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WEAK CHEBYSHEV SUBSPACES AND CONTINUOUS SELECTIONS FOR THE METRIC PROJECTION

BY

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ABSTRACT. Let G be an n-dimensional subspace of $C[a,b]$. It is shown that there exists a continuous selection for the metric projection if for each f in $C[a,b]$ there exists exactly one alternation element g_f, i.e., a best approximation for f such that for some $a \leq x_0 < \cdots < x_n \leq b$,

$$\varepsilon(-1)^i(f-g_f)(x_i) = \|f-g_f\|, \qquad i = 0, \ldots, n, \varepsilon = \pm 1.$$

Further it is shown that this condition is fulfilled if and only if G is a weak Chebyshev subspace with the property that each g in G, $g \neq 0$, has at most n distinct zeros. These results generalize in a certain sense results of Lazar, Morris and Wulbert for $n = 1$ and Brown for $n = 5$.

If G is a nonempty subset of a normed linear space E then for each f in E, we define $P_G(f)$: $= \{g_0 \in G: \|f - g_0\| = \inf\{\|f-g\|: g \in G\}\}$. P_G defines a set-valued mapping of E into 2^G which in the literature is called the *metric projection* onto G. A continuous mapping s of E into G is called a *continuous selection for the metric projection* P_G (or, more briefly, continuous selection) if $s(f)$ is in $P_G(f)$ for each f in E. In this paper we treat the problem of the existence of continuous selections for n-dimensional subspaces G of $C[a,b]$, with $C[a,b]$ as usual the Banach space of real-valued continuous functions on $[a,b]$ under the uniform norm.

A. Lazar, P. Morris and D. Wulbert [4] have characterized the 1-dimensional subspaces of $C(X)$ with X compact Hausdorff, which admit a continuous selection. They raised the problem of characterizing the corresponding n-dimensional subspaces. The only known result for higher dimensional subspaces has been given by A. Brown [1], who has shown the existence of continuous selections for certain 5-dimensional subspaces of $C[-1,1]$.

To obtain continuous selections, Lazar, Morris and Wulbert [4] and Brown [1] proceeded as follows: For each f in $C[a,b]$ they considered all g in $P_G(f)$ which can be written as $g = a_1 g_1 + \cdots + a_n g_n$, where $g_1, \ldots, g_n$ is a basis

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of G, and chose the unique element g in $P_G(f)$ with maximal coefficient a_n. This works in the cases $n = 1$ and $n = 5$.

Using this kind of selection it does not seem possible to get a general theorem for n-dimensional subspaces in $C[a,b]$. With new methods, however, and in the setting of weak Chebyshev subspaces we can give a sufficient condition for the existence of continuous selections.

R. Jones and L. Karlovitz [2, Theorem 4] have shown that an n-dimensional subspace G of $C[a,b]$ is weak Chebyshev if and only if for each f in $C[a,b]$ there exists at least one alternation element g_f (see Definition 1 below) in $P_G(f)$. We show that if for each f in $C[a,b]$ there exists exactly one alternation element g_f in $P_G(f)$, then $s(f) = g_f$ defines a continuous selection (Proposition 2). From Theorem 8 and Theorem 11, which together represent the main result of this paper, it follows that for an n-dimensional weak Chebyshev subspace G each f in $C[a,b]$ has exactly one alternation element in $P_G(f)$ if and only if each $g \in G$, $g \neq 0$, has at most n distinct zeroes. (In particular, g may not vanish on intervals.)

Using this result and Proposition 2, we immediately get an existence theorem for continuous selections for n-dimensional subspaces (Corollary 9). Brown [1] uses essentially stronger conditions to guarantee the existence of continuous selections for 5-dimensional subspaces of $C[-1,1]$. Brown's result disproves a claim of Lazar, Morris and Wulbert [4], who tried to show that for n-dimensional subspaces G in $C(X)$ (X a connected, compact, Hausdorff space) such that 1 is in G and each g in G, $g \neq 0$, does not vanish on an open set in X, there does not exist a nontrivial continuous selection.

Finally, Let us remark that from P. Schwartz [8] it follows that under the assumption of Corollary 9 the continuous selection is unique.

In the following let G be an n-dimensional subspace of $C[a,b]$.

1. Definition. If f is in $C[a,b]$, then g in $P_G(f)$ is called an *alternation element* (A-element) of f if there exist $n + 1$ distinct points $a \leqslant x_0 < \cdots < x_n \leqslant b$ such that

$$\varepsilon(-1)^i(f-g)(x_i) = \|f-g\|, \qquad i = 0, \ldots, n, \ \varepsilon = \pm 1.$$

The points $a \leqslant x_0 < \cdots < x_n \leqslant b$ are called *alternating extreme points* of $f - g$.

First, we want to show that when each f has a unique A-element then we can always define a continuous selection.

2. Proposition. *Suppose for each f in $C[a,b]$ there exists exactly one A-element g_f in $P_G(f)$. Define s: $C[a,b] \to G$ by $s(f) = g_f$ for each $f \in C[a,b]$. Then, s is a continuous selection for P_G: $C[a,b] \to 2^G$.*

Proof. We suppose s is not continuous.

Because of the finite dimensionality of G, there exist $f \in C[a,b]$, $g \in G$ and a sequence $(f_m) \subset C[a,b]$ so that $f_m \to f$, $s(f_m) \to g$, but $g \neq s(f)$.

We will show that g is an A-element of f and this will contradict the uniqueness of the A-element.

By definition, $s(f_m)$ is an A-element of f_m, $m \in N$. Therefore, there are extreme points $a \leqslant x_0^{(m)} < x_1^{(m)} < \cdots < x_n^{(m)} \leqslant b$ of $f_m - s(f_m)$.

We can assume that

$$(1)\quad (-1)^i(f_m - s(f_m))(x_i^{(m)}) = \|f_m - s(f_m)\|, \qquad i = 0, \ldots, n, \ m \in \mathbf{N}.$$

Here it may be necessary to choose a subsequence of (f_m) and perhaps work with $-f$ and $-f_m$ in place of f and f_m. We can also assume (again choosing a subsequence if necessary) that $\lim_{m\to\infty} x_i^{(m)} = x_i$ exists, $i = 0,1, \ldots, n$. Now, since $\lim_{m\to\infty} f_m = f$ and $\lim_{m\to\infty} s(f_m) = g$, we have

$$\begin{aligned} \|f - g\| &= \lim_{m\to\infty} \|f_m - s(f_m)\| \\ &= (-1)^i \lim_{m\to\infty} (f_m - s(f_m))(x_i^{(m)}) \\ &= (-1)^i (f - g)(x_i) \end{aligned}$$

where in the second equality we used (1) and the uniform convergence. This shows that g is an A-element, which is the desired contradiction.

Jones and Karlovitz [2] have characterized those n-dimensional subspaces of $C[a,b]$ which have at least one A-element for each f in $C[a,b]$. For this characterization, we need the following definition:

3. DEFINITION. G is called *weak Chebyshev* if each g in G has at most $n - 1$ changes of sign, i.e., there do not exist points $a \leqslant x_0 < \cdots < x_n \leqslant b$ such that $g(x_i) \cdot g(x_{i+1}) < 0$, $i = 0, \ldots, n - 1$.

Jones-Karlovitz [2] have proved the following theorem:

4. THEOREM. *G is weak Chebyshev if and only if for each f in $C[a,b]$ there exists at least one A-element in $P_G(f)$.*

To get a continuous selection under application of Proposition 2, we examine what additional conditions a weak Chebyshev subspace has to fulfill in order that each f in $C[a,b]$ has exactly one A-element.

We need the following standard definition:

5. DEFINITION. A zero x_0 of f in $C[a,b]$ is said to be a *simple zero* if f changes sign at x_0 or if $x_0 = a$ or $x_0 = b$.

A zero x_0 of f in $C[a,b]$ is said to be a *double zero* if f does not change sign at x_0 and $x_0 \neq a$, $x_0 \neq b$.

In the following, we count simple zeroes as one zero and double zeroes as two zeroes. To prove the following results we need the lemma below.

6. LEMMA. *If f is in $C[a,b]$ and if there exist $n + 1$ points $a \leqslant x_0 < \cdots <$*

$x_n \leqslant b$ *such that*

$$\varepsilon(-1)^i f(x_i) \geqslant 0, \qquad i = 0, \ldots, n, \varepsilon = \pm 1,$$

then f has at least n zeroes y_i *such that*

$$x_0 \leqslant y_0 \leqslant x_1 \leqslant y_1 \leqslant \cdots \leqslant x_{n-1} \leqslant y_{n-1} \leqslant x_n.$$

7. LEMMA. *If G is an n-dimensional weak Chebyshev subspace of* $C[a,b]$ *such that there exists a g in G,* $g \neq 0$, *with at least* $n + 2$ *zeroes, then there exists a* $\tilde{g}$ *in G,* $\tilde{g} \neq 0$, *with at least* $n + 1$ *distinct zeroes.*

PROOF. Let g be in G, $g \neq 0$, with at least $n + 2$ zeroes in $[a,b]$, but only r, $r \leqslant n$, distinct zeroes. Suppose first that $g(a) = g(b) = 0$, and set $\bar{x}: = \max\{x \in [a,b) | g(x) = 0\}$.

Let $a < x_1 < \cdots < x_s \leqslant \bar{x}$ be the simple zeroes of g.

(a) $s + n - 1$ is an even number.

We choose $n - 1 - s$ points $\bar{x} < x_{s+1} < \cdots < x_{n-1} < b$. Since G is weak Chebyshev, by Jones and Karlovitz [2, p. 140] there exists a $\bar{g} \in G$, $\bar{g} \neq 0$, with

$$\varepsilon(-1)^i \bar{g}(x) \geqslant 0, \quad x_{i-1} < x < x_i, \quad i = 1, \ldots, n, \varepsilon = \pm 1,$$

where $x_0 = a$, $x_n = b$. By Lemma 6, $\bar{g}$ has at least $n - 1$ distinct zeroes. We choose ε such that $\operatorname{sgn}(g(x) \cdot \bar{g}(x)) \geqslant 0$ if $x \in [a,x_{s+1}]$. Let $a = y_1 < \cdots < y_t = b$ be the distinct zeroes of g in $[a,b]$.

Then

$$M: = \min_{i=2,\ldots,t} \| g \|_{[y_{i-1},y_i]} > 0.$$

We define $\tilde{g}: = M\bar{g}/(2\|\bar{g}\|)$.

The function $\tilde{g}$ has at least two further distinct zeroes in $[a,b]$, otherwise the function $g - \tilde{g}$ would have at least n changes of sign. This would be a contradiction.

(b) $s + n - 1$ is an odd number.

We choose $x_0 = a$ and $n - s - 2$ points

$$\bar{x} < x_{s+1} < \cdots < x_{n-2} < b.$$

Since G has an $(n - 1)$-dimensional weak Chebyshev subspace (see Sommer and Strauss [11, Theorem 2.6]), by Jones and Karlovitz [2, p. 140] there exists a $\bar{g} \in G$, $\bar{g} \neq 0$ with $\varepsilon(-1)^i \bar{g}(x) \geqslant 0$, $x_{i-1} < x < x_i$, $i = 1, \ldots, n - 1$, $\varepsilon = \pm 1$ where $x_{n-1} = b$.

As before let $\operatorname{sgn}(g(x) \cdot \bar{g}(x)) \geqslant 0$ if $x \in [a,x_{s+1}]$.

Following (a) we construct a function $\tilde{g}$.

As before it follows that either the function $\tilde{g}$ or the function $g - \tilde{g}$ has $n + 1$ distinct zeroes in $[a,b]$.

If not $g(a) = g(b) = 0$, the assertion can be shown in an analogous way. This completes the proof.

8. THEOREM. *If G is an n-dimensional weak Chebyshev subspace of $C[a,b]$ such that each g in G, $g \neq 0$, has at most n distinct zeroes, then each f in $C[a,b]$ has exactly one A-element g_f in $P_G(f)$.*

PROOF. *Assumption.* There exists a function f in $C[a,b]$ which has two A-elements g_1 and g_2 in $P_G(f)$.

Let $a \leqslant x_0 < \cdots < x_n \leqslant b$ be $n + 1$ alternating extreme points of $f - g_1$ and let $a \leqslant y_0 < \cdots < y_n \leqslant b$ be $n + 1$ alternating extreme points of $f - g_2$.

We distinguish two cases:

First case.

$$(-1)^i(f - g_1)(x_i) = \|f - g_1\|, \qquad i = 0, \ldots, n,$$
$$(-1)^i(f - g_2)(y_i) = \|f - g_2\|, \qquad i = 0, \ldots, n.$$

Then

$$\text{(i)} \qquad \begin{aligned} &(-1)^i(g_2 - g_1)(x_i) \geqslant 0, \qquad i = 0, \ldots, n,\\ &(-1)^i(g_2 - g_1)(y_i) \leqslant 0, \qquad i = 0, \ldots, n. \end{aligned}$$

We treat only the case

$$\text{(ii)} \qquad x_{i-2} \leqslant y_i \leqslant x_{i+2}, \qquad i = 0, \ldots, n,$$

where the points x_i for $i = -2, -1, n + 1, n + 2$ are omitted. In the other case, if $y_i < x_{i-2}$ for some i, we choose the points $y_0, \ldots, y_i, x_{i-2}, \ldots, x_n$ fulfilling

$$(-1)^j(g_2 - g_1)(y_j) \leqslant 0, \qquad j = 0, \ldots, i,$$
$$(-1)^j(g_2 - g_1)(x_{j-3}) \leqslant 0, \qquad j = i + 1, \ldots, n + 3.$$

By Lemma 6, $g_2 - g_1$ has at least $n + 3$ zeroes. Applying Lemma 7 we get a contradiction of the hypothesis that elements of G have at most n distinct zeroes.

A similar argument works for $x_{i+2} < y_i$.

Now we prove by induction that $g_1 - g_2$ has at least $n + 1$ distinct zeroes. This is a contradiction of the hypothesis on G. If $x_i = y_i$, $i = 0, \ldots, n$, then $(g_1 - g_2)(x_i) = 0$, $i = 0, \ldots, n$. We may assume $x_i < y_i$ for some $i = \{0, \ldots, n\}$.

We show: $x_j \leqslant y_j, j = 0, \ldots, n$.

If $y_{j_0} < x_{j_0}$ for any $j_0 \in \{0, \ldots, n\}$ we choose

$$y_0, \ldots, y_{j_0}, x_{j_0}, \ldots, x_i, y_i, \ldots, y_n \quad \text{if } j_0 < i$$

and

$$x_0, \ldots, x_i, y_i, \ldots, y_{j_0}, x_{j_0}, \ldots, x_n \quad \text{if } j_0 > i.$$

Because of (i) in both cases $g_1 - g_2$ has at least $n + 2$ zeroes by Lemma 6. Applying Lemma 7 we get a contradiction of the hypothesis on G.

Now we show by induction that $g_1 - g_2$ has at least $n + 1$ distinct zeroes in $[x_0, y_n]$: $n = 1$.

If $x_0 \leqslant y_0 < x_1 \leqslant y_1$ (respectively $x_0 < y_0 = x_1 < y_1$ or $x_0 < x_1 < y_0 < y_1$), then $g_1 - g_2$ has one zero in each interval $[x_0, y_0]$, $[x_1, y_1]$ (respectively $[x_0, y_0)$, $(x_1, y_1]$ or $[x_0, x_1]$, $[y_0, y_1]$).

Let the statement be true for $n - 1$.

If $y_{n-1} < x_n \leqslant y_n$, then by assumption $g_1 - g_2$ has n distinct zeroes in $[x_0, y_{n-1}]$ and a further zero in $[x_n, y_n]$.

If $y_{n-1} = x_n < y_n$, then by assumption $g_1 - g_2$ has n distinct zeroes in $[x_0, y_{n-1})$ and a further zero in $(x_n, y_n]$.

Finally we consider the case $x_n < y_{n-1} < y_n$:

Since $(-1)^n(g_2 - g_1)(x_n) \geqslant 0$, $(-1)^n(g_2 - g_1)(y_{n-1}) \geqslant 0$, and $y_{n-2} \leqslant x_n$ we conclude as in the case $y_{n-1} < x_n \leqslant y_n$.

Second case.

$$(-1)^i(f - g_1)(x_i) = \|f - g_1\|, \qquad i = 0, \ldots, n,$$
$$-(-1)^i(f - g_2)(y_i) = \|f - g_2\|, \qquad i = 0, \ldots, n.$$

We treat only the case that $f - g_1$ and

(iii) $\qquad f - g_2$ have exactly $n + 1$ alternating extreme points.

Otherwise we can apply the first case.

Then

$$\text{(iv)} \qquad \begin{aligned} (-1)^i(g_2 - g_1)(x_i) &\geqslant 0, \qquad i = 0, \ldots, n, \\ (-1)^i(g_2 - g_1)(y_i) &\geqslant 0, \qquad i = 0, \ldots, n. \end{aligned}$$

It is now enough to treat the case

$$\text{(v)} \qquad x_{i-1} \leqslant y_i \leqslant x_{i+1}, \qquad i = 0, \ldots, n,$$

where the points x_{-1} and x_{n+1} are omitted. Otherwise we can conclude as in the first case. Applying the first case to the points $x_0, \ldots, x_{n-1}, y_1, \ldots, y_n$ because of (v) $g_1 - g_2$ has n distinct zeroes $z_1, \ldots, z_n$ in $[x_0, y_n]$.

We first prove: $z_1, \ldots, z_n \in (a, b)$. If $z_1 = x_0$, then $y_0 < x_0$. Otherwise $f - g_2$ has $n + 2$ alternating extreme points $x_0, y_0, \ldots, y_n$. This is a contradiction to (iii). Therefore $z_1 > a$.

If $z_n = y_n$, then $x_n > y_n$. Otherwise $f - g_1$ has $n + 2$ alternating extreme points $x_0, \ldots, x_n, y_n$. This is a contradiction to (iii). Therefore $z_n < b$.

If $g_1 - g_2$ has $n + 1$ distinct zeroes, then we would get a contradiction of the hypothesis on G.

Therefore we know $g_1 - g_2$ has no further zero in $[a, b]$. Because of $a < z_1 < \cdots < z_n < b$ and G weak Chebyshev $g_1 - g_2$ has at most $n - 1$ changes of sign under the points z_i. We show $g_1 - g_2$ has at most $n - 2$ changes of sign:

If $g_1 - g_2$ has $n - 1$ changes of sign under the points z_i, then there exists exactly one zero $z_j \in (a, b)$ such that $g_1 - g_2$ does not change sign at z_j.

Then it holds: If $z_1 > x_0$, then because of (iv)

$$\text{(vi)} \qquad \begin{aligned} &(-1)^0(g_2 - g_1)(x) > 0 \quad \text{if } a \leqslant x < z_1 \quad \text{and finally} \\ &(-1)^n(g_2 - g_1)(x) < 0 \quad \text{if } z_n < x \leqslant b. \end{aligned}$$

If $z_1 = x_0$, then $y_0 < x_0$ and (vi) is also valid.

Now we get a contradiction to (iv):

(1) $$x_n \geqslant y_n.$$

Then $z_n < x_n$ and because of (iv) $(-1)^n(g_2 - g_1)(x_n) \geqslant 0$. This is a contradiction.

(2) $$x_n < y_n.$$

If $z_n < y_n$ we also get a contradiction because of (iv). But if $z_n = y_n$, then $x_n > y_n$ is always valid because of (iii).

We have shown:

If $g_1 - g_2$ has exactly n distinct zeroes, then $g_1 - g_2$ has at most $n - 2$ changes of sign. But in this case there exist $n + 2$ zeroes of $g_1 - g_2$ because of $a < z_1, z_n < b$.

Applying Lemma 7 we get a contradiction to the assumption.

Schwartz [8] has shown that for an n-dimensional subspace G of $C(X)$ with the property that no g in G, $g \neq 0$, vanishes identically on an open subset of X, the set of functions in $C(X)$ having unique best approximation in G is dense in $C(X)$. Therefore there exists at most one continuous selection. By this result, Proposition 2 and Theorem 8 the next corollary follows immediately.

9. Corollary. *If G is an n-dimensional weak Chebyshev subspace such that each g in G, $g \neq 0$, has at most n distinct zeroes, then there exists a unique continuous selection s: $C[a, b] \to G$ for P_G: $C[a, b] \to 2^G$.*

Now we will give some nontrivial examples of subspaces G in $C[a, b]$ fulfilling the assumption of Corollary 9.

10. Examples. (a) $G: = \langle x, x^2, \ldots, x^n \rangle \subset C[0, 1]$. G is Chebyshev in $(0, 1]$ and therefore the assumption of Corollary 9 is fulfilled, but G is not Chebyshev in $[0, 1]$.

(b) For $n \geqslant 2$ and n even, we define $G: = \langle 1, x(1 - x^2), x^2, x^3(1 - x^2), x^4, \ldots, x^{n-1}(1 - x^2), x^n \rangle \subset C[-1, 1]$. The dimension of G is $n + 1$. Each function g in G is a polynomial of degree $\leqslant n+1$ and has therefore at most $n+1$ zeroes in $[-1, 1]$. Such a function g can be written as $g = g_1 - g_2$ where

$$g_1(x) = \sum_{i=0}^{n/2} a_{2i}x^{2i} \quad \text{and} \quad g_2(x) = x(1-x^2)\sum_{i=1}^{n/2} a_{2i-1}x^{2i-2}.$$

Because of the behaviour of $g_1(x)$ and $g_2(x)$ for $x \to \pm\infty$ it can be shown that $g_1 - g_2$ has a zero in $(-\infty, -1] \cup (1, \infty)$. Therefore G is Chebyshev in $(-1, 1]$.

G is not Chebyshev in $[-1, 1]$ because there exists a function

$$g_0(x) = x(1-x^2)\sum_{i=1}^{n/2} a_{2i-1}x^{2i-2} \quad \text{in } G, g \neq 0,$$

having exactly $n+1$ zeroes in $[-1, 1]$.

A similar example has been given by Brown [1] in the case $n = 5$.

(c) $G := \langle |x|, x^3 \rangle \subset C[-1, 1]$. G is weak Chebyshev and each $g \in G$, $g \neq 0$, has at most 2 distinct zeroes, but G is not Chebyshev in $[-1, 1)$ or $(-1, 1]$.

Finally we ask how strong the assumption of Theorem 7 is for the uniqueness of A-elements and we show that this is the weakest condition because the converse of Theorem 7 is true.

11. THEOREM. *If G is an n-dimensional weak Chebyshev subspace of $C[a, b]$ such that for each f in $C[a, b]$ there exists exactly one A-element in $P_G(f)$ then each g in G, $g \neq 0$, has at most n distinct zeroes.*

PROOF. *Assumption.* There exists a $\tilde{g}_0$ in G, $\tilde{g}_0 \not\equiv 0$, with at least $n+1$ distinct zeroes.

We define: $g_0 := \tilde{g}_0 / \|\tilde{g}_0\|$. Then $\|g_0\| = 1$.

Since G is weak Chebyshev, g_0 has at most $n-1$ changes of sign. Therefore $n+1$ distinct zeroes $x_0, \ldots, x_n$ of g_0 exist such that $\varepsilon_i g_0(x) \geqslant 0$, $x \in [x_i, x_{i+1}]$, $i = -1, 0, \ldots, n$, $\varepsilon_i = \pm 1$, $x_{-1} := a$, $x_{n+1} := b$.

We construct a function f in $C[a, b]$, having two A-elements in $P_G(f)$. We define f in the following way:

(1) $$\varepsilon_{-1}(-1)^i f(x_i) = 1, \qquad i = 0, \ldots, n,$$

(2) $$\|f\| = 1,$$

(3) $$0, g_0 \quad \text{in } P_G(f).$$

Then g_0 and 0 are A-elements of f.

Construction of f:

(a) We may assume $g \geqslant 0$ for $x \in [a, x_0]$.

We define: $f(x) := 1$ if $x \in [a, x_0]$, $(-1)^i f(x_i) = 1$, $i = 0, \ldots, n$.

(b) Definition of f in $[x_0, x_1]$:

First case. $g_0(x) \geqslant 0$ if $x \in [x_0, x_1]$.

Let $\tilde{x} := (x_0 + x_1)/2$ and $f(\tilde{x}) := 0$.

Let f be linear in $[x_0, \tilde{x}]$

$$f(x) := g_0(x) - g_0(\tilde{x}) + 2(g_0(\tilde{x}) - 1)\frac{x - \tilde{x}}{x_1 - x_0} \quad \text{if } x \in [\tilde{x}, x_1].$$

Second case. $g_0(x) \leqslant 0$ for $x \in [x_0, x_1]$.

$$f(x) := g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x}))\frac{\tilde{x} - x}{\tilde{x} - x_0} \quad \text{if } x \in [x_0, \tilde{x}],$$

$$f(\tilde{x}) := 0.$$

Let f be linear in $[\tilde{x}, x_1]$.

This construction of f is continued in an analogous way for the intervals $[x_1, x_2], \ldots, [x_{n-1}, x_n], [x_n, b]$. Obviously f is continuous in $[a, b]$.

We show: $|f(x)| \leqslant 1$ if $x \in [x_0, x_1]$.

In the first case:

$$\begin{aligned} -1 &\leqslant g_0(x) - g_0(\tilde{x}) + g_0(\tilde{x}) - 1 \\ &\leqslant g_0(x) - g_0(\tilde{x}) + 2(g_0(\tilde{x}) - 1)\frac{x - \tilde{x}}{x_1 - x_0} \\ &= f(x) \leqslant g_0(x) - g_0(\tilde{x}) \leqslant g_0(x) \leqslant 1 \quad \text{if } \tilde{x} \in [x, x_1]. \end{aligned}$$

In the second case:

$$\begin{aligned} -1 &\leqslant g_0(x) \leqslant g_0(x) - g_0(\tilde{x}) \\ &\leqslant g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x}))\frac{\tilde{x} - x}{\tilde{x} - x_0} \\ &= f(x) \leqslant g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x})) \\ &\leqslant g_0(x) + 1 \leqslant 1 \quad \text{if } x \in [x_0, \tilde{x}]. \end{aligned}$$

Therefore $|f(x)| \leqslant 1$ if $x \in [x_0, x_1]$.

We can show in an analogous way: $|f(x) - g_0(x)| \leqslant 1$ if $x \in [x_0, x_1]$. These estimations hold in each interval because of the construction of f.

Therefore $f - 0$ and $f - g_0$ have $x_0, \ldots, x_n$ as alternating extreme points.

If 0 and g_0 are not in $P_G(f)$, then there would exist a function g in G such that $\|f - g\| < \|f\| = \|f - g_0\| = 1$. Since $(-1)^i f(x_i) = 1 = \|f\| > (-1)^i(f(x_i) - g(x_i))$ it follows $(-1)^i g(x_i) > 0, i = 0, \ldots, n$.

Hence g has at least n changes of sign in $[a, b]$. This is a contradiction to the assumption that G is weak Chebyshev. Therefore 0 and g_0 are two A-elements of f in $P_G(f)$.

This completes the proof.

Finally we show in Proposition 14 that a large class of weak Chebyshev subspaces in $C[a, b]$ whose nonzero functions have only finitely many zeroes fulfill the assumption of Corollary 9 and therefore admit a unique continuous selection.

We need the following definition (cf. Singer [10, p. 126]):

12. DEFINITION. A linear subspace G of a normed linear space E is called k-Chebyshev (where k is an integer with $0 \leqslant k \leqslant \infty$), if for each f in E we have $0 \leqslant \dim P_G(f) \leqslant k$.

Finite-dimensional k-Chebyshev subspaces in $C(X)$ (X compact) are characterized in Singer [10, p. 240]:

13. THEOREM. *If G is an n-dimensional subspace of $C(X)$ (X compact) and k an integer with $0 \leqslant k \leqslant n - 1$. Then G is a k-Chebyshev subspace if and only if there do not exist $n - k$ distinct points $x_1, \ldots, x_{n-k}$ in X and $k + 1$ linearly independent functions $g_0, g_1, \ldots, g_k$ in G, such that*

$$g_i(x_j) = 0, \qquad j = 1, \ldots, n - k, i = 0,1, \ldots, k.$$

Using the methods in the proof of Lemma 7 and Theorem 13 we can show in a straightforward manner that the following Proposition holds:

14. PROPOSITION. *If G is an n-dimensional, weak Chebyshev subspace which is $(n - 1)$-Chebyshev and if each g in G, $g \neq 0$, has only finitely many zeroes, then each g in G, $g \neq 0$, has at most n distinct zeroes.*

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