[2]

To the memory of Vicente Landázuri Narváez
March 12, 1947–January 29, 1976

Abstract. A subgroup X of a finite group G is called **-standard* if $\tilde{X} = X/O(X)$ is quasisimple, $Y = C_G(\tilde{X})$ is tightly embedded in G and $N_G(X) = N_G(Y)$. This generalizes the notion of standard subgroups.

Theorem. *Let G be a finite group with $O(G) = 1$. Suppose X is *-standard in G and $\tilde{X}/Z(\tilde{X}) \cong L_2(2^n)$, $U_3(2^n)$ or $\mathrm{Sz}(2^n)$. Assume $X \lhd G$. Then $O(X) = 1$ and one of the following holds*:

(i) $E(G) \cong X \times X$.
(ii) $X \cong L_2(2^n)$ *and* $E(G) \cong L_2(2^{2n})$, $U_3(2^n)$ *or* $L_3(2^n)$.
(iii) $X \cong U_3(2^n)$ *and* $E(G) \cong L_3(2^{2n})$.
(iv) $X \cong \mathrm{Sz}(2^n)$ *and* $E(G) \cong \mathrm{Sp}(4, 2^n)$.
(v) $X \cong L_2(4)$ *and* $E(G) \cong M_{12}, A_9, J_1, J_2, A_7, L_2(25), L_3(5)$ *or* $U_3(5)$.
(vi) $X \cong \mathrm{Sz}(8)$ *and* $E(G) \cong \mathrm{Ru}$ (*the Rudvalis group*).
(vii) $X \cong L_2(8)$ *and* $E(G) \cong G_2(3)$.
(viii) $X \cong \mathrm{SL}(2, 5)$ *and G has sectional* 2-*rank at most* 4.

In particular, if G is simple, $G \cong M_{12}, A_9, J_1, J_2$, Ru, $U_3(5)$, $L_3(5)$, $G_2(5)$, *or* ${}^3D_4(5)$.

1. **Introduction.** This paper is concerned with those finite groups G containing a standard subgroup of Bender type. Actually we deal with a more general situation as we allow for cores.

A subgroup X of a finite group G is called *-standard if $\tilde{X} = X/O(X)$ is quasisimple, $Y = C_G(\tilde{X})$ is tightly embedded in G and $N_G(X) = N_G(Y)$. A standard subgroup (in the sense of Aschbacher [1]) is clearly *-standard.

We classify finite groups with a *-standard subgroup of Bender type.

Theorem. *Let G be a finite group with $O(G) = 1$. Suppose X is *-standard in G and $\tilde{X}/Z(\tilde{X}) \cong L_2(2^n)$, $U_3(2^n)$, or $\mathrm{Sz}(2^n)$. Assume that $X \ntrianglelefteq G$. Then $O(X) = 1$ and one of the following holds*:

(i) $E(G) \cong X \times X$.
(ii) $X \cong L_2(2^n)$ *and* $E(G) \cong L_2(2^{2n})$, $U_3(2^n)$, *or* $L_3(2^n)$.
(iii) $X \cong U_3(2^n)$ *and* $E(G) \cong L_3(2^{2n})$.
(iv) $X \cong \mathrm{Sz}(2^n)$ *and* $E(G) \cong \mathrm{Sp}(4, 2^n)$.

Received by the editors July 20, 1976.
AMS (*MOS*) *subject classifications* (1970). Primary 20D05, 20G40.
(1)Research supported in part by N.S.F. Grant 76-07280.
(2)Research supported in part by N.S.F. Grant 37982X1.

(v) $X \cong L_2(4)$ *and* $E(G) \cong M_{12}, A_9, J_1, J_2, A_7, L_2(25), L_3(5)$, *or* $U_3(5)$.

(vi) $X \cong \mathrm{Sz}(8)$ *and* $E(G) \cong \mathrm{Ru}$ (*the Rudvalis group*).

(vii) $X \cong L_2(8)$ *and* $E(G) \cong G_2(3)$.

(viii) $X \cong \mathrm{SL}(2, 5)$, *G has sectional* 2-*rank at most* 4, *so by* [12], $E(G) \cong U_3(5), L_3(5), G_2(5)$, *or* ${}^3D_4(5)$.

In particular, if G is simple then $G \cong M_{12}, A_9, J_1, J_2, \mathrm{Ru}, U_3(5), L_3(5), G_2(5)$, *or* ${}^3D_4(5)$.

Let G and X be as in the main theorem with $X \ntrianglelefteq G$ and let T_0 be a Sylow 2-subgroup of Y. Then, except in cases (v) and (vi), $|T_0| = 2$ and T_0 induces an outer automorphism on $E(G)$. This shows that if X is a standard subgroup and $m(C_G(Y)) > 1$, then the conclusion of the main theorem in [3] holds.

The proof of the main theorem involves a "pushing up" procedure. Starting from a Sylow 2-subgroup of $M = N_G(X)$, we attempt to find a Sylow 2-subgroup of G. At each stage of the procedure there occurs a certain 2-transitive group and this permutation group either has a regular normal 2-subgroup or a normal subgroup isomorphic to $L_3(2)$. In all cases except (vi) and (vii) we show that the latter does not occur. When $E(G) = G_2(3)$ an $L_3(2)$ does occur at the first step in the process, while for $E(G) = \mathrm{Ru}$, a factor of $L_3(2)$ occurs in the second step of the process.

The method of proof eventually reduces us to a situation where we may quote a previous characterization theorem. In particular, we will use the work of Goldschmidt [11] and Gilman and Gorenstein [10] in the identification of $E(G)$. In the exceptional cases (v), (vi) and (vii) we also use Aschbacher [2], Dempwolff [6], Assa [4], O'Nan [19], and Harada [14].

The paper is organized so that §2 contains preliminary lemmas and §3 basic reductions together with the first step of the "pushing up" process. Then §§4, 5, 6 deal with the cases $\tilde{X} \cong L_2(2^n)$, $\mathrm{Sz}(2^n)$, $U_3(2^n)$, respectively.

2. **Preliminaries.** The first lemma deals with tightly embedded subgroups in the automorphism group of a Bender group.

(2.1) *Let X be a simple Bender group and* $X \leqslant Y \leqslant \mathrm{Aut}(X)$. *If* $X \nleqslant F$ *and F is a tightly embedded subgroup of Y, then one of the following holds*:

(i) $F \cap X$ *lies in the normalizer of a Sylow* 2-*subgroup of X, has even order, and contains every involution of F.*

(ii) $F \cap X = 1$, $|F| = 2$, *and F induces a field automorphism on X.*

(iii) $F = (F \cap X)\langle t\rangle$, *where* $|F \cap X|$ *is odd, and t induces a field automorphism of order* 2 *on* $X \cong L_2(4)$ *or* $U_3(2^n)$. *If* $X \cong L_2(4)$, *then* $F \cap X \cong Z_3$, *and if* $X \cong U_3(2^n)$, $F \cap X \neq 1$ *is cyclic of order dividing* $2^n + 1$ *and* $F \cap X$ *centralizes* $E(C_X(t)) \cong L_2(2^n)$.

Proof. Suppose $t \in F \cap X$ is an involution. Then t is central in a Sylow

2-subgroup U of X, so that U normalizes F and $U(F \cap X)$ is a group. It follows that $U(F \cap X) \leqslant N_X(U)$ (see (1.6) of [19]) and, consequently, $F \cap X$ fixes a unique point in the usual 2-transitive permutation representation of X. From here we have $F \leqslant N(U)$ as U is the unique Sylow 2-subgroup of the stabilizer of that point. If $F - (F \cap X)$ contained an involution j, then $C_X(j) \leqslant N(F)$, whereas j must induce a field automorphism of X and $C_X(j)$ does not contain a normal Sylow 2-subgroup. We have now verified (i).

Assume now that $|F \cap X|$ is odd and t is an involution in F. So t induces a field automorphism on X and, by [22], $X \cong L_2(2^n)$ or $U_3(2^n)$. So $C_X(t) \cong L_2(2^{n/2})$ or $L_2(2^n)$, respectively, and this group normalizes F. Let V be a Sylow 2-subgroup of $C_X(t)$. We may assume $C_X(t) \cong L_2(q_0)$ with $q_0 \geqslant 4$, as otherwise the result is trivial. So we may write $F \cap X = \langle C_{F \cap X}(v) \colon v \in V^{\#} \rangle$. If $V \leqslant U \in \mathrm{Syl}_2(X)$, then $C_{F\cap X}(v) \leqslant N_X(U)$ for each $v \in V^{\#}$. Say $F \cap X \neq 1$. Then from the structure of $N_X(U)$ we conclude that $X \cong U_3(2^n)$, $n \geqslant 2$, $F \cap X$ is cyclic of order dividing $2^n + 1$, and $[F \cap X, C_X(t)] = 1$.

In any case $[C_X(t), F] \leqslant F \cap X$, and the above implies $[C_X(t), F] = 1$ for $q_0 \geqslant 4$. This implies that $F = (F \cap X)\langle t \rangle$, and we have either (ii) or (iii).

The next several lemmas deal with 2-groups and their automorphism groups.

(2.2) *Let U be a 2-group of order q^2 and Y a cyclic group of order $q - 1$ acting fixed-point-free on U. Let $V \lhd U$ be Y-invariant and such that U/V and V are elementary and equivalent as $\mathbf{F}_2(Y)$-modules. Then U is abelian.*

PROOF. Higman [17].

(2.3) *Let UY be as in* (2.2) *and suppose that T is a 2-group of order q^2, normalized by UY, $[T, U] \leqslant T \cap U = V$, and Y is fixed-point-free on T. Then one of the following holds*:

(i) $[T, U] = 1$.

(ii) *For any $t \in T - V$, $u \in U - V$, $[t, u] \neq 1$.*

PROOF. This is proved using Lie ring methods. See Dempwolff [6, Lemma 1.1].

(2.4) *Let U be a 2-group and $\langle t \rangle \times Y$ acting on U with t an involution and Y cyclic of order $2^n - 1$. Suppose that Y is regular on $C_U(t)^{\#}$. Then one of the following holds*:

(i) *U is isomorphic to a Sylow 2-subgroup of $L_3(2^n)$.*

(ii) *U is isomorphic to a Sylow 2-subgroup of $U_3(2^n)$.*

(iii) *U is homocyclic of rank n and inverted by t.*

(iv) *U is homocyclic of rank n and each involution in $U\langle t \rangle - U$ is U-conjugate to t.*

(v) *U is elementary abelian of order 2^{2n} and each involution in $U\langle t\rangle - U$ is U-conjugate to t.*

PROOF. This is essentially contained in Finkelstein [8, Lemmas 2.1 and 2.2]. However, instead of (iv) and (v) he simply states that U is abelian and each involution in $U\langle t\rangle - U$ is U-conjugate to t. If U is not homocyclic of rank n, then using the action of Y we have $|\Omega_1(U)| \geqslant 2^{2n}$. As $|C_U(t)| = 2^n$, this must be an equality. Here the only involutions in $\Omega_1(U)\langle t\rangle$ are in $\Omega_1(U)$ or in $tC_U(t)$ and

$$t^U \cap \Omega_1(U)\langle t\rangle = t^{\Omega_1(U)} = tC_U(t).$$

Consequently, $U = \Omega_1(U)$ and (v) holds.

(2.5) *Let $A = A_1 \times A_0$ be an elementary abelian 2-group, $|A_0| = 2$, $A \lhd N$, $R = O_2(N)$. Suppose also that N contains a cyclic subgroup K which operates regularly on $(R/A)^\#$ and on $A_1^\#$. If $C_R(A_0) = A$, then $C_N(A_0)$ covers N/R.*

PROOF. Assume $C_R(A_0) = A$. Then the action of K on A forces $A_1 = Z(R)$. Consequently, if $A_0 = \langle t\rangle$, then $t^N \subseteq A_1t$. On the other hand, the hypotheses force $|R/A| = |A_1|$ and $t^R = A_1t$. The result follows.

The following is a useful result of Goldschmidt.

(2.6) *Let $T \in \mathrm{Syl}_2(G)$, W a weakly closed subgroup of T (with respect to G), and A an abelian subgroup of $C_T(W)$, normal in T. Let $\mathcal{S} = \{B \leqslant T\colon B \nleqslant A$, B is conjugate to a subgroup of $A\}$ and set $r = \max\{m(B/C_B(W))\colon B \in \mathcal{S}\}$. Then either*

(i) *$\Omega_1(A)$ is strongly closed in T (with respect to G); or*

(ii) *there exists $B \in \mathcal{S}$ such that $m(B) + r \geqslant m(A)$; also if $t \in T$ is conjugate to an element of A, then $m([A, t]) \leqslant 2r$, with $m[A, t] \leqslant r$ provided $B/C_B(W)$ is elementary for each $B \in \mathcal{S}$.*

PROOF. Theorem 4 of [11].

The following results are the key to the determination of the Sylow 2-subgroup in a group G satisfying the hypotheses of the main theorem.

We consider groups G satisfying the following.

HYPOTHESIS (*). (1) $R \leqslant G$ is elementary and a Sylow 2-subgroup of a tightly embedded subgroup K of G.

(2) There is a subgroup $X \ntrianglelefteq N_G(R)$ such that $X \leqslant C_G(R)$, and if $U \in \mathrm{Syl}_2(X)$, then U is elementary of order $q = 2^n > 4$, and $N_X(U)/C_X(U)$ is cyclic of order $q - 1$ and is regular on $U^\#$.

(3) For $S \in \mathrm{Syl}_2(N_G(R))$ with $U \times R = V \trianglelefteq S$, S/V is faithful on U.

(2.7) *Assume that G satisfies Hypothesis (*). Then one of the following holds:*

(a) *$S \in \mathrm{Syl}_2(G)$ and V is strongly closed in S.*

(b) $S_1 \in \mathrm{Syl}_2(G)$ *with* $|S_1 : S| = 2$, S_1 *acts on* S *interchanging* U *and* R, *and* V *is strongly closed in* S.

(c) *Each of the following holds*:

(i) $V - U = \bigcup\{(R^{\#})^g : g \in G, R^g \leq V\}$.

(ii) $N(V)$ *is* 2-*transitive of degree* q *on* $\Delta = R^G \cap V$.

(iii) *Either* $N(V)^{\Delta}$ *contains the Frobenius group of order* $q(q-1)$ *as a normal subgroup, or* $q = 8$ *and* $(N(V)^{\Delta})' \cong L_3(2)$.

PROOF. Suppose that G satisfies Hypothesis (*), and that (a) and (b) are false. First note that we may regard U as $\mathbf{F}_q$ with $N_X(U)/C_X(U)$ acting as scalar multiplications and S/V acting as field automorphisms.

We first claim that $S \notin \mathrm{Syl}_2(G)$. Otherwise we set $S = T$, $V = W = A$ in (2.6). As $q > 4$, V is weakly closed in S. So the lemma applies and $r \leq 1$. But for $q > 4$ this is impossible. Consequently, $S \notin \mathrm{Syl}_2(G)$.

As V is weakly closed in S, $N_G(S) \leq N_G(V)$ so V contains more than one conjugate of R. Applying (3.6) of [3] (which is independent of any results in this paper) we have (i) and (ii) provided we can show that $R^G \cap V \neq \{R, U\}$. So suppose this latter case occurs. Let $y \in N(S) - S$ with $y^2 \in S$. Then $U = R^y$. Set $S_1 = S\langle y \rangle$. It is easily checked that V is weakly closed in S_1, and, since $R^G \cap V = \{R, R^y\} = R^{S_1}$, we have $S_1 \in \mathrm{Syl}_2(G)$. Again we appeal to (2.6) to get a contradiction. At this point we have established (i) and (ii).

Now consider the 2-transitive group $N(V)^{\Delta}$. The stabilizer of R in $N(V)$ will normalize X and, hence, will normalize $N_X(U) = N_X(V \cap X)$. This implies (using (2) of Hypothesis (*)) that $N(V)^{\Delta}$ satisfies the conditions of Theorem 1.1 of Hering, Kantor and Seitz [16]. We conclude that either $N(V)^{\Delta}$ has a regular normal subgroup, so that (iii) holds, or $N(V)^{\Delta}$ contains $\mathrm{PSL}(2, p)$ acting in its usual 2-transitive representation of degree $p + 1$. Suppose the latter case holds. Then $p + 1 = q = 2^n$ and p is a Mersenne prime. If we consider $N(V)' \cap N(R)$, then this group acts on U inducing a Frobenius group of order $\frac{1}{2}(p-1)p = p(q-2)(q-1)$. This forces $\frac{1}{2}(q-2)$ to divide n, and hence $n = 3$, completing the proof of (iii).

(2.8) *Suppose that* G *satisfies Hypothesis* (*), V *is not strongly closed in a Sylow* 2-*subgroup of* G, *and that conditions* (i)–(iii) *of* (2.7) *hold with* $(N(V))^{\Delta}$ *containing a regular normal subgroup. Let* D *be a* 2-*complement of* $N_X(U)$. *Then there is a Sylow* 2-*subgroup* V_1 *of* $O_{2',2}(N(V))$ *and a* 2′-*group* D_1 *with the following properties*:

(a) $SD_1 \leq N(V_1)$, $V_1 S \in \mathrm{Syl}_2(N(V))$, *and* D_1 *induces* D *on* V.

(b) $V_1 = U_1 R$ *with* $U_1 \cap R = 1$, *where* $U_1 = [D_1, V_1]$.

(c) $U < U_1$, *and* U_1/U *and* U *are equivalent* $\mathbf{F}_2(D_1)$-*modules*.

PROOF. The existence of V_1 and D_1 satisfying (a) is easy. By (2.7)(i) U is characteristic in V and we consider V_1/U. Suppose that $\Omega_1(V_1/U) = V/U$. Then $V = \Omega_1(V_1)$ is weakly closed in V_1S and $V_1S \in \mathrm{Syl}_2(G)$. However, we can now apply (2.6) to conclude that V is strongly closed in V_1S, contradicting our hypothesis.

So $\Omega_1(V_1/U) > V/U$ and, since D_1 is transitive on $(V_1/V)^\#$, each coset of V/U in V_1/U contains an involution. Since D centralizes R we must have D_1V_1 centralizing V/U. It follows that V_1/U is elementary. From here (b) follows as well as the first claim in (c). Finally we get the last statement in (c) by letting $r \in R^\#$ and noting that the map $u_1U \to [u_1, t]$ is a D_1-homomorphism from U_1/U to U. The proof is complete.

We next make the observation that the above may be repeated. Namely, suppose that Hypothesis (*) holds for G, V is not strongly closed in a Sylow 2-subgroup of G, and $N(V)^\Delta$ contains a regular normal subgroup. Choose D_1 and V_1 as in (2.8) and consider $G_1 = \bar{G} = N_G(U)/U$. Then for $\bar{g} \in G_1$, $\bar{V}^g \leqslant \overline{U_1R}$ implies $\bar{V}^g = \bar{V}$ or $\bar{V}^g \cap \bar{V} = 1$. With this we can argue as in (2.7) and (2.8).

Suppose now that the process is repeated until at some stage either the induced 2-transitive group does in fact contain $L_3(2)$ as a normal subgroup or the analogue of V in G_m is strongly closed in a Sylow 2-subgroup of G_m and (a) or (b) of (2.7) holds. Assume that the process terminates in the latter way. Then there is a subgroup D_m and a 2-group V_m of G such that $[D_m, V_m] = U_m \geqslant U$, R normalizes U_m, $R \cap U_m = 1$, $V_m = U_mR$, each D_m-composition factor of U_m is isomorphic to U and D_m induces D on U. Also $S \leqslant N(U_m)$ and $SU_m \in \mathrm{Syl}_2(N_G(V_{m-1}))$.

For $r \in R^\#$, $C_{U_m}(r) = N_{U_m}(R) = U$, so U_m satisfies the hypotheses of (2.4). With this notation we can conclude:

(2.9) *Let G satisfy Hypothesis* (*) *and suppose that the above process does not yield the $L_3(2)$ case at any stage. Let U_m be as above. Then one of the following holds*:

(1) *U_m is isomorphic to a Sylow 2-subgroup of $U_3(q)$ or $L_3(q)$ and $U_mS \in \mathrm{Syl}_2(G)$.*

(2) *$U < U_m$ which is homocyclic of rank n and $U_mS \in \mathrm{Syl}_2(G)$.*

(3) *V is strongly closed in a Sylow 2-subgroup of G.*

(4) *$U_m = U$ is elementary of order q^2.*

PROOF. We may assume $U < U_m$, as otherwise (3) follows as in (2.7). Also we assume that U_m does not satisfy (v) of (2.4), as otherwise (4) holds. Suppose that U_mS is normalized by an element $y \in G - U_mS$ and $y^2 \in U_mS$. We first show that y normalizes U. If $S = RU$, then $U = Z(U_mS)$ and this is clear. Suppose $S > RU$. If $S' \nleqslant U$, then S' contains an element fused to an element in $R^\#$. In this case $U = C_{U_mS}((U_mS)')$ so y normalizes U as claimed. So we suppose that $S' \leqslant U$ and, hence, $(U_mS)' \leqslant U_m$. If U_m

satisfies (i) or (ii) of (2.4) then $(U_mS)' \geqslant [U_m, R]$ which is homocyclic of order 2^{2n} and of rank n. So here $U = \Omega_1((U_mS)')$ if $(U_mS)' = [U_m, R]$ and $U = Z((U_mS)')$ if $(U_mS)' > [U_m, R]$. Either way $y \in N(U)$. If U_m satisfies (iii) or (iv) of (2.4), then $(U_mS)' > U_{m-1}$, so $U = \Omega_1((U_mS)')$ is normalized by y. So in all cases the claim holds.

In particular, y normalizes $C_{U_mS}(U) = U_mR$. But then y normalizes $(U_mR)' = U_{m-1}$. Hence $U_mS \not\in \mathrm{Syl}_2(N_G(U_{m-1}))$ so that we are in case (b) of (2.7). But then (4) holds. Thus we may assume $U_mS \in \mathrm{Syl}_2(G)$. We complete the proof by using (2.4) to get the structure of U_m.

3. **Initial reductions.** Let G be a finite group having a *-standard subgroup M_1 such that $\tilde{M}_1$ is a Bender group and the conclusions of the main theorem are violated. Choose $|G|$ minimal and M_1 minimal in the group G. Let $M = N_G(M_1)$ and $M_0 = C_M(M_1/O(M_1))$.

Choose $T \in \mathrm{Syl}_2(M)$ and set $T_i = T \cap M_i$, $i = 0, 1$. Then $T = T_1T_0T_3$ where T_3 is cyclic. We set $q = |\Omega_1(\tilde{T}_1)|$, so that $q = 2^m$ and $\tilde{M}_1 \cong L_2(2^m)$, $\mathrm{Sz}(2^m)$, or $U_3(2^m)$, unless $\tilde{M}_1$ is a perfect central extension of Sz(8), when we set $q = 8$, $m = 3$, or $\tilde{M}_1 \cong \mathrm{SL}(2, 5)$, when we set $q = 4$, $m = 2$. Let K_1 be a 2-complement in $N_{M_1}(T_0T_1)$ and $K = K_1^{q-1}$. Finally set $A_i = \Omega_1(T_i)$ and $A = A_1A_0$.

The above notation will be maintained throughout the rest of the paper.

(3.1) $M_1 = \langle C_{M_1}(t)\colon t \in \mathrm{Inv}(T_0)\rangle$.

Proof. $C_{M_1}(T_0)$ covers $M_1/O(M_1)$. So if $m(T_0) > 1$ the result is clear. If $m(T_0) = 1$, then it is easy to check that $C_{M_1}(\Omega_1(T_0))$ is a *-standard subgroup, so by minimality of M_1 we again have the result.

(3.2) $F(G) = 1$.

Proof. By hypothesis we have $O(G) = 1$. Suppose $O_2(G) \neq 1$. For each involution $t \in T_0$, the tight embedding property implies that $C_{O_2(G)}(t)$ centralizes M_1. Now (3.1) and the $P \times Q$ lemma imply that $M_1 \leqslant C_G(O_2(G))$. But then $O_2(G) \leqslant T_0$, so $G \leqslant N(O_2(G)) \leqslant N(M_1)$ and $m_1 \lhd G$, a contradiction.

(3.3) *There does not exist a normal subgroup* $1 < N \trianglelefteq G$ *such that* N *has Sylow* 2-*subgroups of class at most* 2.

Proof. If such an N exists, then using (3.2) and the result of Gilman and Gorenstein [10], the structure of N is known. Consideration of the action of T_0 on $E(N)$ gives a contradiction.

Similarly, we have

(3.4) *G does not contain a normal subgroup* $1 < N \trianglelefteq G$ *such that a Sylow*

2-subgroup S of N contains an abelian subgroup A with A strongly closed in S with respect to N.

PROOF. Use (3.2) and Goldschmidt's theorem [11].

(3.5) (a) $G = \langle T_0^G \rangle$.

(b) $|G : O^2(G)| \leq 2$. *If the index is 2, then $G = O^2(G)T_0$ and $T_0 \cap O^2(G) = 1$. In particular, $|T_0| = 2$ in this case.*

PROOF. Set $G_0 = \langle T_0^G \rangle$ and suppose $G_0 < G$. If $M_1 \cap G_0 \nleq Z^*(M_1)$, then $M_1 \cap G_0$ is a *-standard subgroup in N and, by minimality of G, the structure of $E(G_0)$ is known, from which we have a contradiction.

Suppose that $M_1 \cap G_0 \leq Z^*(M_1)$. We claim that $T_0 \in \mathrm{Syl}_2(G_0)$. Otherwise, let $X > T_0$ be a 2-subgroup of G_0 normalizing T_0. Then $X \leq N(M_1)$, so $[M_1, X] \leq M_1 \cap G_0 \leq Z^*(M_1)$. But this forces $X \leq M_0$, impossible. Consequently, $T_0 \in \mathrm{Syl}_2(G_0)$ and $G = G_0 N_G(T_0) = G_0 M$. It follows that $M \cap G_0$ is strongly embedded in G_0, so using Bender's theorem [5] we have a contradiction. This proves (a).

For (b) use the minimality of G.

(3.6) *There exists $g \in G - M$ such that $1 \neq R = T_0^g \cap M \leq T$.*

(i) $R \cap T_0 = 1$.

(ii) *If $m(T_0) > 1$, then g can be chosen such that $R = T_0^g$.*

(iii) *If $|R| > 2$, then $\Omega_1(R) \leq \Omega_1(T_1)T_0$.*

(iv) *If $m(T_0) > 1$, then T_0 is elementary abelian.*

(v) *If $m(T_0) \geq 3$, then $R = T_0^g$ for all such g.*

PROOF. If $m(T_0) = 1$, then we apply (3.2) and the Z^*-theorem of Glauberman. Also, in any case, (i) follows from the tight embedding property. We now assume that $m(T_0) > 1$.

At this point we apply the work of Aschbacher [1]. Theorems 1 and 3 of [1] apply directly, while the proof of Theorem 2 carries over with just one change. Namely at a certain point Aschbacher uses $[M_0, M_0^g] \neq 1$ for any $g \in G$ and his Hypotheses II to conclude that (iii) holds. However, in our case, (iii) follows as in the proof of (2.1). So we may apply the theorems in [1] to obtain (3.6) in the case $m(T_0) > 1$.

(3.7) *Suppose that $m(T_0) > 1$. Then*:

(i) *There is no subgroup $G_0 < G$ such that $T < G_0$, $M_1 = O(M_1)(M_1 \cap G_0)$, and $M_1 \cap G_0$ is a *-standard subgroup of G_0, but $M_1 \cap G_0 \ntrianglelefteq G_0$.*

(ii) $O(M) = 1$.

PROOF. Suppose that $m(T_0) > 1$. First we show that (i) implies (ii). So assume (i) to hold, but (ii) false. Let p be a prime divisor of $|O(M)|$ and P_0 a T-invariant Sylow p-subgroup of $O(M)$. Extend P_0 to a T-invariant Sylow

p-subgroup, P, of $M_0 \cap C(T_0T_1O(M)/O(M))$. As $[M_1, P] \leq [M_1, M_0] \leq O(M)$, $N_{M_1}(P)$ covers $\tilde{M}_1$.

Let $g \in G - M$ be as in (3.6)(ii). Since $P = \langle C_P(t): t \in (T_0^g)^{\#} \rangle$, $P \leq M^g$. It is easily checked that if $\tilde{M}_1 \not\cong L_2(4)$, then T_1T_0/T_0 is the unique group of its isomorphism type in T/T_0. Applying this to T^g/T_0^g we have $T_1T_0 \cong (T_1T_0)^g \leq M_1^gM_0^g$ and the structure of M_1^g forces $P \leq M_0^g$. If $\tilde{M}_1 \cong L_2(4)$ or SL(2, 5), this also holds, so in all cases $N_{M_1^g}(P)$ covers $\tilde{M}_1^g$. Setting $G_0 = N_G(P)$ it is easily checked that $M_1 \cap G_0$ is *-standard in G_0. So it suffices to prove (i).

We apply induction to $G_0/O(G_0)$. Since $m(T_0) > 1$, we must be in case (v) or (vi) of the main theorem. Let $T \subseteq S \in \mathrm{Syl}_2(G_0)$, $S_0 \in \mathrm{Syl}_2(N_G(S))$. First, assume $S \subset S_0$.

If $G_0/O(G_0) \cong A_9$ or S_9, then $T_0 \sim T_1$ in G_0 and $Z_2(S)$ is a klein group which we may take to be $\langle t \rangle \times \langle t^s \rangle$ for $t \in T_0$ and $s \in S$. As $tt^s \in Z(S)$, $S_0 = SC_{S_0}(t)$, a contradiction.

Suppose that $G_0/O(G_0) \cong J_2$ or Aut(J_2). Again we check centralizers to see that for each $t \in T_0^{\#}$ $t^{S_0} \subseteq t^{G_0}$. Using the results in [13] we see that S contains precisely 8 conjugates of T_0 and $t^{G_0} \cap S$ is contained in the union of those conjugates. As S is transitive on $T_0^{G_0} \cap S$, the tight embedding property gives $S_0 \leq SN(T_0)$, and again we have a contradiction.

Next suppose that $\tilde{G}_0 = G_0/O(G_0) \cong \mathrm{Aut}(M_{12})$. Then $T_0 \cap G_0' = \langle t \rangle$ for some involution t and $C_{\tilde{G}_0'}(t) \cong S_5 \times \langle t \rangle$, modulo $O(G_0)$. We have $S \geq T$ and T contains a Sylow 2-subgroup of $C_{G_0'}(t)$, which has the form $T_1\langle a \rangle \times \langle t \rangle$ for some involution a. Set $A = \langle a \rangle \times Z(T_1\langle a \rangle) \times \langle t \rangle$. Then by Theorem 2 of Harada [15], G is of known type. In particular, $G_0O(G) = G$ and certainly $S \in \mathrm{Syl}_2(G)$.

Finally we assume that $\bar{G}_0 = G_0/O(G_0) \cong$ Ru. Here we use information about S available in Dempwolff [6]. In his notation $S = V$ and V contains a normal subgroup W such that $F = W' = A_1$ and $W/A_1 = W/F$ is elementary of order 2^8 on which $N_{\bar{G}_0}(\bar{W}/\bar{F}) \cong \mathrm{GL}(3, 2)$ acts irreducibly. Checking centralizers we see that G_0 controls the fusion of its involution so that S_0 cannot fuse an involution in $T_0^{\#}$ into another G_0-class of involutions.

Using the argument in Lemma 2.2 of [6] we conclude that $S_0 \leq N_G(W)$. So S_0 permutes the involutions in $W - W' = W - F$. However, Lemmas 2.7 and 2.8 of [6] show that S is transitive on $T_0^G \cap W$. Consequently, $S_0 = SN_{S_0}(T_0) = S$, a contradiction.

Now that $S \in \mathrm{Syl}_2(G)$ we can obtain a contradiction by quoting an appropriate characterization theorem giving the structure of $G/O(G)$. For all cases except $G_0/O(G_0) \cong$ Ru we can use the result of Gorenstein and Harada [12]. In the remaining case we quote the recent result of Assa [4]. At this point (3.7) is proved.

(3.8) $\tilde{M}_1 \not\cong L_2(4)$ *or* SL(2, 5).

PROOF. If $m(T_0) = 1$ and $\tilde{M}_1 \cong L_2(4)$, we can quote Theorem 2 of Harada [15] to get a contradiction. If $m(T_0) = 1$ and $\tilde{M}_1 \cong$ SL(2, 5) let $\langle r \rangle = \Omega_1(T_0)$. Then it is easily seen that r is a 2-central involution in G. Since $C(t)$ has Sylow 2-subgroups of sectional rank at most 4 we again have a contradiction.

Suppose $m(T_0) > 1$. By (3.7) $O(M) = 1$, so $O(M_1) = 1$. If M_1 is a standard subgroup of G, then we quote Aschbacher [2], while if M_1 is not standard it is because $E(M_0)$ is conjugate to M_1 and $E(M) = M_1 \times M_1^g$ for some $g \in G$. In particular, T_0 is a klein group and we can quote Smith [21].

(3.9) $T_0 \cap T_1 = 1$.

PROOF. Suppose false. Then $M_1/O(M_1)$ is a perfect central extension of Sz(8) by Z_2 or $Z_2 \times Z_2$. First suppose that $m(T_0) = 1$. Here $T = T_0T_1$ and $\Omega_1(T) = \Omega_1(T_1)$ (as $T_{00} = \Omega_1(T_0) \leqslant T_1$ and $\Omega_1(T_1/T_{00}) = \Omega_1(T_1)/T_{00}$). Also $[T, \Omega_1(T)] = T_0 \cap T_1$. Consequently, $N_G(T) \leqslant N_G(T_0 \cap T_1)$ and it follows that $T \in \mathrm{Syl}_2(G)$. But then T_1 is a strongly closed subgroup of T, contradicting (3.4).

If $m(T_0) > 1$, then T_0 is elementary abelian by (3.6)(iv). Here $\Omega_1(T) = \Omega_1(T_1)T_0$ and the above argument again gives a contradiction.

(3.10) $T \notin \mathrm{Syl}_2(G)$.

PROOF. If $m(T_0) > 1$, then T_0 is elementary by (3.6), so in all cases $V = \Omega_1(T_0T_1) = \Omega_1(Z(T_0T_1))$. Suppose that $t \in T - T_0T_1$ is a conjugate of an involution in T_0. Then $\tilde{M}_1 \cong U_3(q)$ or $L_2(q)$ and $C_{\tilde{M}_1}(t) \cong L_2(q)$ or $L_2(\sqrt{q}\,)$, respectively. Moreover, all involutions in $C_{M_1}(t)t$ are fused to t. Clearly, $C_{M_1}(t)' \leqslant C_G(t)'$ and, by (3.8), $C_{\tilde{M}_1}(t)$ is simple so $C_{M_1}(t)'$ covers $C_{\tilde{M}_1}(t)$. Now we conclude that some conjugate t^g of t induces a nontrivial inner automorphism of M_1.

Assume that $T \in \mathrm{Syl}_2(G)$. If $\tilde{M}_1 \cong L_2(q)$ we use (3.6)(iv) and then (2.6) to conclude that $\Omega_1(T_0T_1) = T_1\Omega_1(T_0)$ is strongly closed in T. This contradicts (3.4). If $\tilde{M}_1 \cong \mathrm{Sz}(q)$, then $T = T_1 \times T_0$ and again $\Omega_1(T)$ is strongly closed and abelian.

Suppose that $\tilde{M}_1 \cong U_3(q)$ and let t^g be as in the first paragraph. The group M^g contains $C_{M_1}(t^g)$ and $C_{M_1/O(M_1)}(t^g)$ has order $(q+1)q^3$ or $\frac{1}{3}(q+1)q^3$. A 2-complement in $N(T_1) \cap C_{M_1}(t^g)$ acts fixed-point-freely on $T_1/\Phi(T_1)$, and from the structure of M^g we conclude $\Omega_1(T_1) \leqslant M_1^g$.

In particular, (3.9) implies that $t^g \notin T_1$. We may assume that $\Omega_1(T_0^g) \leqslant T$ (this is clear if $m(T_0) = 1$, and if $m(T_0) > 1$ we use (3.6)(ii) and (2.1)). Let $\Delta = \Omega_1(T_0)^G \cap V$. Since $N_{M_1}(V)$ contains a cyclic group acting regularly on $\Omega_1(T_1)$ and since $\Delta \not\subseteq \{\Omega_1(T_0), \Omega_1(T_1)\}$, we argue as in (2.7) to conclude that

$N(V)^\Delta$ is 2-transitive of degree q. But $T_0T_1 \leqslant C(V)$ and $|T : T_0T_1| \leqslant n < q$. This is a contradiction.

(3.11) *Let* $T \leqslant S \in \mathrm{Syl}_2(G)$. *Then* $N_S(T) \leqslant N_G(T_1T_0)$.

PROOF. Suppose $\tilde{M}_1 \ncong U_3(q)$. Then from (3.6) and (3.8) it is easy to see that $\Omega_1(T_0T_1)$ is weakly closed in T with respect to T and $T_1T_0 = C_T(\Omega_1(T_0T_1))$. If $\tilde{M}_1 \cong U_3(q)$, then we may assume $T > T_1T_0$. In this case $\Omega_1(Z(\Omega_1(T)')) = J > A_1$ is normalized by $N_S(T)$, and since T_1T_0/J is the unique group of its isomorphism type in T/J, we have the result.

(3.12) T_0 *is elementary abelian.*

PROOF. By (3.6)(iv) we may assume that $m(T_0) = 1$. Choose $y \in N_S(T) - T$. By (3.11) $y \in N(T_1T_0)$. Also $T_0^y \cap T_0 = 1$ and $T_0 \ncong T_1$. The Krull-Schmidt theorem implies that $T_0^y \leqslant T_0Z(T_1)$, and the result follows from the fact that $Z(T_1)$ is elementary.

(3.13) *Let* $L = N_G(A)$ *and* $\Delta = A_0^G \cap A$.

(i) $A - A_1 = \bigcup\{(A_0^\#)^x : x \in G, A_0^x \leqslant A\}$ *is a disjoint union of* q *conjugates of* A_0.

(ii) A_1 *is strongly closed in* A *with respect to* G.

(iii) L *induces a* 2-*transitive group on* Δ.

PROOF. This follows exactly as in the proof of (2.7) once we show $\Delta \neq \{A_0, A_1\}$. Suppose that, in fact, $\Delta = \{A_0, A_1\}$ and let $y \in N_S(T) - S$ for $T \leqslant S \in \mathrm{Syl}_2(G)$ (here we use (3.10)). By (3.11) $y \in N(T_1T_0) \leqslant N(A)$. As $y \notin T$ we must have $A_0^y = A_1$. If $\tilde{M}_1 \ncong L_2(q)$, then $A_1 \leqslant (T_1T_0)'$ and $A_0 \nleqslant (T_1T_0)'$, impossible. Therefore $\tilde{M}_1 \cong L_2(q)$. But now G satisfies the conditions of Hypothesis (*) of §2 ($R = T_0$, $K = M_0$, $X = C_{M_1}(T_0)$, $U = T_1$). So (2.7) implies that A is strongly closed in a Sylow 2-subgroup of G, contradicting (3.4).

(3.14) *Let* $L = N_G(A)$ *be as in* (3.13).

(i) L^Δ *contains* $O_2(L^\Delta)$ *as a regular normal subgroup of order* q.

(ii) $O_2(L^\Delta)K_1^\Delta \trianglelefteq L^\Delta$ *is a* 2-*transitive Frobenius group.*

PROOF. It suffices to show that L^Δ contains a regular normal subgroup. Here we use the proof of (2.7)(iii). If L^Δ does not contain a regular normal subgroup then we must have $(L^\Delta)' \cong L_3(2)$ and $q = 8$. So $\tilde{M} \cong L_2(8)$, Sz(8), or $U_3(8)$. Since L has a 7-element acting nontrivially on A_1, L induces $L_3(2)$ on A_1.

Let $T < S_1 \in \mathrm{Syl}_2(N_G(A))$. Then S_1 contains an element x inducing an automorphism of order 4 on A and satisfying $C_A(x) \leqslant A_1$. From the Jordan form of x acting on A we conclude that $|A_0| = 2$.

First suppose that $\tilde{M}_1 \cong \mathrm{Sz}(8)$. The stabilizer J in L of an element $yA \in (T/A)^{\#}$ induces S_4 on A_1. But also J must stabilize $[T, y]$, a klein group in A_1 and y^2, an involution in A_1. This is impossible.

Next suppose that $\tilde{M}_1 \cong U_3(8)$. We argue as follows, referring the reader to p. 17 of [9] for the structure of T_1. Let $z \in C_{A_1}(x^2)^{\#} \cap [A_1, x^2]$. The square roots in T_0T_1 of z form 9 cosets x_iA, $i = 1, \ldots, 9$, permuted by x. Hence one coset at least, say x_1A, is fixed by x. Then, since $A \leqslant Z(T_0T_1)$, x acts on the 4-element set $\{[x_1, x_i]: i = 2, \ldots, 9\}$, which an easy computation shows is not the case.

Now assume that $\tilde{M}_1 \cong L_2(8)$. Here $T = T_0T_1$ is elementary of order 2^4. We claim that $S_1 \in \mathrm{Syl}_2(G)$. For suppose $g \in N(S_1) - S_1$ with $g^2 \in S_1$. Then $A^g \leqslant S_1$, but $A \neq A^g$. As A^g centralizes $A \cap A^g$, $|A \cap A^g| = 4$ and $A \cap A^g \leqslant A_1$. So there is a conjugate $A_0^x = \langle t^x \rangle \subseteq S_1 - T$. We may assume $t^xA \leqslant Z(S_1/A)$. Then t^x has two nontrivial Jordan blocks on A and, hence, $C_{S_1}(t^x)$ covers S_1/A. This forces $C_{S_1}(t^x)$ to involve D_8, a contradiction. This proves the claim.

Finally we observe that S_1 has sectional 2-rank 4 so that the theorem of Gorenstein and Harada [12] gives a contradiction.

We remark that the only groups G in the main theorem satisfying $(L^{\Delta})' \cong L_3(2)$ are those with $G' \cong G_2(3)$.

Notation (3.15). As in (2.8) we now have the existence of certain subgroups of L. Let L_0 be the subgroup of L stabilizing each element of Δ. Then either $T_1T_0 \in \mathrm{Syl}_2(L_0)$ or $\tilde{M}_1 \cong U_3(q)$, $|T \cap L_0\colon T_0T_1| = 2$, and $T \cap L_0 \in \mathrm{Syl}_2(L_0)$. Choose $R > T \cap L_0$, a 2-subgroup of L so that R^{Δ} is the regular normal subgroup in L^{Δ}. We may assume that $T \leqslant N(R)$. Except in the case $T \cap L_0 > T_0T_1$, we may choose a subgroup $D_1 \leqslant N(R)$ of odd order with D_1 inducing K_1 on T_0T_1 and $T_3 \leqslant N(D_1)$. In those cases set $R_1 = R$. If $T \cap L_0 > T_0T_1$, then $K \leqslant L_0$ and K induces a cyclic group of order $q + 1$ or $\frac{1}{3}(q + 1)$ on T_0T_1 normalized by $\langle R, K_1 \rangle$. From here it is easy to see that R/T_0T_1 is elementary and that R contains a subgroup R_1 of index 2 such that $R_1 > T_0T_1$, R_1 covers $R/R \cap L_0$, and K_1 normalizes R_1 module $O(L_0)$. So here we choose $D_1 \leqslant N(R_1)$ of odd order with D_1 inducing K_1 on T_0T_1 and $T_3 \leqslant N(D_1)$.

Set $R_0 = [R_1, D_1]$.

(3.16) (i) $T_1 \trianglelefteq R$ *and* $[T_1, R_1] \leqslant A_1$.

(ii) $R_0 \cap A_0 = 1$ *and* $R_0A_0T_3 \in \mathrm{Syl}_2(L)$.

(iii) $R_0 = T_1R_2$ *with* $T_1 \cap R_2 = A_1$, R_2 *abelian, and* R_2/A_1 *and* A_1 *are isomorphic* $\mathbf{F}_2(D_1)$*-modules.*

Proof. We have $A \lhd R$, $[A, R] \leqslant A_1$ and $T_1T_0 \lhd R$ (as $T_1T_0 = R_1 \cap$

$C(A)$ and $R \leq R_1T_3$). First we show that $T_1 \lhd R$. If $T_1 = A_1$, this follows from (3.13)(ii). Suppose that $\tilde{M}_1 \cong U_3(q)$. As $q > 2$, $K \leq C(A)$ and $[K, T_1] = T_1$. It follows that if $g \in R$, $A_0^g \leq A$, $T_1 \leq M_1^g$. So $T_1 = T_0T_1 \cap M_1^g$ and $g \in N(T_1)$. In particular, $R \leq N(T_1)$. Now suppose that $\tilde{M}_1 \cong \mathrm{Sz}(q)$. If R/A is not elementary abelian, then since D_1 is transitive on $(R/T)^{\#}$ and on $(T/A)^{\#}$, we have $\Omega_1(R/A) = T/A$. But then $\Omega_1(R) = A$, $R \in \mathrm{Syl}_2(G)$, A is strongly closed in R, and we contradict (3.4). So R/A is elementary abelian. Let X/A be a D_1-invariant complement to T_0T_1/A in R/A. We use the action of D_1 to see that X/A_1 is elementary abelian. Indeed, if X/A_1 is not abelian choose A_2/A_1 a hyperplane in A/A_1 with $X' \not\leq A_2$. Then since D_1 is irreducible on X/A, X/A_2 is extraspecial, contradicting the fact that n is odd. So X/A_1 is abelian, and from the action of D_1 we see that X/A_1 is elementary. Let $x \in X - A_1$ and $t \in T_1 - A_1$. Then $x^t = xa$ for some $a \in A$ as R/A is abelian. Since t centralizes $x^2 \in A_1$, we must have $x^2 = (xa)^2 = x^2a^2[x, a] = x^2[x, a]$. Consequently, $[x, a] = 1$ and, as $x \in X - A_1$, this forces $a \in A_1$. We conclude that $[T_1, X] \leq A_1$ and $T_1 \lhd R$ as claimed.

Now we complete the proof of (i); that is, we show $[T_1, R_1] \leq A_1$. If $T_1 = A_1$ this is obvious. In the other cases we have the result since D_1 acts irreducibly on T_1/A_1, and $T_1/A_1 \cap Z(R/A_1) \neq 1$.

A previous argument shows that R_1/A is elementary if $\tilde{M}_1 \cong \mathrm{Sz}(q)$. We claim that R_1/A is elementary in all cases. If not, then as before $\Omega_1(R_1/A) = T_1A/A$ and $\Omega_1(R_1) = A$. If $\tilde{M}_1 \cong L_2(q)$ it is then easy to see that A is weakly closed in RT_3, $RT_3 \in \mathrm{Syl}_2(G)$, and by (2.6) (using $q > 4$) A is strongly closed in RT_3. This contradicts (3.4). Now assume that $\tilde{M}_1 \cong U_3(q)$ and let $D = D_1^{q-1}$. Then $D \leq L_0$ and, as $q > 2$, $[D, T_1] = T_1$. But also $[D, R_1] \leq T_1T_0$. Consequently, $[R_1, D, R_1] \leq [T_1T_0, R_1] \leq A$ and $[D, R_1, R_1] \leq A$. By the 3-subgroup lemma $[R_1, R_1, D] \leq A$ and so $R_1' \leq A$. That is, R_1/A is abelian and, since D_1 acts irreducibly on T_1A/A, we conclude that $\Phi(R_1/A) = 1$ and R_1/A is elementary.

Choose a D_1-invariant complement X/A to T_1A/A in R_1/A. We next claim that X/A_1 is elementary abelian. If not then there is an element $x \in X$ with $x^2 \in A - A_1$. Then x^2 is R_1-conjugate to an involution in A_0. Therefore, x is R_1-conjugate to a member of T, a contradiction.

We now set $R_2 = [D_1, X]$. Then $A_1 \leq R_2$ and $R_2 \cap A_0 = 1$. As R_1/A_1 is the direct sum of T_1/A, R_2/A_1, and A/A_1, we have $R_0 = T_1R_2$. This proves (ii) and the first two parts of (iii). If $t \in A_0^{\#}$ then the map $r_2A_1 \to [r_2, t]$ is a D_1-isomorphism from R_2/A_1 to A_1. Apply (2.2) to complete the proof of (3.16).

At this stage we have begun the process of building a Sylow 2-subgroup of G. We will complete the proof of the main theorem by taking the cases $\tilde{M}_1 \cong L_2(q)$, $\mathrm{Sz}(q)$, $U_3(q)$ separately.

4. $\tilde{M}_1 \cong L_2(q)$. In this section we assume that $\tilde{M}_1 \cong L_2(q)$. Recall that we are after a contradiction and that, by (3.8), $q > 4$.

For this case the group G satisfies the conditions of (*) in §2 (setting $R = A_0$, $K = M_0$, $U = A_1$). We could immediately apply (2.9) provided we knew that at each stage of the process described in §2 the 2-transitive group did not involve $L_3(2)$. So we first prove this.

Suppose that at some stage $L_3(2)$ does occur. Then $\tilde{M}_1 \cong L_2(8)$ and $T = T_0 \times T_1$. By (3.14) L^Δ does contain a regular normal subgroup, so that the difficulty occurs at stage $m + 1$ of the inductive process, where $m \geqslant 1$. Consequently, there is a subgroup $U_m > A_1$ and a subgroup of odd order D_m, such that $D_m A_0 \leqslant N(U_m)$, D_m acts on A, A_0 as does D_1, each D_m-composition factor of U_m is isomorphic to A_1, and if $U_{m-1} = [U_m, A_0]$, then $N = N_G(U_{m-1}) \cap N_G(U_m A_0)$ induces $L_3(2)$ on $U_m A_0/U_{m-1}$, 2-transitive on $\Omega = (A_0 U_{m-1}/U_{m-1})^N$. Also U_m is normal in N (see (2.7)).

We claim that U_m is homocyclic of rank n, $|A_0| = 2$, and A_0 inverts U_m. To see this, note that for $t \in A_0^\#$, t^N contains elements in $U_m t$. So t inverts elements of U_m, and, using the action of D_m, t inverts an element of each coset of U_{m-1} in U_m. But now (2.4) implies that U_m is abelian, so t inverts U_m and U_m is homocyclic of rank n. As $t \in A_0^\#$ was arbitrary, $A_0 = \langle t \rangle$ and we have the claim.

Next note that $tU_m = t^N$ and the Thompson transfer lemma implies that $t \notin O^2(N)$. In particular, $O^2(N)$ has index 2 in N, is complemented by $\langle t \rangle$, and a Sylow 2-subgroup of N has the form $S = S_0\langle t \rangle$, where $S_0 \cap \langle t \rangle = 1$ and $U_m \leqslant S_0 \in \mathrm{Syl}_2(O^2(N))$. Then $S_0/U_m \cong D_8$. As $U_m > A_1$ has exponent at least 4, U_m is weakly closed in S, and since $U_m A_0 = C_S(\Omega_1(U_m))$, $S \in \mathrm{Syl}_2(G)$. In addition it is clear that t does not fuse into S_0, so by transfer G contains a normal subgroup G_0 of index 2. Clearly, $S_0 \in \mathrm{Syl}_2(G_0)$. At this point we have the structure of G_0 by appealing to [15] or to [19]. In either case we have a contradiction.

We may now apply (2.9) to get the subgroup $U_m > A_1$. Here $S = A_1 A_0 T_3 \in \mathrm{Syl}_2(M)$. By (3.13), (2.9)(3) does not hold.

(4.1) *U_m is not isomorphic to a Sylow 2-subgroup of $U_3(q)$ or $L_3(q)$.*

Proof. Deny. Then U_{m-1} is homocyclic of exponent 4 and, since for each $t \in A_0^\#$, $t^{U_m} = U_{m-1}t$, we have t inverting U_{m-1}. In particular, $A_0 = \langle t \rangle$. By (2.9) $U_m S = U_m A_0 T_3 \in \mathrm{Syl}_2(G)$. Now $A_0 T_3$ is abelian, and if $A_0 T_3$ is cyclic, then we transfer out $A_0 T_3$ and contradict (3.3). So we may assume that $T_3 A_0 = T_3 \times A_0$ and $T_3 \neq 1$. Each involution in $T_3 U_m - U_m$ centralizes a homocyclic subgroup of order $q = 4^{n/2}$ and rank $n/2$ in U_{m-1}. Each involution in U_m has centralizer of order at least q^2. So $t^G \cap U_m T_3 = \varnothing$ and G contains a normal subgroup G_0 of index 2.

By (3.3) and transfer we may assume that $x^{G_0} \cap U_m \neq \emptyset$, where $\langle x \rangle = \Omega_1(T_3)$. Say $y = x^g \in U_m$. Then either $y \in A_1$ and $U_m \leq C(y)$ or $y \in U_m - A_1$, U_m is isomorphic to a Sylow 2-subgroup of $L_3(q)$, and $C_{U_m}(y)$ contains an elementary abelian subgroup of order q^2. However, $t \not\sim tx$ (for the same reason that $t \not\sim x$), and it follows that $B = C_{M_1}(x)/\langle x \rangle$ is a *-standard subgroup in $C_G(x)/\langle x \rangle$ with $C(B/O(B)) \cap C_G(x)$ having $\langle t, x \rangle/\langle x \rangle$ as Sylow 2-subgroup. From the minimality of G we have a contradiction.

(4.2) *U_m is not homocyclic.*

PROOF. Suppose U_m is homocyclic. Then (2.9)(2) implies that $A_1 < U_m$ and $S = U_m A_0 T_3 \in \mathrm{Syl}_2(G)$. So $q \geq 8$ by (3.8). It is now easy to show that U_m is weakly closed in S.

We apply (2.6) to the weakly closed subgroup U_m of S and its subgroup A_1 (so $T = S$, $W = U_m$, $A = A_1$). Let r be the integer given in (2.6).

As $U_m A_0 \leq C(A_1)$, $r \leq 1$. But from (2.6)(ii) and the fact that $q > 4$ we see that, in fact, $r = 0$. By (3.4) A_1 is not strongly closed in S, so there is a conjugate $x \in U_m(A_0^\#)$ of an involution of A_1. Say $t \in A_0^\#$ and $x \in U_m t$. Then $x \notin U_{m-1}t = t^{U_m}$ and so t must invert U_m. As $C_{A_0}(U_m) = 1$, $U_m t$ is the unique coset of U_m in $U_m A_0^\#$ that contains involutions not in A. Also we note that each element of $U_m t - U_{m-1}t$ is conjugate to x.

Suppose $U_m A_0 \in \mathrm{Syl}_2(G)$. If U_m has exponent 4, then $U_m A_0$ has class 2, against (3.3). If U_m has exponent greater than 4, then $U_{m-1}A_0^\#$ consists of involutions so each element of $A_0^\#$ inverts U_{m-1}, forcing $|A_0| = 2$. But now we transfer out A_0 from G_0 and again contradict (3.3).

Thus we may choose $x \in T_3 - U_m A_0$ with $x^2 \in A_0$. x clearly has no conjugates in $U_m A_0$, and if $x^2 \neq 1$ then x^2 is an involution in A_0 and so has no conjugate in $S - U_m A_0$ (check centralizers). Hence $x \notin O^2(G)$ by transfer. By (3.5)(b), $|A_0| = 2$, x is an involution, and $xt \in O^2(G)$, where $A_0 = \langle t \rangle$. But we can transfer out xt also, a contradiction.

(4.3) *U_m is not elementary abelian of order q^2.*

PROOF. Suppose that U_m is elementary abelian of order q^2. Then $U_m = R_2$ and, for $a \in A_0^\#$, $aR_2 - aA_1$ contains no involutions. So $A = \langle A_0^G \cap R \rangle$ and $N_G(R) \leq N_G(A)$. Let $S \in \mathrm{Syl}_2(G)$ with $RT_3 \leq S$. Then $N_S(R) = RT_3$.

Suppose that there are no involutions in $RT_3 - R$. Then $N_S(RT_3) \leq N_S(A) = RT_3$ so $RT_3 \in \mathrm{Syl}_2(G)$. Also $R_2 \lhd RT_3$ must be strongly closed in S, contradicting (3.4). So we may assume that there is an involution $x \in T_3 R - R$, and since R/A is a free $\mathbf{F}_2(\langle x \rangle)$-module, we may take $x \in N(A_0)$. Let $t \in A_0^\# \cap C(x)$.

As $q > 4$, R_2 is weakly closed in $RT_3 = R_2 A_0 T_3 = S_0$. Let $S_1 = N_S(R_2)$. If $S_1 = S_0$, then using (2.6) and (3.4) we obtain a contradiction. So assume $S_1 > S_0$. If $a \in N_{S_1}(S_0) - S_0$ then $A_0^a \cap R_2 A_0 = 1$. As $T_3 R/R$ is cyclic this forces $|A_0| = 2$ and we may assume that $T_3A_0 = T_3 \times A_0$ with T_3 cyclic.

Now, let bars denote images modulo R_2. Since $C_{\bar{S}_1}(\bar{t}) = \langle \bar{t} \rangle \times \bar{T}_3$, we may apply Lemma 2.20 in [18] to conclude that either (i) $\bar{t} \in Z(\bar{S}_1)$, or (ii) $\bar{S}_1$ has a subgroup $\bar{S}_2$ of index 2 with $\bar{S}_2 = \overline{DT_3}$, $\bar{D} = \langle \bar{w}, \bar{t} \rangle$ dihedral (with $(\overline{wt})^2 = 1$) and $\bar{T}_3$ acting on $\bar{D}$ centralizing $\bar{t}$ and normalizing $\langle \bar{w} \rangle$; also $|\bar{T}_3 \cap \bar{D}| = 2$ and the involutions $\bar{w}^i\bar{t}$ are fused in $\bar{S}_2$. Set $\langle s \rangle = \Omega_1(T_3)$. Then $\langle \bar{s} \rangle = Z(\bar{D})$. Let $z \in S_1 - R_2$ be an involution, and suppose $m([R_2, z]) \leqslant 2$. Then $\bar{z} \notin C(\bar{t})$, so we are in case (ii) above. If $\bar{z} \in \overline{DT_3}$, then $z = \bar{z}_1\bar{z}_2$ with $z_1 \in D$, $z_2 \in T_3$, and $|z_2| = 4$. But $\bar{z} \sim \overline{zs}$, so $m([R_2, s]) \leqslant 4$. Hence $q = 16$, $m([R_2, z]) = 2$. If $\bar{z} \notin \overline{DT_3}$, then $\langle \bar{z}, \bar{t} \rangle \geqslant \bar{D}$ is dihedral of order $2|\bar{D}|$ and, hence, we can write $\bar{s}$ as a $\frac{1}{4}|\bar{D}|$th power of a product of $\bar{z}$ and a conjugate. In particular, $m([R_2, s]) \leqslant 4$, so $q = 16$ and $m([R_2, z]) = 2$. Also, $|\bar{D}| = 4$ and so in this case $\overline{DT_3} = C_{\bar{S}_1}(\bar{t})$ and no involution of $\overline{DT_3}$ satisfies $m([R_2, z]) < 2$. At this point one can argue that R_2 is weakly closed in S_1. So $S_1 \in \mathrm{Syl}_2(G)$ and using (2.6) we have R_2 strongly closed in S_1. This is a contradiction.

At this stage we have considered all cases of (2.4) and we conclude that there are no counterexamples to the main theorem with $\tilde{M}_1 \cong L_2(q)$, $q = 2^n > 4$.

5. $\tilde{M}_1 \cong \mathrm{Sz}(q)$. Recall the notation of §3 and assume $\tilde{M}_1 \cong \mathrm{Sz}(q)$. $R \in \mathrm{Syl}_2(N(A))$, $R = T_1R_2A_0$, R_2 is abelian, and $[D_1, R] = T_1R_2$. Let $Y = N_G(R) \leqslant N_G(A_1)$ (as $A_1 = Z(R)$) and consider the induced group Y^* on $\Delta = \{(A/A_1)^Y\}$.

We will obtain a contradiction to the standing assumption that G is a counterexample to the main theorem.

(5.1) *Suppose T_1 is isomorphic to the Sylow 2-subgroup of* Sz(8). *Then* $\mathrm{Aut}(T_1)$ *does not involve* $L_3(2)$.

PROOF. Suppose $X = \mathrm{Aut}(T_1)$ does induce $L_3(2)$. Then $\mathrm{Aut}(T_1)$ induces $L_3(2)$ on T_1/A_1 and on A_1, and looking at the action of an element of order 7 in X we see that the representations of X on T_1/A_1 and on A_1 are contragredient. Choose a basis x_1A_1, x_2A_1, x_3A_1 of T_1/A_1 and a klein group $X_0 < X$ centralizing $\langle x_1A_1, x_2A_1 \rangle$. Then X_0 centralizes $\langle x_1^2, x_2^2 \rangle$, whereas X_0 centralizes no klein group in A_1. This is a contradiction.

(5.2) *If $q = 8$ and $T_0 \cong Z_2 \times Z_2$, then $G \cong$* Ru, *the Rudvalis group.*

PROOF. Dempwolff [6] (see the appendix).

(5.3) $|\Delta| < 5q$.

PROOF. By Lemma 1.8 of [6], $R - A_1$ contains at most $q(2q|A_0| - |A_0| + q - 2)$ involutions. Each conjugate of A contains $q|A_0| - q$ involutions outside

A_1, and by the tight embedding property, $A^g \neq A$ for $g \in Y$ implies $A^g \cap A = A_1$. The result follows.

(5.4) $|\Delta| = q$ *and* Y^* *is* 2-*transitive on* Δ. *Either* $Y^* \cong L_3(2)$ *or* Y^* *contains a regular normal subgroup.*

PROOF. Let N^* be a minimal normal subgroup of Y^*. We note that $|\Delta| = 1 + k(q-1)$ where $k \geq 1$ is an integer. This follows since D_1^* is semiregular on $\Delta - \{A/A_1\}$. Also $|Y^*| = |\Delta|v$ where $q - 1 | v$ and v is odd.

We claim that $N^* \cong L_3(2)$ or N^* is a p-group for some prime p. First note that by (3.3) $R \not\in \mathrm{Syl}_2(Y)$. So $|Y^*|$ is even. By (5.3) $k \leq 5$. Consequently, $k = 1, 3$, or 5. Suppose that $k = 3$ or 5. Then $8 \nmid |Y^*|$. By Feit and Thompson [7] and Gorenstein and Walter [14], if the claim is false then $N^* \cong \mathrm{PSL}(2, q_1)$ for some prime power q_1. Suppose this occurs. Let P^* be a Sylow p-subgroup of Y^* for a primitive divisor p of $q - 1$. If $P^* \cap N^* \neq 1$, then $N_{N^*}(P^* \cap N^*)$ has order twice an odd number. This implies that some involution in Y^* normalizes a conjugate of P^*. But P^* fixes just one point of Δ, and the stabilizer of this point in Y^* has odd order. So $P^* \cap N^* = 1$ and by the Frattini argument P^* normalizes a Sylow 2-subgroup of N^*. But then P^* centralizes this subgroup (as $p \neq 3$) and we again have a contradiction.

Finally consider the case $k = 1$. Here Y^* is 2-transitive. By Hering, Kantor and Seitz [16] either N^* is a p-group or $N^* \cong \mathrm{PSL}(2, q_1)$ for some q_1. As in (2.7) we must have $q_1 = 7$ and $q = 8$ in the latter case.

(5.5) *Y does not induce* $L_3(2)$ *on* R/A_1.

PROOF. Suppose Y induces $L_3(2)$ on R/A_1. Then $q = 8$ and, by (5.2), $|T_0| = 2$ or 8. The nontrivial irreducible constituents of D_1 on R/A_1 are T_1/A_1 and R_2/A_1. These are inequivalent. Also the irreducible $\mathbf{F}_2$-modules of $L_3(2)$ have degrees 1, 3, 3, 8. Suppose that the representation of degree 8 is a Y-composition factor on R/A_1. Since any $\mathbf{F}_2$-module affording this representation is injective and projective, R/A_1 is completely reducible as an $\mathbf{F}_2(Y)$-module. But then there is an involution $t \in A_0$ such that $A_1\langle t\rangle \trianglelefteq Y$. As $t^G \cap A_1\langle t\rangle = t^{R_2} \cap A_1\langle t\rangle$, $Y \leq R_2C(t)$, a contradiction.

Let V/A_1 be a minimal normal subgroup of Y/A_1 contained in R/A_1. Then $|V/A_1| = 2$ or 8 and, as above, the case $|V/A_1| = 2$ gives a contradiction. So $|V/A_1| = 8$ and from the action of D_1 on R/A_1 we have $V = T_1$ or R_2. By (5.1) $V \neq T_1$, so $R_2 \lhd Y$.

As $Y^\Delta \cong L_3(2)$ and $|T_0| = 2$ or 8, D_1 must contain an element g with g inducing an element of order 3 on R and $C_{A_0}(g) \neq 1$. Say $1 \neq t \in C_{A_0}(g)$. Then t normalizes $C_{T_1R_2}(g)$. Now $C_{T_1R_2}(g)$ has order 8 as g induces the regular module for Z_3 on each of A_1, T_1/A_1, and R_2/A_1. So $C_{T_1R_2}(g) = \langle t_1, r_2\rangle$ for some $t_1 \in T_1 - A_1$, $r_2 \in R_2 - A_1$. Therefore $[r_2, t] = t_1^2$.

First suppose that R_2 is elementary abelian. Let $x \in Y$ be such that x^Δ inverts g^Δ and $x \in N(\langle g \rangle)$. Then $x \in N(C_R(g)) = \langle t_1, r_2 \rangle C_{A_0}(g)$. As neither $T_1 t$, nor $R_2 t$ contain involutions not in A, we have $t^x \in t_1 r_2 t A_1$. In particular, $t_1 r_2 t$ must be an involution. This forces $1 = t_1^2 t_1^2 [t_1, r_2]$. So $[t_1, r_2] = 1$ and, by (2.3), $[T_1, R_2] = 1$. At this point Y normalizes $C_R(R_2) = T_1 R_2$ and, arguing as in (5.1) (choosing bases in $T_1 R_2 / R_2$ rather than in T_1 / A_1), we obtain a contradiction.

Thus R_2 is homocyclic and with t_1, r_2, and t as before, $\langle r_2 \rangle = C_{R_2}(\langle g \rangle)$ and t inverts r_2. As t commutes with the action of D_1 on R_2, t inverts R_2. If $T_1 \leq C(R_2)$, then $T_1 R_2 \langle t \rangle$ is the extended centralizer in R of R_2 and $T_1 R_2$, $T_1 R_2 \langle t \rangle \trianglelefteq Y$. For $y \in Y$, $t^y \in T_1 R_2 t$. Also $[T_1, R_2 \langle t \rangle] = 1$ and t inverts R_2, so $t^y \in R_2 t$. Thus $R_2 t = t^Y$ and $T_1 = C_{T_1 R_2}(\langle t^Y \rangle) < Y$. This contradicts (5.1). Therefore $T_1 \not\leq C(R_2)$ and by (2.3) no element in $T_1 - A_1$ commutes with an element in $R_2 - A_1$. The extended centralizer of R_2 in R is $R_2 \langle t \rangle$, so $R_2 \langle t \rangle \trianglelefteq Y$. Let $J / R_2 \langle t \rangle \leq R / R_2 \langle t \rangle$ be a minimal normal subgroup of $Y / R_2 \langle t \rangle$. If $J \leq R_2 A_0$, then A_0 would contain a klein group with each involution inverting R_2. This is ridiculous. So Y induces $L_3(2)$ on $J / R_2 \langle t \rangle$ and $J = R_2 \langle t \rangle T_1$. As D_1 has inequivalent representations on T_1 / A_1 and on R_2 / A_1, the representation of Y on $J / R_2 \langle t \rangle$ is the contragredient of the representation of Y on R_2 / A_1.

We now complete the proof of (5.5) using an argument in Dempwolff [6] (see the end of the proof of Lemma 3.4 in [6]). Let $r_2 \in R_2 - A_1$ with $a = r_2^2$ and let $K = C_Y(a)$. As the representation of Y on $J / R_2 \langle t \rangle$ is contragredient to the representation of Y on A_1, K fixes no involution in $J / R_2 \langle t \rangle = T_1 R_2 \langle t \rangle / R_2 \langle t \rangle$. Choose $t_1 \in T_1$ with $[t_1, r_2] = a$ and $x \in K$ with $(t_1 R_2 \langle t \rangle)^x \neq t_1 R_2 \langle t \rangle$. Say $t_1^x = t_2 t^\alpha r$, with $t_2 \in T_1 - A_1$, $\alpha = 0$ or 1, and $r \in R_2$. Then we have $a = a^x = [t_1^x, r_2^x]$. But $r_2^x \in r_2 A_1$ as R_2 is homocyclic and the squaring map is a Y-isomorphism from R_2 / A_1 to A_1. So

$$a = [t_2 t^\alpha r, r_2^x] = [t_2, r_2^x][t^\alpha, r_2^x] = [t_2, r_2][t^\alpha, r_2].$$

We know $[t, r_2] = a$ and, since $[t_2, r_2] \neq 1$, this forces $\alpha = 0$ and $[t_2, r_2] = a$. But then $t_2 t_1^{-1} \in C_{T_1}(r_2) \leq A_1$, whereas $t_1 R_2 \langle t \rangle \neq t_2 R_2 \langle t \rangle$. This is the final contradiction.

At this stage we know that Y^Δ contains a regular normal subgroup.

(5.6) *$T_1 R_2 \triangleleft Y$ and $R_2 \triangleleft Y$.*

Proof. Let V / A_1 be minimal normal in Y / A_1 with $V \leq R$. If $D_1 \leq C(V / A_1)$, then $V \leq A$, which is impossible. Also D_1 must act irreducibly on V / A_1, so $V = T_1$ or R_2. Suppose that $V = T_1$. If $[R_2, T_1] = 1$, then $R_2 A_0 = C_R(T_1) \triangleleft Y$. In this case $R_2 = R_2 A_0 - (\bigcup_{g \in y} A_0^g)^\# \triangleleft Y$. Suppose that $[R_2, T_1] \neq 1$. Then by (2.3) $x \in T_1 - A_1, y \in R_2 - A_1$ implies that $[x, y] \neq 1$. Consequently neither xy nor xyt centralizes x, where $t \in A_0$. So $A =$

$C_R(T_1) \lhd Y$, which is false. So in all cases $R_2 \lhd Y$.

Let V/R_2 be normal in Y/R_2, with $V \leqslant R_2A_0$. Choose V maximal such that $T_1V \lhd Y$. Say $V > R_2$. Then $V = R_2(A_0 \cap V)$ and $A_0 \cap V$ is tightly embedded in Y. So V contains q^2 conjugates of $A_0 \cap V$ and each element of $V - R_2$ is an involution. It follows that $|A_0 \cap V| = 2$ and $A_0 \cap V = \langle t \rangle$ inverts R_2. Let $U \leqslant Y$ be a D_1-invariant Sylow 2-subgroup of Y, containing R. Then $A_1 = Z(U)$ and we may assume $T_1 \nleqslant C(R_2)$, for otherwise $T_1R_2 = C_R(R_2) \lhd Y$. Say $u \in U$, $x \in T_1 - A_1$, and $x^u = xyt$ with $y \in R_2$. For any $r \in R_2$ we have

$$[x, r] = [x, r]^u = [xyt, r] = [x, r][t, r],$$

whence $[t, r] = 1$. This is certainly false, so $V = R_2$ as required.

We conclude that $T_1R_2 \lhd Y$, proving the result.

Let $U > R$ be a Sylow 2-subgroup of Y, invariant under D_1. Setting $U_1 = [U, D_1]$ we have $U = U_1A_0$ and $U_1 \cap A_0 = 1$.

(5.7) $R_2 = Z(T_1R_2)$.

PROOF. Suppose $[T_1, R_2] \neq 1$. Consider the map $u_1 \to [t, u_1]$, where $u_1 \in U_1$ and $t \in A_0^{\#}$ is fixed. Then considering this map from U_1/T_1R_2 to T_1R_2/A_1 and noting the map commutes with the action of D_1, we see that $[t, U_1]A_1 = T_1$ or R_2 (use the fact that T_1/A_1 and R_2/A_1 are inequivalent irreducible $\mathbf{F}_2(D_1)$-modules). If $[t, U_1]A_1 = T_1$, then U_1 normalizes T_1. As in (5.6) this forces $C_R(T_1) = A$ to be normal in U, which is a contradiction.

Consequently, $[U_1, A_0] = R_2$ and $R_2A_0 - R_2 = \bigcup_{g \in Y} A_0^g$. Then $A_0 = \langle t \rangle$ inverts R_2. We note $U_1/T_1R_2 \cong R_2/A_1 \cong A_1$ as $\mathbf{F}_2(D_1)$-modules. Also U_1/R_2 is elementary abelian, as otherwise $T_1R_2/R_2 = \Omega_1(U_1/R_2)$ and, for $x \in T_1 - R_2$, $xR_2 = u^2R_2$ for some $u \in U_1$. But then $[x, R_2] = [u^2, R_2] = 1$, contradicting (2.3). As $T_1 \ntriangleleft U_1$, (2.3) implies that U_1/A_1 has derived group R_2/A_1 and for each $t_1 \in T_1 - A_1$ and $r_2 \in R_2 - A_1$, there is some element $u \in U_1$ with $t_1^u = t_1r_2a$ with $a \in A_1$. But u centralizes t_1^2, so $t_1^2 = t_1^2r_2^2[t_1, r_2]$ and $r_2^2 = [t_1, r_2]$. So t_1 inverts r_2. But t_1 was arbitrary and, by (2.3), $C_{T_1}(r_2) = A_1$. This is a contradiction.

(5.8) *There are members of* Δ *not in* R_2A_0/A_1. *Consequently*, $\langle x \in \Delta \rangle = R/A_1$.

PROOF. Suppose that $\Delta \subseteq R_2A_0/A_1$. If R_2 is elementary, then there are no involutions in $R_2A_0 - A_1A_0 - R_2$, a contradiction. Consequently, R_2 has exponent 4 and $A_0 = \langle t \rangle$ inverts R_2.

Also Y normalizes $C_R(\langle \Delta \rangle) = T_1$. Consider $N = N_G(T_1)$ and apply (2.9) to obtain the structure of a Sylow 2-subgroup $U_2\langle t \rangle$ of N. By (5.1) $N \cap N(R)$ does not involve $L_3(2)$, so U_2/T_1 is homocyclic of rank n, elementary of order q^2, the Sylow 2-subgroup of $L_3(q)$, or the Sylow 2-subgroup of $U_3(q)$.

We may assume that D_1 normalizes U_2. First we note that U_2 centralizes T_1/A_1 and A_1. Consequently, $\Phi(U_2) \leqslant C_{U_2}(T_1) = U_3$. So U_2/U_3 is elementary abelian. Also $T_1 \nleqslant U_3$, so we may write $U_2/U_3 = T_1U_3/U_3 \times U_4/U_3$, with U_4 D_1-invariant. Each nontrivial D_1-composition factor of U_4 is equivalent to A_1, whereas T_1U_3/U_3 is D_1-equivalent to T_1/A_1, which is not equivalent to A_1. Consequently, t normalizes U_4.

By (2.9) U_4 is elementary of order q^2, homocyclic of rank n, The Sylow 2-subgroup of $U_3(q)$, or the Sylow 2-subgroup of $L_3(q)$. We conclude that U_2/T_1 is elementary or homocyclic of rank n. Also $U_3 \geqslant R_2$. We must have U_4/R_3 of order 1 or q.

If $U_2\langle t\rangle \in \mathrm{Syl}_2(G)$, then we get a contradiction as follows. First we claim that $t^G \cap U_2 = \varnothing$. For suppose $t^g \in U_2$. As each of U_2/T_1 and U_2/U_3 are abelian we have $U_2' \leqslant T_1 \cap U_3 = A_1$. Consequently, $|C_{U_2}(t^g)| \geqslant |U_2|/q$ and it follows that $|U_2| = q^3$. Therefore $U_2 = T_1U_3 = T_1R_2$. But then $R_2 \leqslant C(t^g)$, which is impossible. This proves the claim.

Now apply the Thompson transfer lemma to conclude $t \not\in G'$. Clearly $U_2 \leqslant G'$ by the action of D_1. Notice that U_2/A_1 is abelian, so that U_2 has class 2. This is against (3.3).

We now have that $U_2\langle t\rangle \not\in \mathrm{Syl}_2(G)$. Assume also that U_4 is not abelian. Then for $x \in U_4 - R_2$ and $t_1 \in T_1$,

$$\begin{aligned}(t_1xt)^2 &= t_1(xt)t_1(xt) = t_1^2(xt)[xt, t_1](xt)\\ &= t_1^2(xt)^2[xt, t_1[t_1, xt]] \in (xt)^2A_1\end{aligned}$$

because $t_1^2 \in A_1$ and $[T_1, U_2\langle t\rangle] \leqslant A_1$. As $(xt)^2 \in R_2 - A_1$, so such element is an involution. In particular,

$$t^G \cap U_2\langle t\rangle \subseteq T_1U_3\langle t\rangle = T_1R_2\langle t\rangle = R.$$

It is easy to see that there are no involutions in $T_1^{\#}R_2t$, and the claim above shows that $t^G \cap U_2 = \varnothing$. Consequently, $t^G \cap U_2\langle t\rangle = R_2t = t^{U_2\langle t\rangle}$, and it follows that $U_2\langle t\rangle \in \mathrm{Syl}_2(G)$, a contradiction. Therefore U_4 is a homocyclic of rank n.

Suppose $U_4 = U_3 < C(T_1)$. Let $g \in N(U_2\langle t\rangle) = P$ with $t^g = t_1u_4t$, $t_1 \in T_1 - A_1$, and $u_4 \in U_4$. One checks directly that it is not possible for $C(t^g) \cap T_1U_4$ to cover T_1U_4/U_4 and so this is impossible. Therefore for $g \in P$, $t^g \in U_4\langle t\rangle$ and, hence,

$$C(\langle t^B\rangle) \cap T_1U_4\langle t\rangle = T_1 \lhd B,$$

a contradiction. Therefore $U_4 > U_3 \geqslant R_2$.

Now $Z_2(U_2\langle t\rangle) = T_1R_2$ so P normalizes each of T_1R_2, $R_2 = Z(T_1R_2)$, $T_1U_4 = U_2\langle t\rangle \cap C(R_2)$, and $T_1U_3 = (T_1R_2)Z(T_1U_4)$. Consider $P = N(U_2\langle t\rangle)$. We want to apply (2.9) to P/T_1U_3. First we must show that P does not involve $L_3(2)$.

Suppose that P involves $L_3(2)$. The representations of P on T_1R_2/R_2 and on U_2/T_1U_3 are contragredient. Say $P \leqslant N(U_4)$. Then we argue as in (5.5) to get a contradiction. Namely, choose $u \in U_4 - U_3$ and set $K = C_P(uU_3)$. Choose $t_1 \in T_1 - A_1$ with $[t_1, u] = a$, where $\langle a \rangle = \Omega_1(\langle u \rangle)$. As K fixes no involution in T_1R_2/R_2, we choose $k \in K$ with $t_1^k R_2 \neq t_1 R_2$. Say $t_1^k = t_2 r$, $r \in R_2$, and $u^k = uu_3$, with $u_3 \in U_3$. Then

$$a = a^k = \left[t_1^k, u^k \right] = \left[t_2 r, uu_3 \right] = \left[t_2, u \right].$$

But then $t_1 t_2^{-1} \in C(u)$, contradicting (2.3).

Therefore $P \nleqslant N(U_4)$ and we note that this implies $U_4 \cap U_4^g = U_3$ for $g \in P - N(U_4)$. Otherwise $C(U_4 \cap U_4^g) \geqslant \langle U_4, U_4^g \rangle > U_4$, against (2.3). Choose a Sylow 3-subgroup J of P normalized by t. Then a Sylow 2-subgroup of $N_P(J)$ has the form $Q = \langle t_1, u_4 \rangle \langle x, t \rangle$, where $t_1 \in T_1 - A_1$, $u_4 \in U_4 - U_3$, and $Q/\langle t_1, u_4 \rangle$ is a klein group. Modulo U_3 we must have $u_4^x = t_1 u_4$ and $t^x = tt_1u_4$ or tu_4. Say $t^x = tt_1u_4$ modulo U_3. As $t^{x^2} \in tU_3$ and $t_1^x \in t_1U_3$, we have

$$t \equiv t^{x^2} \equiv (tt_1u_4)^x \equiv tt_1u_4t_1u_4t_1 \equiv tt_1 \pmod{U_3},$$

impossible. Therefore $t^x \in tU_4$, so t inverts U_4. Also $[t_1, u_4] \in A_1 \cap \langle t_1, u_4 \rangle = \langle t_1^2 \rangle$, so, by (2.3), $t_1^{u_4} = t_1^{-1}$. Now $t^x = tu_4u_3$, $u_4^x = t_1u_4u_3'$, for u_3, u_3' in U_3. As t^x must invert u_4^x, we compute and get a contradiction. Therefore P does not involve $L_3(2)$ and (2.9) applies.

By (2.9) and a previous argument we obtain a D_1-invariant subgroup $U_5 > U_4$ such that D_1 is transitive on $(U_5/U_4)^\#$ and $T_1U_5\langle t \rangle \in \mathrm{Syl}_2(P)$. We have $T_1R_2 = \Omega_1(T_1U_4)$, so $[T_1, U_5] \leqslant R_2$. If $u \in U_5$ and $t_1 \in T_1 - A_1$, then $t_1^u = t_1 r_2$ for $r_2 \in R_2$. But u fixes t_1^2, so that $r_2^2 = 1$ and $r_2 \in A_1$. Consequently, $u \in N(T_1)$, contradicting $U_2\langle t \rangle \in \mathrm{Syl}_2(N(T_1))$. This completes the proof of (5.8).

(5.9) R_2 *is elementary abelian.*

PROOF. By (5.8) there are elements $t_1 \in T_1 - A_1$, $r_2 \in R_2$, $t \in A_0^\#$ with t_1r_2t conjugate to an involution in A_0. Say $t^u = t_1r_2t$ for $u \in U_1$. Then

$$T_1 \cap T_1^u = C_{U_1}(t) \cap C_{U_1}(t^u).$$

As T_1 centralizes r_2t, we conclude from the structure of T_1 that $T_1 \cap T_1^u \leqslant A_1\langle t_1 \rangle$.

Choose $t_2 \in T_1 - A_1$. Then $t_2^u = t_2 d$ for some $d \in R_2$. If $t_2 \notin A_1\langle t_1 \rangle$, then, by the above, $d \notin A_1$. On the other hand $d^2 = 1$. So $R_2 - A_1$ contains involutions and, consequently, R_2 is elementary.

(5.10) U_1/R_2 *is elementary abelian.*

PROOF. U_1/A_1 is of class at most 2 and U_1/R_2 is abelian by (2.2). For x, y

in U_1, $[x, y^2] = [x, y]^2$ modulo A_1, so that $[x, y^2] \in A_1$. Therefore

$$[x^2, y^2] = [x, y^2]^x [x, y^2] = [x, y^2]^2 = 1.$$

If U_1/R_2 were homocyclic of exponent 4, then $\Omega_1(U_1/R_2) = T_1R_2/R_2$ and, by the above, T_1 is abelian. This is absurd. U_1/R_2 must be elementary.

(5.11) $R_2 = Z(U_1)$.

PROOF. It suffices to prove that $R_2 \leqslant Z(U_1)$. Suppose otherwise. Consider the semidirect product $R_2(U_1/T_1R_2)D_1$, of order $q^3|D_1|$. By (5.10) and (2.2), we may apply (2.3). As we are assuming that R_2 is not central in U_1, we conclude that if $r_2 \in R_2 - A_1$ and $u \in U_1 - T_1R_2$, then $[u, r_2] \neq 1$.

Recall the proof of (5.9) and the notation. For $t_2 \in T_1 - A_1\langle t_1 \rangle$, $t_2^u = t_2 d$ for $d \in R_2 - A_1$. But $u^2 \in R_2$, so $t_2 = t_2^{u^2} = t_2 d d^u$ and $d = d^u$. This contradicts the above paragraph.

We can now obtain a contradiction in the case $\tilde{M}_1 \cong Sz(q)$. For each involution $t \in A_0^{\#}$, both R_2 and U_1/R_2 are free $\mathbf{F}_2\langle t \rangle$-modules.

Suppose $A_0 = \langle t \rangle$. Then the above implies $t^G \cap U = U_1 t = t^U$, and so $U = U_1\langle t \rangle \in \mathrm{Syl}_2(G)$. By the Thompson transfer lemma $t \notin G'$, although $U_1 \leqslant G'$ (use the action of D_1). As U_1 has class 2 we have a contradiction to (3.3).

Now assume that $|A_0| > 2$. Fix $u \in U_1 - T_1R_2$ and consider the involutions t^u for $t \in A_0^{\#}$. We have $t^u = t t_1 r_2$ for $t_1 \in T_1 - A_1$, $r_2 \in R_2$. As $u^2 \in R_2$, $t^{u^2} \in tA_1$. But $t^{u^2} = t t_1 r_2 t_1^u r_2^u$ and $r_2 = r_2^u$. So $t_1 t_1^u \in A_1$ and $T_1 \cap T_1^u \geqslant A_1\langle t_1 \rangle$. On the other hand, $T_1 \cap T_1^u \leqslant C(A_0) \cap C(A_0^u)$. Choosing $t' \neq t$ in $A_0^{\#}$ we have $(t')^u = t' t_1' r_2'$. We claim that $t_1' A_1 \neq t_1 A_1$. Otherwise $tt' \in A_0$ and $(tt')^u \in tt'R_2$. But as R_2 is a free $\mathbf{F}_2\langle tt' \rangle$-module, this implies that $u \in R_2 C(tt')$, which is false. Therefore $t_1' A_1 \neq t_1 A_1$. Consequently, t_1 does not centralize $(t')^u$. This is a contradiction, as $t_1 \in T_1^u$.

6. $\tilde{M}_1 \cong U_3(q)$. In this section we assume $\tilde{M}_1 \cong U_3(q)$, $q = 2^n \geqslant 4$, and obtain a contradiction. Let D_1 be as in §3 and $D = D_1^{q-1}$. Also $R_0 = T_1R_2$.

(6.1) $|A_0| = 2$.

PROOF. Assume $|A_0| > 2$. By (3.7) $O(M) = 1$. Let K_1 and $K = K_1^{q-1}$ be as in §3. Then $T_1K \leqslant C(A)$ and for each $A_0^g \leqslant A$, $(C_{M_1^g}(K))' \cong L_2(q)$. Set $G_0 = N_G(K)$.

By induction $E(G_0/O(G_0)) \cong A_9$, J_2, or M_{12}. The first case is out by (3.13)(ii). Suppose $E(G_0/O(G_0)) \cong J_2$. Then $|A_0| = q = 4$, R_2 is elementary abelian, and $[A_0R_2, T_1] = 1$ (for this use the 3-subgroups lemma together with $[R_2, K] = 1$, $[T_1, K] = T_1$, and $[R_2, T_1] \leqslant A_1$). Also G_0 contains an involution g interchanging A and R_2. (Sylow 2-subgroups of G_0 are of type

$L_3(4)\langle\sigma\rangle$ where σ is a graph-field automorphism. See [13].) Hence g normalizes $O_2(C_G(\langle A, R_2\rangle)) = T_1$. Since $[g, K] = 1$ and K is irreducible on T_1/A_1, g centralizes T_1/A_1 and, hence, also A_1 which is not the case.

Finally we suppose $E(G_0/O(G_0)) \cong M_{12}$. Then $G_0/O(G_0) \cup \mathrm{Aut}(M_{12})$ and $T_0 \not\leq G_0$. It follows that there is an involution in $N(T_0) \cap C(K)$ not centralizing T_0 (e.g. see the table in [3], which gives centralizers of involutions for $\mathrm{Aut}(M_{12})$). This is impossible.

(6.2) $[T_1, R_2] = 1$.

PROOF. We have $[D, R_2] = 1$ and $[D, T_1] = T_1$. So $[D, R_2, T_1] = 1$ and $[R_2, T_1, D] \leq [A_1, D] = 1$. The 3-subgroups lemma implies the result.

(6.3) (i) *T_1R_2 is characteristic in $T_1R_2A_0$.*
(ii) *R_2 is characteristic in T_1R_2 and in $T_1R_2A_0T_3$.*

PROOF. The abelian subgroups of maximal order in $R_0A_0 = T_1R_2A_0$ are the groups BR_2 where $B \leq T_1$ and $B \cong Z_4^n$. For if $B_1 \leq T_1R_2A_0$ is abelian and $B_1 \not\leq R_0$, then $[B_1, x] \neq 1$ for any $x \in R_2 - A_1$, $|B_1 \cap R_0| \leq 2^{2n}$ and $|B_1| \leq |A_0|2^{2n} < 2^{3n} = |BR_2|$. So $R_0 = J(R_0A_0)$ and R_0 is characteristic in R_0A_0. Also $R_2 = Z(R_0)$. Similarly, $R_0 = J(T_1R_2A_0T_3)$ and we get the result.

Notation. Let $A_0 = \langle t\rangle$ and $Y = N(T_1R_2A_0)$. So $Y \leq N(R_2)$ by (6.3). Set $\Omega = \{(AR_2/R_2)^Y\}$ and let Y^* be the induced group of Y on Ω. Set $S_1 = T_1R_2A_0T_3$ and $S_1 \leq S \in \mathrm{Syl}_2(G)$ with $S \cap Y \in \mathrm{Syl}_2(Y)$.

(6.4) $\Omega \neq \{AR_2/R_2\}$.

PROOF. Suppose that $\Omega = \{AR_2/R_2\}$ so that $A_0R_2 \trianglelefteq Y$. Then

$$Y \leq N(T_1R_2A_0 \cap C(AR_2)) = N(T_1).$$

If R_2 is elementary, then $A_1\langle t\rangle = t^G \cap R_2\langle t\rangle$ and so $S_1 = S$. We can obtain the structure of $S \cap Y$ in case R_2 is homocyclic of rank n as follows. First note that Y cannot involve $L_3(2)$ in its action on R_2T_1/T_1, since using the squaring map Y would then involve $L_3(2)$ in its action on A_1. But then Y involves $L_3(2)$ in its action on T_1 and the argument in (3.14) give a contradiction. Now apply (2.9) to the group Y/T_1 (setting $U = R_2T_1/T_1$ and $R = A_0T_1/T_1$). We may assume $S \cap Y = U_1T_3A_0$ where $T_1R_2 \leq U_1$ and U_1 is invariant under T_3A_0 and there is a subgroup $E_1 \leq N(U_1)$ such that E_1 acts as D_1 on T_1R_2. So $E = E_1^{q-1}$ centralizes U_1/T_1. Also E_1 is irreducible on T_1/A_1 and E does not centralize T_1/A_1. It follows that $U_1/A_1 = T_1/A_1 \times U_2/A_1$ for $U_2/A_1 = C_{U_1/A_1}(E)$. Also $U_2 = C_{U_1}(E)$ and $U_2/A_1 \cong U_1/T_1$. By (2.4) and the fact that R_2 is homocyclic of rank n, U_2 is homocyclic of rank n, or U_2 is isomorphic to a Sylow 2-subgroup of $U_3(q)$ or $L_3(q)$. As in (6.2) $[U_2, T_1] = 1$.

If R_2 is elementary, set $U_2 = R_2$, so that we have the group U_2 in all cases.

We next determine S. If $U_2 = R_2$ is elementary then we have already noticed that $S = S_1 = T_1R_2T_3A_0$. Suppose U_2 is nonabelian. Then

$$N_S(S \cap Y) \leqslant N_S(Z(\Omega_1(S \cap Y))) = N(A_1)$$

and T_1U_2/A_1 is unique of its isomorphic type in $(S \cap Y)/A_1$. Consequently, $T_1U_2 \trianglelefteq N_S(S \cap Y)$. Fix $s \in N_S(S \cap Y) - S \cap Y$. Then $C_{T_1U_2}(t^s) \cong T_1$. If $t^s \in T_1U_2t$, then, as $U_2\langle t\rangle \leqslant C(T_1)$, $t^s \in U_2t$. But

$$t^G \cap U_2t = R_2t = t^{U_2},$$

so $s \in U_2C(t) \leqslant S \leqslant Y$, a contradiction. So $t^s \in T_1U_2(T_3A_0 - A_0)$. t^s normalizes T_1 and U_2, and $C_{T_1U_2/A_1}(t) \cong C_{T_1U_2/A_1}(t^s)$ has order q^3. It follows that t^s centralizes U_2/A_1 and, hence, $U_2\langle t\rangle/A_1$. Then for $x \in U_2\langle t\rangle$,

$$[x^2, t^s] = [x, t^s]^x[x, t^s] = 1$$

so t^s centralizes R_2.

Therefore $R_2^{s^{-1}} \leqslant T_1$ and is inverted by an element of $T_1U_2T_3A_0$. Hence, A_0T_3 is noncyclic and

$$R_2^{s^{-1}}/A_1 = C_{T_1/A_1}(\Omega_1(T_3)).$$

Thus $|N_S(S \cap Y) : S \cap Y| = 2$. We easily see that A_1 is characteristic in $N_S(S \cap Y)$, and so T_1U_2 is characteristic in $N_S(S \cap Y)$. From the action of $N_S(S \cap Y)$ on T_1U_2/A_1 we now see that $N_S(S \cap Y) = S$. Let $\Omega_1(T_3) = \langle v\rangle$. Without loss $t^s \in tvT_1U_2$. Since T_1/A_1 is a free $F_2\langle tv\rangle$-module we may assume that $t^s \in tvU_2$. Also t^s centralizes $U_2\langle t\rangle/A_1$, so tv centralizes U_2/A_1 and, hence, $U_2\langle t\rangle/A_1$. Arguing as in the previous paragraph we obtain $tv \in C(R_2)$. Write $t^s = tvu_2$ with $u_2 \in U_2$. Then $u_2 \in C_{U_2}(R_2) = R_2$. We are assuming that R_2 is not elementary abelian, so for t^s to be an involution we must have $u_2 \in A_1$. Conjugating t^s by an element of T_1 we may assume that $t^s = tv$.

Now by (2.6), T_1U_2/A_1 is strongly closed in S/A_1, whence by Goldschmidt [11], the action of E_1, and the knowledge of the appropriate Schur multipliers, we conclude that $\overline{T_1}\overline{U_2} \trianglelefteq \overline{N(A_1)}$ (where bars refer to images in $N(A_1)/O(N(A_1)))$. Let $x \in S - Y$ and suppose $x \sim t$. T_1U_2/A_1 is an $F_2\langle x\rangle$-module, so if $C/A_1 = C(x) \cap T_1U_2/A_1$, then $|C| \geqslant q^3$. As x centralizes C/A_1, $|C_C(x)| \geqslant q^2$. Any subgroup X of T_1 of order q^2 contains A_1. For otherwise, with H a subgroup of A_1 of index 2 containing $X \cap A_1$, and bars indicating images modulo H, $\overline{T}_1$ is extraspecial of order $2q^2$ and $\overline{X}$ is a subgroup of order $> q$ meeting $Z(\overline{T}_1)$ trivially. This is impossible. Choose g with $x^g = t$ and $C_S(x)^g \leqslant S$. By the above we have $t_1 \in T_1$ satisfying

$$t^{t_1^{-1}xt_1} = t^{xt_1} \in tvA_1,$$

so we may assume that $[t, t^x] = 1$. Considering the action of x on $C_G(t) \cap C_G(t^x)$ we have $|C_{A_1}(x)| = q$ or $\sqrt{q}$. Since $|C| \geqslant q^3$ and all involutions in

A_1x are in $C_{A_1}(x)x$, we conclude that $|C_C(x)| \geq q^2$ or $q^2\sqrt{q}$, respectively, and in either case $\Phi(C_C(x)) \leq C_{A_1}(x)$.

Set $J = \langle x \rangle \times \langle tt^x \rangle C_C(x)$. Then $J^g \leq C_S(t) = T_1T_3A_0$. As $\langle x \rangle \times C_C(x)$ has class at most 2, $(\langle x \rangle \times C_C(x))^g < T_1\langle t, v \rangle$. Suppose $C_{A_1}(x) = A_1$. Then for $r_2 \in R_2 - A_1$, x centralizes the involution $r_2r_2^x$, and so $C_C(x)$ contains an elementary abelian group of order q^2. This implies that $(\langle x \rangle \times C_C(x))^g \leq T_1\langle t, v \rangle$ contains an elementary subgroup of order $2q^2$, a contradiction. Therefore $|C_{A_1}(x)| = \sqrt{q}$. Then $y \in C_C(x)$ implies that

$$|[C_C(x), y]| \leq \sqrt{q} \quad \text{and} \quad |C_C(\langle x, y \rangle)| \geq q^2.$$

Therefore

$$y^g \leq T_1\langle t \rangle \quad \text{and} \quad (C_C(x)\langle x \rangle)^g \leq T_1\langle t \rangle.$$

As $C_C(x)\langle x \rangle$ has order at least $2q^2\sqrt{q}$, we must have $A_1 \leq \Phi(C_C(x))$, whereas $\Phi(C_C(x)) \leq C_{A_1}(x)$ has order at most $\sqrt{q}$. This is a contradiction. We have now shown that $t^G \cap S \subseteq S \cap Y$.

Now $t \not\sim v$ as v centralizes an elementary abelian subgroup of T_1U_2 of order q^2. Hence $t^G \cap \langle v \rangle T_1U_2 = \varnothing$. Next suppose that $v \sim x$ with x extremal and $x \in T_1U_2$ or $x \in S - Y$. Choose $g \in G$ with $v^g = x$, $C_S(v)^g \leq C_S(x)$. Then $t^g \sim_S t$, because we have shown that $t^G \cap S \leq T_1U_2\langle v \rangle t$ and S controls fusion in $t^G \cap T_1U_2\langle v \rangle t$. So $t^g = t$, without loss. Therefore $g \in N(M_1)$ and $v^g \in C(t)$. Then $v^g \in S \cap Y$, and, $x \notin T_1U_2$. This is a contradiction. Therefore v is extremal.

We have $C_S(v) = FT_3\langle t \rangle\langle s_1 \rangle$, where $F = C_{T_1U_2}(v)$ is elementary of order q^2, and $s_1 = 1$ or $s_1 \in S - Y$. By the above v is extremal in S so that $C_S(v) \in \mathrm{Syl}_2(C_G(v))$. Set $S_0 = C_S(v)$ and $I = C_G(v)$. We study the strong closure, J, of $F\langle v \rangle$ in S_0 (with respect to I). We have already seen that $t^G \cap F\langle v \rangle = \varnothing$. Since $t^G \cap Ft = A_1t = t^F$, and since no element of $Ft - A_1\langle t \rangle$ is an involution, we conclude that no element of Ft is conjugate to an element of $F^\#$. Similarly, $t \sim tv$ and $t^G \cap Ftv = A_1tv = (tv)^F$ imply that no element of Ftv is conjugate to an element of F. Let $a \in (F\langle v \rangle)^\#$ and suppose $a^I \cap S_0 \nsubseteq F\langle v \rangle$. Say $b \in (a^I \cap S_0) - F\langle v \rangle$.

No element of $FT_3 - F\langle v \rangle$ is an involution, so, by the above, $b \in S_0 - FT_3\langle t \rangle$. If $F\langle v \rangle$ is not weakly closed in S_0 (with respect to I), then for some $g \in I$, $F\langle v \rangle \neq (F\langle v \rangle)^g \leq S_0$, so $F\langle v \rangle \cap (F\langle v \rangle)^g$ is maximal in $F\langle v \rangle$, and we may assume that b centralizes a maximal subgroup of $F\langle v \rangle$. If $F\langle v \rangle$ is weakly closed in S_0, then this also holds, as is seen from (2.6). We claim that b must centralize F. As b normalizes T_1U_2, b normalizes $C_{T_1U_2/A_1}(F) = \hat{F}/A_1$, where $\hat{F} = T_{11}R_2$, $T_{11} \leq T_1$, and T_{11} is homocyclic of rank n. In particular, $\hat{F}$ is abelian, $\Omega_1(\hat{F}) = F$, and $\mho_1(\hat{F}) = A_1$. Consider the action of b on $\hat{F}/A_1$. Since $t^b \in tvF$, $t^b \in tvA_1$, and so $T_1 \cap T_1^b \leq C_{T_1}(tv) = A_1$. Consequently, $\hat{F}/A_1 = T_{11}/A_1 \times T_{11}^b/A_1$ and $\hat{F}/A_1$ is a free $\mathbf{F}_2\langle b \rangle$-module. If b centralizes

A_1, then $g \in \hat{F}$ implies that gg^b is an involution centralized by b, and it follows that b must centralize F. If b does not centralize A_1, then $[F, b] = [A_1, b]$ has order 2 and $F/A_1 = C(b) \cap \hat{F}/A_1$. In this case there is an element $g \in T_{11}$ with gg^b of order 4 (just choose g so that $(g^2)^b \neq g^2$). But then $gg^bA_1 \notin F/A_1$ although b centralizes gg^bA_1. This is a contradiction, so the claim is proved.

In view of the above claim we conclude that either $J = F\langle v\rangle$ or $J = F\langle v\rangle\langle b\rangle = C_{S_0}(F\langle v\rangle)$. Clearly J is weakly closed in S_0, and using (2.6) we have J strongly closed in S_0. We can then apply Goldschmidt [11] and conclude that $E(I/O(I)) \cong L_2(q^2)$ or $L_2(q) \times L_2(q)$. Let H be a 2-complement in $N(F) \cap C(v)^{(\infty)}$. We consider the group $N = N_0H$, where $N_0 = N_G(F\langle v\rangle) \cap C_G(F)$. Then $N_0 \trianglelefteq N$ and, since $v^G \cap F = \varnothing$, N acts on Fv. Also $P = N_0 \cap T_1U_2$ has order q^4 (check this directly using the fact that $[T_1, U_2] = 1$). As $C_P(v) = F$, P is transitive on Fv. Consequently, N_0 and N are each transitive on Fv.

In N_0 the stabilizer of v must centralize $F\langle v\rangle$. So let $N_1 = C_N(F\langle v\rangle)$. Then $N = N_1PH$ and H acts on PN_1/N_1. Since $C_S(v) \in \mathrm{Syl}_2(C_G(v))$, and since $C_S(v)$ normalizes N_0, we conclude that $C_S(v) \cap N_1 \in \mathrm{Syl}_2(N_1)$. Consequently, $F\langle v\rangle$ has index at most 2 in a Sylow 2-subgroup of N_1, and $\hat{S} = S \cap N_0 \in \mathrm{Syl}_2(N_0)$. Now $N_N(\hat{S})$ covers N/N_0 and we observe that a 2-complement of $N_N(\hat{S})$ induces an abelian group on $\hat{S}$. For let H_1 be a 2-complement in $N_N(\hat{S})$. Then $H_1/C_{H_1}(F) \cong H/C_H(F)$ is abelian. Also $C_{H_1}(F) < N_1$, so $[P, C_{H_1}(F)] \leq F\langle v\rangle\langle g\rangle$, where $g^2 \in F\langle v\rangle$ and $g \in N(F\langle v\rangle)$. Since $C_{H_1}(F)$ centralizes F and normalizes $F\langle v\rangle$, $[P, C_{H_1}(F)] = 1$, proving the claim. Since $E_1^{q+1} \leq FH$ and $[\hat{S}, E_1^{q+1}] = P$, $N_N(\hat{S}) \leq N_N(P)$. Therefore $N_N(\hat{S}) \leq N(P') = N(A_1)$. But $H\langle t\rangle$ acts irreducibly on F and $N_N(\hat{S})$ induces $H/C_H(F)$ on F. This is a contradiction.

We are left with the possibility that U_2 is homocyclic of rank n. If $U_2 = R_2$, then $S = S_1 = T_1R_2T_3A_0$, so suppose $U_2 > R_2$. Checking centralizers we see that

$$R_2t \subseteq t^{N(S\cap Y)} \subseteq U_2t.$$

So

$$N(S \cap Y) \leq N\big(C_{S\cap Y}(t^{S\cap Y})\big) \leq N(T_1) \quad \text{or} \quad N(T_1\Omega_1(T_3)),$$

and as T_1 is characteristic in $T_1\Omega_1(T_3)$, we have $N(S \cap Y) \leq N(T_1)$. Consider the group $I = C_G(T_1)E_1$. We have $(S \cap Y) \cap I = U_2\langle t\rangle$. We apply (2.9) to I. Even though we may have $q = 4$, the arguments used in the proof of (2.9) all carry through in this case as well. We conclude that there is a homocyclic group $\hat{U}_2 \geq U_2$ such that $\hat{U}_2$ has rank n and $\hat{U}_2\langle t\rangle \in \mathrm{Syl}_2(C(T_1))$. We may assume that $T_3 \leq N(\hat{U}_2)$. Letting $S_2 = T_1\hat{U}_2T_3A_0$, we may assume $S_2 \leq S$. Choose $g \in N_S(S_2) - S_2$. Then $g \in N(Z(\Omega_1(S_2))) =$

$N(A_1)$, and since $T_1\hat{U}_2A_0/A_1$ is the unique group of its isomorphism type in S_2/A_1, we have $g \in N(T_1\hat{U}_2A_0)$. Checking centralizers we have $t^g \in \hat{U}_2t$. So g normalizes

$$C_{T_1\hat{U}_2A_0}\left(\langle t^{N_S(S_2)}\rangle\right) = T_1.$$

Now consider $N(T_1)/T_1$. We argue as in (3.14) that $N(T_1)$ does not involve $L_3(2)$. As g normalizes $T_1\hat{U}_2\langle t\rangle/T_1$ we have $\langle E_1, g\rangle$ inducing a 2-transitive group on $T_1\hat{U}_2\langle t\rangle/T_1\mho_1(\hat{U}_2)$. Using the arguments of (2.7) together with an application of the 3-subgroups theorem we conclude that there is a 2-group $\tilde{U}_2 > \hat{U}_2$ with $[\tilde{U}_2, t] = \hat{U}_2$ and $\tilde{U}_2 \leqslant C(T_1)$. However, $\hat{U}_2\langle t\rangle \in \mathrm{Syl}_2(C(T_1))$. This is a contradiction. We conclude that $S = T_1\hat{U}_2T_3A_0$. We now have S in all cases. If $R_2 = U_2$, let $\hat{U}_2 = U_2$. Then $S = T_1\hat{U}_2T_3A_0$.

If T_3A_0 is cyclic, then we have a contradiction as follows. It is easy to check centralizers to get $t^G \cap T_1\hat{U}_2 = \varnothing$. So by the Thompson transfer lemma and the action of D_1, $T_1\hat{U}_2 \in \mathrm{Syl}_2(\langle T_1\hat{U}_2)^G\rangle)$, against (3.3). We may assume that $T_3A_0 = T_3 \times A_0 > A_0$ and let $\langle v\rangle = \Omega_1(T_3)$.

Let $X = N_G(A_1)$ and let bars denote images in $X/A_1 = \bar{X}$. Then using (2.6) we have $\overline{T_1\hat{U}_2}$ strongly closed in $\bar{S}$ unless possibly $\hat{U}_2 = R_2$, in which case $\overline{T_1\hat{U}_2A_0}$ is strongly closed in $\bar{S}$. By Goldschmidt [11] $(\bar{T}_3 \times \bar{A}_0) \cap \langle(\overline{T_1\hat{U}_2})^{\bar{X}}\rangle = 1$. From here we check centralizers of elements of $T_3 \times A_0$ acting on $\overline{T_1\hat{U}_2}$ and use the fact that $\bar{S}/\overline{T_1\hat{U}_2}$ is abelian to see that

$$t^X \cap T_1\hat{U}_2v = t^X \cap T_1\hat{U}_2tv = \varnothing.$$

In particular, $t \not\sim v$ and $t \not\sim tv$ in X. Using the action of $\bar{E}_1$ on $\overline{T_1\hat{U}_2}$ and information on multipliers, we conclude that $\overline{T_1\hat{U}_2O(\bar{X})} \trianglelefteq \bar{X}$.

If $t^g = v$ or tv, then for some Sylow 2-subgroups, H, of M_1^g, $A_1 = \Omega_1(H)$ and we may suppose $g \in N(A_1)$. But we have just seen this to be impossible. So $t \not\sim v$ and $t \not\sim tv$. Consequently, if we set $K = C_G(v)/\langle v\rangle$, then $M_2 = C_{M_1}(v)/\langle v\rangle$ is *-standard in K and $N_K(M_2) \cap C(\tilde{M}_2)$ contains $\langle t, v\rangle/\langle v\rangle$ as a Sylow 2-subgroup. By induction we have the structure of $K/O(K)$. Similarly for $C_G(tv)/\langle tv\rangle$.

We claim that $t^G \cap T_1\hat{U}_2\langle v\rangle = \varnothing$. For suppose $t^g \in T_1\hat{U}_2\langle v\rangle$. Then $A_1 \leqslant C(t^g)$ and we first show that $A_1 \leqslant M_1^g$. If $a \in A_1 - M_1^g\langle t^g\rangle$, then since all involutions in M_1^ga are conjugate we have $a \sim v$ or tv. But then $C_G(v)$ or $C_G(tv)$ contains a conjugate of $T_1\hat{U}_2\langle t, v\rangle$, contradicting the above paragraph. Therefore $A_1 \leqslant M_1^g\langle t^g\rangle$. All involutions in $M_1^gt^g$ are fused to t^g, so $A_1 \leqslant M_1^g$. So we may assume that $g \in X = N_G(A_1)$, whereas we have already shown that $t^X \cap T_1\hat{U}_2\langle v\rangle = \varnothing$. So the claim holds and, similarly, $t^G \cap T_1\hat{U}_2\langle tv\rangle = \varnothing$.

Apply the Thompson transfer lemma to S, with $S_0 = T_1\hat{U}_2T_3$. By the above, $t \notin G'$, although from the structure of $N(T_1\hat{U}_2)$ we have $T_1\hat{U}_2 \leqslant G$. So $S \cap G' = T_1\hat{U}_2\langle l\rangle$ where $\Omega_1(\langle l\rangle) = \langle v\rangle$, $\langle tv\rangle$, or 1. Any involution in

$T_1\hat{U}_2$ is centralized by an abelian subgroup of order q^3. But from the known structure of $C_G(t)$ and $C_G(tv)$, we conclude that $v^G \cap T_1\hat{U}_2 = (tv)^G \cap T_1\hat{U}_2 = \emptyset$. Consequently, we may apply the Thompson transfer theorem once again and obtain a normal subgroup of G with $T_1\hat{U}_2$ as Sylow 2-subgroup. This contradicts (3.3), completing the proof of (6.4).

(6.5) $|\Omega| = q^2$, *Y^* is transitive on Ω, and Y^* contains a regular normal subgroup.*

PROOF. As $A_0^G \cap T_1R_2 = \emptyset$ (check centralizers), $\Omega \subseteq tT_1R_2/R_2$ and $|\Omega| \leqslant q^2$. Using the action of D_1 we see that if Ω is not of order q^2, then $|\Omega| = 1 + \frac{1}{3}(q^2 - 1)$ or $1 + \frac{2}{3}(q^2 - 1)$. In the first case Y^* satisfies the hypotheses, but not the conclusion, of Theorem 1.1 of Hering, Kantor and Seitz [16]. In the second case $|\Omega|$ is odd, so Y^* has cyclic Sylow 2-subgroup and is solvable. By order considerations Y^* is primitive. Also D_1^* is semiregular on $\Omega - \{AR_2/R_2\}$. So if N^* is a minimal normal subgroup of Y^*, N^* is semiregular on Ω and $N^*D_1^*$ is a Frobenius group. But then D_1^* cannot act fixed-point-freely on T_1R_2/R_2. So $|\Omega| = q^2$ and, by definition of Ω, Y^* is transitive on Ω.

It remains to show that Y^* contains a regular normal subgroup. In the action on Ω, $\Omega_1(T_3)$ fixes all (if $T_3 = 1$) or exactly q points of Ω, so if $T_3^* \neq 1$, then $C_{Y^*}(\Omega_1(T_3)^*)$ has Sylow 2-subgroups of order dividing $q|T_3|$. If n is even then $3 \nmid q + 1$, Y^* is 2-transitive and Theorem (1.1) of [16] gives the result. So suppose n is odd. Then $|T_3| = 1$ or 2. Consider $Y_0 = C_Y(A_1) \trianglelefteq Y$. Then Y_0^* has orbits of equal length on Ω and the stabilizer in Y_0^* of AR_2/R_2 is $D^*\Omega_1(T_3)^*$. So $|Y_0^*| = (q + 1)2^a$ or $\frac{1}{3}(q + 1)2^a$ for some integer $a \geqslant 2$. Choose N minimal normal in Y^* with $N \leqslant Y_0$. If $D^* \cap N = 1$, then N is a 2-group and is consequently a regular normal subgroup. So suppose $D^* \cap N \neq 1$. But $|N : C_N(D^* \cap N)|$ is a power of 2, so we obtain a contradiction from Burnside's Theorem.

Notation. Let Y_1 be the kernel of Y on Ω and let U be a Sylow 2-subgroup of the preimage of the regular normal subgroup of Y^*. We may assume $R_2\langle t\rangle \leqslant U \cap Y_1$, that $T_1R_2 \leqslant U$, and that there is a subgroup E_1 of odd order such that T_3 normalizes E_1, E_1 normalizes U, and E_1 induces D_1 on T_1R_2. Then $U = U_1\langle t\rangle$, where $U_1 = [U, E_1]$. Let $\hat{R}_2 = U_1 \cap Y_1$. Standard arguments imply $R_2 = \hat{R}_2$ or $\hat{R}_2/R_2 \cong A_1$ as E_1-modules.

(6.6) *Let $U_0 = [U_1, E]$, where $E = E_1^{q-1}$.*
(i) *U_0 covers $U_1/\hat{R}_2$.*
(ii) *$U_1/\hat{R}_2$ is elementary abelian.*
(iii) $[U_0, \hat{R}_2] = 1$.

PROOF. (i) is clear as $U_1/T_1\hat{R}_2$ and $T_1\hat{R}_2/\hat{R}_2$ are equivalent E_1-modules.

Also, by (2.2), $U_1/\hat{R}_2$ is abelian. As the normal closure of E in Y centralizes R_2, U_0 is in $C(\hat{R}_2)$, proving (iii).

So U_0 has class 2 and for $x, y \in U_0$, $[x^2, y^2] = [x^4, y] \in [\hat{R}_2, y] = 1$. If $U_1/\hat{R}_2$ is not elementary, this would imply T_1 is abelian, which is absurd. This proves (ii).

(6.7) (i) *R_2 is elementary abelian, $R_2 = \hat{R}_2$, and $U_1' = R_2$.*

(ii) *R_2 and U_1 are characteristic in $U_1T_3A_0 = \hat{U}$.*

PROOF. Suppose that R_2 is homocyclic of rank n. As U_0 has class 2 and $\hat{R}_2 \cap U_0 < Z(U_0)$, x, y in U_0 implies that $1 = [x^2, y] = [x, y]^2$. So $[U_0, U_0] < A_1$ and for $x \in U_0$, $|C_{U_1}(x)| \geqslant q^5$. Choose $x \in T_1 - A_1$. Then t normalizes $C_{U_1}(x)$ and $|C_{T_1}(x)| = q^2$. Consequently, $U_1 = T_1C_{U_0}(x)$. As $[U_1, t]R_2 = T_1R_2$, we obtain a contradiction by looking at $[t, C_{U_1}(x)]$. Namely, $C_{U_1}(x)$ covers U_1/T_1R_2, so

$$T_1R_2 = [t, U_1]R_2 = [t, C_{U_1}(x)]R_2.$$

On the other hand, $[t, C_{U_1}(x)] \leqslant C_{T_1R_2}(x)$, which does not cover T_1R_2/R_2. So R_2 is elementary and $R_2 = \hat{R}_2$. Also the above argument shows that $U_1' = R_2$.

By (6.6) $R_2 \leqslant Z(U_1)$. Also $[U_1, t]R_2 = T_1R_2$ and $\hat{U}' < U_1$, so $C_{\hat{U}}(\hat{U}') \geqslant R_2$, is t-invariant and intersects T_1 in A_1. This forces $C_{\hat{U}}(\hat{U}') = R_2$, so R_2 is characteristic in $\hat{U}$. In $\hat{U}/R_2$, U/R_2 is unique of its isomorphism type, proving (ii).

(6.8) $t^G \cap U_1\langle t\rangle = t^{U_1} = U_1t$.

(ii) $T_3A_0 = T_3 \times A_0 > A_0$.

PROOF. Both U_1/R_2 and R_2 are free $\mathbf{F}_2\langle t\rangle$-modules, proving (i) (no conjugate of t centralizes R_2, so $t^G \cap U_1 = \varnothing$). If T_3A_0 is cyclic, then from (6.7)(ii) and (i) we have $S = U_1T_3A_0$. However, we can then transfer out T_3A_0, contradicting (3.3). So T_3A_0 is not cyclic and (ii) holds.

(6.9) $S > U_1T_3A_0$.

PROOF. Suppose that $S = U_1(T_3 \times A_0)$. First assume that $C_S(R_2) > U_1$. Then $C_S(R_2) = U_1\langle k\rangle$, where $\langle k\rangle$ or $\langle kt\rangle$ is $\Omega_1(T_3)$. By (3.5)(a) and transfer, $G = O^2(G)A_0$ with $O^2(G) \cap A_0 = 1$. Clearly $U_1 \leqslant O^2(G)$, so $U_1\langle l\rangle \in \mathrm{Syl}_2(O^2(G))$ with $\Omega_1(\langle l\rangle) = \langle k'\rangle$ and $k' = k$ or kt. If $(k')^{O^2(G)} \cap U_1 = \varnothing$, then we have a contradiction by transfer. So assume $(k')^G \cap U_1 \neq \varnothing$. Involutions in U_1 have centralizers of order at least q^4 (as $U_1' = R_2$). Since U_1/R_2 is a free $\mathbf{F}_2\langle k'\rangle$-module, each involution in U_1k' is conjugate to one in R_2k'. Now $C_{U_1}(k')$ does not cover $C(k') \cap U_1/R_2$, since $C_{T_1}(k') = A_1$. Consequently, elements of $k'^G \cap U_1\langle k'\rangle$ that are extremal in $U_1\langle l\rangle$ are all in U_1. Choosing a Sylow 2-subgroup of $C(k')$ and extending to a Sylow 2-subgroup of G, we see that there is a conjugate U_1^g of U_1 such that $k' \in U_1^g$

and $t \in N(U_1^g)$. Since $t \notin U_1^g$, both U_1^g/R_2^g and R_2^g are free $\mathbf{F}_2\langle t\rangle$-modules. It follows that $|C(t) \cap U_1^g| = q^3$. Modulo $O(M)$, $C(t) \cap U_1^g\langle t\rangle = T_1^*\langle t, v\rangle$, where $T_1^* \leqslant U_1^g$ and has index 2 in T_1. But $[k, T_1]$ is homocyclic of exponent 4, whereas $[k, U_1^g] \leqslant R_2^g$. This is impossible.

We now have $C_S(R_2) = U_1$. Then both U_1/R_2 and R_2 are free modules for $\langle t\rangle$, $\langle v\rangle$, and $\langle tv\rangle$, so we conclude that each involution in $S - U_1$ is conjugate in U_1 to one of t, v, tv. The arguments of the previous paragraph show that $v^G \cap U_1 = (tv)^G \cap U_1 = \varnothing$, and we know that $t^G \cap U_1 = \varnothing$, as t cannot centralize a conjugate of R_2. Suppose $t^G \cap U_1\langle v\rangle = \varnothing$. Then apply the Thompson transfer theorem twice to conclude that $U_1 \in \mathrm{Syl}_2(\langle U_1^G\rangle)$, which contradicts (3.3). So assume that $t^g \in U_1\langle v\rangle$ for some $g \in G$. By the above we may assume that $t^g = v$. Also we may assume g normalizes $C_{M_1}(v)^{(\infty)}$ and A_1. So $(A_1\langle t\rangle)^g = A_1\langle v\rangle$. A Sylow 2-subgroup of $N_G(A_1\langle v\rangle)$ is a conjugate of $T_1R_2T_3\langle t\rangle$. Note that $R_2 \leqslant N(A_1\langle v\rangle)$. We claim that R_2 is strongly closed in $T_1R_2T_3\langle t\rangle$ with respect to G. Suppose $r \in R_2$ and $r^x \in T_1R_2T_3A_0 - R_2$. Then $r^x \notin T_1R_2$, as $\Omega_1(T_1R_2) = R_2$. So $[t, r^x] \in T_1 - A_1$. However, $\langle t, r^x\rangle$ is dihedral, so t inverts $[t, r^x]$. This is impossible. By the above we have now established the claim. So we may take $g \in N(R_2)$. Now use (2.6) and Goldschmidt [11] to conclude that $U_1O(N_G(R_2)) \trianglelefteq N_G(R_2)$. Consequently, we may assume $g \in N(U_1)$.

It now follows that g can be chosen as a 3-element in $N(S)$ and $t \sim v \sim tv$. In particular, $T_3 = \langle v\rangle$. Consider the group $N = N_G(U_1)$ and let bars denote images in $N/C_N(U_1/R_2)$. The group $C_N(R_2)$ is 2-closed and $\bar{E} \leqslant \overline{C_N(R_2)}$. Also, since g normalizes A_1 and $C_{M_1}(v)^{(\infty)}$, we may assume $\bar{g}$ normalizes $\bar{E}_1^{q+1}$. Say $\bar{H}$ is minimal normal in $\bar{N}$ with $H \leqslant C_N(R_2)$. $\bar{H}$ is an elementary l-group for some prime l. Then

$$\bar{H} = C_{\bar{H}}(\bar{t})C_{\bar{H}}(\bar{v})C_{\bar{H}}(\overline{tv}).$$

But $C_{\bar{H}}(\bar{t}) \leqslant \bar{E}$ and $C_{\bar{H}}(\bar{t}) \cap C_{\bar{H}}(\overline{tv}) = 1$. So $\bar{H}$ has order l^3. As $\bar{E}_1^{q+1}$ centralizes $C_{\bar{H}}(\bar{t})$, we use the action of $\bar{g}$ to see that $\bar{E}_1^{q+1}\bar{H} = \bar{E}_1^{q+1} \times \bar{H} = B$. However, U_1/R_2 has at most 2 B-composition factors, as $\bar{E}_1^{q+1} \times C_{\bar{H}}(\bar{t})$ is irreducible on T_1R_2/R_2 and on U_1/T_1R_2. This implies B has rank at most 2, and we have a contradiction. This completes the proof of (6.9).

Let $N_1 = N_G(U_1)$ and let bars denote images in N_1 modulo $C_{N_1}(U_1/R_2)$. Write $\langle v\rangle = \Omega_1(T_3)$. There is an element $s \in N_S(U_1T_3A_0)$ such that $t^s \notin U_1t$. Now s acts on $T_3A_0U_1/U_1 \cong T_3 \times A_0$. So if $|T_3| > 2$ we have $t^s \in U_1tv$. If $|T_3| = 2$ we rechoose T_3, if necessary, so that in all cases $t^s \in U_1tv$. Since each of R_2 and U_1/R_2 is a free $\mathbf{F}_2(\langle t^s\rangle)$-module, we may assume $t^s = tv$. Then s normalizes $\bar{N} = C_{\bar{N}_1}(\overline{\langle t, t^s\rangle})$. So s normalizes $\bar{E}_{00}$, where $E_{00} = E_1^{q+1}$. Also s normalizes $O(\bar{N}_1) \geqslant \bar{E}$.

Now $[\bar{E}_{00}, \bar{E}] = [\bar{E}_{00}, \bar{E}^{\bar{s}}] = 1$, so that $W = \langle \bar{E}_{00}, \bar{E}, \bar{E}^{\bar{s}}\rangle$ centralizes $\bar{E}_{00}$. It follows that $V = U_1/R_2$ may be regarded as an $\mathbf{F}_q$-module for W and that V is either irreducible under the action of W or the sum of two irreducibles. That is, V is the sum of irreducible submodules of dimension 1 or 2. Passing to splitting fields and noting that W has odd order, we see that each of the submodules splits into linear factors. Consequently, W is abelian.

Since t centralizes $\bar{E}$ and t^s inverts $\bar{E}$, we know that $\bar{E}\bar{E}^s = \bar{E} \times \bar{E}^s$. From here we see that V splits into a sum $V = V_1 \oplus V_2$ of inequivalent irreducible modules for $\bar{E}_1\bar{E}_1^s = \bar{E}_{00} \times \bar{E} \times \bar{E}^s = W$.

This decomposition of V gives further information about U_1 as follows. As $\bar{E}_1$ is fixed-point-free on V, t cannot stabilize V_1 and V_2. For otherwise t would centralize $W/C_W(V_i)$, $i = 1, 2$, which implies that $[W, t] \leqslant C_W(V_1) \cap C_W(V_2) = 1$, a contradiction. So t interchanges V_1, V_2. Also s acts on $\{V_1, V_2\}$ as $s \in N(W)$. Consequently, $\bar{v} \in \langle \bar{s}, \bar{t}\rangle'$ must fix each of V_1 and V_2. For $i = 1, 2$ let $\bar{F}_i = C_{\bar{E}\bar{E}^s}(V_i)$. Then $\bar{F} = \bar{F}_1\bar{F}_2 = \bar{E}\bar{E}^s$. Write $J_i/R_2 = V_i$. Then for $j_1 \in J_1$, $j_2 \in J_2$, $[j_1, j_2] = [j_1, j_2^g]$ for each $g \in F_1$. So $[J_2, F_1] \leqslant C(j_1)$, and, as $[J_2, F_1]$ covers J_2/R_2, we conclude that $[J_1, J_2] = 1$.

Let $j_1 \in J_1$. As $|J_1/R_2| = |R_2| = q^2$, $[j_1, J_1] < R_2$. Suppose $q \neq 8$. Then $\bar{F}_2$ is irreducible on J_1/R_2, and $g \in F_2$ implies $[j_1, J_1] = [j_1^g, J_1]$. So here $[J_1, J_1] < R_2$, and using the fact that $[J_1, J_2] = 1$ and the action of E_{00}, we have $|J_1'| = q$. If $q = 8$ an easy Lie ring argument shows that $|J_1'| = 8$. Similarly, $|J_2'| = q$. Setting $Y_i = [J_i, E]$ and using (6.7) we now have $U_1 = Y_1 \times Y_2$ with t interchanging Y_1 and Y_2.

Since t is an involution, $Y_1 \cong Y_2 \cong C_{U_1}(t) = T_1$. Hence $R_2 = \Omega_1(U_1)$. The Krull-Schmidt theorem implies that $\{Y_1Z(U_1), Y_2Z(U_1)\}$ is invariant under $N_G(U_1) = N_1$. Also

$$N_1 = N_{N_1}(Y_1Z(U_1))\langle t\rangle, \qquad N_0 = N_{N_1}(Y_1Z(U_1)) \lhd N_1.$$

We choose T_3 and s so that $\langle T_3, s\rangle \leqslant N_0$. Note that s and v normalize $\bar{E}_{00}$, $\bar{Y}_1$, and Y_2. Also $\bar{F} = \bar{F}_1 \times \bar{F}_2 = \bar{E} \times \bar{E}^{\bar{s}}$. Let $\bar{P}_i$ be a Sylow p-subgroup of $\bar{F}_i$, where $p = 3$ if $q = 8$ and p a primitive divisor of $q + 1$ if $q \neq 8$. Then $\bar{s}$ must centralize one of $\bar{P}_1$ and $\bar{P}_2$ and invert the other. Say $\bar{s}$ centralizes $\bar{P}_1$. Then $\bar{P}_1$ normalizes $[\bar{s}, V_2]$. If $\bar{s} \not\in C(\bar{E}_{00})$, then considering the Frobenius group $[\bar{E}_{00}, \bar{s}]\langle\bar{s}\rangle$, we conclude that V_2 is a free $\mathbf{F}_2(\langle\bar{s}\rangle)$-module. But then $\bar{P}_1$ cannot act on $[V_2, \bar{s}]$. Therefore $\bar{s}$ centralizes $\bar{E}_{00}$ and, as $\bar{E}_{00}\bar{P}_1$ is irreducible on V_2, $\bar{s} \in C(V_2)$. This forces $\bar{s} \in C(\bar{F}_1)$.

Now consider the action of $\langle E_{00}, s, v\rangle$ on R_2 to get $\langle s, v\rangle \in C(R_2)$. Further, an application of the 3-subgroups lemma to $\langle s\rangle$, Y_2 and F_1 shows that $[s, Y_2] = 1$.

We know that $\overline{\Omega_1(T_3)\langle t\rangle}$ is a Sylow 2-subgroup of $C_{\bar{C}}(\bar{t})$, so the Sylow 2-subgroups of $C\langle t\rangle/U_1$ are dihedral or quasidihedral, where $C = N(U_1) \cap C(R_2)$. Now $C \cap C(Y_i) \trianglelefteq C$ for $i = 1, 2$, and from these facts we see that $\langle s, v\rangle U_1 \in \mathrm{Syl}_2(C)$, $\langle s, v\rangle U_1/U_1$ is klein, and C/U_1 has 2-complement

($\bar{s} \not\sim \bar{v}$ as $\bar{s} \in C(\bar{V}_2)$). Let $C_i = C_{N_1}(Y_i)$ so that $\bar{F}_i \leq \bar{C}_i$. Then $[C_1, C_2] \leq C(U_1)$ and $[\bar{C}_1, \bar{C}_2] = 1$. As $\bar{C}_2 = \bar{C}_1^t$, $\bar{C}_1\bar{C}_2 \leq \bar{C}_1 C_{\bar{C}}(t)$. This shows that $\bar{F}_1\bar{F}_2 = O(\bar{C}_1\bar{C}_2) \leq \bar{N}_1$.

Say $q = 8$. We claim that $U_1\langle s, v\rangle\langle t\rangle = S$. For $U_1\langle s, v\rangle \leq C(R_2)$ implies that $t^G \cap U_1\langle s, v\rangle = 1$, and the involutions in $U_1\langle s, v\rangle\langle t\rangle - U_1\langle s, v\rangle$ are in $U_1t \cup U_1tv$. As $(U_1t)^s = U_1tv$ and $t^G \cap U_1t = t^{U_1}$, $U_1\langle s, v\rangle\langle t\rangle$ controls its fusion of conjugates of t, and the claim follows. Suppose $q \neq 8$; then $\bar{F}_1$ is irreducible on Y_2. As $\overline{N_0'} \leq C(\bar{F}_1)$, $\overline{N_0'}$ induces a cyclic group on V_2. Similarly for V_1. So $\overline{N_0'}$ is abelian of odd order and of rank at most 2. Since $C(t)$ covers $C(t) \cap (N_0/N_0')$, N_0/N_0' has Sylow 2-subgroups of rank 2. It is now easy to check that U_1 is characteristic in $S \cap N_1$, so $S = S \cap N_1 \in \mathrm{Syl}_2(G)$.

As $\Omega_1(S \cap N_0) \leq U_1\langle s, v\rangle \leq C(R_2)$, $t^G \cap (S \cap N_0) = \varnothing$, and we apply transfer to obtain a subgroup G_0 of index 2 in G with $U_1\langle v\rangle \leq G_0$. Considering the action of t on a Sylow 2-subgroup of $C_G(v)$, we see that

$$v^G \cap U_1\langle s\rangle = v^G \cap U_1\langle vs\rangle = \varnothing.$$

Again we can apply transfer to get $v \notin G'$. But then $s^t \in svU_1$ implies that $U_1 \in \mathrm{Syl}_2(G')$, contradicting (3.3).

We have now completed the proof of the main theorem.

BIBLIOGRAPHY

1. M. Aschbacher, *Tightly embedded subgroups of finite groups*, J. Algebra **42** (1976), 85–101.

2. ______, *Standard components of alternating type centralized by a 4-group* (to appear).

3. M. Aschbacher and G. Seitz, *On groups with a standard component of known type*, Osaka J. Math. **13** (1976), 439–482.

4. S. Assa, Ph.D. Thesis, Ohio State University, 1973.

5. H. Bender, *Transitive Gruppen gerader Ordnung in denen jede Involution genau einen Punkt festlasst*, J. Algebra **17** (1971), 527–554.

6. U. Dempwolff, *A characterization of the Rudvalis simple group of order* $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ *by the centralizer of noncentral involutions*, J. Algebra **32** (1974), 53–88.

7. W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029.

8. L. Finkelstein, *Finite groups with a standard component of type Janko-Ree*, J. Algebra **36** (1975), 416–426.

9. P. Fong and G. Seitz, *Groups with a (B, N)-pair of rank* 2. I, Invent. Math. **21** (1973), 1–57.

10. R. Gilman and D. Gorenstein, *Finite groups with Sylow* 2*-subgroups of class* 2, Trans. Amer. Math. Soc. **207** (1975), 1–126.

11. D. Goldschmidt, 2*-fusion in finite groups*, Ann. of Math. (2) **99** (1974), 70–117.

12. D. Gorenstein and K. Harada, *Finite groups whose* 2*-subgroups are generated by at most* 4 *elements*, Mem. Amer. Math. Soc., no. 147, 1974.

13. ______, *A characterization of Janko's two new simple groups*, J. Univ. Tokyo **16** (1970), 331–406.

14. D. Gorenstein and J. Walter, *The characterization of finite groups with dihedral Sylow* 2*-subgroups*. I, II, III, J. Algebra **2** (1965), 85–151; 218–270; 354–393.

15. K. Harada, *On finite groups having self-centralizing* 2*-subgroups of small order*, J. Algebra **33** (1975), 144–160.

16. C. Hering, W. Kantor and G. Seitz, *Finite groups with a split (B, N)-pair of rank* 1, J. Algebra **20** (1972), 435–475.

17. G. Higman, *Suzuki 2-groups*, Illinois J. Math. **7** (1963), 79–96.

18. D. R. Mason, *On finite simple groups G in which every element of $\mathfrak{L}(G)$ is of Bender type*, J. Algebra **40** (1976), 125–202.

19. M. O'Nan, *Some evidence for the existence of a new simple group* (to appear).

20. G. Seitz, *Flag-transitive subgroups of Chevalley groups*, Ann. of Math. (2) **97** (1973), 27–56.

21. F. L. Smith, *A general characterization of the Janko simple group J_2* (to appear).

22. M. Suzuki, *On a class of doubly transitive groups*, Ann. of Math. (2) **75** (1962), 105–145.

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

Department of Mathematics, University of Cambridge, Cambridge, England

Department of Mathematics, University of Oregon, Eugene, Oregon 97403