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A GROSS MEASURE PROPERTY

BY

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ABSTRACT. We prove that there exists a subset E of $[0, 1] \times \mathbf{R}^2$ such that the 2-dimensional Gross measure of E is 0, while the 1-dimensional Gross measure of $\{z: (y, z) \in E\}$ is positive for all $y \in [0, 1]$. It is known that for Hausdorff measures no set exists satisfying these conditions.

1. **Introduction.** A special case of [**1**, 2.10.25, 2.10.27] states that for any positive integers k, m, n there exists $c \in \mathbf{R}$ such that

$$\int^* \mathcal{H}^k\{z: (y, z) \in A\}\, d\mathcal{L}^m y \leqslant c\mathcal{H}^{k+m}(A)$$

for all $A \subset \mathbf{R}^m \times \mathbf{R}^n$, where $\mathcal{H}^m$, $\mathcal{L}^m$ denote m-dimensional Hausdorff and Lebesgue measure respectively. It immediately follows that the same type of relation holds with the Hausdorff measures replaced by the spherical, $\mathcal{T}$, Carathéodory or Gillespie measures (provided $k < n$), since the ratios between the Hausdorff measure and any one of these other measures of the same dimension are bounded [**1**, 2.10.6]. It is also known that the inequality holds with $c = 1$ in the case of spherical or Gillespie measures [**1**, 2.10.27, 3.2.45], but not in the case of Hausdorff measures [**2**].

In this paper we establish that no such relation is true for Gross measures (denoted $\mathcal{G}$) by constructing a subset E of $[0, 1] \times \mathbf{R}^2$ for which $\mathcal{G}^2(E) = 0$ (Theorem 4.1), while $\mathcal{G}^1\{z: (y, z) \in E\} = 2$ for all $y \in [0, 1]$ (Lemma 3.1). The method of proof that $\mathcal{G}^2(E) = 0$ uses the structure theory of [**1**, 3.3] and, in particular, incorporates some of the ideas of [**1**, 3.3.19].

One consequence of our result is that some theorems concerning $(\mathcal{H}^m, m)$ rectifiable sets [**1**, 3.2.14] do not hold for $(\mathcal{G}^m, m)$ rectifiable sets. For example our set E shows that [**1**, 3.2.22] is no longer true if $\mathcal{H}$ is replaced by $\mathcal{G}$.

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2. **Preliminaries.** In general we adopt the notation and terminology of [**1**]. Presented in this section are some additional definitions that we use.

Throughout this paper, unless otherwise restricted, $0 < y < 1$, while $n \geqslant 3$

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is an integer. We also note that we do not distinguish between $\mathbf{R}^2$ and $\mathbf{C}$ or between $\mathbf{R} \times \mathbf{R}^2$ and $\mathbf{R}^3$.

For $a, b \in \mathbf{R}^m$ let $[a, b]$ denote the closed line segment with endpoints a, b.

Define $\lambda: \mathbf{R} \to \mathbf{C}$, $\xi: \mathbf{R} \times \mathbf{C} \to \mathbf{R} \times \mathbf{C}$, $\lambda(x) = \exp(x\mathbf{i})$, $\xi(x, z) = (x, z\lambda(x))$ for $x \in \mathbf{R}, z \in \mathbf{C}$.

The following series of definitions culminate in the definition of the set E referred to in the Introduction. For each closed circular disk $S = \mathbf{B}(a, r) \subset \mathbf{R}^2$ and positive integer $i \leq n$, let

$$x_{i,n}(S) = \inf\{x: (x, w) \in S\} + (2i - 1)r/n,$$

$$w_{i,n}(S) = \begin{cases} \sup\{w: \mathbf{B}[(x_{i,n}, w), r/n] \subset S\} & \text{if } i \text{ is odd,} \\ \inf\{w: \mathbf{B}[(x_{i,n}, w), r/n] \subset S\} & \text{if } i \text{ is even,} \end{cases}$$

$$F_n(S) = \{\mathbf{B}[(x_{j,n}, w_{j,n}), r/n]: j = 1, \ldots, n\}.$$

Then inductively define families $G_3, G_4, G_5, \ldots,$ of closed circular disks by taking $G_3 = \{\mathbf{B}(0, 1)\}$, $G_{n+1} = \cup \{F_{n+1}(S): S \in G_n\}$ for $n \geq 3$. Finally let $A = \bigcap_{n=3}^{\infty} \cup G_n$ and $E = \xi([0, 1] \times A)$.

Define $p(x, z) = x$, $q(x, z) = z$ for $(x, z) \in \mathbf{R} \times \mathbf{R}^2$.

For $x \in \mathbf{R}^m \sim \{0\}$ let $\tau(x) = x/|x|$.

For $w \in \mathbf{R}$ define $\rho_w \in \mathbf{O}^*(3, 1)$, $\rho_w(x) = q(x) \cdot \lambda(w)$ for $x \in \mathbf{R}^3$. (Note that throughout this paper a "$\cdot$" between two complex numbers denotes an inner product, not complex multiplication.)

Take $r_n = 6/n!$. Note that diam $S = 2r_n$ for $S \in G_n$.

Let $A_y = E \cap p^{-1}[y]$, $G_{n,y} = \{\xi(\{y\} \times S): S \in G_n\}$, $K_n = \{\xi([(j - 1)r_n, jr_n] \times S): j = 1, 2, \ldots, r_n^{-1}, S \in G_n\}$.

3. **Some lemmas.** In 3.1–3.5 we prove a few lemmas about the Hausdorff and Gross measures of E and certain of its subsets. In the remainder of this section the key results are Lemmas 3.10, 3.13, 3.15, each of which is used in Theorem 4.1 to show that a different subset of E has $\mathcal{G}^2$ measure 0.

3.1. LEMMA. $\mathcal{G}^1(E_y) = \mathcal{H}^1(E_y) = 2$.

PROOF. Since $\mathcal{L}^1[\rho_y(E_y)] = 2$ it follows from [1, 2.10.8] that $\mathcal{G}^1(E_y) \geq 2$. Furthermore, since $G_{n,y}$ covers E_y and $\sum_{S \in G_{n,y}}$ diam $S = 2$ we have $\mathcal{H}^1(E_y) \leq 2$. Finally we recall that $\mathcal{G}^1 \leq \mathcal{H}^1$ [1, 2.10.6].

3.2. LEMMA. *If* $S \in G_{n,y}$ *then* $\mathcal{G}^1(S \cap E_y) = \mathcal{H}^1(S \cap E_y) = 2r_n$.

PROOF. This lemma follows by applying the method used to establish Lemma 3.1.

3.3. LEMMA. $\mathcal{H}^2(E) < \infty$.

PROOF. We observe that K_n covers E, card $K_n = r_n^{-2}$, and that diam $p(S) = r_n$, diam $q(S) \leq 3r_n$ for every $S \in K_n$. Consequently

$$(\pi/4) \sum_{s \in K_n} (\operatorname{diam} S)^2 \leq 5\pi/2.$$

Hence $\mathcal{H}^2(E) \leq 5\pi/2$.

3.4. DEFINITION. For $w \in \mathbf{R}$ let $I(w) = E \cap \{x: q(x) \wedge \lambda(w) = 0\}$.

3.5. LEMMA. $\mathcal{H}^2[I(w)] = 0$ *for all* $w \in \mathbf{R}$.

PROOF. The result follows from Lemma 3.3 and the fact that if $x \in [0, \frac{1}{2}]$ then

$$\mathcal{H}^2\Big[I(w) \cap p^{-1}\Big(\Big[0, \tfrac{1}{2}\Big]\Big)\Big] = \mathcal{H}^2\Big[I(w+x) \cap p^{-1}\Big(\Big[x, x+\tfrac{1}{2}\Big]\Big)\Big],$$

$$\mathcal{H}^2\Big[I(w) \cap p^{-1}\Big(\Big[\tfrac{1}{2}, 1\Big]\Big)\Big] = \mathcal{H}^2\Big[I(w-x) \cap p^{-1}\Big(\Big[\tfrac{1}{2}-x, 1-x\Big]\Big)\Big].$$

3.6. DEFINITIONS. For $\varnothing \neq T \subset S \in G_{n,y}$ let $\beta_n(T) = S$ and $c_n(T)$ denote the center of S; for $x \in S \sim \{c_n(S)\}$ define $\eta_n(x) = \tau(q[x - c_n(S)])$.

Let $\Delta_{n,y}$ denote the set of all closed proper line segments $[a_1, a_2] \in p^{-1}\{y\}$ for which there exists $S \in G_{n,y}$ satisfying $\{q(a_1), q(a_2)\} \subset \operatorname{Bdry} q(S)$, $c_n(S) \notin [a_1, a_2]$.

Let $L \in \Delta_{n,y}$ define

$$\begin{aligned}
\alpha(L) &= \inf\{|\eta_n(x) \wedge \lambda(y)|: x \in L\},\\
R(L) &= \beta_n(L) \cap \{x: [c_n\{x\}, x] \cap L \neq \varnothing\},\\
M(L) &= G_{n+1,y} \cap \{T: T \subset \beta_n(L), \eta_n(T) \subset \eta_n(L)\},\\
m(L) &= \operatorname{card} M(L).
\end{aligned}$$

3.7. LEMMA. *If* $L \in \Delta_{n,y}$, $\alpha(L) \neq 0$ *and* $n \geq 2^4[\alpha(L)]^{-1}(\operatorname{diam} L)^{-1} r_n$, *then*

$$m(L)/(n+1) \geq 2^{-3}\alpha(L)(\operatorname{diam} L) r_n^{-1}.$$

PROOF. Since $\operatorname{card}[G_{n+1,y} \cap \{T: T \subset \beta_n(L), \rho_y(T) \cap \rho_y(L) \neq \varnothing\}] \leq 2m(L) + 3$ and $r_{n+1} = r_n/(n+1)$, it follows

$$[2m(L) + 3]2r_n/(n+1) \geq \operatorname{diam} \rho_y(L).$$

Furthermore, we note that if x, w are the endpoints of L and $\alpha(L) = |\eta_n(x) \wedge \lambda(y)|$, then

$$\begin{aligned}
\operatorname{diam} \rho_y(L) &= |\tau[q(w-x)] \cdot \lambda(y)| \operatorname{diam} L \geq |[\mathrm{i}\eta_n(x)] \cdot \lambda(y)| \operatorname{diam} L\\
&= \alpha(L) \operatorname{diam} L.
\end{aligned}$$

We then combine these last two results with the given bound on n to obtain our conclusion.

3.8. LEMMA. *If* $L \in \Delta_{n,y}$ *and* $n \geqslant 2^{10}(\text{diam } L)^{-2}r_n^2$, *then* $m(L)/(n+1) \geqslant 2^{-9}(\text{diam } L)^2 r_n^{-2}$.

PROOF. Consider $a \in R(L) \cap q^{-1}(\text{Bdry } q[\beta_n(L)])$ satisfying

$$|\eta_n(a) \wedge \lambda(y)| = \sup\{|\eta_n(x) \wedge \lambda(y)|: x \in L\}.$$

Then choose $L_1 \in \Delta_{n,y}$ with $a \in L_1 \subset R(L)$ and $\text{diam } L_1 = (\text{diam } L)/8$. We observe that if w is either of the two points of $\beta_n(L)$ satisfying $|\rho_y(w - c_n\{w\})| = r_n$, then $|a - w| \geqslant (\text{diam } L)/2$; consequently $|\eta_n(a) \wedge \lambda(y)| \geqslant (\text{diam } L)/(4r_n)$. We combine this with the fact that $|\eta_n(a) \wedge \eta_n(x)| \leqslant (\text{diam } L)/(8r_n)$ for all $x \in L_1$, to obtain $\alpha(L_1) \geqslant (\text{diam } L)/(8r_n)$. We then apply Lemma 3.7 to L_1.

3.9. DEFINITION. Let F_1 denote the set of all points x of E for which $\{\eta_n(x): n \geqslant 3\}$ is not dense in $\mathbf{S}^1$.

3.10. LEMMA. $\mathcal{H}^2(F_1) = 0$.

PROOF. Consider any closed proper subarc J of $\mathbf{S}^1$. We will obtain our result by showing that

$$\mathcal{H}^2\left[E \cap \bigcap_{n=3}^{\infty} \{x: \eta_n(x) \notin J\}\right] = 0. \tag{1}$$

To do this we let J_1 denote the closed subarc of $\mathbf{S}^1$ with the same midpoint as J, satisfying $\mathcal{H}^1(J_1) = \mathcal{H}^1(J)/2$. Choose an integer $\nu > 2^{10}(\text{diam } J_1)^{-2}$ for which $r_\nu \leqslant \mathcal{H}^1(J)/4$. Then inductively define the three sequences $B_\nu, B_{\nu+1}, B_{\nu+2}, \ldots, D_{\nu+1}, D_{\nu+2}, D_{\nu+3}, \ldots, C_{\nu+1}, C_{\nu+2}, C_{\nu+3}, \ldots$, by taking $B_\nu = K_\nu$, and for $n \geqslant \nu$ letting

$$D_{n+1} = K_{n+1} \cap \left\{T: T \subset \bigcup B_n\right\},$$
$$B_{n+1} = D_{n+1} \cap \{T: \eta_n(T) \cap (\mathbf{S}^1 \sim J) \neq \varnothing\},$$
$$C_{n+1} = D_{n+1} \cap \left\{T: \eta_n\left[T \cap p^{-1}\{\sup p(T)\}\right] \subset J_1\right\}.$$

We observe that if $x \in T \in C_{n+1}$ and $t = \sup p(T)$, then $(t, q(x)\lambda[t - p(x)]) \in T \cap p^{-1}\{t\}$ and $|\eta_n[(t, q(x)\lambda[t - p(x)])] \wedge \eta_n(x)| = \sin[t - p(x)] \leqslant \sin r_{n+1} \leqslant \sin[\mathcal{H}^1(J)/4]$; consequently $B_{n+1} \cap C_{n+1} = \varnothing$. Furthermore, if $S \in B_n$, $w \in \{\sup p(T): T \in D_{n+1}, T \subset S\}$ and L is the line segment satisfying the conditions $L \in \Delta_{n,w}$, $\beta_n(L) = p^{-1}\{w\} \cap S$, $\eta_n(L) = J_1$, then from Lemma 3.8 it follows that

$$m(L)/(n+1) \geqslant 2^{-9}(\text{diam } L)^2 r_n^{-2} = 2^{-9}(\text{diam } J_1)^2.$$

Therefore, either $B_n = \varnothing$ or

$$\mathcal{H}^2\Big(E \cap \bigcup B_{n+1}\Big)/\mathcal{H}^2\Big(E \cap \bigcup B_n\Big) = (\operatorname{card} B_{n+1})/(\operatorname{card} D_{n+1})$$
$$\leq 1 - (\operatorname{card} C_{n+1})/(\operatorname{card} D_{n+1}) \leq 1 - 2^{-9}(\operatorname{diam} J_1)^2.$$

Thus $\mathcal{H}^2[\bigcap_{n=\nu}^{\infty}(E \cap \bigcup B_n)] = 0$, from which (1) follows.

3.11. Corollary. $\mathcal{H}^1(F_1 \cap E_y) = 0$.

Proof. If $t, w \in [0, 1]$ then

$$F_1 \cap E_w = \{(w, q(x)\lambda(w - t)) : x \in F_1 \cap E_t\};$$

consequently $\mathcal{H}^1(F_1 \cap E_w) = \mathcal{H}^1(F_1 \cap E_t)$. Furthermore, by [1, 2.10.27] and Lemma 3.10 we have

$$\int_0^1 \mathcal{H}^1(F_1 \cap E_y)\, d\mathcal{L}^1 y \leq (4/\pi)\mathcal{H}^2(F_1) = 0.$$

3.12. Definitions. For $a \in \mathbf{R}^3$, $0 < r < r' \leq \infty$, $V \in \mathbf{G}(3, 1)$, $0 < s < 1$, let

$$\mathbf{X}(a, r, V, s) = \mathbf{R}^3 \cap \{x : s^{-1}\operatorname{dist}(x - a, V) < |x - a| < r\},$$
$$\mathbf{Y}(a, r, r', V, s) = \operatorname{Clos}[\mathbf{X}(a, r', V, s) \sim \mathbf{X}(a, r, V, s)].$$

3.13. Lemma. *If* $a \in E_y \sim F_1$, $V \in \mathbf{G}(3, 1)$, $V \subset p^{-1}\{0\}$, $v \in \mathbf{S}^1$, $\mathbf{R}v = q(V)$, $b = |v \cdot \lambda(y)| > 0$ *and* $0 < s < b$, *then*

$$\limsup_{n\to\infty} \operatorname{card}\big[G_{n+1,y} \cap \{T : T \subset \beta_n\{a\} \cap \mathbf{Y}(a, r_n s/8, r_n s, V, s)\}\big]/(n + 1)$$
$$\geq 2^{-7}bs.$$

Proof. Let $V_n = \{0\} \times (\mathbf{R}[\eta_n(a)\mathbf{i}])$, $W_n = \mathbf{Y}(a, r_n s/4, r_n s/2, V_n, s/2)$. Let N denote the set of all integers n such that $n \geq \sup\{64/s^2, 2^8/(bs)\}$, $|v \wedge [\eta_n(a)\mathbf{i}]| \leq s/4$ and $\{x : \operatorname{dist}(x, W_n) \leq 2r_{n+1}\} \subset \mathbf{Y}(a, r_n s/8, r_n s, V, s)$. We note that $\sup N = \infty$ since $a \notin F_1$; consequently it suffices to obtain our result with $\limsup_{n\to\infty}$ replaced by $\inf_{n\in N}$. To do this we choose $n \in N$ and consider $a_1, a_2, a_3 \in p^{-1}\{y\} \cap q^{-1}[\operatorname{Bdry} q(\beta_n\{a\})] \cap \{x : q(x - a) \cdot [\eta_n(a)\mathbf{i}] \geq 0\}$ satisfying $|a_1 - a| = r_n s/2$, $|a_2 - a| = r_n s/4$, $q(a_3 - a) \cdot \eta_n(a) = 0$. Let $L = [a_1, a_2]$. We note that $\bigcup M(L) \subset \{x : \operatorname{dist}[x, R(L)] \leq 2r_{n+1}\}$. Therefore to complete the proof we need only show that $R(L) \subset W_n$ and $m(L)/(n + 1) \geq 2^{-7}bs$.

To obtain the former relation we simply observe that

(2) $\quad |\eta_n(a) \wedge \eta_n(x)| \leq |\eta_n(a) \wedge \eta_n(a_1)| \leq s/2$ for every $x \in R(L)$,

while since $n \geq 64/s^2$ we also have

$$|a_3 - a| = \left(r_n^2 - |a - c_n\{a\}|^2\right)^{1/2} \leq \left(1 - [1 - 2/(n + 1)]^2\right)^{1/2} r_n$$
$$\leq 2r_n/(n + 1)^{1/2} \leq r_n s/4.$$

To compute $m(L)$ we first note that diam $L \geqslant r_n s/4$. To obtain a lower bound on $\alpha(L)$ we let $\delta_1 = \sin^{-1}[|\eta_n(a) \wedge \lambda(y)|]$, $\delta_2 = \sin^{-1}[|(v\mathbf{i}) \wedge \lambda(y)|]$, $\delta_3 = \sin^{-1}[|(v\mathbf{i}) \wedge \eta_n(a)|]$. Then since $\delta_1 \geqslant |\delta_2 - \delta_3|$ it follows that

$$
\begin{aligned}
|\eta_n(a) \wedge \lambda(y)| &= \sin(\delta_1) \geqslant |\sin(\delta_2) - \sin(\delta_3)| \\
&= \big|\, |(v\mathbf{i}) \wedge \lambda(y)| - |(v\mathbf{i}) \wedge \eta_n(a)| \,\big| \\
&= \big|\, |v \cdot \lambda(y)| - |v \wedge [\eta_n(a)\mathbf{i}]| \,\big| \geqslant b - s/4 \geqslant 3b/4,
\end{aligned}
$$

which we combine with the inequalities (2) and $s < b$ to find that $\alpha(L) \geqslant b/4$. Finally we see that Lemma 3.7 is applicable since $n \geqslant 2^8/(bs)$.

3.14. LEMMA. *E_y is purely $(\mathcal{H}^1, 1)$ unrectifiable.*

PROOF. It follows from Corollary 3.11 and Lemma 3.13 that

$$\lim_{s \to 0^+} \limsup_{n \to \infty} \mathcal{H}^1\big[E_y \cap \mathbf{X}(a, r_n s, \ker \rho_{y+\pi/4}, s)\big] r_n^{-1} s^{-2} = \infty$$

for $\mathcal{H}^1$-almost all a in E_y; consequently $\mathcal{L}^1[\rho_{y+\pi/4}(E_y)] = 0$ by [**1**, 3.3.9]. Similarly $\mathcal{L}^1[\rho_{y+3\pi/4}(E_y)] = 0$. Hence E_y is purely $(\mathcal{H}^1, 1)$ unrectifiable by [**1**, 3.2.27].

3.15. LEMMA. *E is purely $(\mathcal{H}^2, 2)$ unrectifiable.*

PROOF. If W is an $(\mathcal{H}^2, 2)$ rectifiable Borel subset of E, then it follows from [**1**, 3.2.29] that $\mathcal{H}^2$-almost all of W is contained in the union of some countable family of 2 dimensional submanifolds of class 1 of $\mathbf{R}^3$. Let B denote a member of such a family and let $M = B \cap W$. We will complete the proof by showing that $\mathcal{H}^2(M) = 0$.

To do this we first observe that for $\mathcal{L}^1$-almost all y in $[0, 1]$ we have by [**1**, 3.2.22(2)] that $M \cap p^{-1}\{y\}$ is $(\mathcal{H}^1, 1)$ rectifiable and hence $\mathcal{H}^1(M \cap p^{-1}\{y\}) = 0$ by Lemma 3.14. It then follows from [**1**, 3.2.22(3)] that

$$\int_M apJ_1 p \, d\mathcal{H}^2 = \int_0^1 \mathcal{H}^1(M \cap p^{-1}\{y\}) \, d\mathcal{L}^1 y = 0.$$

Consequently, $apJ_1 p(x) = 0$ for $\mathcal{H}^2$-almost all x in M, which combined with [**1**, 3.2.19] implies that $\operatorname{Tan}^2(\mathcal{H}^2 L M, b) = p^{-1}\{0\}$ for $\mathcal{H}^2$-almost all b in M. We next choose any $b \in M$ for which $\operatorname{Tan}^2(\mathcal{H}^2 L M, b) = p^{-1}\{0\}$ and observe that by [**1**, 3.1.19(4)] there exists a neighborhood T of b in $\mathbf{R}^3$ such that $q|(B \cap T)$ is univalent, $q(B \cap T)$ is convex, ψ is of class 1 and $D\psi[q(b)] = q^*$, where $\psi[q|(B \cap T)]^{-1}$. From the conditions on ψ we find that there exists a convex neighborhood S of $q(b)$ in $\mathbf{R}^2$ such that $\|D\psi(z) - q^*\| \leqslant \frac{1}{2}$ for all $z \in S$; consequently $\operatorname{Lip}[(\psi - q^*)|S] \leqslant \frac{1}{2}$, which in turn implies that $\operatorname{Lip}[(p \circ \psi)|S] \leqslant \frac{1}{2}$. We let $Z = \psi(S) \cap M$ and note that to finish the proof it suffices to show that $\mathcal{H}^1(Z) < \infty$.

To accomplish this we define $h\colon Z \to E_0$, $h(x) = (0, q(x)\lambda[-p(x)])$. Then since $\operatorname{Lip}[(p \circ \psi)|S] < \frac{1}{2}$ we see that for $x, w \in Z$,

$$|h(x) - h(w)| \geq |q(x) - q(w)| - |p(x) - p(w)|$$
$$\geq \left(\tfrac{1}{2}\right)|q(x) - q(w)| \geq (5^{1/2}/4)|x - w|.$$

Consequently $\operatorname{Lip}(h^{-1}) \leq 4/5^{1/2}$, which we combine with [1, 2.10.11] and Lemma 3.1 to conclude that

$$\mathcal{H}^1(Z) \leq \operatorname{Lip}(h^{-1})\mathcal{H}^1[h(Z)] \leq (4/5^{1/2})\mathcal{H}^1(E_0) < \infty.$$

4. **Principal theorem.** We prove here that $\mathcal{G}^2(E) = 0$. This result and Lemma 3.1 establish the claim made in the Introduction.

4.1. THEOREM. $\mathcal{G}^2(E) = 0$.

PROOF. Consider any $\theta \in \mathbf{O}^*(3, 2)$. We will obtain our result by showing that $\mathbf{L}^2[\theta(E)] = 0$. To do this we first choose $v \in \mathbf{S}^2$ satisfying $\mathbf{R}v = \ker \theta$. If $q(v) \neq 0$ we then take any $\varepsilon > 0$ and apply Lemma 3.5 to obtain a closed proper subarc J of $\mathbf{S}^1$ whose midpoint is $\tau[\mathbf{i}q(v)]$ and which satisfies $\mathcal{H}^2[\bigcup\{I(w)\colon \lambda(w) \in J\}] < \varepsilon$; we then let $F_2 = \bigcup\{I(w)\colon \lambda(w) \in J\}$. On the other hand, if $q(v) = 0$ we take $F_2 = \varnothing$. For $z \in q(E \sim F_2)$ define $\sigma(z) \in \mathbf{S}^2$, $\gamma(z) \in \mathbf{R} \times \mathbf{R}^2$, $g(z) \in [0, \pi]$, by $\sigma(z) = \tau[(1, \mathbf{i}z)]$, $\gamma(z) = (0, q(v) - \mathbf{i}p(v)z)$ and $g(z) = |\arg(\mathbf{i}q[\gamma(z)])|$. Let $H = q(E \sim F_2) \cap \{z\colon g(z) \leq 1\}$, and define $f\colon H \to \mathbf{R} \times \mathbf{R}^2, f(z) = (g(z), z)$ for $z \in H$. Finally let

$$K = E \sim \left[F_1 \cup F_2 \cup f(H) \cup p^{-1}\{0, 1\}\right].$$

We next show that to complete the proof we need only establish that

$$\mathcal{L}^2\left[\theta(K)\right] = 0. \tag{3}$$

For it follows from the definition of γ, g and f that f is Lipschitzian, which together with Lemma 3.15 implies $\mathcal{H}^2[E \cap f(H)] = 0$. Recalling Lemmas 3.1, 3.10 and the definition of F_2, we would then have $\mathcal{H}^2(E \sim K) < \varepsilon$, which together with [1, 2.10.8, 2.10.6] would yield

$$\mathcal{L}^2\left[\theta(E \sim K)\right] \leq \mathcal{G}^2(E \sim K) \leq \mathcal{H}^2(E \sim K) < \varepsilon.$$

Finally this last result and (3) imply $\mathcal{L}^2[\theta(E)] = 0$.

To obtain (3) we consider any $a \in K$, let $b = |\tau[q(\gamma[q(a)])] \cdot \lambda[p(a)]|$, and note that $b > 0$ since $a \notin f(H)$. We will show that

$$\lim_{s \to 0^+} \limsup_{n \to \infty} \mathcal{H}^2\left[E \cap \mathbf{X}(a, r_n s, \ker \theta, s)\right] r_n^{-2} s^{-4} = \infty. \tag{4}$$

We then immediately have (3) by [1, 3.3.9].

To deduce (4) we first choose $\nu_1, \nu_2 \in \mathbf{S}^2$ so that v, ν_1, ν_2 is an orthonormal basis for $\mathbf{R}^3$ and $\sigma[q(a)]$ is a linear combination of v and ν_1. Let $P = \{a + dv + t\nu_2\colon d, t \in \mathbf{R}\}$, $V = \mathbf{R}\gamma[q(a)]$. For $x \in \mathbf{R}^3$ define

$$\Omega(x) = x - \big([(x-a)\cdot \nu_1]/[\sigma[q(a)]\cdot \nu_1]\big)\sigma[q(a)].$$

We note that $\lim \Omega \subset P$ since $[\Omega(x) - a]\cdot \nu_1 = 0$ for all $x \in \mathbf{R}^3$. We also observe that since $\gamma[q(a)] = v - p(v)(1 + [q(a)]^2)^{1/2}\sigma[q(a)]$ is a linear combination of v and ν_1, it follows that $[\Omega(a + x\gamma[q(a)]) - a]\cdot \nu_2 = 0$ for all $x \in \mathbf{R}$ and hence

$$(5) \qquad \Omega(a + V) \subset a + \ker\theta.$$

Let $k = \inf\{|\sigma[q(a)] \wedge \tau(x_1 - x_2)|: x_1, x_2 \in P, x_1 \neq x_2\}$. We note that since $a \notin F_2$ it follows that $\sigma[q(a)] \notin \mathbf{R}v$ and consequently $k \neq 0$. We then choose s satisfying $0 < s < \inf\{4b/k, \operatorname{dist}[p(a), \{0, 1\}]\}$ and let

$$T_n = p^{-1}\{p(a)\} \cap \mathbf{Y}(a, kr_n s/32, kr_n s/4, V, ks/4),$$
$$Y_n = P \cap \mathbf{Y}(a, kr_n s/64, r_n s/4, \ker\theta, s/2).$$

We next show that $\Omega(T_n) \subset Y_n$. To do this we consider any $x, w \in p^{-1}\{p(a)\}, x \neq w$. Then since $|p(\sigma[q(a)])| \geq 2^{-1/2} > \frac{1}{2}$ we have

$$|x - w|/2 \leq |\sigma[q(a)] \wedge (x - w)| \leq |x - w|,$$

while the definition of k yields

$$k|\Omega(x) - \Omega(w)| \leq |\sigma[q(a)] \wedge [\Omega(x) - \Omega(w)]| \leq |\Omega(x) - \Omega(w)|.$$

From these inequalities and the relation $|\sigma[q(a)] \wedge (x - w)| = |\sigma[q(a)] \wedge [\Omega(x) - \Omega(w)]|$ we obtain $\frac{1}{2} \leq |\Omega(x) - \Omega(w)|/|x - w| \leq 1/k$, which we then combine with (5) to conclude $\Omega(T_n) \subset Y_n$.

Let

$$Z_n = \{c_{n+1}(S): S \in G_{n+1,p(a)}, S \subset \beta_n\{a\} \cap T_n\}.$$

For $x \in Z_n$ define

$$\Gamma(x) = \big(p[\Omega(x)], q(x)\lambda(p[\Omega(x) - x])\big),$$
$$Q_n(x) = \Big\{\big(y, z\lambda(y - p[\Gamma(x)])\big): |y - p[\Gamma(x)]| < 2^{-8}kr_n s^2,$$
$$z \in q\big(\beta_{n+1}[\Gamma(x)]\big)\Big\}.$$

Applying [**1**, 2.10.27], Lemma 3.2, and also noting that $p[Q_n(x)] \subset [0, 1]$ because of the choice of s, we deduce that

$$\mathcal{H}^2[E \cap Q_n(x)] \geq (\pi/4)\mathcal{L}^1(p[Q_n(x)])2r_{n+1} = 2^{-8}\pi k r_n^2 s^2/(n+1)$$

for all $x \in Z_n$. Furthermore from Lemma 3.13, with s replaced by $ks/4$, it follows that

$$\limsup_{n\to\infty} (\operatorname{card} Z_n)/(n + 1) \geq 2^{-9}kbs.$$

We then combine these last two results to obtain

$$\limsup_{n\to\infty} \mathcal{H}^2\Big[E \cap \bigcup_{x\in Z_n} Q_n(x)\Big] r_n^{-2} \geq 2^{-17}\pi k^2 b s^3.$$

Consequently to complete the proof of (4) we need only establish that for n sufficiently large,

$$\bigcup_{x\in Z_n} Q_n(x) \subset \mathbf{X}(a, r_n s, \ker\theta, s). \tag{6}$$

To obtain (6) we choose $x \in Z_n$ and observe that since $\Omega(x) \in Y_n$, we have

$$\operatorname{dist}\big[\Omega(x), \mathbf{R}^3 \sim \mathbf{X}(a, r_n s, \ker\theta, s)\big] \geq |\Omega(x) - a|s/2 \geq 2^{-7}kr_n s^2.$$

Furthermore, it follows from the definition of $Q_n(x)$ that if $w \in Q_n(x)$ then $|p[w - \Gamma(x)]| \leq 2^{-8}kr_n s^2$, $|q[w - \Gamma(x)]| \leq r_{n+1} + 2^{-8}kr_n s^2$. We then combine these last three inequalities to deduce that to conclude (6) it suffices to show

$$\lim_{n\to\infty}\big[\sup\{|\Omega(x) - \Gamma(x)| : x \in Z_n\}/r_n\big] = 0. \tag{7}$$

To prove (7) we consider any $n \geq 5$ and $x \in Z_n$. We then let $h = p[\Omega(x) - x]$, $w_1 = q[\Omega(x) - x]$, $w_2 = q[\Gamma(x) - x]$, $u = q[x + \Gamma(x)]/2$. Since $\Omega(x) \in Y_n$ we see that $|h| \leq r_n s/4 \leq r_n$. We also note that $|w_1| = |q(a)| \cdot |h| \leq r_n$, $|w_2| = 2|q(x)|\sin(|h|/2) \leq r_n$, $|q(x - a)| \leq kr_n s/4 \leq r_n$. Then since

$$|u - q(a)| \leq |w_2|/2 + |q(x - a)| \leq 3r_n/2,$$

$|q(a)| \geq \frac{1}{2}$ and $n \geq 5$, it follows that

$$\tau[q(a)] \cdot \tau(u) \geq (1 - 9r_n^2)^{1/2} \geq 1 - r_n.$$

Furthermore, if $h > 0$ then $\tau(w_1) = \tau[q(a)\mathbf{i}]$ and $\tau(w_2) = \tau(u\mathbf{i})$, while if $h < 0$ then $\tau(w_1) = \tau[q(a)(-\mathbf{i})]$ and $\tau(w_2) = \tau[u(-\mathbf{i})]$; consequently in either case $\tau(w_1) \cdot \tau(w_2) \geq 1 - r_n$. Finally we conclude (7) by using this last inequality to compute

$$\begin{aligned} |\Omega(x) - \Gamma(x)|^2 &= |w_1 - w_2|^2 = |w_1|^2 + |w_2|^2 - 2|w_1|\cdot|w_2|\tau(w_1)\cdot\tau(w_2) \\ &\leq (|w_1| - |w_2|)^2 + 2|w_1|\cdot|w_2|r_n \leq (|w_1| - |w_2|)^2 + 2r_n^3, \end{aligned}$$

and then combining this result with the relation

$$\begin{aligned} \big|\, |w_1| - |w_2| \,\big| &= \big|\, |q(a)|\cdot|h| - 2|q(x)|\sin(|h|/2)\big| \\ &\leq (|q(x)| + r_n)|h| - 2|q(x)|\sin(|h|/2) \\ &\leq |h| + r_n\,|h| - 2\sin(|h|/2) \\ &\leq r_n + r_n^2 - 2\sin(r_n/2). \end{aligned}$$

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