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TRANSLATION PLANES OF ORDER q^2: ASYMPTOTIC ESTIMATES

BY

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ABSTRACT. R. H. Bruck has pointed out the one-to-one correspondence between the isomorphism classes of certain translation planes, called subregular, and the equivalence classes of disjoint circles in a finite miquelian inversive plane $IP(q)$. The problem of determining the number of isomorphism classes of translation planes is old and difficult. Let q be an odd prime-power. In this paper, a study of sets of disjoint circles in $IP(q)$ enables the author to find an asymptotic estimate of the number of isomorphism classes of translation planes of order q^2 which are subregular of index 3 or 4. It is conjectured (and proved for $n < 3$) that, given a set of n disjoint circles in $IP(q)$, the numbers of circles disjoint from each of the given n circles is asymptotic to $q^3/2^n$. This conjecture, if true, would allow one to estimate the number of subregular translation planes of order q^2 with any positive index.

Introduction. The study of finite translation planes was first reduced to the study of spreads in projective spaces of odd dimension (see [3]). In particular, we restrict ourselves to dimension three. There exists a construction process which assigns to each spread S of $PG(3, q)$ a translation plane $\pi(S)$ of order q^2, where q is any prime-power. The subregular translation planes are those that arise from subregular spreads of $PG(3, q)$ (see [1]).

If we assume $q > 3$ and ignore one well-studied exceptional case, the classification of subregular translation planes of order q^2 is further reduced to the classification of sets of disjoint circles of a finite miquelian inversive plane $IP(q)$ of order q (see [2, §7]). In particular, there exists a one-to-one correspondence between the isomorphism classes of translation planes of order q^2 which are subregular of index k and the equivalence classes of sets of k disjoint circles in $IP(q)$ under the group of all collineations of $IP(q)$. It is extremely difficult, if not impossible, to obtain an exact count of the number of isomorphism classes of subregular translation planes. This was adequately

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pointed out in [**1**, §7], where asymptotic counts were first employed. In this paper, we restrict ourselves to odd q and find asymptotic estimates of the number of isomorphism classes of translation planes of order q^2 which are subregular of index 3 or 4. In a forthcoming paper, quadruples of disjoint circles in $IP(q)$ satisfying certain orthogonality conditions are completely classified.

1. Preliminary results. An *inversive plane* is a set I of objects, called points, and a collection of subsets of I, called circles, such that: (i) every three distinct points of I lie on exactly one circle of I; (ii) given a circle C of I, a point P on C, and a point Q which is not on C, there exists exactly one circle C' of I such that C' contains P, Q and has only P in common with C; (iii) every circle of I is nonempty, and there exist four points of I not lying on any circle of I.

Let I be an inversive plane. Any two distinct circles of I must be *disjoint*, *tangent*, or *secant* accordingly as they have zero, one, or two points in common. For any two distinct points P and Q of I, the set of all circles of I which pass through both P and Q is called the *bundle with carriers P and Q*. A maximal set of mutually tangent circles at a point P is called a *pencil with carrier P*. Clearly any bundle or pencil covers all the points of I.

An inversive plane is called finite if it contains only a finite number of points. Let I be a finite inversive plane. It is easy to show (see [**4**]) that there exists a positive integer n, called the *order* of I, such that:

(1) I has exactly $n^2 + 1$ points.

(2) I has exactly $n(n^2 + 1)$ circles.

(3) There are exactly $n + 1$ points of I on every circle of I.

(4) There are exactly $n(n + 1)$ circles of I through every point of I.

(5) Every bundle contains exactly $n + 1$ circles.

(6) Every pencil contains exactly n circles.

(7) Every flock contains exactly $n - 1$ circles.

(8) If Q is a point of I not lying on a circle C of I, there are exactly $n(n + 1)/2$ circles through Q secant to C, $n + 1$ circles through Q tangent to C, and $(n - 2)(n + 1)/2$ circles through Q disjoint from C.

(9) If C is a circle of I, there are precisely $n^2(n + 1)/2$ circles of I secant to C, $n^2 - 1$ circles of I tangent to C, and $n(n - 1)(n - 2)/2$ circles of I disjoint from C.

We will be concerned with finite inversive planes of a special type, called *miquelian*. It can be shown (see [**7**]) that a finite miquelian inversive plane must have prime-power order, and there exists a unique (up to isomorphism) miquelian inversive plane of order q, denoted by $IP(q)$, for every prime-power q. We will use the model for $IP(q)$ given in [**1**, §7]. Let $L = PG(1, q^2)$ denote the projective line of order q^2. Then the $q^2 + 1$ points of L can be thought of as the points of $IP(q)$, with the projective sublines of L of order q regarded as the circles of $IP(q)$.

A *collineation* of an inversive plane I is a bijection of the points of I onto itself which sends concircular sets onto concircular sets and preserves incidence. A nonidentity collineation of I that fixes some circle C pointwise is called an *inversion with respect to C*. If an inversion with respect to C exists, it is unique, has order two, and fixes no points other than those of C (see [**5**]). For finite miquelian inversive planes, a unique inversion exists with respect to every circle in the plane.

2. Counting arguments in $IP(q)$. We are now ready to begin counting. We know there are $q^3 + q$ circles in $IP(q)$ and there are $q(q-1)(q-2)/2$ circles disjoint from a given circle of $IP(q)$. We would like to know how many circles in $IP(q)$ are disjoint from each circle in a given disjoint pair, a given disjoint triple, and so on. We begin with a triple, and assume throughout that q is odd.

Let C_1, C_2, C_3 denote three pairwise disjoint circles in $IP(q)$. Let $f(i, j, k)$ denote the number of circles in $IP(q)$ that meet C_1 in i points, C_2 in j points, and C_3 in k points, where i, j, k are integers such that $0 \leqslant i, j, k \leqslant 2$. We would like to find an estimate for $f(0, 0, 0)$. In all our computations, we will be summing from 0 to 2. That is, the symbol $\Sigma_{i,j,k} f(i, j, k)$ will mean $\Sigma^2_{i,j,k=0} f(i, j, k)$, and so on.

LEMMA 1. (i) $q(q+1)(q-3)/4 \leqslant \Sigma_k f(2, 0, k) \leqslant q(q^2-1)/4$.
(ii) $q(q^2-1)/4 \leqslant \Sigma_k f(2, 2, k) \leqslant q(q+1)^2/4$.

PROOF. Fix points P, Q on circle C_1. Let $J(P, Q)$ denote the bundle of circles with carriers P and Q. Let $g_{P,Q}(j, k) \equiv g(j, k)$ denote the number of circles in $J(P, Q) \setminus \{C_1\}$ that meet C_2 in j points and meet C_3 in k points, where j, k are integers such that $0 \leqslant j, k \leqslant 2$. By fact (5) above, $\Sigma_{j,k} g(j, k) = q$.

Since q is odd, it can be shown (see [**6**, Lemma 3.2]) that, given a circle C of $IP(q)$ and two distinct points R and S both nonincident with C, there are exactly zero or two circles in $J(R, S)$ tangent to C. Applying this result to our situation, we see that $\Sigma_k g(1, k) = 0$ or 2.

Since $J \equiv J(P, Q) \setminus \{C_1\}$ covers the $q + 1$ points of C_2, and at most two circles in J are tangent to C_2, we obtain

$$\frac{q-1}{2} \leqslant \sum_k g(2, k) \leqslant \frac{q+1}{2}.$$

In fact, the above analysis shows that either $(q+3)/2$ or $(q+1)/2$ circles in J meet C_2 and, hence,

$$\frac{q-3}{2} \leqslant \sum_k g(0, k) \leqslant \frac{q-1}{2}.$$

Using the fact that $\Sigma_{P,Q} g_{P,Q}(j, k) = f(2, j, k)$ if we sum over the $\binom{q+1}{2}$ possible pairs P, Q of distinct points lying on C_1, the result now follows by setting $j = 0$ and $j = 2$.

By symmetry we obtain results similar to (i) and (ii) for any permutation of the ordered triples $(2, 0, k)$, $(2, 2, k)$, respectively. Next we obtain an estimate for the number of circles disjoint from two of the given three disjoint circles.

LEMMA 2.

$$\frac{q^2 - 13q^2 + 21q + 7}{4} \leqslant \sum_k f(0, 0, k) \leqslant \frac{q^3 - q^2 - 3q - 5}{4}.$$

PROOF. Fix a point Q nonincident with both C_1 and C_2, and let the points R, S vary over the circles C_1, C_2, respectively. Let $\langle R, S, Q\rangle$ denote the unique circle determined by the three distinct points R, S, Q. Let $g_Q(i, j)$ denote the number of distinct circles through Q that meet C_1 in i points and C_2 in j points, where i, j are integers such that $0 \leqslant i, j \leqslant 2$. As R, S vary, we obtain $(q + 1)^2$ (not necessarily distinct) circles $\langle R, S, Q\rangle$, all of which meet both C_1 and C_2.

By definition of an inversive plane, we obtain at most $2(q + 1)$ distinct circles in the above manner which are tangent to either C_1 or C_2, any such circle being counted at most twice. Thus at least $(q + 1)^2 - 4(q + 1) = (q - 3)(q + 1)$ of the circles $\langle R, S, Q\rangle$ are scant to both C_1 and C_2, and each such circle is counted four times. Hence

$$(q-3)(q+1)/4 \leqslant g_Q(2, 2) \leqslant (q+1)^2/4.$$

Using (8) from §1 and the facts that $0 \leqslant g_Q(2, 1), g_Q(1, 0) \leqslant q + 1$, we now see that

$$(q^2 - 4q - 5)/4 \leqslant g_Q(2, 0) \leqslant (q^2 + 4q + 3)/4$$

and, therefore,

$$(q^2 - 10q - 11)/4 \leqslant g_Q(0, 0) \leqslant (q^2 + 2q + 1)/4.$$

Allowing Q to vary and using the fact that $1 + \Sigma_k f(0, 0, k) = (q + 1)^{-1} \Sigma_Q g_Q(0, 0)$, the result now follows.

Once again we obtain similar results to the above for any permutation of $(0, 0, k)$. To obtain our estimate of $f(0, 0, 0)$ we need to count the number of circles secant to each of C_1, C_2, C_3.

LEMMA 3.
$$\frac{q^3 - 3q^2 - 23q - 19}{8} \leqslant f(2, 2, 2) \leqslant \frac{(q + 1)^3}{8}.$$

PROOF. Let R, S, T be points of C_1, C_2, C_3, respectively. Let $\langle R, S, T\rangle$ denote the unique circle determined by R, S, T. Allowing R, S, T to vary over their respective circles, let $x(3)$ denote the number of distinct circles $\langle R, S, T\rangle$ containing exactly three points of $C_1 \cup C_2 \cup C_3$. Similarly define $x(4)$, $x(5)$, $x(6)$. The above circles $\langle R, S, T\rangle$ are said to be of type 3, 4, 5 or 6, respectively. Allowing R, S, T to vary and counting the circles $\langle R, S, T\rangle$ (with multiplicities), we see that

$$(\#) \qquad x(3) + 2 \cdot x(4) + 4 \cdot x(5) + 8 \cdot x(6) = (q + 1)^3.$$

Since q is odd, it can be shown (see [6, proof of Lemma 2.2]) that, given two distinct nontangent circles C and D that have a common tangent circle, through every point on C, but not on D, there exist exactly two circles tangent to both C and D. Concentrating on $T \in C_3$ in our situation, we see that $x(3) \leqslant 2(q + 1)$. A similar argument applied to type 4 circles and the use of symmetry show that $x(4) \leqslant 6(q + 1)$.

A circle $\langle R, S, T\rangle$ of type 5 will be secant to two of the C_i's and tangent to the third. Say that $\langle R, S, T\rangle$ is secant to C_1 and C_2 while tangent to C_3. Define

$$L(T) = \{C_3\} \cup \{\text{all circles tangent to } C_3 \text{ at } T\}.$$

Since the $q + 1$ points of C_1 are covered by the pencil $L(T)$, a symmetry argument shows that $x(5) \leqslant 3(q + 1)^2/2$. The lemma now follows from (#) above and the obvious fact that $f(2, 2, 2) = x(6) \leqslant (q + 1)^3/8$.

We are now ready to count the number of circles disjoint from each of the given three disjoint circles C_1, C_2, C_3.

THEOREM 1. $f(0, 0, 0) = (q^3/8)\lambda$, *where*
$$1 - 37/q + 39/q^2 + 21/q^3 \leqslant \lambda \leqslant 1 + 13/q + 33/q^2 + 13/q^3.$$

PROOF. From Lemmas 1 and 2,

$$(*) \qquad \begin{aligned} q(q + 1)(q - 3)/4 &\leqslant f(2, 0, 0) + f(2, 1, 0) + f(2, 2, 0) \leqslant q(q^2 - 1)/4; \\ q(q^2 - 1)/4 &\leqslant f(2, 2, 0) + f(2, 2, 1) + f(2, 2, 2) \leqslant q(q + 1)^2/4; \\ (q^3 - 13q^2 + 21q + 7)/4 &\leqslant f(0, 0, 0) + f(1, 0, 0) + f(2, 0, 0) \\ &\leqslant (q^3 - q^2 - 3q - 5)/4. \end{aligned}$$

Letting S be a point on C_2, there are at most $(q + 1)/2$ circles tangent to C_2 at S and also secant to C_1. Thus

$$0 \leqslant f(2, 1, 0) \leqslant (q + 1)^2/2$$

and, similarly,

$$0 \leqslant f(2, 2, 1) \leqslant (q + 1)^2/2.$$

Suppose that R is a point of C_1, and let $L(R)$ denote the pencil of circles with carrier R which contains the circle C_1. Since the $q + 1$ points of C_2 are covered by the $q - 1$ circles in $L(R) \setminus \{C_1\}$, at most $(q - 3)/2$ circles in $L(R) \setminus \{C_1\}$ are disjoint from C_2 and therefore

$$0 \leqslant f(1, 0, 0) \leqslant (q - 3)(q + 1)/2.$$

The theorem now follows from Lemma 3 and (*) above.

In the above theorem, $\lambda \to 1$ as $q \to \infty$. Hence the following corollary is immediate.

COROLLARY. *$f(0, 0, 0)$ is asymptotic to $q^3/8$.*

REMARK. It should be noted that

$$f(0, 0, 0),\quad f(0, 0, 2),\quad f(0, 2, 0),\quad f(2, 0, 0),\quad f(2, 2, 0),$$
$$f(2, 0, 2),\quad f(0, 2, 2),\quad \text{and}\quad f(2, 2, 2)$$

are all asymptotic to $q^3/8$, and hence the $q^3 + q - 3$ circles of $IP(q)$ other than C_1, C_2, C_3 are rather evenly distributed concerning their intersection patterns with the given three circles. This fact, as well as other considerations, leads to the following conjecture, which we have now proved to be true for $n \leqslant 3$.

CONJECTURE. Given a set of n pairwise disjoint circles $C_1, \ldots, C_n$ in $IP(q)$, where q is an odd prime-power, the number of distinct circles that are disjoint from each of the given n circles is asymptotic to $q^3/2^n$.

3. Applications to translation planes. Our calculations above can now be used to give an asymptotic estimate of the number of triples (or quadruples) of disjoint circles in $IP(q)$.

THEOREM 2. *Let q be an odd prime-power. Then*

(i) *the number of triples of disjoint circles in $IP(q)$ is asymptotic to $q^9/48$, and*

(ii) *the number of quadruples of disjoint circles in $IP(q)$ is asymptotic to $q^{12}/1536$.*

PROOF. Follows immediately from (2), (9) of §1, Lemma 2, and the corollary to Theorem 1.

As stated at the start of this paper, in order to interpret these results in terms of subregular translation planes, we must introduce the concept of equivalence classes to our sets of disjoint circles. Let G be that subgroup of the collineation group of $IP(q)$ which is generated by the inversions and the collineations induced by the projective linear group of the line $PG(1, q^2)$. It is well known (see [**1**, Theorem 7.5(ii)]) that $|G| = 2q^2(q^4 - 1)$. Let C_1, C_2, C_3 be a triple of disjoint circles in $IP(q)$, where q is an odd prime-power. Let

$$H = \{\theta \in G\colon \theta \text{ permutes the } C_i\text{'s among themselves}\},$$
$$K = \{\theta \in G\colon \theta \text{ fixes each of the } C_i\text{'s}\}.$$

R. H. Bruck has shown (see [**1**, Theorem 7.21 and §8]) that, for large q, practically all triples of disjoint circles in $IP(q)$ are nonlinear, and most (nonlinear) triples have $|H| = 2$. Hence $|G\colon H|$ is asymptotic to q^6 for practically all triples. Thus, from Theorem 2(i), the number of equivalence classes of triples of disjoint circles in $IP(q)$ under the group G is asymptotic to $q^3/48$.

When q is a prime, G is the group of all collineations of $IP(q)$. For this case, as explained in the beginning of the paper, we have now shown that the number of isomorphism classes of translation planes of order q^2 which are subregular of index 3 is asymptotic to $q^3/48$. When $q = p^e$, where p is a prime and $e > 1$, we must enlarge G by using the field automorphism $x \to x^p$ to obtain the group of all collineations of $IP(q)$. This group has order $e \cdot |G|$. Hence, in this case, the number of isomorphism classes of translation planes of order q^2 which are subregular of index 3 is asymptotic to $q^3/48e$. These results agree with those obtained by Bruck [**1**], but the methods of computation are entirely different. In addition, while Bruck's methods were not extendable to index 4 or more, our computations can be used to count the number of subregular translation planes of index 4, and, modulo the conjecture given above, to count the number of subregular translation planes of any positive index.

THEOREM 3. *Let q be an odd prime. Then the number of isomorphism classes of translation planes of order q^2 which are subregular of index* 4 *is asymptotic to* q^6/λ, *where* $16 \leqslant \lambda \leqslant 3072$.

PROOF. Let C_1, C_2, C_3, C_4 be a quadruple of disjoint circles in $IP(q)$, and let H, K be defined analogously to that above. Theorem 2(ii) now implies that the number of equivalence classes of quadruples under G is asymptotic to something at least as big as $q^6/3072$.

Since the number of equivalence classes of linear quadruples is asymptotic to $q^3/48$ (see [**1**, Theorem 7.5]), we can ignore linear quadruples and assume that C_1, C_2, C_3 is a nonlinear triple. Bruck has shown [**1**] that the order of a subgroup of G fixing each circle in a nonlinear triple is at most 8. Hence $|K| \leqslant 8$, $|H| \leqslant 192$, and the result now follows from Theorem 2(ii).

REMARK. Taking a lesson from the case of triples of disjoint circles (i.e. subregular translation planes of index 3), $|H|$ is probably close to 1 for most quadruples, and therefore λ is probably much closer to 3072 than to 16. Also, if $q = p^e$ where p is an odd prime and $e > 1$, we obtain a similar result upon dividing by e as before.

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