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ON A DEGENERATE PRINCIPAL SERIES OF REPRESENTATIONS OF $U(2, 2)$

BY

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ABSTRACT. A degenerate principal series of representations $T(\rho, m; \cdot)$, $(\rho, m) \in \mathbf{R} \times \mathbf{Z}$, of $U(2, 2)$, is realized on the Hilbert space of all square integrable functions on the space X of 2×2 Hermitian matrices. Using Fourier analysis, gamma functions, and Mellin analysis, we spectrally analyze the operator equation $AT(\rho, m; g) = T(\rho, m; g)A$ for all $g \in \mathfrak{G} = U(2, 2)$ on an invariant subspace of $L^2(X)$, and obtain the first main result: For $\rho \neq 0$ or m odd, $T(\rho, m; \cdot)$ is irreducible. Then we define certain integral transforms on $L^2(X)$ the analytic continuation of which leads to the second main result: $T(0, 2n; \cdot)$ is reducible.

1 **Introduction.** The irreducibility of representations of the nondegenerate principal series for real semisimple Lie groups has received considerable attention and known general results are very extensive. However, for degenerate principal series the known results are quite specialized (cf. Stein [8], Gross [3]) and no general theory has yet appeared.

In this paper we are concerned with a degenerate principal series of representations $T(\chi; \cdot)$ of the (reductive) group $\mathfrak{G} = U(2, 2)$ consisting of all 4×4 complex matrices

$$(1.1) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, \text{ and } d \in \mathbf{C}^{2\times 2},$$

such that $gpg^* = p$, where

$$(1.2) \qquad p = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

These representations are described as follows. Let $\mathfrak{B}$ be the parabolic subgroup of elements

$$(1.3) \qquad b = \begin{pmatrix} a^{*-1} & a^{*-1}x \\ 0 & a \end{pmatrix}, \qquad a \in GL(2, \mathbf{C}), x = x^* \in \mathbf{C}^{2\times 2},$$

and for $(\rho, m) \in \mathbf{R} \times \mathbf{Z}$ let $\chi = \chi_{\rho,m}$ be the character

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$$\chi(b) = |\det a|^{i\rho}[\det a]^m \tag{1.4}$$

of $\mathfrak{B}$ where $[\det a] = \det a/|\det a|$. Then the representations $= T(\rho, m; \cdot)$ of $\mathfrak{G}$ induced from χ comprise the degenerate principal series of interest here. For our purposes, we realize these representations in the space $L^2(X)$, where X is the space of 2×2 Hermitian matrices.

The main thrust of this work is to give a complete resolution of irreducibility and reducibility of these representations. Specially, we prove

(i) For $\rho \neq 0$ or m odd, $T(\rho, m; \cdot)$ is irreducible,

(ii) $T(0, 2n; \cdot)$ is reducible.

The proofs of these results require the development of considerable techniques of harmonic analysis on the space X. In particular, the proof of (ii) necessitates the construction of a certain translation-invariant operator on $L^2(X)$ analogous to a "Hilbert transform" relative to the natural action of $GL(2, \mathbb{C})$ on X. These ideas are of analytic interest in their own right.

Let us briefly describe the ideas involved. The irreducibility proof proceeds in two stages. First, in the spirit of Gelfand-Naĭmark, we apply the Fourier transform F on to the commuting relation $AT(\rho, m; g) = T(\rho, m; g)A$ restricted to the subgroup $\mathfrak{B}'$, transpose of $\mathfrak{B}$. Secondly, we analyze $\hat{A}\hat{T}(\rho, m; g) = \hat{T}(\rho, m; g)\hat{A}$ and for this we apply Mellin analysis. However, since X has no natural multiplication structure and because the inverse of a Hermitian matrix has an unmanageable denominator, we look at this relation on the invariant subspace

$$L_0^2(X) = \left\{ f \in L^2(X) \middle| f(w) = f(uxu^{-1}) \text{ for all } u \in SU(2) \right\}.$$

In the end we have simple algebraic equations to solve, and the solution leads directly to the irreducibility theorem.

In contrast to the proof of irreducibility in which one needs only to show the commuting operator is necessarily a scalar multiple of the identity operator, to prove reducibility one has to exhibit a nontrivial commuting operator which in our situation involves the construction of a "singular integral" by analytic continuation. In particular, we are interested in the operators

$$\hat{B}(\rho, 1): f(x) \to |\det x|^{-i\rho}[\det x] f(x),$$

and the corresponding operators

$$B(\rho, 1) = F^{-1}\hat{B}(\rho, 1)F.$$

Although one knows by general consideration that $B(\rho, 1)$ is convolution by some distribution on X, for our purposes we need more explicit information. This we obtain by a suitable analytic continuation construction, modelled after that of Stein [14], which goes as follows: extend ρ to be complex. Formally

$$(B(\rho, 1)f)(x) = r^*(\rho)\int f(x-y)|\det y|^{i\rho-2}\,dy$$

where $r^*(\rho)$ is a product of Γ-functions and a certain analytic function. We observe that the functions $|\det x|^{-i\rho}[\det x]$ and $|\det y|^{i\rho-2}$ on X are locally integrable only for ρ in the disjoint regions $I_m(\rho) \geqslant 0$ and $I_m(\rho) \leqslant -2$ respectively; nevertheless we prove that $r^*(\rho)\int f(x-y)|\det y|^{i\rho-2}\,dy$ is the analytic continuation of $(B(\rho, 1)f)(x) = (F^{-1}\hat{B}(\rho, 1)Ff)(x)$ for $f \in C_c^\infty(X)$. Thus we have an explicit realization of $B(\rho, 1)$ from which it is shown that $B(0, 1)$ is a nontrivial commuting operator. This renders the second result.

In a paper in preparation, the result of this work will be extended to the group $SO(n, m)$ and the Kunze-Stein theory of intertwining operators, analytic continuation, uniformly bounded representations of these degenerate principal series will be examined. Finally, I would like to express my sincere gratitude to Professors Kenneth I. Gross and Ray A. Kunze for their guidance and advice in this work.

2. **Representation of degenerate principal series and their intertwining operators.** To give explicit realization of the representations, we let

$$\mathfrak{U} = \left\{ u(x) = \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \middle| x \in X \right\},$$

$$\mathcal{C} = \left\{ c(a) = \begin{pmatrix} a^{*-1} & 0 \\ 0 & a \end{pmatrix} \middle| a \in GL(2, \mathbf{C}) \right\},$$

$$\mathfrak{V} = \left\{ v(x) = \begin{pmatrix} I & 0 \\ x & I \end{pmatrix} \middle| x \in X \right\}.$$

$\mathfrak{U}$, $\mathcal{C}$, and $\mathfrak{V}$ are subgroups of $\mathfrak{G}$, such that $\mathfrak{U}$ and $\mathfrak{V}$ both are isomorphic to the group X, and $\mathcal{C}$ is isomorphic to $GL(2, \mathbf{C})$. $\mathfrak{B}$ is the semidirect product of $\mathfrak{U}$ and $\mathcal{C}$. For $b \in \mathfrak{B}$ let $\mu(b) = (\Delta_{\mathfrak{B}}(b))^{-1}$ where $\Delta_{\mathfrak{B}}$ is the modular function of $\mathfrak{B}$. By calculation, $\mu(u(x)c(a)) = |\det a|^{-4}$. For sake of simplicity we often write $\mu(a)$ and $\chi(a)$ for $\mu(u(x)c(a))$ and $\chi(u(x)c(a))$ respectively.

The product $\mathfrak{B}\mathfrak{V}$ is open and dense in $\mathfrak{G}$ and the complement of $\mathfrak{B}\mathfrak{V}$ in $\mathfrak{G}$ is of measure zero. For any $x \in X$ we write

$$x = \begin{pmatrix} x_1 + x_4 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_1 - x_4 \end{pmatrix}, \quad x_i \in \mathbf{R}.$$

Let $dx = dx_1dx_2dx_3dx_4$ be the measure on X. The representation $T(\rho, m; \cdot)$ induced from the character $\chi = \chi_{\rho,m}$ of $\mathfrak{B}$ is realized as a multiplier representation on $L^2(X)$; namely for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{G}$ and $f \in L^2(X)$

$$
\begin{aligned}
(T(\rho, m; g)f)(x) &= (T(\chi; g)f)(x) \\
(2.1) \qquad &= |\det(xb+d)|^{i\rho-2}[\det(xb+d)]^m f((xb+d)^{-1}(xa+c)) \\
&= (\chi\mu^{1/2})(xb+d)f((xb+d)^{-1}(xa+c)) \quad \text{a.e. } x.
\end{aligned}
$$

For $x, y \in X$, define an inner product $(x|y) = \operatorname{tr}(xy)$ on X. Then X is isomorphic to 4-dimensional Euclidean space. Let F be the Fourier transform on $L^2(X)$ given by

$$(Ff)(x) = \hat{f}(x) = \frac{1}{\pi^2}\int_X e^{i(y|x)} f(y)\, dy.$$

For a bounded operator B on $L^2(X)$, let $\hat{B} = FBF^{-1}$

We shall discuss the Fourier transformed intertwining relations of the representations.

The group action $\tilde{a}$: $(x, a) \to x\tilde{a} = a^*xa$ of $GL(2, C)$ on X has 3 orbits of nonzero measure. These orbits are denoted by X_1, X_2 and X_3, and are representated by $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ respectively. Let δ_1, δ_2 and δ_3 be the respective characteristic functions.

PROPOSITION (2.2). (1) *If* $m_1 \neq m_2$, $T(\rho_1, m_1; \cdot)$ *and* $T(\rho_2, m_2; \cdot)$ *are not equivalent*. (2) *Let B be a bounded operator on $L^2(X)$ such that*

$$(2.3) \qquad BT(\rho_1, m_1; b) = T(\rho_2, m_1; b)B \quad \textit{for all } b \in \mathfrak{B}' = \mathcal{C}\mathcal{V}.$$

Then $\hat{B}$ is of the form of multiplication by

$$\beta(x) = (C_1\delta_1(x) + C_2\delta_2(x) + C_3\delta_3(x))|\det x|^{-i(\rho_2-\rho_1)/2}$$

where C_1, C_2 and C_3 are constants.

PROOF. (1) is quite obvious if we observe that $T(\rho, m; e^{i\theta}I) = e^{im\theta}\mathrm{Id}$ where Id is the identity operator on $L^2(X)$ and $T(\rho, m; g) = T(\rho, m_2; g)$ if $m_1 \equiv m_2$ mod 2 and $g \in SU(2, 2)$. More rigorously, let B be a bounded operator on $L^2(X)$ such that $BT(\rho_1, m_1; b) = T(\rho_2, m_2; b)B$ for all $b \in \mathfrak{B}'$. Then,

$$(2.4) \qquad \hat{B}\hat{T}(\rho_1, m_1; b) = \hat{T}(\rho_2, m_2; b)\hat{B} \quad \text{for all } b \in \mathfrak{B}'.$$

By formula (2.1) and a straightforward calculation,

$$\hat{T}(\rho, m; c(a)v(y))f(x) = |\det a|^{i\rho+2}[\det a]^m e^{-i(y|a^*xa)} f(a^*xa).$$

In particular, when $a = I$, (2.4) implies that $\hat{B}$ commutes with the operators of multiplication by $e^{-i(y|x)}$ for all y. Hence, as is well known, $\hat{B}$ must be of multiplication type; i.e. there exists an essentially bounded measurable function $\beta(x)$ such that $(\hat{B}f)(x) = \beta(x)f(x)$ for $f \in L^2(X)$. Then, by letting $\rho = \rho_2 - \rho_1$, $m = m_2 - m_1$, (2.4) reduces to

$$\beta(x) = |\det a|^{i\rho}[\det a]^m \beta(a^*xa) \quad \text{a.e.}$$

In particular, by choosing

$$a = \begin{pmatrix} e^{i\pi/k} & 0 \\ 0 & e^{i\pi/k} \end{pmatrix}$$

we have

$$\beta(x) = (e^{i2\pi/k})^m \beta(x) \quad \text{a.e.}$$

Hence, if $m \neq 0$, $\beta(x) = 0$ a.e., so part (1) is proved.

Now we assume $m = 0$. Let $\psi(x) = \beta(x)|\det x|^{i\rho/2}$ for $x \in X^*$ (i.e. $\det x \neq 0$). Then $\psi(a^* x a) = \psi(x)$. Since $|\det x|^{-2}\,dx$ and $\psi(x)|\det x|^{-2}\,dx$ are both invariant measure on the homogeneous spaces X_1, X_2 and X_3, by the uniqueness of invariant measure we obtain $\psi(x) = C_1\delta_1(x) + C_2\delta_2(x) + C_3\delta_3(x)$. Hence

$$\beta(x) = (C_1\delta_1(x) + C_2\delta_2(x) + C_3\delta_3(x))|\det x|^{-\rho/2} \quad \text{a.e.,}$$

and the proposition is proved.

3. **Mellin analysis and the irreducibility theorem.** In this section we shall prove that for $\rho \neq 0$ or m odd, any bounded operator A such that $AT(\rho, m; g) = T(\rho, m; g)A$ for all $g \in \mathfrak{G}$ is a scalar multiple of identity operator. Hence we obtain one of our main results, namely:

THEOREM (3.1). *For $\rho \neq 0$ or m odd, the representation $T(\rho, m; \cdot)$ is irreducible.*

Since $\mathfrak{B}' = \mathcal{C}\mathcal{V}$ and $p = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ generate $\mathfrak{G}$, $T(\chi; \cdot)$ is completely determined by its restriction to $\mathfrak{B}' = \mathcal{C}\mathcal{V}$ and p. Therefore, by Proposition (2.2) we need only to consider the commutation equation

$$\hat{A}\hat{T}(\chi; p) = \hat{T}(\chi; p)\hat{A}$$

where $\hat{A}$ is multiplication by $\alpha(x) = C_1'\delta_1(x) + C_2'\delta_2(x) + C_3'\delta_3(x)$. It is convenient to rewrite the above equation as

$$\hat{A}FT(\chi; p)F^{-1} = FT(\chi; p)F^{-1}\hat{A}. \tag{3.2}$$

Consider the subspace

$$L_0^2(X) = \left\{ f \in L^2(X) \,\middle|\, f(x) = f(uxu^{-1}) \text{ for all } u \in SU(2) \right\} \tag{3.3}$$

which is easily seen to be invariant under F, $T(\chi; p)$ and $\hat{A}$. We shall Mellin analyze the operator equation (3.2) on this subspace and see that this partial spectral analysis is enough to give complete irreducibility results for the representations $T(\chi; \cdot)$. We note here that analysis on this subspace is not enough to obtain reducibility results, and for this we resort to other techniques that appear in the next sections. The proof of Theorem (3.1) takes a few steps.

Let

$$\mathbf{T} = \left\{ u(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \, 0 \leqslant \theta \leqslant 2\pi \right\}$$

be the maximal torus of $SU(2)$. The homogeneous space $SU(2)/\mathbf{T}$, which is the 2-sphere, consists of cosets $\dot{u}_p = u_p\mathbf{T}$ and $\dot{u}(\theta_1, \theta_2) = u(\theta_1, \theta_2)\mathbf{T}$, where

$$u_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$u(\theta_1, \theta_2) = \begin{bmatrix} \cos\theta_1 & (\sin\theta_1)e^{i\theta_2} \\ -(\sin\theta_1)e^{-i\theta_2} & \cos\theta_1 \end{bmatrix}$$

for $0 \leqslant \theta_1 < \pi/2, 0 \leqslant \theta_2 \leqslant 2\pi$.

Let Ω be the set of all real 2×2 diagonal matrices

$$\Omega = \left\{ \omega = (\omega_1, \omega_2) = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \middle| \, \omega_1, \omega_2 \in \mathbf{R} \right\}$$

since a 2×2 Hermitian matrix can be diagonalized by an element of $SU(2)$ into a real diagonal matrix. More explicitly, for any $x \in X$, there exists a $u \in SU(2)$ such that $u^{-1}xu = \omega \in \Omega$. Ω is homeomorphic to $\mathbf{R}^2$. We let $d\omega = d\omega_1 d\omega_2$ be the Lebesgue measure on Ω. Since $u(\theta)$ commutes with all $\omega \in \Omega$ and $u_p^{-1}\omega u_p = u_p^{-1}(\omega_1, \omega_2)u_p = (\omega_2, \omega_1)$ one can prove that for ω such that $\omega_1 \neq \omega_2$ the map $(\omega, \dot{u}) \to u\omega u^{-1}$ is continuous and one-to-one. Let

$$\Omega_0 = \{\omega = (\omega_1, \omega_2) \in \Omega | \, \omega_1 > \omega_2\}.$$

Since the set of ω for which $\omega_1 = \omega_2$ is of measure 0. We identify $\Omega_0 \times SU(2)/\mathbf{T}$ with X. This gives a new coordinates system on X namely $(\omega_1, \omega_2, \theta_1, \theta_2)$. The coordinate transformation is given by

$$x_1 = \tfrac{1}{2}(\omega_1 + \omega_2),$$

$$x_2 = \tfrac{1}{2}(\omega_1 - \omega_2)\sin 2\theta_1 \cos\theta_2,$$

$$x_3 = \tfrac{1}{2}(\omega_1 - \omega_2)\sin 2\theta_1 \sin\theta_2,$$

$$x_4 = \tfrac{1}{2}(\omega_1 - \omega_2)\cos 2\theta_1.$$

Thus, one may think of $(\omega_1, \omega_2, \theta_1, \theta_2)$ as spherical coordinates of X. Define the measure $d\dot{u} = d\dot{u}(\theta_1, \theta_2) = \frac{1}{4}\sin 2\theta_1 \, d\theta_1 \, d\theta_2$ on $SU(2)/\mathbf{T}$. The following integration formula will be used later.

$$(3.4) \qquad \int_X f(x)dx = \int_{\Omega_0}\int_{SU(2)/\mathbf{T}} f(u\omega u^{-1})(\omega_1 - \omega_2)^2 \, d\dot{u} \, d\omega.$$

Let $L_1^2(\Omega) = \{\phi \in L^2(\Omega, d\omega) | \phi(\omega') = \phi(\omega_2, \omega_1) = -\phi(\omega_1, \omega_2) = \phi(\omega)\}$, i.e. $L_1^2(\Omega)$ is the Hilbert space of "antisymmetric" functions in $L^2(\Omega)$.

PROPOSITION (3.5). *For $f \in L_0^2(X)$, define the map Φ by $(\Phi f)(\omega) = \sqrt{\pi}\,(\omega_1 - \omega_2)f(\omega)/2$. Then Φ is a unitary map of $L_0^2(X)$ onto $L_1^2(\Omega)$ with Φ^{-1} given by $(\Phi^{-1}\phi)(x) = 2(\omega_1 - \omega_2)^{-1} \cdot \phi(\omega)/\sqrt{\pi}$ for $x = u\omega u^{-1}$ and $\omega_1 \neq \omega_2$. Moreover, we have*

$$\Phi F \Phi^{-1}\phi(\omega) = -i(2\pi)^{-1}\int_\Omega e^{i(\eta|\omega)}\phi(\eta)\,d\eta,$$

$$\Phi T(\chi; p)\Phi^{-1}\phi(\omega) = \chi(\omega)(\det\omega)^{-1}\phi(-\omega^{-1}) \quad \textit{for } \omega_1\omega_2 \neq 0,$$

$$\Phi \hat{A} \Phi^{-1}\phi(\omega) = \alpha(\omega)\phi(\omega).$$

PROOF. The fact that Φ is a well-defined bijective map is a straightforward consequence of its definition and the antisymmetric property of $\phi \in L_1^2(\Omega)$. Let us prove that Φ is unitary. Define $\Omega_0' = \{\omega' |\, \omega \in \Omega_0\} = \{\omega \in \Omega |\, \omega_1 < \omega_2\}$. Then $\Omega = \Omega_0 \cup \Omega_0'$, and $\Omega_0 \cap \Omega_0' = \{\omega \in \Omega |\, \omega_1 = \omega_2\}$ which is of measure zero in Ω. Now

$$\begin{aligned}
\|f\|^2 &= \int_{\Omega_0}\int_{SU(2)/\mathbf{T}} (f(u\omega u^{-1})|\, f(u\omega u^{-1}))(\omega_1 - \omega_2)^2\, d\dot{u}\, d\omega \\
&= \int_{\Omega_0}\int_{SU(2)/\mathbf{T}} (f(\omega)|f(\omega))(\omega_1 - \omega_2)^2\, d\dot{u}\, d\omega \\
&= \int_{\Omega_0} \frac{\pi}{2}\, |f(\omega)|^2(\omega_1 - \omega_2)^2\, d\omega \\
&= \int_{\Omega_0} \frac{\pi}{4}\, |f(\omega)|^2(\omega_1 - \omega_2)^2\, d\omega + \int_{\Omega_0} \frac{\pi}{4}\, |f(\omega)|^2(\omega_1 - \omega_2)^2\, d\omega \\
&= \int_{\Omega_0} \frac{\pi}{4}\, |f(\omega)|^2(\omega_1 - \omega_2)^2\, d\omega + \int_{\Omega_0'} \frac{\pi}{4}\, |f(\omega)|^2(\omega_1 - \omega_2)^2\, d\omega \\
&= \int_\Omega |(\Phi f)(\omega)|^2\, d\omega.
\end{aligned}$$

Therefore, Φ is a unitary map of $L_0^2(X)$ onto $L_1^2(\Omega)$. We next compute $\Phi F \Phi^{-1}$ by using a well-known formula,

$$\begin{aligned}
&\pi^{-1}(\omega_1 - \omega_2)(\eta_1 - \eta_2)\int_{SU(2)/\mathbf{T}} e^{i(v\eta v^{-1}|\omega)}\, d\dot{v} \\
&\qquad = -\frac{i}{2}\left(e^{i(\eta|\omega)} - e^{i(\eta'|\omega)}\right).
\end{aligned} \tag{3.6}$$

In fact, let

$$v = \begin{bmatrix} \cos\theta_1 & (\sin\theta_1)e^{i\theta_2} \\ -(\sin\theta_1)e^{-i\theta_2} & \cos\theta_1 \end{bmatrix},$$

then

$$\pi^{-1}(\omega_1-\omega_2)(\eta_1-\eta_2)\int_{SU(2)/\mathbf{T}} e^{i(v\eta v^{-1}|\omega)}\,d\dot{v}$$
$$=(4\pi)^{-1}\int_0^{\pi/2}\int_0^{2\pi}(\omega_1-\omega_2)(\eta_1-\eta_2)$$
$$\cdot\, e^{i[(\eta|\omega)\cos^2\theta_1+(\eta'|\omega)\sin^2\theta_1]}\sin 2\theta_1\,d\theta_2\,d\theta_1.$$

Since

$$\tfrac{1}{2}\left[(\eta|\omega)-(\eta'|\omega)\right]=\tfrac{1}{2}(\omega_1-\omega_2)(\eta_1-\eta_2)$$

and

$$(\eta|\omega)\cos^2\theta_1+(\eta'|\omega)\sin^2\theta_1$$
$$=\tfrac{1}{2}\left[(\eta|\omega)+(\eta'|\omega)\right]+\tfrac{1}{2}\left[(\eta|\omega)-(\eta'|\omega)\right]\cos 2\theta_1,$$

(3.6) follows by integration.

Using formulas (3.4) and (3.6),

$$(\Phi Ff)(\omega)=\pi^{-2}\int_{\Omega_0}\left((\omega_1-\omega_2)(\eta_1-\eta_2)\int_{SU(2)/\mathbf{T}} e^{i(v\eta v^{-1}|\omega)}\,d\dot{v}\right)(\Phi f)(\eta)\,d\eta,$$
$$\Phi F\Phi^{-1}\phi(\omega)=-i(2\pi)^{-1}\int_{\Omega_0}e^{i(\eta|\omega)}\phi(\eta)\,d\eta+i(2\pi)^{-1}\int_{\Omega_0}e^{i(\eta'|\omega)}\phi(\eta')\,d\eta$$
$$=-i(2\pi)^{-1}\int_{\Omega}e^{i(\eta|\omega)}\phi(\eta)\,d\eta.$$

Formulas for $\Phi T(\chi;p)\Phi^{-1}$ and $\Phi\hat{A}\Phi^{-1}$ are straightforward computations. Thus the proposition is proved.

Since our analysis will only be on the space $L_1^2(\Omega)$, we shall write F, F^{-1}, $T(\chi;p)$ and $\hat{A}$ for $\Phi F\Phi^{-1}$, $\Phi F^{-1}\Phi^{-1}$, $\Phi T(\chi;p)\Phi^{-1}$ and $\Phi\hat{A}\Phi^{-1}$ respectively.

Let Ω^* denote the collection of $\omega\in\Omega$ for which $\omega_1\omega_2\neq 0$. The complement of Ω^* in Ω is of measure 0. Hence, we will not distinguish $L^2(\Omega)$ from $L^2(\Omega^*)$, nor $L_1^2(\Omega)$ from $L_1^2(\Omega^*)$. Let $\hat{\Omega}^*$ denote the group of unitary characters of the multiplicative group Ω^*. The generic element of $\hat{\Omega}^*$ is

$$\lambda(\omega)=|\omega_1|^{i\xi_1}\left(\frac{\omega_1}{|\omega_1|}\right)^{\varepsilon_1}|\omega_2|^{i\xi_2}\left(\frac{\omega_2}{|\omega_2|}\right)^{\varepsilon_2}$$
$$=|\omega_1|^{i\xi_1}[\omega_1]^{\varepsilon_1}|\omega_2|^{i\xi_2}[\omega_2]^{\varepsilon_2}$$

where ξ_1, ξ_2 are in $\mathbf{R}$ and ε_1, ε_2 are either 0 or 1. We also denote λ by $(\xi_1,\varepsilon_1;\xi_2,\varepsilon_2)$ and identify $\hat{\Omega}^*$ with $(\mathbf{R}\times\mathbf{Z}_2)\times(\mathbf{R}\times\mathbf{Z}_2)$.

For any $\phi\in L^2(\Omega)$ and $\lambda\in\hat{\Omega}^*$ define the Mellin transform M by

$$\tilde{\phi}(\lambda)=(M\phi)(\lambda)=(8\pi)^{-1}\int_{\Omega}|\det\omega|^{1/2}\lambda(\omega)\phi(\omega)|\det\omega|^{-1}\,d\omega.$$

From the above two formulas, one sees immediately that M is the tensor product of the usual real Mellin transform; that is, it is the usual Mellin transform applied to both coordinates ω_1 and ω_2. Thus, M is unitary on $L^2(\Omega)$. Let $\mathcal{H}$ be the image of $L_1^2(\Omega)$ under M. We are about to complete the following commutative diagram. Before computing $\tilde{F} = MFM^{-1}$, $\tilde{F}^{-1} = MF^{-1}M^{-1}$, $\tilde{T}(\chi;\ p) = MT(\chi;\ p)M^{-1}$, $\tilde{\hat{A}} = M\hat{A}M^{-1}$, let us observe the following facts.

$$\begin{array}{ccc} L_1^2(\Omega) & \xrightarrow{\ F,\ F^{-1},\ T(\chi;p),\ \hat{A}\ } & L_1(\Omega) \\ {\scriptstyle M}\big\downarrow & & \big\downarrow{\scriptstyle M} \\ \mathcal{H} & \overset{\tilde{F},\ \tilde{F}^{-1},\ \tilde{T}(\chi;p),\ \tilde{\hat{A}}}{\dashrightarrow} & \mathcal{H} \end{array}$$

Let ζ_1, ζ_2, ζ_3, and ζ_4 be the characters of Ω^* given by

$$\begin{cases} \zeta_1(\omega) = 1, \\ \zeta_2(\omega) = [\omega_1], \\ \zeta_3(\omega) = [\omega_2], \\ \zeta_4(\omega) = [\omega_1][\omega_2], \end{cases} \quad \text{or} \quad \begin{cases} \zeta_1 = (0, 0; 0, 0), \\ \zeta_2 = (0, 1; 0, 0), \\ \zeta_3 = (0, 0; 0, 1), \\ \zeta_4 = (0, 1; 0, 1). \end{cases}$$

1. The function α restricted to Ω can be written as a linear combination of ζ_j. In fact

$$\alpha(\omega) = \tfrac{1}{4}(C_1' + 2C_2' + C_3')\zeta_1(\omega) + \tfrac{1}{4}(C_1' - C_3')\zeta_2(\omega)$$
$$+ \tfrac{1}{4}(C_1' - C_3')\zeta_3(\omega) + \tfrac{1}{4}(C_1' - 2C_2' + C_3')\zeta_4(\omega);$$

(3.7) $$\alpha(\omega) = C_1\zeta_1(\omega) + C_2\zeta_2(\omega) + C_3\zeta_3(\omega) + C_4\zeta_4(\omega), \qquad C_2 = C_3,$$

which establishes fact 1.

REMARK (3.8). $\alpha(x)$ is a constant function if and only if $C_1' = C_2' = C_3'$ or equivalently if and only if $C_2 = C_3 = C_4 = 0$ in (3.7).

2. The function $(\det \omega)|\det \omega|^{-1}$ restricted to Ω^* is a character. Since $\det \omega = |\omega_1\omega_2|[\omega_1\omega_2] = |\omega_1|\ |\omega_2|[\omega_1][\omega_2]$,

$$(\det \omega)|\det \omega|^{-1} = [\omega_1][\omega_2] = \zeta_4(\omega).$$

3. $\chi = \chi_{(\rho,m)}$ restricted to Ω^* is a character for

$$\chi_{(\rho,m)}(\omega) = |\det \omega|^{i\rho}(\det \omega/|\det \omega|)^m$$
$$= \begin{cases} |\omega_1|^{i\rho}|\omega_2|^{i\rho} & \text{if } m \text{ is even}, \\ |\omega_1|^{i\rho}[\omega_1]|\omega_2|^{i\rho}[\omega_2] & \text{if } m \text{ is odd}. \end{cases}$$

We denote χ by $(\rho, m;\ \rho, m)$ with the understanding that m denotes the integer 0 or 1 congruent to m modulo 2.

Before writing down the formulas of $\tilde{F}$, $\tilde{F}^{-1}$, $\tilde{T}(\chi; p)$ and $\tilde{\hat{A}}$ we introduce a function on Ω^* that comes up in the Mellin analysis of the Fourier transform.

Let $\Gamma(z)$ be the usual gamma-function. For any pair $(\xi, \varepsilon) \in \mathbf{R} \times \mathbf{Z}_2$, let

$$k_1(\xi, \varepsilon) = \begin{cases} 2^{i\xi}\Gamma\left(\frac{1}{4} + i\xi/2\right)/\Gamma\left(\frac{1}{4} - i\xi/2\right), & \varepsilon = 0, \\ i2^{i\xi}\Gamma\left(\frac{3}{4} + i\xi/2\right)/\Gamma\left(\frac{3}{4} - i\xi/2\right), & \varepsilon = 1. \end{cases}$$

For $\lambda = (\xi_1, \varepsilon_1; \xi_2, \varepsilon_2) \in \Omega^*$, define

$$k_2(\lambda) = k_2(\xi_1, \varepsilon_1; \xi_2, \varepsilon_2) = ik_1(\xi_1, \varepsilon_1)k_1(\xi_2, \varepsilon_2).$$

PROPOSITION (3.9). *For any $\phi \in L_1^2(\Omega)$, let $\tilde{\phi} = M\phi \in \mathcal{H}$, the image of ϕ under the Mellin transform M. Then for any $\lambda \in \hat{\Omega}^*$*

$$(\tilde{F}\tilde{\phi})(\lambda) = k_2(\lambda)\tilde{\phi}(\lambda^{-1}),$$

$$(\tilde{F}^{-1}\tilde{\phi}) = -\lambda(-I)k_2(\lambda)\tilde{\phi}(\lambda^{-1}),$$

(3.10)
$$\left(\tilde{\hat{A}}\tilde{\phi}\right)(\lambda) = \sum_{j=1}^{4} C_j\tilde{\phi}(\lambda\zeta_j) \quad \textit{with } C_2 = C_3,$$

$$\left(\tilde{T}(\chi; p)\tilde{\phi}\right)(\lambda) = \lambda(-I)\tilde{\phi}(\chi\lambda^{-1}\zeta_4).$$

PROOF. In the following we use the fact that $|\det \omega|^{-1}\, d\omega$ is an invariant measure on Ω^*, as well as the identities $\chi(-I) = 1$, $\chi^{-1} = \chi$ and $(\det \omega)|\det \omega|^{-1} = \zeta_4(\omega)$.

$$\begin{aligned}
\left(\tilde{T}(\chi; p)\tilde{\phi}\right)(\lambda) &= (MT(\chi; p)\phi)(\lambda) \\
&= (8\pi)^{-1}\int_\Omega |\det \omega|^{1/2}\lambda(\omega)\chi(\omega)(\det \omega)^{-1}\phi(-\omega^{-1})|\det \omega|^{-1}\, d\omega \\
&= (8\pi)^{-1}\int_\Omega |\det(-\omega^{-1})|^{1/2}\lambda(-\omega^{-1})\chi(-\omega^{-1}) \\
&\qquad \cdot \left(\det(-\omega^{-1})\right)^{-1}\phi(\omega)|\det \omega|^{-1}\, d\omega \\
&= (\chi\lambda)(-I)(8\pi)^{-1}\int_\Omega |\det \omega|^{-1/2}(\det \omega) \\
&\qquad \cdot (\chi^{-1}\lambda^{-1})(\omega)\phi(\omega)|\det \omega|^{-1}\, d\omega \\
&= \lambda(-I)(8\pi)^{-1}\int_\Omega |\det \omega|^{1/2}(\chi^{-1}\lambda^{-1}\zeta_4)(\omega)\phi(\omega)|\det \omega|^{-1}\, d\omega \\
&= \lambda(-I)\tilde{\phi}(\chi^{-1}\lambda^{-1}\zeta_4) = \lambda(-I)\tilde{\phi}(\chi\lambda^{-1}\zeta_4).
\end{aligned}$$

$$\begin{aligned}
\left(\tilde{\hat{A}}\tilde{\phi}\right)(\lambda) &= \left(M\hat{A}\phi\right)(\lambda) = (8\pi)^{-1}\int_\Omega |\det \omega|^{1/2}\lambda(\omega)\alpha(\omega)\phi(\omega)|\det \omega|^{-1}\, d\omega \\
&= (8\pi)^{-1}\int_\Omega |\det \omega|^{1/2}\lambda(\omega)\left(\sum_{i=1}^{4} C_j\zeta_j(\omega)\right)\phi(\omega)|\det \omega|^{-1}\, d\omega \\
&= \sum_{j=1}^{4} C_j\int_\Omega |\det \omega|^{1/2}(\lambda\zeta_j)(\omega)\phi(\omega)|\det \omega|^{-1}\, d\omega = \sum_{j=1}^{4} C_j\tilde{\phi}(\lambda\zeta_j).
\end{aligned}$$

The calculations of $\tilde{F}$ and $\tilde{F}^{-1}$ are quite standard, at least in the case of the real line. The formal calculation involves a simple interchange of the order of integration. The rigorous proof requires a regularization procedure similar to that which we use in proving Lemma (4.6) of the next section. Therefore, we shall not give its computation here.

From (3.2) $\hat{A}FT(\chi;\ p)F^{-1} = FT(\chi;\ p)F^{-1}\hat{A}$ on $L_1^2(\Omega)$ if and only if $\tilde{\hat{A}}\tilde{F}\tilde{T}(\chi; p)\tilde{F}^{-1} = \tilde{F}\tilde{T}(\chi; p)\tilde{F}^{-1}\tilde{\hat{A}}$ on $\mathcal{H}$.

Proposition (3.11). *$\tilde{\hat{A}}\tilde{F}\tilde{T}(\chi;\ p)\tilde{F}^{-1} = \tilde{F}\tilde{T}(\chi;\ p)\tilde{F}^{-1}\tilde{\hat{A}}$ on $\mathcal{H}$ if and only if for $j = 1, 2, 3, 4$*

(3.12-j)
$$C_j k_2(\lambda)k_2(\chi\zeta_4\lambda) = C_j k_2(\zeta_j\lambda)k_2(\chi\zeta_4\zeta_j\lambda)$$

a.e. for λ in $(\mathbf{R} \times \mathbf{Z}_2) \times (\mathbf{R} \times \mathbf{Z}_2)$.

Proof. Since $\chi^{-1} = \chi$, $\zeta_j^{-1} = \zeta_j$ and $\zeta_4(-1) = 1$,

$$\left(\tilde{\hat{A}}\tilde{F}\tilde{T}(\chi; p)\tilde{F}^{-1}\tilde{\phi}\right)(\lambda) = -\sum_{i=1}^{4} C_i k_2(\zeta_i\lambda)k_2(\chi\zeta_4\zeta_i\lambda)\tilde{\phi}(\chi\zeta_4\zeta_i\lambda^{-1}),$$

$$\left(\tilde{F}\tilde{T}(\chi; p)\tilde{F}^{-1}\tilde{\hat{A}}\tilde{\phi}\right)(\lambda) = -\sum_{i=1}^{4} C_i k_2(\lambda)k_2(\chi\zeta_4\lambda)\tilde{\phi}(\chi\zeta_4\zeta_i\lambda^{-1}).$$

Since ζ_1 is the identity character (i.e. $\zeta_1\lambda = \lambda$), (3.12-1) always holds. Therefore, to prove Theorem (3.1), we will find what χ have the property that (3.12-2), (3.12-3) and (3.12-4) hold only when $C_2 = C_3 = C_4 = 0$ (cf. Remark (3.8)). Recall that we already know that $C_2 = C_3$.

In the calculation below, we are going to use another formula for $k_1(\xi, \varepsilon)$; namely,

$$k_1(\xi, \varepsilon) = \begin{cases} \dfrac{2}{\sqrt{2\pi}}\Gamma(1/2 + i\xi)\cos(1/4 + i\xi/2)\pi, & \varepsilon = 0, \\ \dfrac{2i}{\sqrt{2\pi}}\Gamma(1/2 + i\xi)\sin(1/4 + i\xi/2)\pi, & \varepsilon = 1, \end{cases}$$

which is obtained from using Legendre's duplication formula and identity $\Gamma(1/2 + Z)\Gamma(1/2 - Z) = \pi\sec(\pi Z)$.

Case 1. m is odd; i.e. $\chi = (\rho, 1; \rho, 1)$. (3.12-2) becomes

$$\begin{aligned} &C_2 k_2(\xi_1, \varepsilon_1; \xi_2, \varepsilon_2)k_2(\rho + \xi_1, \varepsilon_1; \rho + \xi_2, \varepsilon_2) \\ &\quad = C_2 k_2(\xi_1, 1 + \varepsilon_1; \xi_2, \varepsilon_2)k_2(\rho + \xi_1, 1 + \varepsilon_1; \rho + \xi_2, \varepsilon_2) \end{aligned}$$

or

$$C_2 k_1(\xi_1, \varepsilon_1)k_1(\rho + \xi_1, \varepsilon_1) = C_2 k_1(\xi_1, 1 + \varepsilon_1)k_1(\rho + \xi_1, 1 + \varepsilon_1)$$

or

$$C_2 \cos\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \cos\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi$$
$$= -C_2 \sin\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \sin\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi$$

or

$$C_2\left(\cos\left(\frac{1}{2} + i\frac{\rho + 2\xi_1}{2}\right)\pi + \cos i\frac{\rho}{2}\pi\right)$$
$$= C_2\left(\cos\left(\frac{1}{2} + i\frac{\rho + 2\xi_1}{2}\right)\pi - \cos i\frac{\rho}{2}\pi\right)$$

or

$$C_2 \cos\frac{i\rho\pi}{2} = 0.$$

Hence $C_2 = 0$ for any $\rho \in \mathbf{R}$.

If $\varepsilon_1 = \varepsilon_2$, (3.12-4) reduces to

$$C_4 \cos\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \cos\left(\frac{1}{4} + i\frac{\xi_2}{2}\right)\pi$$
$$\cdot \cos\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi \cos\left(\frac{1}{4} + i\frac{\rho + \xi_2}{2}\right)\pi$$
$$= C_4 \sin\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \sin\left(\frac{1}{4} + i\frac{\xi_2}{2}\right)\pi$$
$$\cdot \sin\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi \sin\left(\frac{1}{4} + i\frac{\rho + \xi_2}{2}\right)\pi.$$

This further reduces to

$$C_4 \cos\left(\frac{1}{2} + i\frac{\rho + \xi_1 + \xi_2}{2}\right)\pi \cos\left(i\frac{\rho}{2}\pi\right) = 0 \quad \text{a.e. } \xi_1, \xi_2.$$

This implies $C_4 = 0$, for any $\rho \in \mathbf{R}$.

Case 2. m is even; i.e. $\chi = (\rho, 0; \rho, 0)$. (3.12-2) becomes

$$C_2 \cos\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \sin\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi$$
$$= C_2 \sin\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \cos\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi,$$
$$C_2 \sin i\frac{\rho}{2}\pi = 0.$$

Hence $C_2 = 0$ for $\rho \neq 0$. (3.12-4) becomes

$$C_4 k_2(\xi_1, \varepsilon_1; \xi_2, \varepsilon_2) k_2(\rho + \xi_1, 1 + \varepsilon_1; \rho + \xi_2, 1 + \varepsilon_2)$$
$$= C_4 k_2(\xi_1, 1 + \varepsilon_1; \xi_2, 1 + \varepsilon_2) k_2(\rho + \xi_1, \varepsilon_1; \rho + \xi_2, \varepsilon_2).$$

If $\rho = 0$, the above equality holds independent of C_4. But, if $\rho \neq 0$, $\varepsilon_1 = \varepsilon_2$, then

$$C_4 \cos\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \cos\left(\frac{1}{4} + i\frac{\xi_2}{2}\right)\pi$$
$$\cdot \sin\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi \sin\left(\frac{1}{4} + i\frac{\rho + \xi_2}{2}\right)\pi$$
$$= C_4 \sin\left(\frac{1}{4} + i\frac{\xi_1}{2}\right)\pi \sin\left(\frac{1}{4} + i\frac{\xi_2}{2}\right)\pi$$
$$\cdot \cos\left(\frac{1}{4} + i\frac{\rho + \xi_1}{2}\right)\pi \cos\left(\frac{1}{4} + i\frac{\rho + \xi_2}{2}\right)\pi$$

or

$$C_4 \cos\left(\frac{1}{2} + i\frac{2\rho + \xi_1 + \xi_2}{2}\right)\pi = C_4 \cos\left(\frac{1}{2} + i\frac{\xi_1 + \xi_2}{2}\right)\pi \quad \text{a.e. } \xi_1, \xi_2.$$

We now conclude that when $\rho \neq 0$, $C_4 = 0$.

In summary, for $\rho \neq 0$ or m odd, $C_2 = C_3 = C_4 = 0$. In view of Remark (3.8) we have that for $\rho \neq 0$ or m odd, $\alpha(x)$ is a constant function. Thus Theorem (3.1) is proved.

4. Analytic continuation and some singular integrals. Roughly speaking the commuting operators are limits of intertwining operators. This and other reasons lead to further study of the intertwining operators. For $j = 1, 2, 3$ and $\rho \in \mathbf{R}$, let

(4.1) $$\big(\hat{B}_j(\rho)f\big)(x) = \delta_j(x)|\det x|^{-i\rho} f(x) \quad \text{for } f \in L^2(X)$$

where δ_j is the characteristic function of the orbit X_j. Let

(4.2) $$B_j(\rho) = F^{-1}\hat{B}_j(\rho)F$$

where F is the Fourier transform on $L^2(X)$. As it is well known $B_j(\rho)$ is convolution by a *distribution*. We extend ρ to be complex and consider the three regions which are specified by the inequalities $\operatorname{Im}(\rho) \geqslant 0$, $0 \geqslant \operatorname{Im}(\rho) \geqslant -2$ and $\operatorname{Im}(\rho) \leqslant -2$ respectively. One observes immediately that $|\det y|^{i\rho-2}$ is locally integrable for $\operatorname{Im}(\rho) \leqslant -2$. Hence, if $\operatorname{Im}(\rho) < -2$, $C\int_X f(x - y)|\det y|^{i\rho-2}\,dy$ is well defined for $f \in C_C^\infty(X)$. Furthermore, as a function of ρ, $C\int_X f(x - y)|\det y|^{i\rho-2}$ is an analytic function. On the other hand, for $f \in C_C^\infty(X)$ and $\operatorname{Im}(\rho) \geqslant 0$, $F^{-1}\hat{B}_j(\rho)Ff$ exists by a well-known property of Fourier transform, and $(F^{-1}\hat{B}_j(\rho)f)(x)$ is an analytic function.

$$\gamma(\rho) = \frac{1}{2\pi^3}\Gamma(2 - i\rho)\Gamma(1 - i\rho),$$

$$k_1(\rho, y) = \gamma(\rho)(e^{-\rho\pi}\delta_1(y) + \delta_2(y) + e^{\rho\pi}\delta_3(y))|\det y|^{i\rho-2},$$

$$(4.3)\quad k_2(\rho, y) = -\gamma(\rho)(2\delta_1(y) + (e^{-\rho\pi} + e^{\rho\pi})\delta_2(y) + 2\delta_3(y))|\det y|^{i\rho-2},$$

$$k_3(\rho, y) = \gamma(\rho)(e^{\rho\pi}\delta_1(y) + \delta_2(y) + e^{-\rho\pi}\delta_3(y))|\det y|^{i\rho-2}$$

where Γ is the usual gamma-function. Our main result of this section is

THEOREM (4.4) *Let* $f \in C_C^\infty(X)$ *and* $x \in X$. *Then for* $j = 1, 2, 3$ *the function*

$$\rho \to \left(F^{-1}\hat{B}_j(\rho)Ff\right)(x)$$
$$= \pi^{-4}\int_X\left(\delta_j(x)|\det y|^{-i\rho}\int_X f(z)e^{i(z|y)}\,dz\right)e^{-i(y|x)}\,dy$$

originally defined for $\operatorname{Im}(\rho) \geqslant 0$, *has an analytic continuation into the whole complex plane and is equal to the function*

$$\rho \to \int_X f(x-y)k_j(\rho, y)\,dy \quad \textit{when } \operatorname{Im}(\rho) \leqslant -2.$$

This theorem will imply the reducibility theorem in the next section. The analytic continuation technique is seen in [**8**]. In this section, ρ will be a complex number except in $B_j(\rho)$. $\hat{B}_j(\rho), j = 1, 2, 3$, are given in (4.1). To avoid ambiguity we will not use $B_j(\rho)$ given by (4.2) for complex ρ.

For any $x \in X$, let τ_x denote the translation by x, i.e. $(\tau_x f)(y) = f(y - x)$. Clearly, τ_x is a bijective map of $C_C^\infty(X)$ onto itself and

$$\int_X f(x-y)k_j(\rho, y)\,dy = \int_X(\tau_{-x}f)(-y)k_j(\rho, y)\,dy.$$

Since $(F\tau_x f)(y) = e^{i(x|y)}(Ff)(y)$,

$$\left(F^{-1}\hat{B}_jFf\right)(x) = \left(\tau_{-x}F^{-1}\hat{B}_j(\rho)Ff\right)(0) = \left(F^{-1}\hat{B}_j(\rho)F\tau_{-x}f\right)(0).$$

Therefore, Theorem (4.4) follows from

PROPOSITION (4.5). *Let* $f \in C_C^\infty(X)$. *The meromorphic function*

$$I_j^f(\rho) = \int_X f(-y)k_j(\rho, y)\,dy$$

originally defined for $\operatorname{Im}(\rho) \leqslant -2$ *has an analytic continuation into the whole complex plane such that*

$$I_j^f(\rho) = \left(F^{-1}\hat{B}_j(\rho)Ff\right)(0)$$
$$= \pi^{-2}\int_X\delta_j(y)|\det y|^{-i\rho}(Ff)(y)\,dy \quad \textit{for } \operatorname{Im}(\rho) \geqslant 0.$$

Recall $X^* = \{x \in X \,|\, \det x \neq 0\} = X_1 \cup X_2 \cup X_3$. Therefore, for $\phi \in C_C^\infty(X^*) \subset C_C^\infty(X)$ and $j = 1, 2, 3$ the function

$$I_j^{\phi}(\rho) = \int_X \phi(-y) k_j(\rho, y)\, dy$$

is a meromorphic function with poles $-i$, $-2i$, $-3i\ldots$. (They are poles of $\gamma(\rho)$.)

LEMMA (4.6). *For ρ such that* $\mathrm{Im}(\rho) \geqslant 0$ *and* $\phi \in C_C^\infty(X^*)$,

$$\int_X \phi(-y) k_j(\rho, y)\, dy = \pi^{-2} \int_X \delta_j(x) |\det x|^{-i\rho} (F\phi)(x)\, dx$$
$$= \big(F^{-1} \hat{B}_j(\rho) F\phi\big)(0).$$

Let $f \in C_C^\infty(X)$. For $1 > \varepsilon > 0$ choose $\phi_\varepsilon \in C_C^\infty(X^*)$ such that $\phi_\varepsilon \neq f$ only on a set X_ε of measure less than ε. Set

$$I_j^\varepsilon(\rho) = \int_X \phi_\varepsilon(-y) k_j(\rho, y)\, dy, \qquad j = 1, 2, 3.$$

LEMMA (4.7). $I_j^\varepsilon(\rho)$ *converges uniformly on any compact subset of the strip* $\{\rho \in \mathbf{C} \mid -2 \leqslant \mathrm{Im}(\rho) \leqslant 0\}$ *that is bounded away from* $\rho = -i$ *and* $\rho = -2i$.

PROOF OF PROPOSITION (4.5) Since $f \in C_C^\infty(X)$, $F^{-1}\hat{B}_j(\rho)Ff$ exists for $\mathrm{Im}(\rho) \geqslant 0$. Hence,

$$\pi^{-2} \int_X \delta_j(x) |\det x|^{-i\rho} (Ff)(x)\, dx = \big(F^{-1} \hat{B}_j(\rho) Ff\big)(0)$$

is continuous for $\mathrm{Im}(\rho) \geqslant 0$ and analytic for $\mathrm{Im}(\rho) > 0$. Also

$$I_j^f(\rho) = \int f(-y) k_j(\rho, y)\, dy$$

is continuous for $\mathrm{Im}(\rho) \leqslant -2$, except $\rho = -ni$, and meromorphic for $\mathrm{Im}(\rho) < -2$. By Lemma (4.7), $\lim_{\varepsilon\to 0} I_j^\varepsilon(\rho)$ is continuous on the strip and meromorphic inside the strip $\{\rho \in \mathbf{C} \mid -2 \leqslant \mathrm{Im}(\rho) \leqslant 0\}$. We shall show that this limit coincides with $(F^{-1}\hat{B}_j(\rho)Ff)(0)$ and $I_j^f(\rho)$ on upper boundary and lower boundary respectively. It is clear from our choice of ϕ_ε that, for $1 \leqslant p < \infty$, $\phi_\varepsilon \to f$ in L^p as $\varepsilon \to 0$ and hence $F\phi_\varepsilon \to Ff$ in both $L^1(X)$ and $L^2(X)$ as $\varepsilon \to 0$. By Lemma (4.6) and the Lebesgue dominated convergence theorem, when $\mathrm{Im}(\rho) = 0$

$$\lim_{\varepsilon\to 0} I_j^\varepsilon = \lim_{\varepsilon\to 0} \int_X \phi_\varepsilon(-y) k_j(\rho, y)\, dy$$
$$= \lim_{\varepsilon\to 0} \pi^{-2} \int \delta_j(x) |\det x|^{-i\rho} (F\phi_\varepsilon)(x)\, dx$$
$$= \pi^{-2} \int \lim_{\varepsilon\to 0} \big(\delta_j(x) |\det x|^{-i\rho} (F\phi_\varepsilon)(x)\big)\, dx$$
$$= \pi^{-2} \int \delta_j(x) |\det x|^{-i\rho} (Ff)(x)\, dx$$

and when $\mathrm{Im}(\rho) = -2$

$$\lim_{\varepsilon\to 0} I_j^\varepsilon(\rho) = \lim_{\varepsilon\to 0}\int_X \phi_\varepsilon(-y)k_j(\rho,y)\,dy$$
$$= \int_X f(-y)k_j(\rho,y)\,dy = I_j^f(\rho).$$

Therefore, Proposition (4.5) follows by analytic continuation.

We shall prove Lemma (4.7) first, then Lemma (4.6).

Proof of Lemma (4.7) Since $\Gamma(2-i\rho)$ and $\Gamma(1-i\rho)$ have only simple poles at $\rho = -i$, $\rho = -2i$ in the strip, we can choose a rational function $P(\rho)$ of ρ such that $|P(\rho)| \leqslant 1$ and

$$\left|P(\rho)\frac{\Gamma(2-i\rho)\Gamma(1-i\rho)}{2\pi^3}e^{|a|\pi}\right| \leqslant 1.$$

For $n = 1, 2, 3$ consider $P(\rho)I_n^\varepsilon(\rho)$.

When $\mathrm{Im}(\rho) = -2$,

$$|P(\rho)I_n^\varepsilon(\rho)| \leqslant \int|\phi_\varepsilon(y)|\,dy = \|\phi_\varepsilon\|_1.$$

When $\mathrm{Im}(\rho) = 0$,

$$|P(\rho)I_n^\varepsilon(\rho)| = |P(\rho)|\,|I_n^\varepsilon(\rho)| \leqslant \int|F\phi_\varepsilon(x)|\,dx = \|F\phi_\varepsilon\|_1.$$
$$|P(\rho)I_n^{\varepsilon_1}(\rho) - P(\rho)I_n^{\varepsilon_2}(\rho)| \leqslant \mathrm{Max}\left(\|\phi_{\varepsilon_1}-\phi_{\varepsilon_2}\|_1, \|F\phi_{\varepsilon_1}-F\phi_{\varepsilon_2}\|_1\right)$$

on the strip by the maximal principle. Since ϕ_ε converges in L^1-norm, $P(\rho)I_n^\varepsilon(\rho)$ and thus $I_n^\varepsilon(\rho)$ converges uniformly on any bounded subset of the strip. The lemma is proved.

The proof of Lemma (4.6) involves regularization of divergent integrals. The following lemma, which is verified through long computation, is needed in the regularization process.

Write $x = v\eta v^{-1}$ where $\eta \in \Omega_0$ and $\dot v \in SU(2)/\mathbf{T}$. Then $dx = (\eta_1 - \eta_2)^2\,d\dot v\,d\eta$ (cf. (3.4)). For $j = 1, 2, 3$ and $0 < \varepsilon < 1$, let

$$k_j^\varepsilon(\rho,y) = \pi^{-4}\int_X e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\det x|^{-i\rho}e^{-i(y|x)}\,dx. \tag{4.8}$$

Lemma (4.9). *For any fixed $\rho \neq -ni$ and each j, $|k_j^\varepsilon(\rho, y)|$ is uniformly bounded on every compact subset of X^* and*

$$\lim_{\varepsilon\to 0} k_j^\varepsilon(\rho,y) = k_j(\rho,y).$$

Proof of Lemma (4.6).

$$\pi^{-2}\int_X \delta_j(x)|\det x|^{-i\rho}(F\phi)(x)\,dx$$
$$= \pi^{-4}\int_X\left(\delta_j(x)|\det x|^{-i\rho}\int_X e^{i(y|x)}\phi(y)\,dy\right)dx.$$

Set $x = v\eta v^{-1}$; then $|\det x| = |\eta_1\eta_2|$. Write $\rho = \sigma + i\tau$, $\tau \geqslant 0$. For any positive number ε

$$\left|e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\det x|^{-i\rho}e^{i(y|x)}\phi(y)\right|$$
$$=\left|e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\eta_1\eta_2|^{\tau}\phi(y)\right|$$

and

$$\int_{\Omega_0} e^{-\varepsilon(|\eta_1|+|\eta_2|)}|\eta_1\eta_2|^{\tau}(\eta_1-\eta_2)^2\,d\eta < \infty.$$

Since $\phi \in C_C^\infty(X^*)$, by formula (3.3)

$$e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\det x|^{-i\rho}e^{i(y|x)}\phi(y)$$

is integrable with respect to $dy\,dx = (\eta_1-\eta_2)^2\,dy\,d\dot{v}\,d\eta$. By the Lebesgue dominated convergence theorem and Fubini theorem,

$$\begin{aligned}
&\pi^{-2}\int_X \delta_j(x)|\det x|^{-i\rho}(F\phi)(x)\,dx\\
&\quad=\lim_{\varepsilon\to 0}\pi^{-2}\int_X e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\det x|^{-i\rho}(F\phi)(x)\,dx\\
&\quad=\lim_{\varepsilon\to 0}\pi^{-4}\int_X e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\det x|^{-i\rho}\int_X e^{i(y|x)}\phi(y)\,dy\,dx\\
&\quad=\lim_{\varepsilon\to 0}\int_X \phi(-y)\left(\pi^{-4}\int_X e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(x)|\det x|^{-i\rho}e^{-i(y|x)}\,dx\right)dy\\
&\quad=\lim_{\varepsilon\to 0}\int_X \phi(-y)k_j(\rho,y)\,dy\\
&\quad=\int_X \phi(-y)\lim_{\varepsilon\to 0}k_j^{\varepsilon}(\rho,y)\,dy=\int_X\phi(-y)k_j(\rho,y)\,dy.
\end{aligned}$$

The last two equalities follow from Lemma (4.9). Thus, Lemma (4.6) is proved.

Now we turn to prove Lemma (4.9). In the proof we shall use the following well-known formula of Γ-functions. Let z be a complex number such that $-\pi/2 < \arg z < \pi/2$ then

$$\int_0^\infty t^{w-1}e^{-zt}\,dt = \Gamma(w)z^{-w}.$$

PROOF OF LEMMA (4.9). First, it is easy to verify that $k_j^\varepsilon(uyu^{-1}) = k_j^\varepsilon(y)$ for $u \in SU(2)$, and hence k_j^ε is determined by its value on Ω. Let $\omega \in \Omega$, then, by formula (3.4),

$$\begin{aligned}
k_j^\varepsilon(\rho,\omega) = \pi^{-4}\int_{\Omega_0}\int_{SU(2)/\mathbf{T}} &e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}\\
&\cdot e^{-i(\omega|v\eta v^{-1})}(\eta_1-\eta_2)^2\,d\dot{v}\,d\eta.
\end{aligned}$$

By formula (3.6),

$$\int_{SU(2)/T} e^{-i(\omega|v\eta v^{-1})}\,dv$$

$$= -\frac{\pi}{2i}(\omega_1-\omega_2)^{-1}(\eta_1-\eta_2)^{-1}(e^{-i(\omega|\eta)}-e^{-i(\omega|\eta')}).$$

Hence,

$$k_j^{\varepsilon}(\rho,\omega) = \frac{i}{2\pi^3(\omega_1-\omega_2)}\int_{\Omega_0} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}(\eta_1-\eta_2)$$

$$\cdot\,(e^{-i(\omega|\eta)}-e^{-i(\omega|\eta')})\,d\eta,$$

$$\int_{\Omega_0} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}(\eta_1-\eta_2)e^{-i(\omega|\eta')}\,d\eta$$

$$= \int_{\Omega_0'} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta')|\det\eta'|^{-i\rho}(\eta_2-\eta_1)e^{-i(\omega|\eta)}\,d\eta$$

$$= -\int_{\Omega_0} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}(\eta_1-\eta_2)e^{-i(\omega|\eta)}\,d\eta.$$

Therefore,

$$k_j^{\varepsilon}(\rho,\omega) = \frac{i}{2\pi^3(\omega_1-\omega_2)}\int_{\Omega_0} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}(\eta_1-\eta_2)e^{-i(\omega|\eta)}\,d\eta$$

$$+\int_{\Omega_0'} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}(\eta_1-\eta_2)e^{-i(\omega|\eta)}\,d\eta$$

$$= \frac{i}{2\pi^3(\omega_1-\omega_2)}\int_{\Omega} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\det\eta|^{-i\rho}(\eta_1-\eta_2)e^{-i(\omega|\eta)}\,d\eta$$

$$= \frac{i}{2\pi^3(\omega_1-\omega_2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\varepsilon(|\eta_1|+|\eta_2|)}\delta_j(\eta)|\eta_1\eta_2|^{-i\rho}(\eta_1-\eta_2)$$

$$\cdot e^{-i(\omega_1\eta_1+\omega_2\eta_2)}\,d\eta_1\,d\eta_2.$$

Since

$$\delta_1(\eta) = \begin{cases} 1 & \textit{if } \eta_1>0,\ \eta_2>0,\\ 0 & \textit{otherwise},\end{cases}$$

$$\delta_2(\eta) = \begin{cases} 1 & \textit{if } \eta_1\eta_2<0,\\ 0 & \textit{otherwise},\end{cases}$$

and

$$\delta_3(\eta) = \begin{cases} 1 & \textit{if } \eta_1<0,\ \eta_2<0,\\ 0 & \textit{otherwise},\end{cases}$$

$$k_1^\varepsilon(\rho,\omega) = i\gamma(\rho)(\omega_1-\omega_2)^{-1}\Big((\varepsilon+i\omega_1)^{i\rho-2}(\varepsilon+i\omega_2)^{i\rho-1} - (\varepsilon+i\omega_1)^{i\rho-1}(\varepsilon+i\omega_2)^{i\rho-2}\Big),$$

$$k_2^\varepsilon(\rho,\omega) = i\gamma(\rho)(\omega_1-\omega_2)^{-1}\Big((\varepsilon+i\omega_1)^{i\rho-2}(\varepsilon-i\omega_2)^{i\rho-1} + (\varepsilon+i\omega_1)^{i\rho-1}(\varepsilon-i\omega_2)^{i\rho-2} - (\varepsilon-i\omega_1)^{i\rho-2}(\varepsilon+i\omega_2)^{i\rho-1} - (\varepsilon-i\omega_1)^{i\rho-1}(\varepsilon+i\omega_2)^{i\rho-2}\Big),$$

$$k_3^\varepsilon(\rho,\omega) = -i\gamma(\rho)(\omega_1-\omega_2)^{-1}\Big((\varepsilon-i\omega_1)^{i\rho-2}(\varepsilon-i\omega_2)^{i\rho-1} - (\varepsilon-i\omega_1)^{i\rho-1}(\varepsilon-i\omega_2)^{i\rho-2}\Big).$$

We note here that we use only the principal branch of logarithm; i.e. for example

$$(4.10)\qquad \begin{aligned}(\varepsilon+i\omega_1)^{i\rho-2} &= \exp\big((i\rho-2)\mathrm{Log}(\varepsilon+i\omega_1)\big)\\ &= \exp\big((i\rho-2)(\log|\varepsilon+i\omega_1| + i\arg(\varepsilon+i\omega_1))\big)\end{aligned}$$

where $-\pi < \arg(\varepsilon+i\omega_1) < \pi$.

By (4.10), $(\varepsilon+i\omega_1)^{i\rho-2}$ is a continuous function of ω_1 and ε. Hence, if ε is bounded, say $0<\varepsilon<1$, then $|(\varepsilon+i\omega_1)^{i\rho-2}|$ is uniformly bounded on every compact subset excluding 0. A similar conclusion holds for other terms. Therefore, $|k_j^\varepsilon(\rho,\omega)|$ is uniformly bounded on every compact subset of Ω^*. So $|k_j^\varepsilon(\rho,y)|$ is uniformly bounded on every compact subset of X^*. Thus, the first part of Lemma (4.9) is proved.

Clearly, $\lim_{\varepsilon\to 0} k_j^\varepsilon(\rho,y)$ also satisfies the invariance condition. Therefore, we only have to calculate $\lim_{\varepsilon\to 0} k_j^\varepsilon(\rho,\omega)$ for $\omega\in\Omega_0^*$. Since

$$\lim_{\varepsilon\to 0} i\arg(\varepsilon+i\omega_1) = \begin{cases} i\pi/2 & \text{if } \omega_1>0,\\ -i\pi/2 & \text{if } \omega_1<0,\end{cases}$$

by (4.10)

$$\lim_{\varepsilon\to 0}(\varepsilon+i\omega_1)^{i\rho-2} = \begin{cases} |\omega_1|^{i\rho-2}e^{-(\rho+2i)\pi/2} & \text{if } \omega_1>0,\\ |\omega_1|^{i\rho-2}e^{(\rho+2i)\pi/2} & \text{if } \omega_1<0.\end{cases}$$

The other limits have similar formulas. Therefore, (1) if $\omega_1>\omega_2>0$

$$\begin{aligned}\lim_{\varepsilon\to 0} k_1^\varepsilon(\rho,\omega) &= i\gamma(\rho)(\omega_1-\omega_2)^{-1}|\omega_1\omega_2|^{i\rho-2}\big(e^{-(\rho+2i)\pi/2}e^{-(\rho+i)\pi/2}\omega_2 - \omega_1 e^{-(\rho+i)\pi/2}e^{-(\rho+2i)\pi/2}\big)\\ &= i\gamma(\rho)(\omega_1-\omega_2)^{-1}|\omega_1\omega_2|^{i\rho-2}e^{-(2\rho+3i)\pi/2}(\omega_2-\omega_1)\\ &= \gamma(\rho)e^{-\rho\pi}|\omega_1\omega_2|^{i\rho-2}\end{aligned}$$

and similarly,

$$\lim_{\varepsilon\to 0} k_2^\varepsilon(\rho,\omega) = -2\gamma(\rho)|\omega_1\omega_2|^{i\rho-2},$$

$$\lim_{\varepsilon\to 0} k_3^\varepsilon(\rho,\omega) = \gamma(\rho)e^{\rho\pi}|\omega_1\omega_2|^{i\rho-2};$$

(2) if $\omega_1 > 0 > \omega_2$,

$$\begin{aligned}\lim_{\varepsilon\to 0} k_1^\varepsilon(\rho,\omega) &= i\gamma(\rho)(\omega_1-\omega_2)^{-1}|\omega_1\omega_2|^{i\rho-2}\big(e^{-(\rho+2i)\pi/2}e^{(\rho+i)\pi/2}(-\omega_2)\\ &\qquad - e^{-(\rho+i)\pi/2}e^{(\rho+2i)\pi/2}\omega_1\big)\\ &= i\gamma(\rho)(\omega_1-\omega_2)^{-1}|\omega_1\omega_2|^{i\rho-2}\big(e^{-i\pi/2}(-\omega_2) - e^{i\pi/2}\omega_1\big)\\ &= \gamma(\rho)|\omega_1\omega_2|^{i\rho-2}\end{aligned}$$

and similarly,

$$\lim_{\varepsilon\to 0} k_2^\varepsilon(\rho,\omega) = -\gamma(\rho)(e^{-\rho\pi}+e^{\rho\pi})|\omega_1\omega_2|^{i\rho-2},$$

$$\lim_{\varepsilon\to 0} k_3^\varepsilon(\rho,\omega) = \gamma(\rho)|\omega_1\omega_2|^{i\rho-2};$$

(3) if $0 > \omega_1 > \omega_2$

$$\begin{aligned}\lim_{\varepsilon\to 0} k_1^\varepsilon(\rho,\omega) &= i\gamma(\rho)|\omega_1\omega_2|^{i\rho-2}\big(e^{(\rho+2i)\pi/2}e^{(\rho+i)\pi/2}(-\omega_2)\\ &\qquad - e^{(\rho+i)\pi/2}e^{(\rho+2i)\pi/2}(-\omega_1)\big)\\ &= i\gamma(\rho)|\omega_1\omega_2|^{i\rho-2}e^{\rho\pi}\big(e^{i3\pi/2}(-\omega_2) - e^{i3\pi/2}(-\omega_1)\big)\\ &= \gamma(\rho)e^{\rho\pi}|\omega_1\omega_2|^{i\rho-2}\end{aligned}$$

and similarly,

$$\lim_{\varepsilon\to 0} k_2^\varepsilon(\rho,\omega) = -2\gamma(\rho)|\omega_1\omega_2|^{i\rho-2},$$

$$\lim_{\varepsilon\to 0} k_3^\varepsilon(\rho,\omega) = \gamma(\rho)e^{-\rho\pi}|\omega_1\omega_2|^{i\rho-2}.$$

Combine all these three cases and we have

$$\begin{aligned}\lim_{\varepsilon\to 0} k_1^\varepsilon(\rho,\omega) &= \gamma(\rho)(e^{-\rho\pi}\delta_1(\omega)+\delta_2(\omega)+e^{\rho\pi}\delta_3(\omega))|\omega_1\omega_2|^{i\rho-2}\\ &= k_1(\rho,\omega),\end{aligned}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} k_2^\varepsilon(\rho,\omega) &= \gamma(\rho)(-2\delta_1(\omega)-(e^{-\rho\pi}+e^{\rho\pi})\delta_2(\omega)-2\delta_3(\omega))|\omega_1\omega_2|^{i\rho-2}\\ &= k_2(\rho,\omega),\end{aligned}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} k_3^\varepsilon(\rho,\omega) &= \gamma(\rho)(e^{\rho\pi}\delta_1(\omega)+\delta_2(\omega)+e^{-\rho\pi}\delta_3(\omega))|\omega_1\omega_2|^{i\rho-2}\\ &= k_3(\rho,\omega).\end{aligned}$$

By the invariance condition, $\lim_{\varepsilon\to 0} k_j^{\varepsilon}(\rho, y) = k_j(\rho, y)$. Thus Lemma (4.9) is proved.

5. **Reducibility theorem.** In this section we shall use the different realization of $B_j(\rho) = F^{-1}\hat{B}_j(\rho)F$ provided by analytic continuation to show that

$$\begin{aligned} B(0, 1) &= B_1(0) - B_2(0) + B_3(0) \\ &= F^{-1}\big(\hat{B}_1(0) - \hat{B}_2(0) + \hat{B}_3(0)\big)F = F^{-1}\hat{B}(0, 1)F \end{aligned}$$

is a nontrivial commuting operator for $T(0, 2n; \cdot)$. Therefore, the representation $T(0, 2n; \cdot)$, $n \in \mathbf{Z}$, is reducible.

From formula (1.2) and the fact that $\det x$ is real, for $f \in L^2(X)$

$$\big(T(\rho, 2n; \rho)f\big)(x) = |\det x|^{i\rho-2} f(-x^{-1}) \quad \text{a.e.}$$

In particular,

(5.1) $$\big(T(0, 2n; p)f\big)(x) = \big(T_0(\rho)f\big)(x) = |\det x|^{-2} f(-x^{-1}) \quad \text{a.e.}$$

For any complex ρ and $f \in C_C^{\infty}(X)$, define

(5.2) $$\begin{aligned} \big(\hat{B}(\rho, 0)f\big)(x) &= |\det x|^{-i\rho} f(x) \\ &= (\delta_1(x) + \delta_2(x) + \delta_3(x))|\det x|^{-i\rho} f(x) \\ &= \big((\hat{B}_1(\rho) + \hat{B}_2(\rho) + \hat{B}_3(\rho))f\big)(x) \end{aligned}$$

and

(5.3) $$\begin{aligned} \big(\hat{B}(\rho, 1)f\big)(x) &= |\det x|^{-i\rho}[\det x]\, f(x) \\ &= (\delta_1(x) - \delta_2(x) + \delta_3(x))|\det x|^{-i\rho} f(x) \\ &= \big((\hat{B}_1(\rho) - \hat{B}_2(\rho) + \hat{B}_3(\rho))f\big)(x). \end{aligned}$$

Clearly, when $\operatorname{Im}(\rho) = 0$, $\hat{B}(\rho, 0)$ and $\hat{B}(\rho, 1)$ are unitary operators, and in particular $\hat{B}(0, 0) = I$ is the identity operator. Notice that $\hat{B}(0, 1)$ is a nontrivial operator in the commuting algebra of $\hat{T}(\chi; \cdot)|_{\mathfrak{B}'}$ where $\mathfrak{B}' = \mathcal{C}\mathcal{V}$ (cf. Proposition (2.1)). When $\operatorname{Im}(\rho) = 0$, let $B(\rho, 1) = F^{-1}\hat{B}(\rho, 1)F$. Thus, $B(0, 1)$ is a nontrivial operator in the commuting algebra of $T(\chi; \cdot)|_{\mathfrak{B}'} = T(\rho, m; \cdot)|_{\mathfrak{B}'}$ (i.e. $T(\rho, m; cv)B(0, 1) = B(0, 1)T(\rho, m; cv)$ for all $cv \in \mathfrak{B}' = \mathcal{C}\mathcal{V}$). Moreover, we have

THEOREM (5.4) *$T(0, 2n; p)B(0, 1) = B(0, 1)T(0, 2n; p)$ and hence the representation $T(0, 2n; \cdot)$ is reducible.*

The proof of this theorem involves proving a more general lemma. First let us extend the use of the notation $B(\rho, 1)$ to complex ρ; more explicitly we mean that for $f \in C_C^{\infty}(X)$ and $x \in X$

$$\big(B(\rho, 1)f\big)(x) = \big(F^{-1}\hat{B}(\rho, 1)Ff\big)(x)$$

originally defined for $\operatorname{Im}(\rho) \geqslant 0$ has analytic continuation into the whole complex plane and when $\operatorname{Im}(\rho) \leqslant -2$

$$(B(\rho, 1)f)(x) = \int f(x-y)h_1(\rho, y)\,dy \tag{5.5}$$

where

$$h_1(\rho,y) = k_1(\rho,y) - k_2(\rho,y) + k_3(\rho,y)$$
$$= \gamma(\rho)(e^{\rho\pi} + e^{-\rho\pi} + 2)|\det y|^{i\rho-2}$$

(cf. Theorem (4.4) and formulas (4.3)).

LEMMA (5.6) *Let* $\rho \neq -i, -2i, \ldots$. *For* $f \in C_C^\infty(X^*)$ *and* $x \in X^*$

$$|\det x|^{-2}(B(\rho, 1)T_0(p)f)(-x^{-1}) = |\det x|^{-i\rho}(B(\rho, 1)\hat{B}(\rho, 0)f)(x). \tag{5.7}$$

PROOF OF THEOREM (5.4) Let $\rho = 0$ in (5.7) then for $f \in C_C(X^*)$, $x \in X^*$

$$(T_0(p)B(0, 1)T_0(p)f)(x) = (B(0, 1)f)(x).$$

Since X^* is dense in X, $C_C^\infty(X^*)$ is dense in $C_C^\infty(X)$ and hence also in $L^2(X)$. Therefore

$$T_0(p)B(0, 1)T_0(p) = B(0, 1).$$

Since $T(0, 2m; p) = T_0(p)$, and $T(0, 2m; p)^{-1} = T_0(p)^{-1} = T_0(p)$, $T_0(p)B(0, 1) = B(0, 1)T_0(p)$ and the theorem is thus proved.

Now we turn to prove Lemma (5.6).

PROOF OF LEMMA (5.6) First we observe that $C_C^\infty(X)$ is invariant under $T_0(p)$ and $\hat{B}(\rho, 0)$; hence the equation (5.7) is well defined. For $x \in X^*$ both sides of (5.7) are meromorphic functions of ρ on the whole complex plane by Theorem (4.9).

We are going to show that (5.7) holds for $\operatorname{Im}(\rho) \leqslant -2$ and hence for all complex ρ by analytic continuation.

Let $\operatorname{Im}(\rho) \leqslant -2$. From formulas (5.1) and (5.5) by making the change of variables $y \to y^{-1}$ and noticing $dy^{-1} = |\det y|^{-4}\,dy$, then

$$|\det x|^{-2}(B(\rho, 1)T_0(p)f)(-x^{-1})$$
$$= |\det x|^{-2}\int_X f\left((x^{-1}+y)^{-1}\right)\left|\det(x^{-1}+y)\right|^{-2} h_1(\rho,y)\,dy$$
$$= |\det x|^{-2}\int_X f(y)h_1(\rho, y^{-1} - x^{-1})|\det y|^{-2}\,dy$$

and

$$|\det x|^{-i\rho}(B(\rho, 1)\hat{B}(\rho, 0)f)(x)$$
$$= |\det x|^{-i\rho}\int_X |\det(x-y)|^{-i\rho} f(x-y)h_1(\rho, y)\,dy$$
$$= |\det x|^{-\rho}\int_X f(y)|\det y|^{-i\rho}h_1(\rho, x-y)\,dy.$$

Hence it is left to verify that

$$|\det x|^{-2}|\det y|^{-2}h_1(\rho, y^{-1} - x^{-1}) = |\det x|^{-i\rho}|\det y|^{-i\rho}h_1(\rho, x - y)$$

or to verify that

$$(5.8) \qquad \begin{aligned} &|\det x|^{-2}|\det y|^{-2}|\det(y^{-1} - x^{-1})|^{i\rho-2} \\ &\qquad = |\det x|^{-i\rho}|\det y|^{-i\rho}|\det(x - y)|^{i\rho-2}. \end{aligned}$$

If

$$x = \begin{pmatrix} x_1 + x_4 & -x_2 - x_3 \\ -x_2 + ix_3 & x_1 - x_4 \end{pmatrix} \in X$$

then

$$x^{-1} = (\det x)^{-1}\begin{pmatrix} x_1 - x_4 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 + x_4 \end{pmatrix}.$$

For $x, y \in X$, let $\langle x|y\rangle = x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4$. Then

$$\begin{aligned} \det(x - y) &= (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 - (x_4 - y_4)^2 \\ &= \det x + \det y - 2\langle x|y\rangle, \end{aligned}$$

$$\det(y^{-1} - x^{-1}) = \det y^{-1} + \det x^{-1} - 2\langle y^{-1}|x^{-1}\rangle,$$

$$\langle y^{-1}|x^{-1}\rangle = (\det x)^{-1}(\det y)^{-1}\langle x|y\rangle.$$

Hence

$$\begin{aligned} \det(y^{-1} - x^{-1}) &= (\det x \det y)^{-1}(\det x + \det y - 2\langle x|y\rangle) \\ &= (\det x \det y)^{-1}\det(x - y). \end{aligned}$$

Thus (5.8) holds and hence the lemma is proved for $\mathrm{Im}(\rho) \leqslant -2$. By analytic continuation the lemma is proved in general.

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