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HERMITIAN LIE ALGEBRAS AND METAPLECTIC REPRESENTATIONS. I

BY

SHLOMO STERNBERG[1] AND JOSEPH A. WOLF[2]

ABSTRACT. A notion of "hermitian Lie algebra" is introduced which relates ordinary and graded Lie algebra structures. In the case of real-symplectic and arbitrary-signature-unitary Lie algebras, it leads to an analysis of the minimal dimensional coadjoint orbits, and then to the metaplectic representations and their restrictions to unitary groups of arbitrary signature and parabolic subgroups of these unitary groups.

1. Introduction. We are going to study examples of the following "hermitian Lie algebra" structure: $\mathfrak{l}$ is a real Lie algebra, represented by linear transformations of a complex vector space V, and $H: V \times V \to \mathfrak{l}_{\mathbf{C}}$ is an $\mathfrak{l}$-equivariant hermitian form. Here "hermitian" means that $H(u, v)$ is linear in u and conjugate-linear in v with

$$H(v, u) = H(u, v)^{-}, \qquad {}^{-} = \text{conjugation of } \mathfrak{l}_{\mathbf{C}} \text{ over } \mathfrak{l},$$

and "equivariant" means that

$$[\xi, H(u, v)] = H(\xi u, v) + H(u, \xi v) \quad \text{for } \xi \in \mathfrak{l} \text{ and } u, v \in V$$

where [,] is extended as usual from $\mathfrak{l}$ to $\mathfrak{l}_{\mathbf{C}}$.

2 Im $H: V \times V \to \mathfrak{l}$ is antisymmetric and $\mathbf{R}$-bilinear, so one tries to use it to make $\mathfrak{l} + V$ into a Lie algebra by: the usual bracket $\mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$, the representation $\mathfrak{l} \times V \to V$ (i.e., $[\xi, u] = \xi u = -[u, \xi]$), and $V \times V \to \mathfrak{l}$ given by

$$[u, v] = 2 \operatorname{Im} H(u, v) = i^{-1}\{H(u, v) - H(v, u)\} \quad \text{for } u, v \in V.$$

This defines a Lie algebra if and only if the Jacobi identity holds, and that is the case just when it holds for any three elements of V:

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0 \text{ for } \quad u, v, w \in V. \tag{1.1}$$

In other words, (1.1) gives a Lie algebra structure on $\mathfrak{l} + V$ just when

$$\begin{aligned} &\{H(u, v)w + H(v, w)u + H(w, u)v\} \\ &\qquad - \{H(v, u)w + H(w, v)u + H(u, w)v\} = 0. \end{aligned} \tag{1.2}$$

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Similarly 2 Re H: $V \times V \to \mathfrak{l}$ is symmetric and $\mathbf{R}$-bilinear, so one tries to use it to make $\mathfrak{l} + V$ into a $\mathbf{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, $\mathfrak{g}_+ = \mathfrak{l}$ and $\mathfrak{g}_- = V$, by the usual bracket $\mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$, the representation $\mathfrak{l} \times V \to V$, and $V \times V \to \mathfrak{l}$ given by

$$(1.3) \qquad [u, v]_G = 2 \operatorname{Re} H(u, v) = H(u, v) + H(v, u) \quad \text{for } u, v \in V.$$

See [4] for the definition and basic properties of graded Lie algebras. The graded version of Jacobi's identity says that left multiplication is a derivation:

$$[x, [y, z]_G]_G = [[x, y]_G, z]_G + (-1)^{(\deg x)(\deg y)}[y, [x, z]_G]_G.$$

Again, (1.3) defines a graded Lie algebra if and only if the graded Jacobi identity holds, that is, the case just when it holds for any three elements of V, and the latter is equivalent to

$$(1.4) \qquad \begin{aligned} &\{H(u, v)w + H(v, w)u + H(w, u)v\} \\ &\qquad + \{H(v, u)w + H(w, v)u + H(u, w)v\} = 0. \end{aligned}$$

Notice that we obtian both a Lie algebra and a graded Lie algebra, i.e., that both Jacobi identities are satisfied, just when

$$(1.5) \qquad H(u, v)w + H(v, w)u + H(w, u)v = 0 \quad \text{for } u, v, w \in V.$$

In order to indicate a source of hermitian forms H, and to be able to describe what we do in this paper, we define a basic class of hermitian Lie algebras in which both Jacobi identities hold. These are the unitary algebras

$$\{\mathfrak{u}(k, l) \oplus \mathfrak{u}(1)\} \oplus \mathbf{C}^{k,l} \quad \text{and} \quad \{(\mathfrak{u}(k, l)/\mathfrak{u}(1)) \oplus \mathfrak{u}(1)\} \oplus \mathbf{C}^{k,l}$$

where $\mathbf{C}^{k,l}$ is complex $(k + l)$-space with hermitian scalar product

$$\langle z, w\rangle = -\sum_1^k z_j \bar{w}_j + \sum_{k+1}^{k+l} z_j \bar{w}_j,$$

and where

$$\mathfrak{u}(k, l) = \{\xi: \mathbf{C}^{k,l} \to \mathbf{C}^{k,l} \text{linear}: \langle \xi z, w\rangle + \langle z, \xi w\rangle = 0\}$$

is the Lie algebra of its unitary group.

$\mathfrak{u}(k, l)$ has complexification $\mathfrak{gl}(k + l; \mathbf{C})$, the Lie algebra of all complex $(k + l) \times (k + l)$ matrices. Let $*$ denote adjoint relative to $\langle\ ,\ \rangle$, that is, $\langle \xi z, w\rangle = \langle z, \xi^* w\rangle$. Then $\mathfrak{u}(k, l) = \{\xi \in \mathfrak{gl}(k + l; \mathbf{C}): \xi^* = -\xi\}$, and $\xi^- = -\xi^*$ is complex conjugation of $\mathfrak{gl}(k + l; \mathbf{C})$ over $\mathfrak{u}(k, l)$. Now

$$(1.6) \qquad H_0: \mathbf{C}^{k,l} \times \mathbf{C}^{k,l} \to \mathfrak{gl}(k + l; \mathbf{C}) \quad \text{by } H_0(u, v)w = i\langle w, v\rangle u$$

is hermitian and $\mathfrak{u}(k, l)$-equivariant, for

$$\langle H_0(u, v)w, z\rangle = i\langle w, v\rangle\langle u, z\rangle = i\langle w, v\rangle\overline{\langle z, u\rangle} = -\langle w, H_0(v, u)z\rangle$$

and, for $\xi \in \mathfrak{u}(k, l)$,

$$[\xi, H_0(u, v)]w = i\{\langle w, v\rangle \xi u - \langle \xi w, v\rangle u\} = H_0(\xi u, v)w + H_0(u, \xi v)w.$$

As to the Jacobi identities, note

$$H_0(u, v)w + H_0(v, w)u + H_0(w, u)v = i\{\langle w, v\rangle u + \langle u, w\rangle v + \langle v, u\rangle w\}.$$

The trick here is to give $\mathfrak{u}(1)_{\mathbf{C}} = \mathfrak{gl}(1; \mathbf{C})$ the conjugate of its usual complex structure, so that

(1.7) $$H_1: \mathbf{C}^{k,l} \times \mathbf{C}^{k,l} \to \mathfrak{gl}(1; \mathbf{C}) \quad \text{by } H_1(u, v) = i\langle v, u\rangle$$

is also hermitian. This done,

(1.8) $$H: \mathbf{C}^{k,l} \times \mathbf{C}^{k,l} \to \mathfrak{u}(k, l)_{\mathbf{C}} \oplus \mathfrak{u}(1)_{\mathbf{C}} \quad \text{by } H = H_0 \oplus H_1$$

is hermitian and $\mathfrak{u}(k, l)$-equivariant, and satisfies (1.5), i.e., satisfies both Jacobi identities. Thus we have both ordinary and graded Lie algebra structures on our unitary algebras $\{\mathfrak{u}(k, l) \oplus \mathfrak{u}(1)\} + \mathbf{C}^{k,l}$.

Now let $\mathfrak{u}(k, l) \oplus \mathfrak{u}(1)$ act on $\mathbf{C}^{k,l}$ by $(\xi, c)u = \xi(u) + cu$. In effect, this action is the projection

(1.9) $$\mathfrak{u}(k, l) \oplus \mathfrak{u}(1) \to (\mathfrak{u}(k, l)/\mathfrak{u}(1)) \oplus \mathfrak{u}(1) \cong \mathfrak{u}(k, l).$$

It gives us ordinary and graded Lie algebra structures on quotient unitary algebras $\mathfrak{u}(k, l) + \mathbf{C}^{k,l}$. In this latter formulation, the hermitian Lie algebra structure is obscured, for H seems not to be hermitian. Here note that $\mathfrak{u}(2, 2) + \mathbf{C}^{2,2}$ is the spin-conformal algebra of Wess and Zumino (cf. [4]), which is of some interest in recent physical literature.

In §2 we give the general construction of unitary algebras. Then we show how a hermitian Lie algebra is associated to every homogeneous bounded domain. Here the domain $\{m \times m \text{ complex matrices } Z: I - Z^*Z \gg 0\}$ gives a unitary algebra. Finally we associate a hermitian Lie algebra to the nilradical in a certain class of parabolic subgroups of classical Lie groups.

In §3 we return to the unitary algebras described above, and show that the $H_0(u, u)$ give the lowest dimensional nonzero coadjoint orbits for $\mathfrak{u}(k, l)$. Our method generalizes a technique introduced by Carey and Hannabuss [3] for $\mathfrak{u}(2, 2)$. We work out the transitivity properties of a certain maximal parabolic subgroup on these orbits. In the $\mathfrak{u}(2, 2)$ case, that parabolic is the Poincaré group with scale, and the Poincaré group has the same transitivity, so these orbits correspond to the zero mass six dimensional orbits in the dual of the Poincaré algebra. We then discuss the relation between these orbits and these of the symplectic algebra $\mathfrak{sp}(k + l; \mathbf{R})$. For this purpose we use a graded Lie algebra associated to the symplectic algebra which is known as orthosymplectic algebra O Sp$(k + l, 1)$.

In §4 we use the graded Lie algebra associated to give a new and very simple construction of the metaplectic representation of the 2-sheeted covering group Mp$(m; \mathbf{R})$ of the real symplectic group Sp$(m; \mathbf{R})$. If $k + l =$

m, then the imaginary part $\operatorname{Im}\langle\ ,\ \rangle$ of the scalar product on $\mathbf{C}^{k,l}$ embeds $U(k, l)$ into $\mathrm{Sp}(m; \mathbf{R})$. Restricted to the inverse image of $U(k, l)$ in $\mathrm{Mp}(m; \mathbf{R})$, the metaplectic representation decomposes as a multiplicity-free discrete sum of representations which remain irreducible on every maximal parabolic subgroup. Mack and Todorov **[16]** obtained one case of this for $U(2, 2)$. Kazhdan **[13]** also reduced the metaplectic representation for $U(k, k)$.

In §5 we discuss the representations of the 2-sheeted covering group of $U(k, l)$, which come from the metaplectic representation of §4, in terms of holomorphic or antiholomorphic discrete series representations.

In the continuation of this paper we plan to relate the metaplectic representation μ of $\mathrm{Mp}(m; \mathbf{R})$ and its $U(k, l)$-reduction, $\nu = \sum_{d \in \mathbf{Z}} \nu_d$, to the reduction described in §3 for the orbit picture by the method of geometric quantization.

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2. Examples. In this section we construct three classes of examples of hermitian Lie algebras. The first, the class of "unitary algebras," satisfies (1.5) and thus provides hermitian-related Lie algebras and graded Lie algebras; the unitary algebras mentioned in the Introduction are special cases. The second class consists of an algebra for every bounded homogeneous domain, and it overlaps the first class as regards the classical bounded symmetric domains

$$D_{k,l} = \{k \times l \text{ complex matrices } Z\colon I - Z^*Z \gg 0\}.$$

The third class corresponds to the nilradicals of the maximal parabolic subgroups of certain classical Lie groups.

2A. *Unitary algebras.* Fix nonnegative integers $a + b = c$ and $k + l = m$, and let $\mathbf{C}^{k,l} \oplus \mathbf{C}^{a,b}$ denote complex $(m + c)$-space with the hermitian scalar product

$$\langle z, w\rangle = -\sum_{i}^{k} z_j\bar{w}_j + \sum_{k+1}^{m} z_j\bar{w}_j - \sum_{m+1}^{m+a} z_j\bar{w}_j + \sum_{m+a+1}^{m+c} z_j\bar{w}_j. \tag{2.1}$$

Let $\mathfrak{u}(k, l; a, b)$ denote the Lie algebra of the unitary group of $\mathbf{C}^{k,l} \oplus \mathbf{C}^{a,b}$. It consists of all linear transformations ξ such that $\langle \xi z, w\rangle + \langle z, \xi w\rangle = 0$. In matrices, it consists of all

$$\begin{bmatrix} A_{11} & A_{12} & Z_{11} & Z_{12} \\ A_{12}^* & A_{22} & Z_{21} & Z_{22} \\ -Z_{11}^* & Z_{21}^* & B_{11} & B_{12} \\ Z_{12}^* & -Z_{22}^* & B_{12}^* & B_{22} \end{bmatrix} \begin{matrix} \}k \\ \}l \\ \}a \\ \}b \end{matrix}$$

$$\underbrace{}_{k} \quad \underbrace{}_{l} \quad \underbrace{}_{a} \quad \underbrace{}_{b}$$

with $A_{jj}^* = -A_{jj}$ and $B_{jj}^* = -B_{jj}$. To simplify this denote

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}^{\#} = \begin{pmatrix} -U_{11}^* & U_{21}^* \\ U_{12}^* & -U_{22}^* \end{pmatrix}.$$

Then

$$\text{(2.2)} \qquad \mathfrak{u}(k, l; a, b) = \left\{ \begin{pmatrix} A & Z \\ Z^{\#} & B \end{pmatrix} \begin{matrix} \}m \\ \\ \}c \end{matrix} \; : \; \begin{matrix} A = A^{\#}, \text{ i.e., } A \in \mathfrak{u}(k, l) \\ \\ B = B^{\#}, \text{ i.e., } B \in \mathfrak{u}(a, b) \end{matrix} \right\}.$$

Now $\mathfrak{u}(k, l; a, b)$ evidently is of the form $\mathfrak{l} + V$ where

$$\mathfrak{l} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathfrak{u}(k, l), B \in \mathfrak{u}(a, b) \right\} \cong \mathfrak{u}(k, l) \oplus \mathfrak{u}(a, b),$$

$$V = \left\{ \begin{pmatrix} 0 & Z \\ Z^{\#} & 0 \end{pmatrix} : Z \text{ is } m \times c \right\} \approx \mathbf{C}^{k,l} \otimes \mathbf{C}^{a,b}$$

and the Lie algebra $\mathfrak{l}$ acts on the complex vector space V by

$$\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & Z \\ Z^{\#} & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & AZ - ZB \\ (AZ - ZB)^{\#} & 0 \end{pmatrix}.$$

2.3. Proposition. $\mathfrak{l}_{\mathbf{C}} = \{ \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) : A$ *is* $m \times m, B$ *is* $c \times c \}$. *Give* $\mathfrak{l}_{\mathbf{C}}$ *the complex structure that is linear in A and conjugate linear in B. Then the map H:* $V \times V \to \mathfrak{l}_{\mathbf{C}}$ *defined by*

$$H(z, w) = izw, \quad i.e.$$

$$\text{(2.4)} \qquad H\left(\begin{pmatrix} 0 & Z \\ Z^{\#} & 0 \end{pmatrix}, \begin{pmatrix} 0 & W \\ W^{\#} & 0 \end{pmatrix} \right) = \begin{pmatrix} iZW^{\#} & 0 \\ 0 & iZ^{\#}W \end{pmatrix}$$

is hermitian, is $\mathfrak{l}$*-equivalent, and defines both Lie algebra and graded Lie algebra structures because it satisfies* (1.5). *Further, the Lie algebra structure is the original one on* $\mathfrak{u}(k, l; a, b)$.

Proof. The statement on $\mathfrak{l}_{\mathbf{C}}$ is standard, and the complex structure there is specified so that $H(z, w)$ is linear in z, conjugate linear in w. An elementary calculation shows $(iZW^{\#})^{\#} = iWZ^{\#}$, and this proves H hermitian. $\mathfrak{l}$-equivariance is seen by direct calculation, using

$$i[A, ZW^{\#}] = i\{(AZ - ZB)W^{\#} - Z(W^{\#}A - BW^{\#})\}$$
$$= i\{(AZ - ZB)W^{\#} + Z(AW - WB)^{\#}\} \quad \text{for } A = A^{\#}, B = B^{\#}$$

and its counterpart

$$i[B, Z^{\#}W] = i\{(AZ - ZB)^{\#}W + Z^{\#}(AW - WB)\}$$
$$\text{for } A = A^{\#}, B = B^{\#}$$

To check (1.5) we calculate

$$(2.5)\quad \begin{aligned}[H(z, w), u] &= i\left[\begin{pmatrix} ZW^{\#} & 0 \\ 0 & Z^{\#}W \end{pmatrix}, \begin{pmatrix} 0 & U \\ U^{\#} & 0 \end{pmatrix}\right] \\ &= i\begin{bmatrix} 0 & ZW^{\#}U - UZ^{\#}W \\ Z^{\#}WU^{\#} - U^{\#}ZW^{\#} & 0 \end{bmatrix},\end{aligned}$$

so

$$[H(z, w), u] + [H(w, u), z] + [H(u, z), w] = \begin{pmatrix} 0 & T \\ T^{\#} & 0 \end{pmatrix},$$

where

$$\frac{1}{i}T = ZW^{\#}U - UZ^{\#}W + WU^{\#}Z - ZW^{\#}U + UZ^{\#}W - WU^{\#}Z = 0.$$

And finally $2 \operatorname{Im} H(z, w) = [z, w]$ directly from (2.4). Q.E.D.

The unitary algebras mentioned in the Introduction are the case $a = 1$ and $b = 0$. For there V is identified with $\mathbf{C}^{k,l}$ and (2.5) specializes to

$$H(z, w)\colon u \mapsto i\{ZW^{\#}U - UZ^{\#}W\} = i\langle u, w\rangle z - i\langle w, z\rangle u.$$

Here note the necessity of mapping to $\mathfrak{u}(k, l)_{\mathbf{C}} \oplus \mathfrak{u}(1)_{\mathbf{C}}$ to ensure that H be hermitian, even though that algebra acts on $\mathbf{C}^{k,l}$ as $\mathfrak{u}(k, l)_{\mathbf{C}}$.

When $k = l$ our graded Lie algebra structure has a more refined graduation. Split $\mathbf{C}^{k,k} = W_1 + W_{-1}$, sum of two complementary k-dimensional isotropic subspaces, and use bases of $W_{\pm 1}$ that are dual to each other. So we have a basis of $\mathbf{C}^{k,k}$ in which its hermitian scalar product has matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. In that basis,

$$\mathfrak{u}(k, k) = \left\{\begin{pmatrix} M & E \\ F & -M^* \end{pmatrix}\colon M \text{ is } k \times k \text{ and } E, F \in \mathfrak{u}(k)\right\},$$

which gives the vector space direct sum splitting $\mathfrak{u}(k, k) = \mathfrak{g}_{-2} + \mathfrak{g}_0 + \mathfrak{g}_2$, where

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} : F \in \mathfrak{u}(k) \right\}, \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} M & 0 \\ 0 & -M^* \end{pmatrix} : M \text{ is } k \times k \right\},$$
(2.6)
$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} : E \in \mathfrak{u}(k) \right\}.$$

Similarly V, which maps $\mathbf{C}^{a,b}$ to $\mathbf{C}^{k,k}$, splits as $\mathfrak{g}_{-1} + \mathfrak{g}_1$ where

(2.7) $\mathfrak{g}_{-1} = \{ z \in V : z : \mathbf{C}^{a,b} \to W_{-1} \}$ and $\mathfrak{g}_1 = \{ z \in V : z : \mathbf{C}^{a,b} \to W_1 \}$.

Now we have a vector space direct sum decomposition

(2.8) $$\mathfrak{u}(k, k; a, b) = \sum \mathfrak{g}_i \quad \text{with } \mathfrak{g}_i = 0 \text{ for } |i| > 2$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, in general, $H: \mathfrak{g}_i \times \mathfrak{g}_j \to (\mathfrak{g}_{i+j})_{\mathbf{C}}$ for i, j odd. So the graded Lie algebra product also satisfies $[\mathfrak{g}_i, \mathfrak{g}_j]_G \subset \mathfrak{g}_{i+j}$, refining the $\mathbf{Z}_2$-grading to a $\mathbf{Z}$-grading.

2B. *Bounded homogeneous domains*. We now show how every bounded homogeneous domain in complex euclidean space gives a $\mathbf{Z}$-graded algebra of the type of (2.8).

Recall the structure of bounded homogeneous domains and their automorphism algebras from Pjateckii-Shapiro **[20]**, Murakami **[18]** and Takeuchi **[30]**. Start with Ω: a nonempty convex cone in $\mathbf{R}^n$ that does not contain a line and F: a hermitian map $\mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}^n$ such that each $F(w, w) \in \bar{\Omega}$ with $F(w, w) = 0$ only for $w = 0$. These data define a *Siegel domain of Type* II:

(2.9) $$D = \{ (z, w) \in \mathbf{C}^n \oplus \mathbf{C}^m : \operatorname{Im} z - F(w, w) \in \Omega \}.$$

D is called a *tube domain* or *Siegel domain of Type* I if $m = 0$, i.e., if D is the tube $\mathbf{R}^n + i\Omega$ over Ω. In general the *tube part* of D is given by $w = 0$. D is called *homogeneous* if

$$G(D) = \{\text{complex analytic diffeomorphisms of } D\},$$

which is a Lie group, acts transitively on D. If D is homogeneous then

$$A(D) = \{ g \in G(D) : g \text{ is affine on } \mathbf{C}^n \oplus \mathbf{C}^m \}$$

is already transitive. Now the point of all this is that every bounded homogeneous domain (in complex euclidean space) is analytically equivalent to a Siegel domain of Type II.

Fix a Siegel domain D as in (2.9). Then its affine automorphism group $A(D)$ has Lie algebra

(2.10) $$\mathfrak{a}(D) = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

where

(i) $\mathfrak{g}_0$ consists of the linear vector fields $Az \cdot \partial/\partial z + Bw \cdot \partial/\partial w$ that preserve Ω and F, i.e., $A: \mathbf{C}^n \to \mathbf{C}^n$ is real-linear and each $\exp(tA)\Omega = \Omega$, $B: \mathbf{C}^m \to \mathbf{C}^m$

is complex-linear, and $AF(u, v) = F(Bu, v) + F(u, Bv)$ for all $u, v \in \mathbf{C}^m$;

(ii) $\mathfrak{g}_1$ consists of the infinitesimal affine vector fields $2iF(w, b)\cdot\partial/\partial z + b\cdot\partial/\partial w$, $b \in \mathbf{C}^m$; and

(iii) $\mathfrak{g}_2$ consists of the infinitesimal real translation $a\cdot\partial/\partial z$, $a \in \mathbf{R}^n$.

If D is homogeneous, then its anlaytic automorphism group $G(D)$ has Lie algebra

$$\mathfrak{g}(D) = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{a}(D) = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \tag{2.11}$$

where, writing $p_{r,s}$ to denote a polynomial of degree r in z and s in w,

(iv) $\mathfrak{g}_{-1}$ consists of all $\xi = p_{1,1}\cdot\partial/\partial z + \{p_{1,0} + p_{0,2}\}\cdot\partial/\partial w$ such that $[\xi, \mathfrak{g}_2] \subset \mathfrak{g}_1$ and $[\xi, \mathfrak{g}_1] \subset \mathfrak{g}_0$; and

(v) $\mathfrak{g}_{-2}$ consists of all $\eta = p_{2,0}\cdot\partial/\partial z + p_{1,1}\cdot\partial/\partial w$ such that $[\eta, \mathfrak{g}_1] \subset \mathfrak{g}_{-1}$ and such that, if $\tau \in \mathfrak{g}_2$ then $[\xi, \tau] = Az\cdot\partial/\partial z + Bw\cdot\partial/\partial w \in \mathfrak{g}_0$ with $\operatorname{Im}\operatorname{Trace}\left(\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right) = 0$.

Fix $\theta \in \mathfrak{g}_0$ corresponding to $A = 0$ and $B = -iI$,

$$\theta = -iw\cdot\partial/\partial w \tag{2.12}$$

and identify $\mathfrak{g}_1$ with $\mathbf{C}^m$ under

$$\mathbf{C}^m \ni b \mapsto \xi_b = iF(w, b)\cdot\frac{\partial}{\partial z} + \frac{1}{2}b\cdot\frac{\partial}{\partial w}. \tag{2.13}$$

The map $\operatorname{ad}(\theta)\colon \mathfrak{g}(D) \to \mathfrak{g}(D)$ preserves each $\mathfrak{g}_i$, so it maps ξ_b to some $\xi_{b'}$, and $[\theta, b\cdot\partial/\partial w] = ib\cdot\partial/\partial w$ shows $b' = ib$. By direct calculation

$$\left[\theta, p_{11}\cdot\frac{\partial}{\partial z}\right] = ip_{11}\cdot\frac{\partial}{\partial z}, \quad \left[\theta, p_{10}\cdot\frac{\partial}{\partial w}\right] = ip_{10}\cdot\frac{\partial}{\partial w},$$

$$\left[\theta, p_{02}\cdot\frac{\partial}{\partial w}\right] = -ip_{02}\cdot\frac{\partial}{\partial w},$$

so $\operatorname{ad}(\theta)^2\xi = -\xi$ for $\xi \in g_{-1}$. And θ commutes with every vector field of the form $p(z)\cdot\partial/\partial z + B(z)w\cdot\partial/\partial w$, in particular, with every element $\mathfrak{g}_{-2} + \mathfrak{g}_0 + \mathfrak{g}_2$. In summary, using (2.13),

2.14. Lemma. *$\operatorname{ad}(\theta)$ is trivial on $\mathfrak{g}_{\text{even}} = \mathfrak{g}_{-2} + \mathfrak{g}_0 + \mathfrak{g}_2$, is multiplication by i on $\mathfrak{g}_1$, and has square -1 on $\mathfrak{g}_{-1}$.*

Now we use $\operatorname{ad}(\theta)$ as complex structure on $\mathfrak{g}_{\text{odd}} = \mathfrak{g}_{-1} + \mathfrak{g}_1$. This is consistent with (2.13).

2.15. Theorem. *Define $H\colon \mathfrak{g}_{\text{odd}} \times \mathfrak{g}_{\text{odd}} \to (\mathfrak{g}_{\text{even}})_{\mathbf{C}}$ by $H(\xi, \eta) = [[\theta, \xi], \eta] + i[\xi, \eta]$. Then H is a $\mathfrak{g}_{\text{even}}$-equivariant hermitian form whose imaginary part is the Lie algebra bracket, and $H|_{\mathfrak{g}_1\times\mathfrak{g}_1} = -F$ in the identification* (2.13).

Proof. If $\xi, \eta \in \mathfrak{g}_{\text{odd}}$, then $[\xi, \eta] \in \mathfrak{g}_{\text{even}}$, so $[[\xi, \eta], \theta] = 0$ and thus $[[\theta, \xi], \eta] = [[\theta, \eta], \xi]$ by the Jacobi identity on $\mathfrak{g}(D)$. It follows that $H(\eta, \xi)^- = H(\xi, \eta)$. Also, since $\operatorname{ad}(\theta)$ is the complex structure on $\mathfrak{g}_{\text{odd}}$,

$$H(\operatorname{ad}(\theta)\xi, \eta) = \left[\operatorname{ad}(\theta)^2\xi, \eta\right] + i\left[\operatorname{ad}(\theta)\xi, \eta\right] = iH(\xi, \eta)$$

shows $H(\xi, \eta)$ to be **C**-linear in ξ. We have proved H to be hermitian, and $\mathfrak{g}_{\text{even}}$-equivariance follows by Jacobi and $[\theta, \mathfrak{g}_{\text{even}}] = 0$.

Finally, if $u, v \in \mathbf{C}^m$ and $\xi_u, \xi_v \in \mathfrak{g}_1$ are the elements corresponding as in (2.13),

$$[\xi_u, \xi_v] = \frac{1}{2} iF(u, v) \cdot \frac{\partial}{\partial z} - \frac{1}{2} iF(v, u) \cdot \frac{\partial}{\partial z} = -\operatorname{Im} F(u, v) \cdot \frac{\partial}{\partial z}.$$

So the hermitian forms H and $-F$ on $\mathfrak{g}_1 \times \mathfrak{g}_1$ have the same imaginary part, and thus are the same. Q.E.D.

Looking back to (1.5), and applying ordinary Jacobi identity to the $[\operatorname{Im} H(\xi, \eta), \zeta]$, we see that H gives us a graded Lie algebra structure if and only if

$$(2.16)\qquad [[[\theta, \xi], \eta], \zeta] + [[[\theta, \eta], \zeta], \xi] + [[[\theta, \zeta], \xi], \eta] = 0 \quad \text{for } \xi, \eta, \zeta \in \mathfrak{g}_{\text{odd}}.$$

Monique Parker observed that (2.16) can never hold when $\theta = [\operatorname{ad}(\theta)\xi, \xi]$ for some $\xi \in \mathfrak{g}_{\text{odd}}$, for then

$$[[[\theta, \xi], \xi], \xi] = [\theta, \xi] \neq 0.$$

She further noted, in view of this, that $\mathfrak{sl}(2; \mathbf{R})$ carries a hermitian Lie algebra structure

$$\mathfrak{l} \text{ spanned by } \theta = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V \cong \mathbf{C} \text{ under } \begin{pmatrix} y & x \\ x & -y \end{pmatrix} \leftrightarrow x + iy,$$

$$H: V \times V \to \mathfrak{l}_{\mathbf{C}} \text{ by } H(\xi, \eta) = [[\theta, \xi], \eta] + i[\xi, \eta],$$

for which the graded Jacobi Identity (1.4) fails.

Despite this, (2.16) holds trivially when $\xi, \eta, \zeta \in \mathfrak{g}_1$, and so we do have a graded Lie algebra structure on the infinitesimal affine automorphism algebra $\mathfrak{a}(D) = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$.

2C. *Nilradicals of parabolic subalgebras.* Let F denote one of the fields **R** (real), **C** (complex) or **Q** (quaternion). Given $k, l \geqslant 0$ let $\mathbf{F}^{k,l}$ denote the right vector space $\mathbf{F}^{k+l}$ with hermitian scalar product $\langle z, w\rangle = -\sum_1^k z_j \bar{w}_j + \sum_{k+1}^{k+l} z_j \bar{w}_j$. $U(k, l; \mathbf{F})$ denotes the unitary group of $\mathbf{F}^{k,l}$ and $\mathfrak{u}(k, l; \mathbf{F})$ is its Lie algebra.

We recall some information from [**38**, §§2 and 3] concerning the structure of parabolic subgroups. A subspace $E \subset \mathbf{F}^{k,l}$ is *totally isotropic* if $\langle E, E\rangle = 0$. The maximal parabolic subgroups of $U(k, l; \mathbf{F})$ are the

$$P_E = \{g \in U(k, l; \mathbf{F}): gE = E\},$$

E totally isotropic nonzero. Here there are $\min(k, l)$ conjugacy classes as

$s = \dim E$ runs from 1 to $\min(k, l)$. Denote

$$\mathfrak{n}_E\text{: Lie algebra of the unipotent radical of } P_E.$$

Then $\mathfrak{n}_E$ is isomorphic to the algebra $\mathfrak{n}_{s;\, k-s, l-s}(\mathbf{F})$ defined below.

$\mathbf{F}^{a\times b}$ denotes the real vector space of $a \times b$ matrices over $\mathbf{F}$. If $A \in \mathbf{F}^{a\times b}$ then $A^* \in \mathbf{F}^{b\times a}$ denotes its conjugate (over $\mathbf{R}$) transpose. $\mathbf{F}^{s\times(t,u)}$ denotes $\mathbf{F}^{s\times t} \oplus \mathbf{F}^{s\times u}$ with the map $H\colon \mathbf{F}^{s\times(t,u)} \times \mathbf{F}^{s\times(t,u)} \to \mathbf{F}^{s\times s}$ by

$$H((Z', Z''), (W', W'')) = \tfrac{1}{2}\{Z'W'^* - Z''W''^*\}. \tag{2.17}$$

Split $\mathbf{F}^{s\times s} = \operatorname{Re} \mathbf{F}^{s\times s} \oplus \operatorname{Im} \mathbf{F}^{s\times s}$ under the projections

$$\operatorname{Re} D = \tfrac{1}{2}(D + D^*) \quad \text{and} \quad \operatorname{Im} D = \tfrac{1}{2}(D - D^*).$$

Then, by definition,

$$\mathfrak{n}_{s;t,u}(\mathbf{F}) = \operatorname{Im} \mathbf{F}^{s\times s} + \mathbf{F}^{s\times(t,u)}$$
$$\text{with } [(D, Z), (E, W)] = (\operatorname{Im} H(Z, W), 0).$$

Now H is $\mathbf{F}$-hermitian here, the double Jacobi identity (1.5) is trivially satisfied, and so $\mathfrak{n}_E$ acquires a good "hermitian Lie algebra structure" over $\mathbf{F}$.

3. Minimal orbits. In this section we study the $2(m-1)$-dimensional orbits of $U(k, l)$, $k + l = m$, on the dual of its Lie algebra. We describe them in terms of the map H_0 of (1.6). This description leads to an easy analysis of the transitivity properties of parabolic subgroups. It also starts us on the identification of the associated unitary representations.

Identify the Lie algebra $\mathfrak{u}(k, l)$ with its real dual space $\mathfrak{u}(k, l)^*$ by

$$\xi \leftrightarrow f_\xi \quad \text{where } f_\xi(\eta) = \operatorname{trace}(\xi\eta). \tag{3.1}$$

Of course, if $\xi, \eta \in \mathfrak{su}(k, l)$, then their inner product under the Cartan-Killing form is $2m \cdot \operatorname{trace}(\xi\eta)$. The identification (3.1) is equivariant for adjoint and coadjoint representations, and so preserves orbit structure. We then examine the analogous construction for the symplectic group $\mathrm{Sp}(m; \mathbf{R})$ and relate the two.

3.2. Lemma. *Let A be a nonscalar $m \times m$ complex matrix and $Z(A)$ its centralizer in* $\mathrm{GL}(m; \mathbf{C})$. *Then* $\dim_{\mathbf{C}} Z(A) \leqslant m^2 - 2m + 2$ *with equality if and only if* $A = B + \lambda I$ *with* $\lambda \in \mathbf{C}$ *and B of rank* 1.

Proof. Split $\mathbf{C}^m = \sum V_j$, sum of the generalized eigenspaces of A. Each V_j is $Z(A)$-stable, so $\dim_{\mathbf{C}} Z(A) \leqslant \sum(\dim_{\mathbf{C}} V_j)^2$, which is $< (m-1)^2 + 1$ unless (i) there is just one V_j or (ii) there are just two V_j, and one of them has dimension 1.

We need the elementary fact: if E is a matrix in Jordan normal form, and if E' is obtained from E by changing an off-diagonal 1 to 0, then $\dim_{\mathbf{C}} Z(E) < \dim_{\mathbf{C}} Z(E')$.

In case (i) above, $A = B + \lambda I$ where $\lambda \in \mathbf{C}$ and B is nilpotent. By the elementary fact just mentioned, the dimension of $Z(A) = Z(B)$ is maximized when B has rank 1. That maximum is $m^2 - 2m + 2$.

In case (ii) above, $A = (B + \lambda I) + N$ where $\lambda \in \mathbf{C}$, B is semisimple and rank 1, N is nilpotent and $BN = NB$. Then N annihilates the nonzero eigenspace of B and preserves the zero eigenspace, so $Z(A) = Z(B) \cap Z(N)$. By the elementary fact above, the dimension is maximized when $N = 0$, and that maximum is $m^2 - 2m + 2$. Q.E.D.

If $\xi \in \mathfrak{u}(k, l)$, viewed as an $m \times m$ complex matrix where $m = k + l$, then the complex dimension of the $GL(m; \mathbf{C})$-centralizer of ξ is equal to the real dimension of the $U(k, l)$-centralizer. For that $GL(m; \mathbf{C})$-centralizer is a complex Lie group stable under complex conjugation of $GL(m; \mathbf{C})$ over $U(k, l)$. In view of the correspondence (3.1), now Lemma 3.2 specializes to

3.3. LEMMA. *The nontrivial coadjoint orbits* $\mathrm{Ad}^*(U(k, l)) \cdot f_\xi$ *have (real) dimension* $\geqslant 2m - 2$, *with equality if and only if* $\xi = \xi_0 + \lambda I$ *where* $\lambda \in \mathbf{C}$ *and* ξ_0 *has rank* 1 *as* $m \times m$ *matrix.*

If $m > 1$ and $(k, l) \neq (1,1)$, then $\xi_0 \in \mathfrak{u}(k, l)$ and $\lambda \in i\mathbf{R}$ in Lemma 3.3. For $m - 1 > \min(k, l)$, so the kernel of ξ_0 contains a nonisotropic vector v, $\xi(v) = \lambda v$ forces $\lambda \in i\mathbf{R}$ and thus $\lambda I \in \mathfrak{u}(k, l)$, and then $\xi_0 = \xi - \lambda I \in \mathfrak{u}(k, l)$. And, of course, if $(k, l) = (1, 1)$ then all nontrivial coadjoint orbits have dimension $2m - 2 = 2$.

Now observe the connection with the map H_0 of (1.6):

3.4. LEMMA. *The rank* 1 *elements of* $\mathfrak{u}(k, l)$ *are the* ξ_u: $w \mapsto i\langle w, u\rangle u$, $0 \neq u \in \mathbf{C}^{k,l}$, *and the* $-\xi_u$.

PROOF. $\xi_u = H_0(u, u)$, which belongs to $\mathfrak{u}(k, l)$ because H_0 is hermitian. And evidently ξ_u has rank 1. Conversely let $\xi \in \mathfrak{u}(k, l)$ have rank 1 and let v span its range. If $w \perp v$ then $\langle \xi w, x\rangle = -\langle w, \xi x\rangle = 0$ for all $x \in \mathbf{C}^{k,l}$, so ξ annihilates $v^\perp$. Now $\xi(x) = c\langle x, v\rangle v$ for some $0 \neq c \in \mathbf{C}$, and $\langle \xi x, x\rangle \in i\mathbf{R}$ forces $c \in i\mathbf{R}$, so $c = \pm ir^2$ with r real. Thus $\xi = \pm \xi_{rv}$. Q.E.D.

Define $U(k, l)$-equivariant maps

(3.5a) $\mathcal{Q}_\lambda: \mathbf{C}^{k,l} \to \mathfrak{u}(k, l)^*$ by $\mathcal{Q}_\lambda(u)(\xi) = i(\langle \xi u, u\rangle + \lambda \operatorname{trace}(\xi))$.

Since $\xi\xi_u$: $w \mapsto i\langle w, u\rangle\xi(u)$ has trace $i\langle \xi u, u\rangle$, (3.5a) is equivalent to

(3.5b) $\mathcal{Q}_\lambda(u) = f_{\xi_u} + i\lambda \operatorname{trace}$ where $\xi_u = H_0(u, u)$: $w \mapsto i\langle w, u\rangle u$.

Now we can state our orbit structure theorem.

3.6. THEOREM. *Suppose* $m > 1$ *and* $(k, l) \neq (1,1)$. *Then the nontrivial coadjoint orbits of* $U(k, l)$ *on* $\mathfrak{u}(k, l)^*$, *of minimal dimension* $2m - 2$, *are the*

$$\mathcal{O}_{\pm 1, r, \lambda} = \pm \mathcal{Q}_\lambda\left(\left\{u \in \mathbf{C}^{k,l}: u \neq 0, \|u\|^2 = r\right\}\right) = \mathcal{O}_{\pm 1, r, 0} \oplus (\pm i\lambda \operatorname{trace})$$

for r, λ real. Orbits $\mathcal{O}_{\mathcal{E},r,\lambda} = \mathcal{O}_{\mathcal{E}',r',\lambda'}$ just when $(\mathcal{E}, r, \lambda) = (\mathcal{E}', r', \lambda')$, $\mathcal{O}_{\pm 1,r,1}$ is an elliptic semisimple orbit for $r \neq 0$ and $\mathcal{O}_{\pm 1,0,0}$ is a nilpotent orbit. The stability groups on these orbits are given by $g \in U(k, l)$ fixes $\pm Q_\lambda(u)$ if and only if $g(u) = au$ with $|a| = 1$.

PROOF. According to the comment just after (3.5a), $Q_0(u) = f_{\xi_u}$ and $Q_\lambda(u) = Q_0(u) + i\lambda$ trace. Lemma 3.4 now shows that the coadjoint orbits of minimal dimension $2m - 2$ are those of the $\pm Q_\lambda(u)$, $0 \neq u \in \mathbf{C}^{k,l}$. A nonzero orbit $U(k, l) \cdot v$ is specified by $\|v\|^2$, so equivariance tells us that the coadjoint orbits of minimal dimension $2m - 2$ are just the $\mathcal{O}_{\pm 1,r,\lambda}$.

Suppose $\mathcal{O}_{\mathcal{E},r,\lambda} = \mathcal{O}_{\mathcal{E}',r',\lambda'}$. Then we have nonzero $u, v \in \mathbf{C}^{k,l}$ with $\|u\|^2 = r$, $\|v\|^2 = r'$ and $\mathcal{E} Q_\lambda(u) = \mathcal{E}' Q_{\lambda'}(v)$. Since $m > 1$ and $(k, l) \neq (1, 1)$, the comment after Lemma 3.3 gives $\mathcal{E}\xi_u = \mathcal{E}'\xi_v$. Now $\mathcal{E}\langle w, u\rangle u = \mathcal{E}'\langle w, v\rangle v$ for all $w \in \mathbf{C}^{k,l}$. Thus $u = tv$ where $0 \neq t \in \mathbf{C}$, and $|t|^2 \mathcal{E}\langle w, v\rangle v = \mathcal{E}\langle w, u\rangle u = \mathcal{E}'\langle w, v\rangle$ gives $|t|^2 = 1$ and $\mathcal{E} = \mathcal{E}'$. Now $\mathcal{E} = \mathcal{E}'$ and $r = r'$, so

$$\mathcal{E}(r + \lambda m) = \mathcal{E} Q_\lambda(u)(-iI) = \mathcal{E}' Q_{\lambda'}(v)(-iI) = \mathcal{E}'(r' + \lambda' m)$$

implies $\lambda = \lambda'$.

If u is isotropic, then $\pm Q_0(u) = \pm \xi_u$ has square zero, so the orbit $\mathcal{O}_{\pm 1,0,0}$ is nilpotent. If u is not isotropic, then

$$\operatorname{trace}(\xi_u^2) = i\langle iu\langle u, u\rangle, u\rangle = -\langle u, u\rangle^2 < 0,$$

so $\pm\xi_u$ are elliptic semisimple elements of $\mathfrak{u}(k, l)$, and it follows that every $\pm Q_\lambda(u)$ is elliptic semisimple.

Finally, $g \in U(k, l)$ sends $\pm Q_\lambda(u)$ to $\pm Q_\lambda(gu)$. We saw that $\mathcal{E} Q_\lambda(u) = \mathcal{E}' Q_{\lambda'}(v)$ implies $\mathcal{E} = \mathcal{E}'$, $\lambda = \lambda'$ and $v = au$ with $|a|^2 = 1$. In particular, g fixes $\pm Q_\lambda(u)$ just when $g(u) = au$ with $|a|^2 = 1$. Q.E.D.

The situation is slightly different for $U(1, 1)$, but there it is immediate from standard results on $SL(2; \mathbf{R}) \cong SU(1, 1)$: *The nontrivial coadjoint orbits of $U(1, 1)$ on $\mathfrak{u}(1, 1)^*$ all have minimal dimension* 2, *and they are* (i) *the elliptic orbits*

$$\mathcal{O}_{\pm 1,r,\lambda} = \pm Q_\lambda\left(\left\{u \in \mathbf{C}^{1,1}: \|u\|^2 = r\right\}\right)$$

where $\lambda, r \in \mathbf{R}$ and $r > 0$, (ii) *the essentially parabolic orbits*

$$\mathcal{O}_{\pm 1,0,\lambda} = \pm Q_\lambda\left(\left\{u \in \mathbf{C}^{1,1}: u \neq 0 \text{ and } \|u\|^2 = 0\right\}\right),$$

where $\lambda \in \mathbf{R}$, and (iii) *the essentially hyperbolic orbits*

$$\left\{f_\xi: \xi \in \mathfrak{su}(1, 1) \text{ and } \det(\xi) = r\right\} \otimes (i\lambda \operatorname{trace})$$

where $\lambda, r \in \mathbf{R}$ and $r < 0$.

The maximal parabolic subgroups of $U(k, l)$ are the

(3.7a) $$P_E = \{ g \in U(k, l) \colon gE = E \},$$
E nonzero totally isotropic subspace of $\mathbf{C}^{k,l}$.

The maximal unimodular subgroup of P_E is

(3.7b) $$P'_E = \{ g \in U(k, l) \colon gE = E \text{ and } |\det(g|_E)| = 1 \}.$$

These groups are near to transitive on the coadjoint orbits $\mathcal{O}_{\pm 1,r,\lambda}$. In fact, setting $S_r = \{u \in \mathbf{C}^{k,l} \colon u \neq 0 \text{ and } \|u\|^2 = r\}$,

3.8. PROPOSITION. *The coadjoint $U(k, l)$-orbit $\mathcal{O}_{\pm 1,r,\lambda} = \pm Q_\lambda(S_r)$ breaks into three P_E-orbits, each in the closure of the next nonempty one*:

(i) $\pm Q_\lambda(S_r \cap E)$,

(ii) $\pm Q_\lambda((S_r \cap E^\perp) \setminus (S_r \cap E))$, *and*

(iii) $\pm Q_\lambda(S_r \setminus (S_r \cap E^\perp))$,

which are nonempty except for (i) *when* $r \neq 0$, (ii) *when* $r < 0$ *and* $\dim E = k$, (ii) *when* $r > 0$ *and* $\dim E = l$, (ii) *when* $r = 0$ *and* $\dim E = \min(k, l)$. *In particular, if E is a maximal totally isotropic subspace of* $\mathbf{C}^{k,l}$, *then* $\mathcal{O}_{\pm 1,r,\lambda}$ *breaks into two P_E-orbits*:

if $r \neq 0$: $\pm Q_\lambda(S_r \cap E^\perp)$ *and* $\pm Q_\lambda(S_r \setminus (S_r \cap E^\perp))$;

if $r = 0$: $\pm Q_\lambda(S_0 \cap E)$ *and* $\pm Q_\lambda(S_0 \setminus (S_0 \cap E))$,

and further, $\pm Q_\lambda(S_r \cap E^\perp)$ *is empty when* $r < 0$ *and* $k \lessdot l$, *and when* $r > 0$ *and* $k \gtrdot l$, *in particular, when* $r \neq 0$ *and* $k = l$.

P'_E *is transitive on all of these orbits except for* (i) *and* (iii) *in the case where* $\dim E = 1$.

PROOF. The corresponding facts for the action of P_E and P'_E on S_r come from Witt's Theorem. Apply $\pm Q_\lambda$. Q.E.D.

Let us complete the results of Proposition 3.8 with respect to $U(1, 1)$ and the essentially hyperbolic orbits. Here $\dim E = 1$ necessarily, and P_E has two orbits on $\{f_\xi \colon \xi \in \mathfrak{su}(1, 1), \det(\xi) = r\} \otimes (i\lambda \text{ trace})$, $r < 0$, given by $\xi \in \mathfrak{p}_E$ and $\xi \notin \mathfrak{p}_E$, the first sitting in the closure of the second. To see this, view $\mathfrak{su}(1, 1)$ as $\mathbf{R}^{1,2}$ under the Killing form, note that the Killing form corresponds to a negative multiple of det, and observe that $\mathrm{Ad}(U(1, 1))$ goes over to $\mathrm{SO}(1, 2)$.

$U(k, l)$ acts, as a subgroup of $\mathrm{GL}(m; \mathbf{C})$, on the projective space $P^{m-1}(\mathbf{C})$ associated to $\mathbf{C}^{k,l}$. Its orbits are: an open orbit consisting of the negative definite lines, a closed orbit consisting of the isotropic lines, and an open orbit consisting of the positive definite lines. If $r \neq 0$ then $\pm Q_\lambda(u) \mapsto u\mathbf{C}$ is a G-equivariant bijection of $\mathcal{O}_{\pm 1,r,\lambda}$ onto the negative (if $r < 0$) or positive (if $r > 0$) orbit in $P^{m-1}(\mathbf{C})$. This puts a G-invariant complex structure on the minimal nontrivial elliptic coadjoint orbit $\mathcal{O}_{\pm 1,r,\lambda}$. If $s = |r|^{1/2} > 0$ then $\pm Q_\lambda(se_1)$ represents $\mathcal{O}_{\pm 1,r,\lambda}$ $(r < 0)$ and $\pm Q_\lambda(se_m)$ represents $\mathcal{O}_{\pm 1,r,\lambda}$ $(r > 0)$, so we also have indefinite hermitian symmetric space descriptions

$$\begin{aligned} \mathcal{O}_{\pm 1,-s^2\lambda} &\cong U(k,l)/U(1) \times U(k-1,l), \\ \mathcal{O}_{\pm 1,s^2\lambda} &\cong U(k,l)/U(k,l-1) \times U(1). \end{aligned} \tag{3.9}$$

These provide the same complex structures.

Define $\mathcal{E}_j \in \mathfrak{u}(k,l)^*_{\mathbf{C}}$ by $\mathcal{E}_j(\xi) = \xi_{jj}$, jth diagonal entry of ξ. So a simple root system of $\mathfrak{u}(k,l)_{\mathbf{C}}$, relative to the diagonal Cartan subalgebra, is given by the restriction of the $\mathcal{E}_j - \mathcal{E}_{j+1}$, $1 \leqslant j \leqslant m-1$. Now notice

$$\begin{cases} \text{if } r < 0\text{: } Q_\lambda(se_1) = i\left(r\mathcal{E}_1 + \lambda\sum_1^m \mathcal{E}_j\right) & \text{where } s = |r|^{1/2}, \\ \text{if } r > 0\text{: } Q_\lambda(se_m) = i\left(r\mathcal{E}_m + \lambda\sum_1^m \mathcal{E}_j\right) & \text{where } s = |r|^{1/2}. \end{cases} \tag{3.10}$$

If $r < 0$, now $\exp(\pm iQ_\lambda(se_1))$ is a well-defined character on $U(1) \times U(k-1,l)$ if and only if the coefficients of $\mathcal{E}_1$ and of $\sum_1^m \mathcal{E}_j$ are integers, i.e., $\lambda + r$ and λ are integers. Same situation for $r > 0$. So we have the integrability condition:

3.11. LEMMA. *Let $v \in \mathbf{C}^{k,l}$ with $\|v\|^2 = r \neq 0$, and let $U(k,l)_v$ denote the isotropy subgroup of $U(k,l)$ at $\pm Q_\lambda(v) \in \mathcal{O}_{\pm 1,r,\lambda}$. Then $\pm iQ_\lambda(v)$ integrates to a well-defined unitary character on $U(k,l)_v$ if, and only if, both r and λ are integers.*

Assume the integrality condition of Lemma 3.11 for the minimal coadjoint orbit $\mathcal{O}_{\pm 1,r,\lambda}$. Associated to the character on the isotropy subgroup, we have a G-homogeneous hermitian complex line bundle $\mathbf{L} \to \mathcal{O}_{\pm 1,r,\lambda}$. Using the complex structure, which simply is a choice of totally complex polarization, the method of [**40**, Lemma 7.1.4] gives us the structure of G-homogeneous holomorphic line bundle on $\mathbf{L} \to \mathcal{O}_{\pm 1,r,\lambda}$. Of course, the natural symplectic form on $\mathcal{O}_{\pm 1,r,\lambda}$ is the curvature of that bundle. This prequantizes $\mathcal{O}_{\pm 1,r,\lambda}$ for $r \neq 0$.

By contrast, the essentially-nilpotent orbits $\mathcal{O}_{\pm 1,r,\lambda}$ are not amenable to prequantization. In fact,

3.12. LEMMA. *If $m > 1$ and $(k,l) \neq (1,1)$, then $\mathcal{O}_{\pm 1,0,\lambda}$ does not admit a polarization.*

PROOF. Let $\mathfrak{q} \subset \mathfrak{u}(k,l)_{\mathbf{C}}$ be a complex polarization for $\mathcal{O}_{\pm 1,0,\lambda}$. Then $\mathfrak{q}$ is a complex polarization for $\mathcal{O}_{\pm 1,0,0} = \mathrm{Ad}^*(U(k,l)) \cdot (\pm Q_0(e_1 + e_m))$. Following Ozeki and Wakimoto [**19**], $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{u}(k,l)_{\mathbf{C}}$. Since $f_{e_1+e_m}$ is nilpotent, a result of Rothschild and Wolf [**27**] says that the functionals in $\mathcal{O}_{\pm 1,0,0}$ annihilate $\mathfrak{q}$, and then a result of Wolf [**41**] says $\mathfrak{q} = \mathfrak{p}_{\mathbf{C}}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{u}(k,l)$. Now $\mathfrak{p}$ is contained in a maximal parabolic subalgebra of $\mathfrak{u}(k, l)$. According to (3.7) that maximal one has form $\mathfrak{p}_E = \{\xi \in \mathfrak{u}(k, l): \xi E \subset E\}$ where $E \subset \mathbf{C}^{k,l}$ is a totally isotropic subspace of some dimension s, $1 \leqslant s \leqslant [m/2]$. Also [38, Theorem 2.10], $\dim \mathfrak{p}_E = m^2 - 2ms + 3s^2$. Now $\mathfrak{p}$ has codimension $\geqslant 2ms - 3s^2$ in $\mathfrak{u}(k, l)$. Since it polarizes an orbit of dimension $2(m-1)$, $\mathfrak{p}$ has codimension exactly $m - 1$. Now $m - 1 \geqslant 2ms - 3s^2$ with $1 \leqslant s \leqslant [m/2]$. The only solution is $s = 1$, $m = 2$. That is just the excluded case $(k, l) = (1, 1)$. Q.E.D.

We now examine the relation between the minimal orbits of $\mathfrak{u}(k, l)$ and the graded Lie algebra structure of $\mathfrak{u}(k, l) + \mathbf{C}^{k,l}$ on the one hand, and similar constructions for the symplectic algebras $\mathfrak{sp}(k + l, \mathbf{R})$ on the other. Let V be $\mathbf{R}^{2n}$ with symplectic form $\{\ ,\ \}$, let $\mathrm{Sp}(V)$ be the group of all real linear transformations which preserve $\{\ ,\ \}$ and let $\mathfrak{sp}(V)$ denote the Lie algebra of $\mathrm{Sp}(V)$. We define a symmetric linear map $V \otimes V \to \mathfrak{sp}(V)$ sending $u \otimes v \to [u, v]_G$ where

$$2[u, v]_G w = \{u, w\}v + \{v, w\}u. \tag{3.13}$$

For $\xi \in \mathfrak{sp}(V)$ and $u \in V$ we define

$$[\xi, u]_G = -[u, \xi]_G = \xi u,$$

while for ξ and $\eta \in \mathfrak{sp}(V)$ we set $[\xi, \eta]_G$ equal to the usual commutator bracket: $[\xi, \eta]_G = [\xi, \eta]$. As pointed out in [4], this makes $\mathfrak{sp}(V) + V$ into a $\mathbf{Z}_2$ graded Lie algebra, where $\mathfrak{sp}(V)$ is the even part and V is the odd part. In fact, this is a real form of one of the infinite simple algebras in the Kac-Kaplansky classification. (Over the complex numbers, this family of algebras has the distinguishing property that, among all finite dimensional graded Lie algebras, it is the only one with nontrivial odd part such that all finite dimensional representations are completely reducible; cf. [5] for this result.) We can consider the quadratic map of $V \to \mathfrak{sp}(V)$ associated with the symmetric map $[\ ,\ ]_G$, i.e. the map $v \to [v, v]_G$. We also have the map Φ: $\mathfrak{sp}(V) \to \mathfrak{sp}(V)^*$ given by

$$\Phi(\xi)\eta = \tfrac{1}{2}\mathrm{tr}_{\mathbf{R}}\xi\eta$$

where $\mathrm{tr}_{\mathbf{R}}$ denotes the usual trace of a real linear transformation of V. The subscript $\mathbf{R}$ emphasizes that we are computing the trace over the real numbers. In the situation that we have been considering previously in this section, V also has the structure of a complex vector space, and we can consider complex linear transformations, and the corresponding trace over the complex numbers which we temporarily denote by $\mathrm{tr}_{\mathbf{C}}$. For a complex linear transformation, ζ, the relation between these two traces is clearly given by

$$\tfrac{1}{2}\mathrm{tr}_{\mathbf{R}}\zeta = \mathrm{Re}(\mathrm{tr}_{\mathbf{C}}\zeta).$$

(We shall have use of the relation between the two traces in describing the relations between the minimal orbits of the unitary and the symplectic algebras.) We shall denote the composite map $v \mapsto \Phi([v, v]_G)$ of $V \to \mathfrak{sp}(V)^*$ by R. A direct computation shows that

$$R(v)\eta = \tfrac{1}{2}\{v, \eta v\}.$$

The map R clearly commutes with the action of $\mathrm{Sp}(V)$, i.e. $R(av) = aR(v)$, where $aR(v)$ denotes the image of $R(v)$ under the element $a \in \mathrm{Sp}(V)$ acting by the coadjoint representation. Now $\mathrm{Sp}(V)$ acts transitively on $V - \{0\}$, and hence R must map $V - \{0\}$, onto an orbit of $\mathrm{Sp}(V)$ in $\mathfrak{sp}(V)^*$. The orbits in $\mathfrak{sp}(V)^*$ have an induced symplectic structure, and so, of course, does $V - \{0\}$. We claim

3.14. PROPOSITION. *The map R is a symplectic diffeomorphism of $V - \{0\}$ onto a minimal positive dimensional orbit in $\mathfrak{sp}(V)^*$. There exists precisely one other orbit of dimension $2n$, which is the image of $V - \{0\}$ under $-R$. All other nonzero orbits have strictly greater dimension.*

PROOF. It is known (cf. **[10]**) that each of the simple complex algebras other than $\mathfrak{sl}(n)$ has exactly one coadjoint orbit of minimal positive dimension. In the complex space $\mathfrak{sp}(V)^* \otimes \mathbf{C}$ this orbit is the orbit of the element $R(v)$, $v \neq 0$, under the complex symplectic group. It is easy to see that the intersection of this orbit with the real subspace has two components, one consisting of the vectors $R(v)$ and the other consisting of the vectors $-R(v)$, $v \neq 0$. Now $V - \{0\}$ is a symplectic homogeneous space for the group $\mathfrak{sp}(V)$, and hence there exists some symplectic diffeomorphism, S, of $V - \{0\}$ onto one of these two orbits which is equivariant with respect to the action of $\mathrm{Sp}(V)$. Now multiplication by ± 1 commutes with the action of $\mathrm{Sp}(V)$ on $\mathfrak{sp}(V)^*$. Hence $S^{-1}R$ maps $V - \{0\}$ onto itself and commutes with the action of $\mathrm{Sp}(V)$. An elementary argument shows that this implies that $S^{-1}R$ must be multiplication by a nonzero scalar. This implies that the symplectic form induced by R is some constant multiple of the original symplectic form, and all that is left for the proof of the proposition is to establish that this constant equals one. To evaluate the constant, we need only compare the two symplectic forms at one point. Let ω denote the symplectic form on the orbit, let ξ and η be elements of $\mathfrak{sp}(V)$ and let ξ_v and η_v be the corresponding tangent vectors at v. Then, by definition,

$$\omega(\xi_v, \eta_v) = -\langle v, [\xi, \eta]\rangle.$$

Let us choose $\xi \in \mathfrak{sp}(V)$ such that $\xi e = e$ and $\xi f = -f$ and η such that $\eta e = f$ and $\eta f = 0$. Then $[\xi, \eta]_v = -2f$. If ζ is any element of $\mathfrak{sp}(V)$, let ζ_v denote the value of the image vector field of ζ at the point $v \in V$. The equivariance of the map R implies that

$$dR\zeta_v = \zeta_{R(v)}.$$

Now, taking $v = e$,

$$\xi_v = e, \quad \eta_v = f \quad \text{and} \quad [\xi, \eta]_v = -2f.$$

Thus

$$(R^*\omega)_v(e, f) = \omega(dRe, dRf) = \omega(dR\xi_v, dR\eta_v) = -\langle R(v), [\xi, \eta]\rangle$$
$$= -\tfrac{1}{2}\{[\xi, \eta]v, v\} = -\tfrac{1}{2}\{-2f, e\} = 1,$$

proving the proposition.

Now assume that $V = \mathbf{C}^{k,l}$ as above. We have an injection ι: $\mathfrak{u}(k, l) \to \mathfrak{sp}(V)$ and a dual projection π: $\mathfrak{sp}(V)^* \to \mathfrak{u}(k, l)^*$. We have

$$H_0(v, v)w = iv\langle w, v\rangle = iv\{iv, w\} + v\{v, w\}$$

so

$$\iota H_0(v, v) = \tfrac{1}{2}[v, v]_G + \tfrac{1}{2}[iv, iv]_G. \tag{3.15}$$

Thus

$$(\pi\Phi\iota)H_0(v, v)(\xi) = \tfrac{1}{2}\{v, \xi v\} + \tfrac{1}{2}\{iv, \xi iv\} = i\langle \xi v, v\rangle \quad \text{for } \xi \in \mathfrak{u}(k, l)$$

since $\mathrm{Re}\langle \xi u, u\rangle = 0$ for any $\xi \in \mathfrak{u}(k, l)$. We thus have

$$\pi\Phi\iota H_0(v, v) = \pi R(v, v) + \pi R(iv, iv) = Q_0(v). \tag{3.16}$$

4. Reduction of the metaplectic representation. In this section, we use a graded Lie algebra construction to obtain explicit realizations of the metaplectic representations μ of the 2-sheeted cover $\mathrm{Mp}(m; \mathbf{R})$ of the symplectic group $\mathrm{Sp}(m; \mathbf{R})$. In these realizations, we restrict μ to the unitary groups $U(k, l)$, $k + l = m$, proving

(i) the space of μ is the discrete direct sum of the isotypic subspaces for the circle group that is the center of $U(k, l)$,

(ii) the representation ν of $U(k, l)$ on such a subspace is irreducible,

(iii) ν remains irreducible when restricted to certain parabolic subgroups and their maximal unimodular subgroups.

This extends results of Bargmann [2], Itzykson [9], and Mack and Todorov [16]. It is related to work of Gross and Kunze [7] and Gelbart [6].

Fix an integer $m > 0$ and let V denote $\mathbf{R}^{2m}$ together with a nondegenerate antisymmetric bilinear form $\{u, v\}$. This gives us the *Heisenberg group* $H_m = H(V)$,

$$H(V) = V + \mathbf{R} \quad \text{with } (u, s)(v, t) = \left(u + v, s + t + \tfrac{1}{2}\{u, v\}\right),$$

the *real symplectic group* $\mathrm{Sp}(m; \mathbf{R}) = \mathrm{Sp}(V)$,

$$\mathrm{Sp}(V) = \mathrm{Aut}(V) = \left\{g \in \mathrm{GL}(2m; \mathbf{R}) : \{gu, gv\} = \{u, v\}, \text{all } u, v \in \mathbf{R}^{2m}\right\},$$

and the semidirect product obtained by viewing $\mathrm{Sp}(V)$ as all automorphisms of $H(V)$ that act trivially on $\mathbf{R}$,

$$\mathrm{Sp}(V)\cdot H(V) \quad \text{with } (g,u,s)(h,v,t) = \left(gh, h^{-1}(u)+v, s+t+\tfrac{1}{2}\{h^{-1}u, v\}\right).$$

Their respective Lie algebras are

$$\mathfrak{h}_m = \mathfrak{h}(V) = V + \mathbf{R} \quad \text{with } \left[(u,s),(v,t)\right] = (0, \{u,v\}),$$

$$\mathfrak{sp}(m;\mathbf{R}) = \mathfrak{sp}(V) = \left\{\xi \in \mathfrak{gl}(2m;\mathbf{R}) \colon \{\xi u, v\} + \{u, \xi v\} = 0, \text{all } u, v\right\},$$

$$\mathfrak{sp}(V) + \mathfrak{h}(V) \quad \text{with } \left[(\xi,u,s),(\eta,v,t)\right] = \left([\xi,\eta], \xi v - \eta u, \{u,v\}\right).$$

$\mathrm{Sp}(m;\ \mathbf{R}) = \mathrm{Sp}(V)$ admits a unique two-sheeted covering group $\mathrm{Mp}(m;\ \mathbf{R}) = \mathrm{Mp}(V)$, called the *metaplectic group*, and so we also have the semidirect product $\mathrm{Mp}(V)\cdot H(V)$.

The equivalence classes $[\pi]$ of irreducible unitary representations π of $H(V)$ fall into two classes. First, there are the unitary characters

$$\chi_f \colon (u,s) \mapsto e^{if(u)} \quad \text{where } f \in V^*,$$

which are trivial on the center $\mathbf{R}$ of $H(V)$. Then, for $0 \neq t \in \mathbf{R}$, one has a class $[\pi_t]$ with nontrivial central character,

$$\pi_t(u,s) = e^{its}\pi_t(u,0).$$

The π_t are infinite dimensional, and evidently $[\pi_t]$ is stable under the action (of conjugation) of $\mathrm{Sp}(V)$. One tries to extend π_t to a unitary representation $\tilde{\pi}_t$ of $\mathrm{Sp}(V)\cdot H(V)$ on the same Hilbert space: given $g \in \mathrm{Sp}(V)$, $(u,s) \mapsto \pi_t(g(u,s)g^{-1})$ is equivalent to π_t, so there is a unitary operator $\tilde{\pi}_t(g)$, unique up to scalar multiple, such that $\pi_t(g(u,s)g^{-1}) = \tilde{\pi}_t(g)\pi_t(u,s)\tilde{\pi}_t(g)^{-1}$ for all $(u,s) \in H(V)$. One knows [29] that the $\tilde{\pi}_t(g)$ cannot be normalized to give a representation of $\mathrm{Sp}(V)\cdot H(V)$, but that they can be normalized to give a representation–denote it by π_t–of $\mathrm{Mp}(V)\cdot H(V)$. Now define (cf. [37]) the *metaplectic representation* μ_t of $\mathrm{Mp}(V)$ by $\mu_t = \pi_t|_{\mathrm{Mp}(V)}$. The unitary equivalence class $[\mu_t]$ depends only on t.

4.1. Lemma. *Metaplectic representation classes* $[\mu_t] = [\mu_{t'}]$ *if and only if* $tt' > 0$, *and* $[\mu_{-t}]$ *is the dual (contragredient) of* $[\mu_t]$.

Proof. The multiplicative group $\mathbf{R}^+$ has an action α by automorphisms on $\mathrm{Mp}(V)\cdot H(V)$: $\alpha_b(\tilde{g}, u, s) = (\tilde{g}, bu, b^2 s)$. Here $\pi_t \circ \alpha_b|_{H(V)} = \pi_{b^2 t}$ and $\mu_t \circ \alpha_b|_{\mathrm{Sp}(V)} = \mu_t$, so α_b gives an equivalence of μ_t and $\mu_{b^2 t}$. If $tt' > 0$, then $t' = b^2 t$ for some $b > 0$, so $[\mu_t] = [\mu_{t'}]$.

We see $\pi_t^* = \pi_{-t}$ from a glance at central characters. Since $\tilde{\pi}_t^*$ restricts to π_t^* on $H(V)$ and μ_t^* on $\mathrm{Sp}(V)$, now $[\mu_t^*] = [\mu_{-t}]$.

In Theorem 4.23 below, we calculate that $d\mu_1$ sends a certain $J_{m,0} \in \mathfrak{sp}(V)$ to an operator with discrete spectrum $\{i(d + m/2) \colon 0 \leqslant d \in \mathbf{Z}\}$. Now $d\mu_{-1}(J_{m,0}) = d\mu_1^*(J_{m,0})$ has discrete spectrum $\{-i(d + m/2) \colon 0 \leqslant d \in \mathbf{Z}\}$, which is different, so $[\mu_{-1}] \neq [\mu_1]$. If $tt' < 0$, say $t > 0 > t'$, now $[\mu_t] = [\mu_1] \neq [\mu_{-1}] = [\mu_{t'}]$. Q.E.D.

Lemma 4.1 tells us that there are exactly two metaplectic representation classes, $[\mu] = [\mu_1]$ and its dual $[\mu^*] = [\mu_{-1}]$.

We recall the $\mathbf{Z}_2$-graded Lie algebra that will give us an elementary explicit construction of the metaplectic representations. We saw, in §3, that

$$u \otimes v \mapsto \xi_{u,v} \quad \text{where } 2\xi_{u,v}(w) = \{u, w\}v + \{v, w\}u \tag{4.2}$$

maps $V \otimes V$ to $\mathfrak{sp}(V)$, and in fact is a vector space isomorphism from the symmetric tensors onto $\mathfrak{sp}(V)$. We defined a multiplication $[\ ,\ ]_G$ on the vector space $\mathfrak{sp}(V) + V$ by

$$\begin{cases} \text{if } \xi, \eta \in \mathfrak{sp}(V) \text{ then } [\xi, \eta]_G = [\xi, \eta] = \xi\eta - \eta\xi, \\ \text{if } \xi \in \mathfrak{sp}(V) \text{ and } u \in V \text{ then } [\xi, u]_G = -[u, \xi]_G = \xi(u), \\ \text{if } u, v \in V \text{ then } [u, v]_G = \xi_{u,v}. \end{cases} \tag{4.3}$$

Define $\mathcal{H}$ to be the space of all holomorphic $f: \mathbf{C}^m \to \mathbf{C}$ such that $\int |f(z)|^2 \exp(-|z|^2)\, d\lambda(z) < \infty$ where $|z|^2 = \sum_1^m |z_j|^2$ and λ is Lebesgue measure on $\mathbf{C}^m$. Then $\mathcal{H}$ is a Hilbert space with inner product

$$(f, g) = \pi^{-m} \int f(z)\overline{g(z)} \exp(-|z|^2)\, d\lambda(z). \tag{4.4}$$

Given $n = (n_1, \ldots, n_m)$ integral, denote $n! = \Pi n_a!$ and $z^n = z_1^{n_1} \cdots z_m^{n_m}$ as usual. Then the functions

$$\varphi_n(z) = z^n/\sqrt{n!}\,, \qquad n = (n_1, \ldots, n_m) \text{ with } 0 \leqslant n_a \in \mathbf{Z}, \tag{4.5}$$

form a complete orthonormal set in $\mathcal{H}$.

Since

$$\frac{\partial}{\partial z_a}\left(f\bar{g}\exp(-|z|^2)\right) = \frac{\partial f}{\partial z_a}\, \bar{g}\exp(-|z|^2) - f \cdot \overline{z_a g}\exp(-|z|^2)$$

for $f, g \in \mathcal{H}$, integration by parts says $(\partial/\partial z_a)^* = z_a$, multiplication by z_a.

Fix a basis $\{p_1, \ldots, p_m; q_1, \ldots, q_m\}$ of V with $\{p_a, q_b\} = \delta_{ab}$. The Bargmann-Fock realization of $[\pi_1]$ as a representation of $H(V)$ on $\mathcal{H}$ is given, on the Lie algebra level, by

$$\begin{aligned} d\pi_1\colon (0, 1) \mapsto i, \quad (p_a, 0) &\mapsto \frac{1}{\sqrt{2}}\left(z_a - \frac{\partial}{\partial z_a}\right), \\ (q_a, 0) &\mapsto \frac{-i}{\sqrt{2}}\left(z_a + \frac{\partial}{\partial z_a}\right). \end{aligned} \tag{4.6}$$

Define a map T on the graded Lie algebra $\mathfrak{sp}(V) + V$ by

(4.7)
$$T: p_a \mapsto i\sqrt{\frac{i}{2}}\left(z_a - \frac{\partial}{\partial z_a}\right), \quad q_a \mapsto \sqrt{\frac{i}{2}}\left(z_a + \frac{\partial}{\partial z_a}\right),$$
$$\xi_{u,v} \mapsto \{T(u)T(v) + T(v)T(u)\}.$$

Notice that $T(u) = i\sqrt{i/2}\ d\pi_1(u, 0)$ for every $u \in V$. We need this for the proof of Theorem 4.10 below.

4.8. Lemma. *T is a graded Lie algebra representation.*

Proof. First, $T[u, v]_G = T\xi_{u,v} = [Tu, Tv]_G$ for u, v in the odd part V of $\mathfrak{sp}(V) + V$. Now we claim that it suffices to prove $T(\xi(u)) = [T(\xi), T(u)]$ for $\xi \in \mathfrak{sp}(V)$ and $u \in V$. For then $T[\xi, u]_G = [T\xi, Tu]_G$ directly, and, since $\mathfrak{sp}(V)$ acts effectively on V,

$$\begin{aligned}[][[T\xi, T\eta]_G, Tu]_G &= [T\xi, [T\eta, Tu]] - [T\eta, [T\xi, Tu]] \\ &= [T\xi, T\eta(u)] - [T\eta, T\xi(u)] = T\xi(\eta(u)) - T\eta(\xi(u)) \\ &= T([\xi, \eta](u)) = [T[\xi, \eta]_G, Tu]_G\end{aligned}$$

shows $T([\xi, \eta]_G) = [T\xi, T\eta]_G$ for $\xi, \eta \in \mathfrak{sp}(V)$.

To verify $T(\xi(u)) = [T\xi, Tu]$, we write out the last part of (4.7) in terms of our basis of V. Setting $\widetilde{uv} = T(\xi_{u,v})$ this says

(4.9)
$$\begin{cases} \widetilde{p_a p_b} = T(\xi_{p_a, p_b}) = -\frac{i}{2}\left(\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} - z_b\frac{\partial}{\partial z_a} + z_a z_b - \delta_{ab}\right), \\ \widetilde{p_a q_b} = T(\xi_{p_a, q_b}) = \frac{1}{2}\left(\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} + z_b\frac{\partial}{\partial z_a} - z_a z_b\right), \\ \widetilde{q_a q_b} = T(\xi_{q_a, q_b}) = \frac{i}{2}\left(\frac{\partial^2}{\partial z_a \partial z_b} + z_a\frac{\partial}{\partial z_b} + z_b\frac{\partial}{\partial z_a} + z_a z_b + \delta_{ab}\right). \end{cases}$$

Now

$$\begin{aligned}[][\widetilde{p_a p_b}, Tp_c] &= \frac{1}{2}\sqrt{\frac{i}{2}}\left[\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} - z_b\frac{\partial}{\partial z_a} + z_a z_b - \delta_{ab}, z_c + \frac{\partial}{\partial z_c}\right] \\ &= 0 = T(0) = T(\xi_{p_a, p_b}(p_c)),\end{aligned}$$

$$\begin{aligned}[][\widetilde{p_a p_b}, Tq_c] &= -\frac{i}{2}\sqrt{\frac{i}{2}}\left[\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} - z_b\frac{\partial}{\partial z_a} + z_a z_b - \delta_{ab}, z_c + \frac{\partial}{\partial z_c}\right] \\ &= \delta_{ac}T(p_b) + \delta_{bc}T(p_a) = T(\xi_{p_a, p_b}(q_c)),\end{aligned}$$

$$\begin{aligned}[][\widetilde{p_a q_b}, Tp_c] &= \frac{i}{2}\sqrt{\frac{i}{2}}\left[\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} + z_b\frac{\partial}{\partial z_a} - z_a z_b, z_c - \frac{\partial}{\partial z_c}\right] \\ &= -\delta_{bc}T(p_a) = T(\xi_{p_a, q_b}(p_c)), \quad \text{etc.}\end{aligned}$$

gives the required verification. Q.E.D.

4.10. THEOREM. *The metaplectic representation class* $[\mu]$ *is given, on the Lie algebra level, by* $d\mu = T|_{\mathfrak{sp}(V)}$ *as in* (4.9).

PROOF. Following (4.9) and Lemma 4.8, $T|_{\mathrm{sp}(V)}$ is a Lie algebra homomorphism whose image consists of operators on $\mathcal{H}$ that are essentially skew adjoint from the dense domain $\mathcal{H}^0 = \{$finite linear combinations of the $z^n\}$. If $\xi \in \mathrm{sp}(V)$ and $\mu \in V$ then, on $\mathcal{H}^0$, with $c = (i\sqrt{i/2}\,)^{-1}$,

$$[T\xi, d\pi_1(u, s)] = [T\xi, cT(u) - is] = c[T\xi, Tu] = cT(\xi(u))$$
$$= d\pi_1(\xi u, 0) = [d\mu(\xi), d\pi_1(u, s)].$$

As π_1 is irreducible and $\mathcal{H}^0$ is a domain of essential skew adjointness for the image of $d\mu$, now $T|_{\mathrm{sp}(V)} - d\mu$ is a linear map $\mathrm{sp}(V) \to \mathbf{C}$, and so $T(\xi) = d\mu(\xi)$ on $\mathcal{H}^0$ for every $\xi \in [\mathrm{sp}(V), \mathrm{sp}(V)] = \mathrm{sp}(V)$. Closing the operators, the assertion follows. Q.E.D.

We use Theorem 4.10 and the explicit formulas (4.9) to see

$$\text{(4.11)}\qquad \begin{cases} d\mu(\xi_{p_a,p_b} + \xi_{q_a,q_b}) = i\left(z_a \dfrac{\partial}{\partial z_b} + z_b \dfrac{\partial}{\partial z_a} + \delta_{ab}\right), \\ d\mu(\xi_{p_a,q_b} - \xi_{p_b,q_a}) = z_b \dfrac{\partial}{\partial z_a} - z_a \dfrac{\partial}{\partial z_b}, \end{cases}$$

$$\text{(4.12)}\qquad \begin{cases} d\mu(\xi_{p_a,p_b} - \xi_{q_a,q_b}) = -i\left(\dfrac{\partial^2}{\partial z_a \partial z_b} + z_a z_b\right), \\ d\mu(\xi_{p_a,q_b} + \xi_{p_b,q_a}) = \dfrac{\partial^2}{\partial z_a \partial z_b} - z_a z_b. \end{cases}$$

Fix a decomposition $k + l = m$, k and l nonnegative integers, and define a closed subspace of $\mathcal{H}$ for every integer d:

$$\text{(4.13)}\quad \mathcal{H}_d\text{: closed span of the } \varphi_n(z) = z_n/\sqrt{n!} \text{ with } \sum_1^k n_a - \sum_{k+1}^{k+l} n_b = d.$$

Then we have

4.14. LEMMA. *Let* $\mathcal{H}'$ *be a closed subspace of* $\mathcal{H}$ *stable under the operators* (4.11) *for* $a \leqslant b \leqslant k$ *and for* $k < a \leqslant b$, *and stable under the operators* (4.12) *for* $a \leqslant k < b$. *Then* $\mathcal{H}'$ *is the closed span of some of the spaces* $\mathcal{H}_d$ *of* (4.13).

PROOF. Taking $a = b$ in (4.11), $\mathcal{H}^0 \cap \mathcal{H}'$ is stable under the $z_a(\partial/\partial z_a)$: $z^n \mapsto n_a z^n$, so $\mathcal{H}'$ is the closed span of some monomials $\varphi_n(z) = z^n/\sqrt{n!}$. Now fix $\varphi_n \in \mathcal{H}' \cap \mathcal{H}_d$ and let $\varphi_{n'} \in \mathcal{H}_d$. By hypothesis and (4.12), $\mathcal{H}'$ is stable under all $\partial^2/\partial z_a \partial z_b$ and all $z_a z_b$ for $a \leqslant k < b$. Set $r = \Sigma_1^k n_c - \Sigma_1^k n'_c$. If $r > 0$ we apply r of these $\partial^2/\partial z_a \partial z_b$ to φ_n, and if $r < 0$ we apply $|r|$ of these $z_a z_b$ to φ_n, sending φ_n to $\varphi_{n''} \in \mathcal{H}' \cap \mathcal{H}_d$ with $\Sigma_1^k n''_c = \Sigma_1^k n'_c$. Thus we may assume $\Sigma_1^k n_c = \Sigma_1^k n'_c$, and so also $\Sigma_{k+1}^m n_c = \Sigma_{k+1}^m n'_c$. By hypothesis and (4.11) with $a < b \leqslant k$, we may apply $z_a(\partial/\partial z_b)$ and $z_b(\partial/\partial z_a)$ repeatedly to φ_n sending it to $\varphi_{n''}$ with $n''_c = n'_c$ for $c \leqslant k$. Now we may assume $n_c = n'_c$ for $c \leqslant k$, and by the same argument we may also assume $n_c = n'_c$ for $c > k$. Q.E.D.

In a trivial way, Lemma 4.14 gives us the standard

4.15. COROLLARY. *The metaplectic representation μ is the direct sum $\mu^+ \oplus \mu^-$ of two irreducible representations, where the corresponding subspaces of $\mathcal{H}$ are $\mathcal{H}^+$: closed span of the z^n with $|n| = \Sigma n_a$ even and $\mathcal{H}^-$: closed span of the z^n with $|n|$ odd.*

PROOF. The spaces $\mathcal{H}^\pm$, defined in the statement, are closed and mutually orthogonal by (4.5) and invariant by (4.9) and Theorem 4.10. Taking $a = b$ in (4.11), every closed invariant subspace is spanned by monomials z^n. Now let z^n and $z^{n'}$ belong to the same $\mathcal{H}^\pm$, that is $|n| \equiv |n'| \bmod 2$. Using (4.12), we apply operators $z_a z_b$ to z^n if $|n| < |n'|$, to $z^{n'}$ if $|n| > |n'|$, and so may assume $|n| = |n'|$. Now the assertion follows from the case $k = 0$ of Lemma 4.14. Q.E.D.

Our fixed $k + l = m$ gives $V = (\mathbf{R}^{2m}, \{\ ,\ \})$ the structure of $\mathbf{C}^{k,l}$ with complex structure

$$J_{k,l}: p_a \mapsto -q_a \mapsto -p_a \quad \text{for } 1 \leqslant a \leqslant k, \qquad p_b \mapsto q_b \mapsto -p_b \quad \text{for } k+1 \leqslant b \leqslant k+l, \tag{4.16}$$

and hermitian scalar product

$$\langle u, v\rangle = \{u, J_{k,l}(v)\} + i\{u, v\} \quad \text{for } u, v \in V. \tag{4.17}$$

Notice that

$$J_{k,l} = \frac{1}{2}\left\{\sum_1^k (\xi_{p_a,p_a} + \xi_{q_a,q_a}) - \sum_{k+1}^{k+l} (\xi_{p_b,p_b} + \xi_{q_b,q_b})\right\} \in \mathfrak{sp}(V). \tag{4.18}$$

We obtain $U(k, l)$ as the $\mathrm{Sp}(V)$-centralizer of $J_{k,l}$, and so

$$\mathfrak{u}(k, l) = \{\xi \in \mathfrak{sp}(V): [\xi, J_{k,l}] = 0\}. \tag{4.19}$$

Now let us be explicit about (4.19):

4.20. LEMMA. *$\mathfrak{u}(k, l)$ has basis consisting of the $\xi_{p_a,p_b} + \xi_{q_a,q_b}$ ($a \leqslant b \leqslant k$ and $k < a \leqslant b$), the $\xi_{p_a,p_b} - \xi_{q_a,q_b}$ ($a \leqslant k < b$), the $\xi_{p_a,q_b} - \xi_{p_b,q_a}$ ($a < b \leqslant k$ and $k < a < b$), and the $\xi_{p_a,q_b} + \xi_{p_b,q_a}$ ($a \leqslant k < b$).*

PROOF. One verifies directly that $J_{k,l}$ commutes with each of the m^2 linearly independent elements of $\mathfrak{sp}(V)$ listed in the lemma. As $\dim \mathfrak{u}(k, l) = m^2$, this proves the assertion. Q.E.D.

We are going to reduce the metaplectic representation μ to

(4.21) $MU(k, l)$: inverse image of $U(k, l)$ by $\mathrm{Mp}(m; \mathbf{R}) \to \mathrm{Sp}(m; \mathbf{R})$.

A glance at compact Cartan subgroups shows that the "meta-unitary" groups $\mathrm{MU}(k, l)$ are connected, so $\mu|_{MU(k,l)}$ does not factor through $U(k, l)$ since μ does not factor through $\mathrm{Sp}(m; \mathbf{R})$. Nevertheless one knows [**39**, Proposition 4.16] that π_1 extends to a representation of $U(k, l) \cdot H(V)$ on $\mathcal{H}$. From this,

(4.22) $\nu = \det_{\mathbf{C}}^{1/2} \otimes (\mu|_{MU(k,l)})$ factors through $U(k, l)$,

and we will speak of ν as a unitary representation of $U(k, l)$.

4.23. THEOREM. *The space $\mathcal{H}_d$ of* (4.13) *is the $i(d + (k - l)/2)$ eigenspace of $d\mu(J_{k,l})$ and the $i(d + k)$ eigenspace of $d\nu(J_{k,l})$. It is invariant under $\mu(MU(k, l))$ and $\nu(U(k, l))$, which are irreducible on it. In particular, $\nu = \sum_{d \in \mathbf{Z}} \nu_d$, discrete direct sum, where ν_d is the irreducible unitary representation of $U(k, l)$ on $\mathcal{H}_d$.*

PROOF. As in (4.18), a complex diagonal matrix

$$\begin{bmatrix} ix_1 & & \\ & \ddots & \\ & & ix_m \end{bmatrix}$$

in $\mathfrak{u}(k, l)$ corresponds to

$$\frac{1}{2} \sum_{1}^{k} x_a(\xi_{p_a,p_a} + \xi_{q_a,q_a}) - \frac{1}{2} \sum_{k+1}^{k+l} x_b(\xi_{p_b,p_b} + \xi_{q_b,q_b}),$$

and the case $a = b$ of (4.11) shows that its $d\mu$-image is

$$i\left\{ \sum_{1}^{k} x_a\left(z_a \frac{\partial}{\partial z_a} + \frac{1}{2}\right) - \sum_{k+1}^{k+l} x_b\left(z_b \frac{\partial}{\partial z_b} + \frac{1}{2}\right)\right\}.$$

Now, by (4.22), the $d\nu$-image is

$$i\left\{ \sum_{1}^{k} x_a\left(z_a \frac{\partial}{\partial z_a} + 1\right) - \sum_{k+1}^{k+l} x_b\left(z_b \frac{\partial}{\partial z_b}\right)\right\}.$$

In particular,

$$d\nu(J_{k,l}) = i\left\{ \sum_{1}^{k} z_a \frac{\partial}{\partial z_a} - \sum_{k+1}^{k+l} z_b \frac{\partial}{\partial z_b} + k\right\}$$

and

$$d\mu(J_{k,l}) = d\nu(J_{k,l}) - im/2.$$

This gives the eigenspace assertions. Stability is immediate, and irreducibility follows from Lemmas 4.14 and 4.20. Q.E.D.

The duals μ^* and ν^* are representations on the conjugate Hilbert space $\overline{\mathcal{H}}$, so Theorem 4.23 gives us

4.23.* THEOREM. *$\overline{\mathcal{H}}_d$ is the $-i(d + (k - l)/2)$ eigenspace of $d\mu^*(J_{k,l})$ and the $-i(d + k)$ eigenspace of $d\nu^*(J_{k,l})$, and ν^* represents $U(k, l)$ on $\mathcal{H}_d$ by the irreducible representation ν_d^*.*

At this point we pause to record the action of $d\nu$:

$$d\nu(\xi) \text{ is equal to } d\mu(\xi) \text{ in (4.11) and (4.12), except for}$$

$$\text{(4.24)} \qquad d\nu(\xi_{p_a p_a} + \xi_{q_a q_a}) = 2i(z_a(\partial/\partial z_a) + 1) \quad \text{for } a \leqslant k,$$
$$= 2i\, z_a(\partial/\partial z_a) \quad \text{for } k < a.$$

Recall the correspondence (3.7) between nonzero totally isotropic subspaces $E \subset \mathbf{C}^{k,l}$ and maximal parabolic subgroups $P_E \subset U(k, l)$. We now are going to show that ν_d and ν_d^* remain irreducible on restriction to P_E–and even to its maximal unimodular subgroup P_E' when $\dim_{\mathbf{C}} E > 1$. The idea is to adapt Kobayashi's reproducing kernel argument [14] to the "nearly transitive" situation provided by Proposition 3.8. The kernels are given by the elementary

4.25. LEMMA. *Let $\mathcal{H}'$ be a closed subspace of $\mathcal{H}$. If $\{\psi_\alpha\}_{\alpha \in A}$ is a complete orthonormal set in $\mathcal{H}'$, then $\sum_{\alpha \in A} \psi_\alpha(z)\overline{\psi_\alpha(\zeta)}$ converges absolutely, uniformly on compact sets, to a function $K'(z, \zeta)$ on $\mathbf{C}^m \times \mathbf{C}^m$ that is independent of choice of $\{\psi_\alpha\}_{\alpha \in A}$. $K'(z, \zeta)$ is holomorphic in z and antiholomorphic in ζ, and if $f \in \mathcal{H}$ then*

$$\text{(4.26)} \qquad f'(z) = \pi^{-m} \int_{\mathbf{C}^m} K'(z, \zeta) f(\zeta) \exp(-|\zeta|^2)\, d\lambda(\zeta)$$

is its orthogonal projection to $\mathcal{H}'$.

PROOF. If $\mathcal{H}' = \mathcal{H}$ we recall that the $\varphi_n(z) = z^n/\sqrt{n!}$ form a complete orthonormal set, and we calculate

$$\sum_n |\varphi_n(z)\overline{\varphi_n(\zeta)}| = \sum_n \frac{\left|(z_1\bar{\zeta}_1)^{n_1} \cdots (z_m\bar{\zeta}_m)^{n_m}\right|}{n!}$$
$$= \sum_{n_1 \geq 0} \frac{|z_1\bar{\zeta}_1|^{n_1}}{n_1!} \sum_{n_2 \geq 0} \frac{|z_2\bar{\zeta}_2|^{n_2}}{n_2!} \cdots \sum_{n_m \geq 0} \frac{|z_m\bar{\zeta}_m|^{n_m}}{n_m!}$$
$$= e^{|z_1\bar{\zeta}_1|} e^{|z_2\bar{\zeta}_2|} \cdots e^{|z_m\bar{\zeta}_m|} = e^{|z_1\bar{\zeta}_1| + \cdots + |z_m\bar{\zeta}_m|}.$$

Thus $\sum_n \varphi_n(z)\overline{\varphi_n(\zeta)}$ converges absolutely, uniformly on compacta, to a function $K(z, \zeta)$ that is holomorphic in z and antiholomorphic in ζ. If $\{\psi_\beta\}_{\beta \in B}$ is another complete orthonormal set, we expand $\psi_\beta = \sum_n a_{\beta,n}\varphi_n$ to see that $\Sigma_{\beta \in B} \psi_\beta(z)\overline{\psi_\beta(\zeta)}$ converges absolutely, uniformly on compacta, to $K(z, \zeta)$. Finally, if $f \in \mathcal{H}$ then

$$\pi^{-m} \int K(x, \zeta) f(\zeta) \exp\left(-|\zeta|^2\right) d\lambda(\zeta) = \sum_n (f, \varphi_n)_{\mathcal{H}} \varphi_n(z) = f(z).$$

In the general case, increase $\{\psi_\alpha\}_{\alpha \in A}$ to a complete orthonormal set $\{\psi_\beta\}_{\beta \in B}$ in $\mathcal{H}$. Then

$$\sum_{\alpha \in A} \left|\psi_\alpha(z)\overline{\psi_\alpha(\zeta)}\right| \leqslant \sum_{\beta \in B} \left|\psi_\beta(z)\overline{\psi_\beta(\zeta)}\right|$$

gives the required convergence, and everything else follows as in the case $\mathcal{H}' = \mathcal{H}$. Q.E.D.

Now we come to

4.27. THEOREM. *Let P_E be the maximal parabolic subgroup of $U(k, l)$ corresponding to a nonzero totally isotropic subspace $E \subset \mathbf{C}^{k,l}$, and let P_E' denote its maximal unimodular subgroup. Then $\nu_d|_{P_E}$ and $\nu_d^*|_{P_E}$ are irreducible, and if* $\dim_{\mathbf{C}} E > 1$ *then, further, $\nu_d|_{P_E'}$ and $\nu_d^*|_{P_E'}$ are irreducible.*

PROOF. Let B be a closed subgroup of $U(k, l)$ and suppose that $\mathcal{H}_d = \mathcal{H}_d' \oplus \mathcal{H}_d''$, orthogonal direct sum of closed $\nu_d(B)$-invariant subspaces. Then the corresponding kernel functions of Lemma 4.25 satisfy $K_d(z, \zeta) = K_d'(z, \zeta) + K_d''(z, \zeta)$, and all three are invariant under the action $\nu_d \otimes \nu_d^*$ of B on $\mathcal{H}_d \otimes \overline{\mathcal{H}}_d$.

Now suppose that, for r real, B has a dense open orbit on $S_r = \{u \in \mathbf{C}^{k,l}: u \neq 0 \text{ and } \|u\|^2 = r\}$. For z in that dense open orbit, $K_d'(z, z)$ is a constant multiple of $K_d(z, z)$. Since the kernels are real analytic, we now have a real analytic function $c: \mathbf{R} \to [0, 1]$ such that $K_d'(z, z) = c(\|z\|^2)K_d(z, z)$ for all $z \in \mathbf{C}^m$. We conclude that $K_d'(z, z)$ is invariant under $(\nu_d \otimes \nu_d^*)(U(k, l))$.

As $K_d'(z, \zeta)$ is holomorphic in z and antiholomorphic in ζ, it is determined by its restriction $K_d'(z, z)$ to the diagonal of $\mathbf{C}^m \times \mathbf{C}^m$, and thus also is $(\nu_d \otimes \nu_d^*)(U(k, l))$-invariant. Now the range $\mathcal{H}_d'$ of the corresponding operator (4.26) is $\nu_d(U(k, l))$-invariant. As ν_d is irreducible we conclude that $\mathcal{H}_d'$ is 0 or $\mathcal{H}_d$. This proves irreducibility of $\nu_d|_B$, and thus also of $(\nu_d|_B)^* = \nu_d^*|_B$.

In Proposition 3.8, we noted that the transitivity required of B always is satisfied by P_E, and is satisfied by P_E' when $\dim_{\mathbf{C}} E > 1$. Q.E.D.

4.28. REMARKS. As is clear from the proof of Theorem 4.27, we have irreducibility of $\nu_d|_B$ and $\nu_d^*|_B$ whenever B enjoys the appropriate "nearly transitive" conditions on the S_r or the corresponding coadjoint orbits.

If $k = l = 1$, then P_E' is commutative, so $\nu_d|_{P_E'}$ and $\nu_d^*|_{P_E'}$ are reducible. This suggests that $\nu_d|_{P_E'}$ and $\nu_d^*|_{P_E'}$ may reduce whenever $\dim_{\mathbf{C}} E = 1$, in particular, whenever $\min(k, l) = 1$. That is also suggested by the lack of near transitivity in these cases.

In the case $k = l = 2$ when we take P'_E to be the Poincaré group (cf. [4]) these representations are exactly the representations corresponding to relativistic elementary particles of mass zero and varying spins.

More generally, if B acts essentially transitively on S_r then the open orbit on S_r corresponds to an open subset of an orbit in $\mathfrak{u}(k, l)^*$ which is a symplectic homogeneous space for B and, in fact, corresponds to an orbit of B acting on $\mathfrak{b}^*$ where $\mathfrak{b}$ is the Lie algebra of B. If B is a semidirect product of a linear group L with a vector group $\mathbf{R}^s$, then the orbits of B on $\mathfrak{b}^*$ have a simple description in terms of the Mackey-Wigner "little group" construction. In particular, an orbit, $\mathfrak{M}$, of B acting on $\mathfrak{b}^*$ is fibered over an orbit, $\mathcal{O}$, of L acting on $(\mathbf{R}^s)^*$ and, for each such orbit, corresponds to an orbit of L_a acting on $\mathfrak{l}_a^*$, where L_a is the isotropy subgroup of a point $a \in \mathcal{O}$ and $\mathfrak{l}_a$ is its Lie algebra. Let us call the corresponding orbit in $\mathfrak{l}_a^*$ the "little orbit". According to Mackey's theorem [17], every irreducible unitary representation of B is induced from a representation (ρ, χ_a) of some $L_a \cdot \mathbf{R}^s$, where ρ is an irreducible representation of L_a and χ_a is the character on $\mathbf{R}^s$ given by the infinitesimal character a, extended to $L_a \cdot \mathbf{R}^s$ as in [39]. According to Rawnsley [24], some invariant polarizations on $\mathfrak{M}$ come from polarizations on the little orbit invariant under L_a. Furthermore, in case the little orbit has an invariant polarization which then gives rise to a unitary representation of L_a, then the Mackey induced representation is obtained by quantizing relative to the associated polarization of $\mathfrak{M}$. One would expect that in this way the labelling of the irreducible representation, $\nu_{d|B}$, is completely determined by the orbit description of §3. We shall return to this point in Part II.

In general, let π be an irreducible unitary representation of a Lie group G and H a closed subgroup of G such that $\pi|_H$ is of type I. If π is associated to a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ by the Kostant-Kirillov orbit method, then the transitivity properties of H on $\mathcal{O}$ should be reflected in the decomposition of $\pi|_H$ into irreducible representations. This is supported by work of Pukánszky [21], [22] and Vergne [33] on exponential solvable groups, and by Kobayashi's reproducing kernel arguments [14], [15] interpreted as applying to the case of a totally complex polarization; and the use of Proposition 3.8 in our proof of Theorem 4.27 is a striking example. That example takes an interesting form when we consider the algebraic reducibility properties of a parabolic subalgebra $\mathfrak{p}_E \subset \mathfrak{u}(k, l)$ and its subalgebra $\mathfrak{p}'_E$ on the dense subspace

$$\mathcal{H}_d^0 = \{\text{polynomials in } \mathcal{H}_d\} = \{U(k) \times U(l)\text{-finite vectors in } \mathcal{H}_d\}$$

of vectors finite under the maximal compact subgroup, and we describe this situation in an Appendix at the end of this paper. In that Appendix we shall see that the lower dimensional orbits of P_E and P'_E are reflected in the existence of invariant subspaces of $\mathcal{H}_d^0$ under $d\nu(\mathfrak{p}_E)$ or $d\nu(\mathfrak{p}'_E)$. These subspaces are all of finite codimension and are dense in $\mathcal{H}_d$, and so do not have any effect on the unitary irreducibility.

5. Structure of the representations. In this section we look at the relation between the representations ν_d and ν_d^* of $U(k, l)$, and the associated bounded symmetric domain of $k \times l$ matrices. First, we analyze the restriction of ν_d to the maximal compact subgroup $U(k) \times U(l)$, and use that information to prove that ν_d and ν_d^* are discrete series representations only in restricted circumstances. Then we recall the extended holomorphic discrete series of Wallach, Rossi and Vergne, and discuss it in terms of lowest highest weight. As a result, we show that ν_d^* belongs to this extended holomorphic discrete series in the cases where one expects it to occur on a holomorphic line bundle, but an example shows that this is not the case in general.

We start by examining the restriction of ν_d from $U(k, l)$ to the maximal compact subgroup $U(k) \times U(l)$. Denote

(5.1) $\quad \mathcal{H}_{r,s}$: span of the $\varphi_n(z) = z^n/\sqrt{n!}$ with $r = \sum_1^k n_a$ and $s = \sum_1^l n_{k+b}$.

So $\mathcal{H}_d = \Sigma_{r-s=d}\, \mathcal{H}_{r,s}$. Set

$$\delta_a = \tfrac{1}{2}\left(\xi_{p_a p_a} + \xi_{q_a q_a}\right) \quad \text{for } 1 \leqslant a \leqslant k,\ \delta_{k+b}$$
$$= -\tfrac{1}{2}\left(\xi_{p_{k+b} p_{k+b}} + \xi_{q_{k+b} q_{k+b}}\right) \quad \text{for } 1 \leqslant b \leqslant l,$$

and denote

$$\mathfrak{t}\text{: Cartan subalgebra of } \mathfrak{u}(k, l) \text{ spanned by the } \delta_j.$$

The correspondence with matrices is

$$\begin{bmatrix} ix_1 & & \\ & \ddots & \\ & & ix_m \end{bmatrix} \leftrightarrow \sum_1^m x_j\,\delta_j,$$

and so we have a simple root system

(5.2) $\quad \overset{\alpha_1}{\circ}\!-\cdots-\!\overset{\alpha_{m-1}}{\circ} \quad$ where $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$, $\varepsilon_j\left(\sum_1^m x_c\,\delta_c\right) = ix_j$.

Here α_k is the noncompact simple root, and the compact simple root system is

$$\overset{\alpha_1}{\circ}\!-\cdots-\!\overset{\alpha_{k-1}}{\circ} \oplus \overset{\alpha_{k-1}}{\circ}\!-\cdots-\!\overset{\alpha_{k+l-1}}{\circ}.$$

5.3. Lemma. *The spaces $\mathcal{H}_{r,s}$ $(r - s = d)$ are stable under $\nu_d(U(k) \times U(l))$, and $\nu_d(U(k) \times U(l))$ acts irreducibly on $\mathcal{H}_{r,s}$ by the representation of highest weight $\kappa + r\varepsilon_1 - s\varepsilon_m$ if $kl > 0$, $\kappa + r\varepsilon_1$ if $l = 0$, $-s\varepsilon_m$ if $k = 0$, where $\kappa = \varepsilon_1 + \cdots + \varepsilon_k$.*

PROOF. $d\nu(\Sigma_1^m x_j\delta_j) = i\{\Sigma_1^k x_a(z_a(\partial/\partial z_a) + 1) - \Sigma_{k+1}^{k+l} x_b(z_b(\partial/\partial z_b))\}$ multiplies $\varphi_n(z) = z^n/\sqrt{n!}$ by $i\{\Sigma_1^k x_a(n_a + 1) - \Sigma_{k+1}^{k+l} x_b n_b\}$. Invariance is clear as in Lemma 4.20, and the method of Lemma 4.14 proves that $U(k) \times U(l)$ is irreducible on $\mathcal{H}_{r,s}$. So now $U(k) \times U(l)$ acts on $\mathcal{H}_{r,s}$ by the irreducible representation with weight system

$$\left\{ \sum_1^k (n_a + 1)\varepsilon_a - \sum_{k+1}^{k+l} n_b\varepsilon_b : \sum_1^k n_a = r \text{ and } \sum_{k+1}^{k+l} n_b = s \right\}.$$

The highest weight is the one to which no root $\alpha_j, j \neq k$, can be added to give another weight, and that is $r\varepsilon_1 - s\varepsilon_m + \kappa$. Q.E.D.

Now we start to locate the series of ν_d:

5.4. THEOREM. *The representation ν_d belongs to the discrete series of $U(k, l)$ if and only if $kl = 0$, $k = 1$ and $d \geq l$, or $l = 1$ and $-d \geq k$.*

PROOF. Suppose that ν_d is a discrete series representation of $U(k, l)$. Then, for some $\lambda \in it^*$ with $\langle \lambda, \alpha \rangle \neq 0$ for every root α, ν_d is equivalent to the discrete series representation π_λ. Let Σ^+ and Σ_K^+ denote the positive root systems for $U(k, l)$ and $U(k) \times U(l)$ described just before Lemma 5.3. Define new positive root systems

$$\Delta^+ = \{\text{roots } \alpha : \langle \lambda, \alpha \rangle > 0\} \quad \text{and} \quad \Delta_K^+ = \{\alpha \in \Delta^+ : \alpha \text{ compact}\}.$$

Let W_K denote the Weyl group for $U(k) \times U(l)$. Then π_λ specifies λ only modulo the action of W_K, and some $w \in W_K$ carries Δ_K^+ to Σ_K^+, so we may suppose $\Delta_K^+ = \Sigma_K^+$. Now Lemma 5.3 applies to $\pi_\lambda|_{U(k) \times U(l)}$.

W. Schmidt [28, Theorem 1.3] characterized π_λ as the unique irreducible unitary representation π of $U(k, l)$, such that $\pi|_{U(k) \times U(l)}(a)$ contains the irreducible representation of $U(k) \times U(l)$ with highest weight

$$\lambda - 2\rho_K^\lambda + \rho^\lambda \quad \text{where } 2\rho_K^\lambda = \sum_{\alpha \in \Delta_K^+} \alpha \text{ and } 2\rho^\lambda = \sum_{\beta \in \Delta^+} \beta,$$

and (b) does not contain any irreducible representation of $U(k) \times U(l)$ with highest weight of the form $\lambda - 2\rho_K^\lambda + \rho^\lambda - A$, where A is a nontrivial sum from Δ^+. Using $\pi_\lambda \simeq \nu_d$, Lemma 5.3, and $\alpha_1 + \cdots + \alpha_{m-1} = \varepsilon_1 - \varepsilon_m$, we conclude that

$$\lambda = \kappa + d\varepsilon_1 + 2\rho_K^\lambda - \rho^\lambda \quad \text{if } d \geq 0, \lambda = \kappa + d\varepsilon_m + 2\rho_K^\lambda - \rho^\lambda \quad \text{if } d \leq 0,$$

and that $\alpha_1 + \cdots + \alpha_{m-1} \in \Delta^+$. Here note that $\Delta_K^+ = \Sigma_K^+$ implies $2\rho_K^\lambda = \Sigma_1^k(k + 1 - 2j)\varepsilon_j + \Sigma_{k+1}^{k+l}(l + 1 - 2(j - k))\varepsilon_j$.

Suppose $\alpha_k \in \Delta^+$. Then $\Delta^+ = \Sigma^+$. In the normalization $\|\varepsilon_j\|^2 = 1$ we compute

$$\langle \lambda, \alpha_k \rangle = d\delta_{k,1} - m + 2 \quad \text{if } d \geq 0, = -d\delta_{k+1,m} - m + 2 \text{ if } d \leq 0.$$

Then $\langle \lambda, \alpha_k \rangle > 0$ gives us:

either $k = 1$ and $d \geqslant m - 1 = l$, or $l = 1$ and $-d \geqslant m - 1 = k$. Conversely, if those conditions hold then Σ^+ is the positive root system defined by $\lambda = \kappa + (d\varepsilon_1$ or $d\varepsilon_m) + 2\rho_K^\lambda - \rho^\lambda$ and ν_d is the discrete series representation π_λ as seen by its $U(k) \times U(l)$ restriction.

Now suppose $\alpha_k \notin \Delta^+$. Define $a \leqslant k$ and $c \geqslant 0$ by the conditions $-(\alpha_a + \cdots + \alpha_{k+c}) \in \Delta^+$, $-(\alpha_{a-1} + \cdots + \alpha_{k+c}) \notin \Delta^+$, $-(\alpha_a + \cdots + \alpha_{k+c+1}) \notin \Delta^+$. Then the set $\Delta^+ \setminus \Delta_K^+$ of positive noncompact roots is

$$-(\alpha_a + \cdots + \alpha_{k+c}); -(\alpha_{a+1} + \cdots + \alpha_{k+c}), -(\alpha_a + \cdots + \alpha_{k+c-1}); \\ \cdots; -\alpha_k; (\alpha_{a-1} + \cdots + \alpha_{k+c}), (\alpha_a + \cdots + \alpha_{k+c+1}); \\ \cdots; (\alpha_1 + \cdots + \alpha_{m-1}).$$

Thus the Δ^+-simple roots are

compact simple: $\alpha_1, \ldots, \alpha_{a-2}$; $\alpha_a, \ldots, \alpha_{k-1}$; $\alpha_{k-1}, \ldots, \alpha_{k+c}$; $\alpha_{k+c+2}, \ldots, \alpha_{m-1}$;

noncompact simple: $-(\alpha_a + \cdots + \alpha_{k+c})$, $\alpha_{a-1} + \cdots + \alpha_{k+c}$, $\alpha_a + \cdots + \alpha_{k+c+1}$,

where impossible indices mean that the root does not occur. Now compute

$$\begin{aligned} \left\langle 2\rho_K^\lambda, -(\alpha_a + \cdots + \alpha_{k+c})\right\rangle &= \langle 2\rho_K, -\varepsilon_a + \varepsilon_{k+c+1}\rangle \\ &= k + l - 2(k + c - a + 1), \\ \left\langle 2\rho_K^\lambda, \alpha_{a-1} + \cdots + \alpha_{k+c}\right\rangle &= \langle 2\rho_K, \varepsilon_{a-1} - \varepsilon_{k+c+1}\rangle \\ &= -(k + l) + 2(k + c - a + 2), \\ \left\langle 2\rho_K^\lambda, \alpha_a + \cdots + \alpha_{k+c+1}\right\rangle &= \langle 2\rho_K, \varepsilon_a - \varepsilon_{k+c+2}\rangle \\ &= -(k + l) + 2(k + c - a + 2). \end{aligned}$$

Write $r\varepsilon_1 - s\varepsilon_m$ for the lowest highest weight $d\varepsilon_1$ or $d\varepsilon_m$. Using $\langle \rho^\lambda$, simple root$\rangle = 1$, now

(i) $\langle \lambda, -(\alpha_a + \cdots + \alpha_{k+c})\rangle = -\delta_{a,1}r - \delta_{c+1,l}s + m - 2(k + c - a + 2)$,

(ii) $\langle \lambda, \alpha_{a-1} + \cdots + \alpha_{k+c}\rangle = \delta_{a-1,1}r + \delta_{c+1,l}s - m + 2(k + c - a + 2)$,

(iii) $\langle \lambda, \alpha_a + \cdots + \alpha_{k+c+1}\rangle = \delta_{a,1}r + \delta_{c+2,l}s - m + 2(k + c - a + 2)$.

Of course, here the second equation occurs only if $a > 1$, and the third only if $c < l - 1$; and $\alpha_1 + \cdots + \alpha_{m-1} \in \Delta^+$ says that either $a > 1$ or $c < l - 1$ or both.

If $a > 2$ the right-hand sides of (i) and (ii) both are positive with sum zero; so $1 \leqslant a \leqslant 2$. If $c < l - 2$ the right-hand sides of (i) and (iii) are positive with sum zero; so $l - 2 \leqslant c \leqslant l - 1$. Now there are three cases: $(a, c) = (2, l - 2), (2, l - 1)$ or $(1, l - 2)$.

If $(a, c) = (2, l - 2)$, then $k + c - a + 1 = m - 4$, and (i), (ii) and (iii) say: $-m + 4 > 0, s + m - 4 > 0, r + m - 4 > 0$. Since $r = 0$ or $s = 0$, this is impossible.

If $(a, c) = (2, l - 1)$ or $(1, l - 2)$, then $k + c - a + 2 = m - 1$, and (i) says $m + s < 2$ or $m + r < 2$, in particular $m < 2$. Then $m = 1$ and $kl = 0$, which was excluded. Q.E.D.

In general, ν_d and ν_d^* will be "close" to the discrete series, but for this we need some machinery.

Our group $U(k, l)$ acts by automorphisms on the bounded symmetric domain

$$\mathfrak{D} = \mathfrak{D}_{k,l} = \{k \times l \text{ complex matrices } Z: I - Z^*Z >> 0\}$$

as linear fractional transformations

$$U(k, l) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}: Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

Similarly, the universal covering group $p: G \to U(k, l)$ acts by $g: Z \mapsto p(g)(Z)$. The respective isotropy subgroups at 0 are

$$U(k) \times U(l) \subset U(k, l) \quad \text{and} \quad K = p^{-1}\{U(k) \times U(l)\} \subset G.$$

Since the actions are transitive, this gives coset space representations

$$U(k, l)/U(k) \times U(l) \cong \mathfrak{D} \cong G/K.$$

Here the holomorphic tangent space to $\mathfrak{D}$ at 0 corresponds to the sum of the positive noncompact root spaces of the positive root system Σ^+ of (5.2).

Let τ_γ denote the irreducible unitary representation of K whose highest weight relative to Σ_K^+ is γ. There is an associated G-homogeneous hermitian holomorphic vector bundle $\mathbf{E}_\gamma \to \mathfrak{D}$, and it gives us a Hilbert space

(5.5a) $\qquad \mathcal{H}(\mathbf{E}_\gamma)$: L_2 holomorphic sections of $\mathbf{E}_\gamma \to \mathfrak{D}$.

One knows **[8]** that

(5.5b) $\qquad \mathcal{H}(\mathbf{E}_\gamma) \neq 0 \Leftrightarrow \langle \gamma + \rho, \alpha_1 + \cdots + \alpha_{m-1} \rangle < 0.$

The natural action of G on $\mathcal{H}(\mathbf{E}_\gamma)$ is an irreducible unitary representation, specifically the relative discrete series representation $\pi_{\gamma+\rho}$. The representations so obtained for G form the *holomorphic relative discrete series*, and their duals form the *antiholomorphic relative discrete series*. The ones that factor through $U(k, l)$–for the τ_γ that factor through $U(k) \times U(l)$–form the holomorphic and antiholomorphic discrete series of $U(k, l)$.

Suppose that Δ^+ is any positive root system of $\mathfrak{u}(k, l)_{\mathbf{C}}$ relative to its diagonal Cartan subalgebra, and that π is any irreducible unitary representation of G. We will say that $\pi|_K$ *has lowest highest weight* γ *relative to* Δ^+ if $\pi|_K$ has a subrepresentation τ_γ of highest weight γ relative to Δ_K^+, but $\pi|_K$ does not have a subrepresentation $\tau_{\gamma - A}$ where A is a nontrivial sum from Δ^+. The proof of Theorem 5.4 was based on W. Schmid's characterization of discrete series representations by lowest highest weight. Now let us see how this applies to ν_d, ν_d^*, and holomorphic and antiholomorphic discrete series. We denote positive root systems by

$\Sigma^+ = \Sigma_K^+ \cup \Sigma_{G/K}^+$: defined by the simple root system (5.2),
$\bar{\Sigma}^+ = \Sigma_K^+ \cup -\Sigma_{G/K}^+$: with simple system
$\{\alpha_{k-1}, \ldots, \alpha_1, -\Sigma_1^{m-1}\alpha_j, \alpha_{k+1}, \ldots, \alpha_{m-1}\}$.

Then we have, by an elementary calculation with roots and weights,

5.6. Proposition. *The representation ν_d has lowest highest weight relative to Σ^+; it is (recall $\kappa = \varepsilon_1 + \cdots + \varepsilon_k$)*

$$\gamma_d = d\varepsilon_1 + \kappa \quad \text{if } d \geqslant 0, \quad = d\varepsilon_m + \kappa \quad \text{if } d \leqslant 0. \tag{5.7a}$$

Similarly, ν_d^* has lowest highest weight relative to $\bar{\Sigma}^+$, given by

$$\gamma_d^* = -d\varepsilon_k - \kappa \quad \text{if } d \geqslant 0, \quad = -d\varepsilon_{k+1} - \kappa \quad \text{if } d \leqslant 0. \tag{5.7b}$$

On the other hand, if $\mathbf{E}_\gamma \to \mathfrak{D}$ satisfies (5.5b), *then the corresponding holomorphic relative discrete series representation $\pi_{\gamma+\rho}$ (ρ for Σ^+) has lowest highest weight γ relative to $\bar{\Sigma}^+$, and the corresponding antiholomorphic discrete series representation*

$$\pi_{\gamma+\rho}^* = \pi_{-(\gamma+\rho)} = \pi_{\gamma^* - 2(\rho-\rho_K)+\rho}, \qquad \tau_{\gamma^*} = (\tau_\gamma)^*,$$

has lowest highest weight γ^* relative to Σ^+.

In applying Proposition 5.6 and a certain extension of it, we will need a technical result.

5.8. Theorem. *Let π and π' be irreducible unitary representations of G. If $\pi|_K$ and $\pi'|_K$ have the same lowest highest weight relative to Σ^+, or have the same lowest highest weight relative to $\bar{\Sigma}^+$, then they are unitarily equivalent.*

Remark. As the proof will show, Theorem 5.8 holds whenever G is a reductive Lie group. $\mathrm{Ad}(g)\colon \mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ is an inner automorphism of $\mathfrak{g}_{\mathbf{C}}$ for every $g \in G$, and K is the Ad_G^{-1}-image of a maximal compact subgroup of the adjoint group. Further, here the sum A from the definition of lowest highest weight need only consist of any noncompact root in Σ^+ or $\bar{\Sigma}^+$.

An immediate consequence of Proposition 5.6 and Theorem 5.8, which one expects from the proof of Theorem 5.4, is

5.9. Corollary. *The discrete series representations ν_d of $U(1, l)$, $d \geqslant l$, and of $U(k, 1)$, $d \leqslant -k$, belong to the antiholomorphic discrete series.*

The proof of Theorem 5.8 uses the "factor of automorphy" that trivializes the holomorphic bundles $\mathbf{E}_\gamma \to \mathfrak{D}$. After proving Theorem 5.8, we will use this factor of automorphy to describe an extension of the holomorphic and antiholomorphic discrete series, and then will turn to the corresponding extension of Corollary 5.9.

$U(k, l)$ has complexification $GL(m; \mathbf{C})$, $m = k + l$. In $\mathfrak{gl}(m; \mathbf{C})$, let $\mathfrak{p}_+$ and $\mathfrak{p}_-$ denote the respective sums of the positive and negative noncompact root spaces for Σ^+. Thus

$$\exp(\mathfrak{p}_+) = P_+ = \left\{\begin{pmatrix} I & B \\ O & I \end{pmatrix}: B \text{ is } k \times l \text{ complex}\right\},$$

$$\exp(\mathfrak{p}_-) = P_- = \left\{\begin{pmatrix} I & O \\ C & I \end{pmatrix}: C \text{ is } l \times k \text{ complex}\right\}.$$

There is a well-defined map

$$\kappa: U(k, l) \to \{U(k) \times U(l)\}_{\mathbf{C}} = GL(k; \mathbf{C}) \times GL(l; \mathbf{C})$$
$$\text{by:} \quad g \in P_+ \cdot \kappa(g) \cdot P_-.$$

Under the coverings $p: G \to U(k, l)$ and $p: K_{\mathbf{C}} \to GL(k; \mathbf{C}) \times GL(l; \mathbf{C})$ it lifts to a map

$$\tilde{\kappa}: G \to K_{\mathbf{C}} \quad \text{such that } p(g) \in P_+ \cdot p(\tilde{\kappa}(g)) \cdot P_-.$$

Let E_γ denote the representation space of τ_γ, and extend τ_γ to a holomorphic representation of $K_{\mathbf{C}}$ on E_γ. The *factor of automorphy* is

(5.10) $\Phi_\gamma: G \to GL(E_\gamma)$ defined by $\Phi_\gamma(g) = \tau_\gamma(\tilde{\kappa}(g))$.

Its main properties are given by

5.11. LEMMA (TIRAO [31]). *Φ_γ is the unique continuously differentiable function $\Phi: G \to GL(E_\gamma)$ such that* (i) $\Phi(1) = I$, *identity transformation of* E_γ, (ii) $\Phi(k_1 g k_2) = \tau_\gamma(k_1) \cdot \Phi(g) \cdot \tau_\gamma(k_2)$ *for* $g \in G$ *and* $k_i \in K$, (iii) $\xi(\Phi) = 0$ *for all* $\xi \in \mathfrak{p}$.

Now we can prove

5.12. LEMMA. *Let π be an irreducible unitary representation of G, say on $\mathcal{H}$, such that $\pi|_K$ has an irreducible subrepresentation τ_γ relative to Σ_K^+, but does not have a subrepresentation $\tau_{\gamma-\alpha}$ for any noncompact root $\alpha \in \Sigma^+$. Then $\pi|_K$ contains τ_γ with multiplicity* 1, *and $d\pi(\mathfrak{p}_-)$ annihilates the τ_γ-isotypic subspace $E \subset \mathcal{H}$. Further, if p_E: $\mathcal{H} \to E$ denotes orthogonal projection, and Φ_γ: $G \to$* GL(E) *as in* (5.10), *then $\Phi_\gamma(g) = p_E \cdot \pi(g)|_E$ and $\Phi_\gamma(g^{-1}) = \Phi_\gamma(g)^*$ for all $g \in G$.*

PROOF. If $d\pi(\mathfrak{p}_-) \cdot E \neq 0$, let α be minimal among the noncompact positive roots such that $d\pi(\xi) \cdot E \neq 0$ where $\xi \in \mathfrak{g}_{\mathbf{C}}^{-\alpha}$. Fix ξ and let β be maximal among the weights of τ_γ such that E contains a β-weight vector w with $d\pi(\xi) \cdot w \neq 0$. Fix w. If $\sigma \in \Sigma_K^+$ and $\eta \in \mathfrak{g}_{\mathbf{C}}^{\sigma}$ then

$$d\pi[\eta, \xi] \cdot w = 0 \quad \text{by minimality of } \alpha \text{ and } \alpha - \sigma < \alpha,$$

and

$$d\pi(\xi) \cdot d\pi(\eta) \cdot w = 0 \quad \text{by maximality of } \beta \text{ and } \beta + \sigma > \beta,$$

so

$$d\pi(\eta) \cdot d\pi(\xi) \cdot w = d\pi[\eta, \xi] \cdot w + d\pi(\xi) \cdot d\pi(\eta) \cdot w = 0.$$

In other words, $d\pi(\xi) \cdot w$ is a highest weight vector for a representation, necessarily $\tau_{\gamma-\alpha}$, in $\pi|_K$. That contradicts the hypothesis on τ_γ. We conclude that $d\pi(\mathfrak{p}_-) \cdot E = 0$.

Since $\mathfrak{g}_{\mathbf{C}} = \mathfrak{p}_+ + \mathfrak{k}_{\mathbf{C}} + \mathfrak{p}_-$, its universal enveloping algebra has factorization $U(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{p}_+) \cdot U(\mathfrak{k}_{\mathbf{C}}) \cdot U(\mathfrak{p}_-)$. Now, if E' is any $\pi(K)$-invariant subspace of E,

$$\mathcal{H} = d\pi(U(\mathfrak{g}_{\mathbf{C}})) \cdot E' = d\pi(U(\mathfrak{p}_+)) \cdot E'.$$

If $\Xi \in U(\mathfrak{p}_+)$ is a monomial of positive degree then $d\pi(\Xi) \cdot E'$ cannot contain a γ-weight vector. We conclude $E' = E$. Now $\pi|_K$ contains τ_γ with multiplicity 1.

Define Φ: $G \to GL(E)$ by $\Phi(g) = p_E \cdot \pi(g)|_E$. Then Φ is real analytic, $\Phi(1) = I$, and for $g \in G$ and $k_i \in K$ one has

$$\begin{aligned}\Phi(k_1 g k_2) &= p_E \cdot \pi(k_1) \cdot \pi(g) \cdot \pi(k_2)|_E \\ &= \pi(k_1) \cdot p_E \cdot \pi(g)|_E \cdot \pi(k_2)|_E = \tau_\gamma(k_1) \cdot \Phi(g) \cdot \tau_\gamma(k_2).\end{aligned}$$

Further, if $\xi \in \mathfrak{p}_-$, then $\xi(\Phi)(g) = p_E \cdot \pi(g) \cdot d\pi(\xi)|_E = 0$, so $\xi(\Phi) = 0$. Now $\Phi = \Phi_\gamma$ by Lemma 5.11. We can write this as

$$\Phi_\gamma(g) = p_E \cdot \pi(g)|_E = \{p_E \cdot \pi(g) \cdot p_E\}|_E \quad \text{for } g \in G.$$

Since π is unitary now the adjoint is given on E by

$$\Phi_\gamma(g)^* = p_E^* \cdot \pi(g)^* \cdot p_E^* = p_E \cdot \pi(g^{-1}) \cdot p_E = \Phi_\gamma(g^{-1}).$$

That completes the proof of Lemma 5.12. Q.E.D.

Now let us go back and prove Theorem 5.8. Suppose that $\pi|_K$ and $\pi'|_K$ have the same lowest highest weight, say γ, relative to Σ^+. Let $\mathcal{H}$ and $\mathcal{H}'$ be their respective representation spaces, $E \subset \mathcal{H}$ and $E' \subset \mathcal{H}'$ the τ_γ-isotypic subspaces, f: $E \to E'$ a K-equivariant isometry, $v \in E$ and $v' = f(v)$ highest weight vectors of length 1. For finite sums we compute

$$\begin{aligned}\left\|\sum c_i \pi(g_i) v\right\|^2_{\mathcal{H}} &= \sum c_i \bar{c}_j \left\langle \pi(g_j^{-1} g_i) v, v \right\rangle_{\mathcal{H}} \\ &= \sum c_i \bar{c}_j \left\langle \Phi_\gamma(g_j^{-1} g_i) v, v \right\rangle_E \quad \text{by Lemma 5.12 for } \pi \\ &= \sum c_i \bar{c}_j \left\langle \Phi_\gamma(g_j^{-1} g_i) v', v' \right\rangle_{E'} \quad \text{under } f: E \to E' \\ &= \sum c_i \bar{c}_j \left\langle \pi'(g_j^{-1} g_i) v', v' \right\rangle_{\mathcal{H}'} \quad \text{by Lemma 5.12 for } \pi' \\ &= \left\|\sum c_i \pi'(g_i) v'\right\|^2_{\mathcal{H}'}.\end{aligned}$$

In other words, $\Sigma c_i \pi(g_i)v \mapsto \Sigma c_i \pi'(g_i)v'$ is an isometry from the dense subspace {finite linear combinations of the $\pi(g)v$, $g \in G$} of $\mathcal{H}$ onto the corresponding dense subspace of $\mathcal{H}'$. Clearly G-equivariant, it extends by continuity to a unitary equivalence of π with π'.

We have proved Theorem 5.8 for representations with the same lowest highest weight relative to Σ^+. Now suppose that π and π' are irreducible with the same lowest highest weight γ relative to $\bar{\Sigma}^+$. Then π^* and π'^* have the same lowest highest weight γ^*, $\tau_{\gamma^*} = (\tau_\gamma)^*$, relative to Σ^+, so they are equivalent, and it follows that π and π' are equivalent. This completes the proof of Theorem 5.8. Q.E.D.

The total space of $\mathbf{E}_\gamma \to G/K = \mathfrak{D}$ is $\mathbf{E}_\gamma = G \times_K E_\gamma$, consisting of all classes $[g, v] = \{(gk, \tau_\gamma(k)^{-1}v) \in G \times E_\gamma \colon k \in K\}$, with projection $[g, v] \mapsto g(0) \in \mathfrak{D}$. An explicit holomorphic trivialization is given by

$$\mathbf{E}_\gamma \ni [g, v] \leftrightarrow (Z, \Phi_\gamma(g)v) \in \mathfrak{D} \times E_\gamma, \qquad Z = g(0). \tag{5.13}$$

See Tirao [31] for a proof. As usual we identify a section $s\colon g(0) \mapsto [g, \varphi(g)]$ with the function $\varphi\colon G \to E_\gamma$, which satisfies $\varphi(gk) = \tau_\gamma(k)^{-1} \cdot \varphi(g)$. In view of Lemma 5.11, the section goes over to a well-defined function

$$f\colon \mathfrak{D} \to E_\gamma \quad \text{by } f(Z) = \Phi_\gamma(g) \cdot \varphi(g), \qquad Z = g(0),$$

and f is holomorphic if and only if the section is holomorphic. Furthermore, the section s has global square norm

$$\|s\|^2 = \int_{G/K} \|\varphi(g)\|^2_{E_\gamma}\, d(gK) = \int_{G/K} \|\Phi_\gamma(g)^{-1} \cdot f(g(0))\|^2_{E_\gamma}\, d(gK)$$

so $\mathcal{H}(\mathbf{E}_\gamma)$ is carried to the Hilbert space

$$\mathcal{H}(\mathfrak{D}; E_\gamma) = \Big\{ f\colon \mathfrak{D} \to E_\gamma \text{ holomorphic:} \quad \int_{G/K} \|\Phi_\gamma(g)^{-1} \cdot f(g(0))\|^2_{E_\gamma}\, d(gK) < \infty \Big\}. \tag{5.14}$$

The nontriviality condition (5.5b) for $\mathcal{H}(\mathbf{E}_\gamma)$ is also the nontriviality condition for $\mathcal{H}(\mathfrak{D}; E_\gamma)$, and one knows [8], [31] that it is equivalent to the condition that $\mathcal{H}(\mathfrak{D}; E_\gamma)$ contain the constant functions $f_v\colon Z \mapsto v$, $v \in E_\gamma$. The section corresponding to f_v is $\varphi_v\colon g \mapsto \Phi_\gamma(g)^{-1} \cdot v$, and [8], [31]

$$\|f_v\|^2_{\mathcal{H}(\mathfrak{D}; E_\gamma)} = \|\varphi_v\|^2_{\mathcal{H}(\mathbf{E}_\gamma)} = c \cdot \dim E_\gamma \cdot |\tilde{\omega}(\gamma + \rho)|^{-1} \|v\|^2_{E_\gamma}$$

where $c > 0$ depends only on normalizations of Haar measure and

$$\tilde{\omega}(\gamma + \rho) = \prod_{\alpha \in \Sigma^+} \frac{\langle \gamma + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Given $\varphi \in \mathcal{H}(\mathbf{E}_\gamma)$, its projection to the lowest highest K-weight space is φ_u,

$u = \varphi(1)$. So the above formula gives us

(5.15) $$\langle \varphi, \varphi_v \rangle_{\mathcal{H}(\mathbf{E}_\gamma)} = c \cdot \dim E_\gamma \cdot |\tilde{\omega}(\gamma + \rho)|^{-1} \langle \varphi(1), v \rangle_{E_\gamma}.$$

Write π for the action of G on $\mathcal{H}(\mathbf{E}_\gamma)$: $[\pi(x)\varphi](g) = (x^{-1}g)$. Choose a highest weight vector $v_\gamma \in E_\gamma$, $\|v_\gamma\| = 1$, and define

(5.16) $$\psi_\gamma \colon G \to \mathbf{C} \quad \text{by } \psi_\gamma(g) = \left\langle v_\gamma, \Phi_\gamma(g)^{-1} \cdot v_\gamma \right\rangle_{E_\gamma}.$$

Set $c' = c \cdot \dim E_\gamma \cdot |\tilde{\omega}(\gamma + \rho)|^{-1}$ for the moment. Then (5.15) gives us

$$\begin{aligned} \langle \pi(g)\varphi_{v_\gamma}, \varphi_{v_\gamma} \rangle_{\mathcal{H}(\mathbf{E}_\gamma)} &= \langle \varphi_{v_\gamma}, \pi(g^{-1})\varphi_{v_\gamma} \rangle_{\mathcal{H}(\mathbf{E}_\gamma)} = c' \langle v_\gamma, [\pi(g^{-1})\varphi_{v_\gamma}](1) \rangle_{E_\gamma} \\ &= c' \langle v_\gamma, \varphi_{v_\gamma}(g) \rangle_{E_\gamma} = c' \langle v_\gamma, \Phi_\gamma(g)^{-1} v_\gamma \rangle_{E_\gamma} = c' \psi_\gamma(g). \end{aligned}$$

Now

(5.17) $$\left\langle \pi(g)\varphi_{v_\gamma}, \varphi_{v_\gamma} \right\rangle_{\mathcal{H}(\mathbf{E}_\gamma)} = c \cdot \dim E_\gamma \cdot |\tilde{\omega}(\gamma + \rho)|^{-1} \cdot \psi_\gamma(g).$$

(5.17) is the starting point for Rossi and Vergne in their extension **[25]** of the holomorphic discrete series. Denote

(5.18) $\mathcal{L}(\mathbf{E}_\gamma)$: finite linear combinations of the $\pi(g)\varphi_{v_\gamma}$, $g \in G$.

It is dense in $\mathcal{H}(\mathbf{E}_\gamma)$ because π is irreducible, and (5.17) says that the norm on $\mathcal{L}(\mathbf{E}_\gamma)$ is given by

(5.19) $$\left\| \sum c_i \pi(g_i) \varphi_{v_\gamma} \right\|^2_{\mathcal{H}(\mathbf{E}_\gamma)} = c \cdot \dim E_\gamma \cdot |\tilde{\omega}(\gamma + \rho)|^{-1} \sum \bar{c}_j c_i \psi_\gamma(g_j^{-1} g_i).$$

Now they drop the $c \cdot \dim E_\gamma \cdot |\tilde{\omega}(\gamma + \rho)|^{-1}$ factor:

5.20 LEMMA. (ROSSI AND VERGNE **[25]**). *Do not assume the nontriviality condition* (5.5b); *instead make the weaker assumption that* ψ_γ (*of* (5.16)) *is of positive type*: $\sum_{i,j=1}^n \bar{c}_j c_i \psi_\gamma(g_j^{-1} g_i) \geqslant 0$ *whenever* $\{c_1, \ldots, c_n\} \subset \mathbf{C}$ *and* $\{g_1, \ldots, g_n\} \subset G$ *with* $n > 0$. *Then* $\mathcal{L}(\mathbf{E}_\gamma)$ (*of* (5.18)) *is a pre-Hilbert space with* $\langle \sum a_i \pi(x_i)\varphi_{v_\gamma}, \sum b_i \pi(y_j)\varphi_{v_\gamma} \rangle = \sum \bar{b}_j a_i \psi(y_j^{-1} x_i)$. *Its completion* $\mathcal{H}'(\mathbf{E}_\gamma)$ *is a Hilbert space of holomorphic sections of* $\mathbf{E}_\gamma$ *on which the natural action of G is an irreducible unitary representation* π'. *If* (5.5b) *holds, then* $\mathcal{H}'(\mathbf{E}_\gamma) = \mathcal{H}(\mathbf{E}_\gamma)$ *and* π' *is the corresponding relative holomorphic discrete series representation.*

PROOF. Using (5.16), $\langle \pi(g)\varphi_{v_\gamma}, \varphi_{v_\gamma} \rangle_{\mathcal{L}(\mathbf{E}_\gamma)} = \psi_\gamma(g) = \langle v_\gamma, \Phi_\gamma(g)^{-1} v_\gamma \rangle_{E_\gamma} = \langle \Phi_\gamma(g^{-1})^{-1} v_\gamma, v_\gamma \rangle_{E_\gamma} = \langle [\pi(g)\varphi_{v_\gamma}](1), v_\gamma \rangle_{E_\gamma}$. Now, by linearity,

(5.21) $$\left\langle \varphi, \varphi_{v_\gamma} \right\rangle_{\mathcal{L}(\mathbf{E}_\gamma)} = \left\langle \varphi(1), v_\gamma \right\rangle_{E_\gamma} \quad \text{for every } \varphi \in \mathcal{L}(\mathbf{E}_\gamma).$$

If $\|\varphi\|_{\mathcal{L}(\mathbf{E}_\gamma)} = 0$ then, taking left translates, (5.21) says φ has all values $\perp v_\gamma$, so $\varphi = 0$ by irreducibility of τ_γ. Now $\mathcal{L}(\mathbf{E}_\gamma)$ is a pre-Hilbert space. Also from (5.21), $\varphi \mapsto \varphi(g)$ is continuous on $\mathcal{L}(\mathbf{E}_\gamma)$, hence also on $\mathcal{H}'(\mathbf{E}_\gamma)$, and so $\mathcal{H}'(\mathbf{E}_\gamma)$ consists of holomorphic sections of $\mathbf{E}_\gamma$. The remaining details are routine. Q.E.D.

The unitary representations π' of G on $\mathcal{H}'(\mathbf{E}_\gamma)$, of Lemma 5.20, constitute the extension ("analytic continuation") of the holomorphic discrete series due to Wallach ([**34**], [**35**], [**36**]) and Rossi and Vergne [**25**], [**26**].

5.22. LEMMA. *Let π' be an extended holomorphic relative discrete series representation of G on $\mathcal{H}'(\mathbf{E}_\gamma)$. Then $\pi'|_K$ has lowest highest weight γ relative to $\overline{\Sigma}^+$.*

PROOF. $\pi|_K$ acts on $E'_\gamma = \{\varphi_v : v \in E_\gamma\} \subset \mathcal{H}'(\mathbf{E}_\gamma)$ by τ_γ, and if $\xi \in \mathfrak{p}_+$ then $d\pi'(\xi) \cdot E'_\gamma = 0$ directly from the definitions of Φ_γ and $\varphi_v(g) = \Phi_\gamma(g)^{-1}v$. Q.E.D.

Recall from Proposition 5.6 that ν_d^* has lowest highest weight $\gamma_d^* = -d\varepsilon_k - \kappa$ if $d \geqslant 0$, $= -d\varepsilon_{k+1} - \kappa$ if $d \leqslant 0$, relative to $\overline{\Sigma}^+$, where $\kappa = \varepsilon_1 + \cdots + \varepsilon_k$.

5.23. THEOREM. *ν_d^* is an extended holomorphic discrete series representation of $U(k, l)$ if and only if the function $\psi_{\gamma_d^*}$ of (5.16) is of positive type, and in that case ν_d^* is the representation of $U(k, l)$ on $\mathcal{H}'(\mathbf{E}_{\gamma_d^*})$.*

In particular, in all the line bundle cases ($d = 0$, or $d > 0$ and $k = 1$, or $d < 0$ and $l = 1$) ν_d^* is an extended holomorphic discrete series representation of $U(k, l)$.

PROOF. If ν_d^* is equivalent to an extended holomorphic discrete series representation, say on $\mathcal{H}'(\mathbf{E}_\gamma)$, then $\gamma = \gamma_d^*$ by Lemma 5.22, so $\psi_{\gamma_d^*}$ is of positive type. If $\psi_{\gamma_d^*}$ is of positive type and π' denotes the representation of $U(k, l)$ on $\mathcal{H}'(\mathbf{E}_\gamma)$, then ν_d^* is equivalent to π' by Lemma 5.22 and Theorem 5.8.

If $d = 0$ then $\gamma_d^* = -\kappa = -(\varepsilon_1 + \cdots + \varepsilon_k)$, so $\langle \gamma_d^*, \alpha_1 + \cdots + \alpha_{m-1} \rangle = -1$. It follows [**26**, Theorem 4.7] that $\psi_{\gamma_d^*}$ is of positive type.

If $k = 1$ and $d \geqslant 0$, then $\gamma_d^* = -(d + 1)\varepsilon_1$, and a result of Wallach [**34**], [**35**] announced in [**36**, §4] ensures that $\psi_{\gamma_d^*}$ is of positive type. If $l = 1$ and $d \leqslant 0$ we tensor ν_d^* by $\det_{\mathbf{C}}^{+1}$, replacing γ_d^* by $(d - 1)\varepsilon_m$, and again use Wallach's result. Q.E.D.

Let us look at the case $d > 0$ in $U(2, 1)$. Here $\gamma_d^* = (\varepsilon_1 + \varepsilon_2) - d\varepsilon_2$, so $\langle \gamma_d^*, \alpha_1 \rangle = d$ and $\langle \gamma_d^*, \alpha_2 \rangle = 1 - d$. Then Wallach's [**36**, Theorem 4.1] shows that ν_d^* does not belong to the extended holomorphic discrete series.

Finally let us note that ν_d and ν_d^* always are realized on some spaces of sections of a holomorphic vector bundle over $\mathfrak{D}$. For if π is an irreducible unitary representation of G, say on $\mathcal{H}$, and γ is its lowest highest weight as in Theorem 5.19, then

$$\mathcal{H} \ni f \mapsto \varphi_f, \qquad \varphi_f : G \to E_\gamma \quad \text{by } \varphi_f(g) = p_{E_\gamma}\pi(g^{-1})f$$

intertwines π with the action of G on some space of sections of $\mathbf{E}_\gamma \to \mathfrak{D}$.

Appendix: Algebraic reducibility of ν_d on a parabolic. Recall $\mathcal{H}_d^0 = \{\text{polynomials in } \mathcal{H}_d\} = \{U(k) \times U(l)\text{-finite vectors in } \mathcal{H}_d\}$, dense subspace of $\mathcal{H}_d$ consisting of all vectors finite under the maximal compact subgroup.

Let E be a maximal totally isotropic subspace of $\mathbf{C}^{k,l}$. We may assume $0 < k \leqslant l$ and that E is the real span of $\{p_a + p_{k+a}\}_{1 \leqslant a \leqslant k} \cup \{J_{k,l}(p_a + p_{k+a}) = -q_a + q_{k+a}\}_{1 \leqslant a \leqslant k}$. Write uv for $\xi_{u,v}$. By direct calculation, every $\xi \in \mathfrak{sp}(V)$ on the list

(A.1)

$$\begin{cases} (p_a q_b - p_b q_a) + (p_{k+a} q_{k+b} - p_{k+b} q_{k+a}), & 1 \leqslant a < b \leqslant k, \\ (p_a p_b + q_a q_b) - (p_{k+a} p_{k+b} + q_{k+a} q_{k+b}), & 1 \leqslant a \leqslant b \leqslant k, \\ (p_a q_{k+b} + p_{k+b} q_a) + (p_{k+a} q_{b+k} - p_{k+b} q_{k+a}), & 1 \leqslant a \leqslant k, 1 \leqslant b \leqslant k, \\ (p_a p_{k+b} - q_a q_{k+b}) + (p_{k+a} p_{k+b} + q_{k+a} q_{k+b}), & 1 \leqslant a \leqslant k, 1 \leqslant b \leqslant k, \\ (p_a q_{2k+c} + p_{2k+c} q_a) + (p_{k+a} q_{2k+c} - p_{2k+c} q_{k+a}), & \\ \qquad 1 \leqslant a \leqslant k, 1 \leqslant c \leqslant l-k, & \\ (p_a p_{2k+c} - q_a q_{2k+c}) + (p_{k+a} p_{2k+c} + q_{k+a} q_{2k+c}), & \\ \qquad 1 \leqslant a \leqslant k, 1 \leqslant c \leqslant l-k, & \\ p_{2k+c} q_{2k+d} - p_{2k+d} q_{2k+c}, & 1 \leqslant c < d \leqslant l-k, \\ p_{2k+c} p_{2k+d} + q_{2k+c} q_{2k+d}, & 1 \leqslant c \leqslant d \leqslant l-k, \end{cases}$$

satisfies $\xi(p_j + p_{k+j}) \in E$ for $1 \leqslant j \leqslant k$. If ξ is listed in (A.1), then Lemma 4.20 shows $\xi \in \mathfrak{u}(k, l)$, so also $\xi(J_{k,l}(p_j + p_{k+j})) \in E$, whence $\xi \in \mathfrak{p}_E$. Since (A.1) is a list of $3k^2 + 2k(l-k) + (l-k)^2$ linearly independent elements of $\mathfrak{p}_E$, and since [**38**, Theorem 2.10] $\mathfrak{p}_E$ has dimension $3k^2 + 2k(l-k) + (l-k)^2$, now (A.1) is a basis of $\mathfrak{p}_E$. The maximal unimodular subgroup P_E' is [**38**, Theorem 2.10] $P_E' = \{g \in P_E : |\det(g|_E)| = 1\}$ so $\mathfrak{p}_E' = \{\xi \in \mathfrak{p}_E : \operatorname{trace}(\xi|_E) \in i\mathbf{R}\}$. Now

$$\mathfrak{p}_E' = \{\xi \in \mathfrak{p}_E : \text{sum of the coefficients of the } p_a q_{k+a} + p_{k+a} q_a \text{ is zero}\}. \tag{A.2}$$

Combine (4.11), (4.12) and (A.2). The result is that $d\nu(\mathfrak{p}_E)_{\mathbf{C}}$ has basis (over $\mathbf{C}$) consisting of all

$$\begin{aligned} & z_a \frac{\partial}{\partial z_b} - z_{k+b} \frac{\partial}{\partial z_{k+a}} + \delta_{ab}, && \frac{\partial^2}{\partial z_a \partial z_{k+b}} - z_{k+a} \frac{\partial}{\partial z_{k+b}}, \\ & z_{k+b}\left(z_a - \frac{\partial}{\partial z_{k+a}}\right), && \frac{\partial^2}{\partial z_a \partial z_{2k+c}} - z_{k+a} \frac{\partial}{\partial z_{2k+c}}, \\ & z_{2k+c}\left(z_a - \frac{\partial}{\partial z_{k+a}}\right), && z_{2k+c} \frac{\partial}{\partial z_{2k+d}}, \end{aligned} \tag{A.3}$$

for $1 \leqslant a, b \leqslant k$ and $1 \leqslant c, d \leqslant l - k$. The subalgebra $d\nu(\mathfrak{p}'_E)_{\mathbf{C}}$ consists of all linear combination of these, for which the sum of the coefficients of the $\partial^2/\partial z_a \partial z_{k+a} - z_{k+a}(\partial/\partial z_{k+a})$ equals the sum of the coefficients of the $z_{k+a}(z_a - \partial/\partial z_{k+a})$.

The operators in (A.3) are not quite right for the algebraic part of the proof of Lemma 4.14, where we worked with the monomials z^n. So we modify the monomials:

A.4. LEMMA. *For each multi-index n there is a unique polynomial $\psi_n \in \mathcal{H}$ of the form $\psi_n(z) = z^n + (\text{lower terms})^3$ such that*

(i) $z_a(\partial/\partial z_a) - z_{k+a}(\partial/\partial z_{k+a}) + 1$ *multiplies* ψ_n *by* $n_a - n_{k+a} + 1$ $(1 \leqslant a \leqslant k)$,

(ii) $\partial^2/\partial z_a \partial z_{k+a} - z_{k+a}(\partial/\partial z_{k+a})$ *multiplies* ψ_n *by* $-n_{k+a}$ $(1 \leqslant a \leqslant k)$,

(iii) $z_{2k+c}(\partial/\partial z_{2k+c})$ *multiplies* ψ_n *by* n_{2k+c} $(1 \leqslant c \leqslant l - k)$.

PROOF. Fix n and denote $d_a = n_a - n_{k+a}$ for $1 \leqslant a \leqslant k$.

Suppose first that $k = 1$ and $m = 2k$. Using (i), the desired polynomial must have form

$$\psi_n(z) = z^n + \sum_{r \geqslant 1} b_r z_1^{n_1 - r} z_{k+1}^{n_1 - d_1 - r}.$$

Writing $b_0 = 1$, (ii) just says

$$-(n_1 - d_1)b_r = -(n_1 - d_1 - r)b_r + (n_1 - r + 1)(n_1 - d_1 - r + 1)b_{r-1}$$

for $1 \leqslant r \leqslant \min(n_1, n_1 - d_1)$. This recursively defines the b_r, and in fact gives us

$$\psi_n(z) = \sum_{r \geqslant 0} \left\{ (-1)^n \frac{1}{r!} \prod_0^{r-1} (n_1 - j)(n_1 - d_1 - j) \right\} z_1^{n_1 - r} z_{k+1}^{n_1 - d_1 - r}.$$

Next suppose that $k > 1$ and $m = 2k$. The case $k = 1$ provides a polynomial

$$\beta_n(z_1, z_{k+1}) = z_1^{n_1} z_{k+1}^{n_1 - d_1} + (\text{lower terms})$$

that satisfies (i) and (ii) for $a = 1$. Induction on k gives us a polynomial

$$\gamma_n(z_2, \ldots, z_k; z_{k+2}, \ldots, z_{2k})$$
$$= z_2^{n_2} \cdots z_k^{n_k} z_{k+2}^{n_2 - d_2} \cdots z_{2k}^{n_k - d_k} + (\text{lower terms})$$

that satisfies (i) and (ii) for $2 \leqslant a \leqslant k$. Now

$$\psi_n(z) = \beta_n(z_1, z_{k+1}) \cdot \gamma_n(z_2, \ldots, z_k; z_{k+a}, \ldots, z_{2k})$$

is the unique polynomial of the required form that satisfies (i) and (ii).

Now consider the general case. As just seen, there is a unique polynomial

[3]Here we say that a multi-index $n' < n$ if $n'_j \leqslant n_j$ for all j and $n'_j < n_j$ for at least one j.

$$\alpha_n(z_1, \ldots, z_{2k}) = z_1^{n_1} \cdots z_{2k}^{n_{2k}} + (\text{lower terms})$$

that satisfies (i) and (ii). Now

$$\psi_n(z) = \alpha_n(z_1, \ldots, z_{2k}) \cdot z_{2k+1}^{n_{2k+1}} \cdots z_m^{n_m}$$

is the unique polynomial of the required form that satisfies (i), (ii) and (iii). Q.E.D.

Let $\mathcal{V}$ be a subspace of $\mathcal{H}_d^0$. If $0 \neq f \in \mathcal{V}$, then there is at least one expansion $f(z) = az^n + \sum_{n'} a_{n'} z^{n'}$, $a \neq 0$, where the sum runs over multi-indices n' such that $n'_j < n_j$ for at least one j. Lemma A.4 gives another expansion $f(z) = a\psi_n(z) + \sum_n b_{n'}\psi_{n'}(z)$ with the same conditions on the n'. If $\mathcal{V}$ is $d\nu(\mathfrak{p}_E)$-invariant, then it is stable under the operators (i), (ii) and (iii) of Lemma A.4, and we conclude $\psi_n \in \mathcal{V}$. If $\mathcal{V}$ is only $d\nu(\mathfrak{p}'_E)$-invariant, it is stable under all the operators (i) and (iii) and the difference of any two operators (ii), and so we have some $\psi_n + \sum c_{n''}\psi_{n''} \in \mathcal{V}$ where the sum runs over multi-indices n'' with

$$n''_a - n''_{k+a} = n_a - n_{k+a} \quad \text{for } 1 \leqslant a \leqslant k, n''_{2k+c} = n_{2k+c} \quad \text{for } 1 \leqslant c \leqslant l-k,$$
$$n''_{k+a} - n''_{k+b} = n_{k+a} - n_{k+b} \quad \text{for } 1 \leqslant a < b \leqslant k, n''_j < n_j \quad \text{for some index } j.$$

In view of the first three of these relations, the fourth becomes $n''_j < n_j$ for $1 \leqslant j \leqslant 2k$, and thus we have some

$$f' = \psi_n + \sum_r c_r \psi_{n-re} \in \mathcal{V}$$

where $n - re$ denotes $(n_1 - r, n_2 - r, \ldots, n_{2k} - r, n_{2k+1}, \ldots, n_m)$ and the sum runs from $r = 1$ to $r = \max\{n_1, \ldots, n_{2k}\}$. If $\sum s_a = \sum t_a$, then the sentence just after (A.3) tells us that $d\nu(\mathfrak{p}'_E)_{\mathbf{C}}$ contains

$$P_{s,t} = \sum_{a=1}^k s_a\left(\frac{\partial^2}{\partial z_a \partial z_{k+a}} - z_{k+a}\frac{\partial}{\partial z_{k+a}}\right) + \sum_{a=1}^k t_a\left(z_a z_{k+a} - z_{k+a}\frac{\partial}{\partial z_{k+a}}\right)$$
$$= \sum_{a=1}^k (s_a + t_a)\left(\frac{\partial^2}{\partial z_a \partial z_{k+a}} - z_{k+a}\frac{\partial}{\partial z_{k+a}}\right) + \sum_{a=1}^k t_a\left(z_a z_{k+a} - \frac{\partial^2}{\partial z_a \partial z_{k+a}}\right).$$

That operator sends

$$f' = \psi_n + \sum_{r \geqslant 1} c_r \psi_{n-re} = \sum_{r \geqslant 0} c_r \psi_{n-re}$$

to

$$P_{s,t}(f') = \sum_{r \geqslant 0} c_r \sum_{j=1}^k (s_j + t_j)(-n_{k+j} + r)\psi_{n-re}$$
$$+ \sum_{j=1}^k t_j z_j z_{k+j} f' - \sum_{j=1}^k t_j \frac{\partial^2 f'}{\partial z_j \partial z_{k+j}}.$$

If $k > 1$, and if A is the difference of the $d\nu$-images of the two operators (ii) of Lemma A.4 for indices a, b, then A multiplies the first term by $n_{k+b} - n_{k+a}$, multiplies $t_j z_j z_{k+j} f'$ by $(n_{k+b} + \delta_{b,j}) - (n_{k+a} + \delta_{a,j})$, and multiplies $t_j(\partial^2 f'/\partial z_j \partial z_{k+j})$ by $(n_{k+b} - \delta_{b,j}) - (n_{k+a} - \delta_{a,j})$. Since $\mathcal{V}$ is $d\nu(\mathfrak{p}'_E)$-invariant, now

$$\sum_{r>0} c_r \sum_{j=1} (s_j + t_j)(-n_{k+j} + r)\psi_{n-re} \in \mathcal{V}.$$

Choosing $s_j = t_j = \delta_{j,1}$, we conclude $\psi_n \in \mathcal{V}$.

In both the $d\nu(\mathfrak{p}_E)$-invariant case and the $d\nu(\mathfrak{p}'_E)$-invariant case, we actually obtained ψ_n from $f = az^n + \Sigma_{n'} a_{n'} z^{n'}$ as the image of f by $d\nu(\Xi)$ for some element Ξ of the complex universal enveloping algebra $\mathfrak{U}(\mathfrak{p}_E)$ or $\mathfrak{U}(\mathfrak{p}'_E)$. Iterating this,

A.5. Lemma. *Let $\mathcal{V}$ be a subspace of $\mathcal{H}^0_d$ that is $d\nu(\mathfrak{p}'_E)$-invariant if $k > 1$, $d\nu(\mathfrak{p}_E)$-invariant if $k = 1$. Then every element of $\mathcal{V}$ is of the form $\Sigma b_n \psi_n$, ψ_n as in Lemma A.4; if $b_n \neq 0$ then $\psi_n = d\nu(\Xi)(\Sigma b_n \psi_n) \in \mathcal{V}$ for an appropriate element Ξ in $\mathfrak{U}(\mathfrak{p}'_E)$ if $k > 1$, in $\mathfrak{U}(\mathfrak{p}_E)$ if $k = 1$.*

In order to discuss the action of the universal enveloping algebra on a polynomial ψ_n, we denote

$$\mathcal{H}^0_{r,s} = \operatorname{span}\left\{ \psi_n : \sum_1^k n_a = r \text{ and } \sum_{k+1}^k n_b = s \right\} \subset \mathcal{H}^0_{r-s}.$$

Evidently $\mathcal{H}^0_{r,s}$ has finite dimension$\binom{k+r-1}{r}\binom{l+s-1}{s}$.

A.6. Lemma. *Let $\mathfrak{p}^{(1)}_E$ denote $\mathfrak{p}'_E$ if $k > 1$, $\mathfrak{p}_E$ if $k = 1$. If $\psi_n \in \mathcal{H}^0_{r,s}$ then $d\nu(\mathfrak{U}(\mathfrak{p}^{(1)}_E)) \cdot \psi_n$ contains $\mathcal{H}^0_{r+t,s+t}$ for every integer $t \geqslant 0$. In particular, $d\nu(\mathfrak{U}(\mathfrak{p}^{(1)}_E)) \cdot \psi_n$ has finite codimension in $\mathcal{H}^0_{r-s}$, bounded by*

$$\sum_{t=1}^{\min(r,s)} \dim \mathcal{H}^0_{r-t,s-t} = \sum_{t=1}^{\min(r,s)} \binom{k+r-t-1}{r-t}\binom{l+s-t-1}{s-t}.$$

Proof. If an element of $d\nu(\mathfrak{p}^{(1)}_E)_{\mathbf{C}}$ carries ψ_n to a polynomial with leading term $z^{n'}$, then Lemmas A.4 and A.5 tell us $\psi_{n'} \in d\nu(\mathfrak{U}(\mathfrak{p}^{(1)}_E)) \cdot \psi_n$.

Now write ε_j for $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the j-place and run through (A.3) as follows:

$$z_{2k+c}\left(z_a - \frac{\partial}{\partial z_{k+a}}\right) \text{ gives } \psi_{n+\varepsilon_a+\varepsilon_{2k+c}} \qquad (1 \leqslant a \leqslant k, 1 \leqslant c \leqslant l),$$

$$z_{k+b}\left(z_a - \frac{\partial}{\partial z_{k+a}}\right) \text{ gives } \psi_{n+\varepsilon_a+\varepsilon_{k+b}} \qquad (1 \leqslant a, b \leqslant k, a \neq b),$$

$$z_{k+a}\left(z_a - \frac{\partial}{\partial z_{k+a}}\right) + \left(\frac{\partial}{\partial z_a} - z_{k+a}\right)\frac{\partial}{\partial z_{k+a}} \text{ gives } \psi_{n+\varepsilon_a+\varepsilon_{k+a}} \qquad (1 \leqslant a \leqslant k).$$

Iterating, we obtain every $\psi_{n'} \in \mathcal{H}_{r-s}$ such that $n'_j \geqslant n_j$ for every j. In particular, we obtain some element $\psi_{n'} \in \mathcal{H}^0_{r+t,s+t}$ for every $t \geqslant 0$. Furthermore, with any such $\psi_{n'}$:

$$\frac{\partial^2}{\partial z_a \partial z_{k+b}} - z_{k+a}\frac{\partial}{\partial z_{k+b}} \text{ gives } \psi_{n'+\varepsilon_{k+a}-\varepsilon_{k+b}} \quad (1 \leqslant a, b \leqslant k, a \neq b),$$

and so

$$z_b\frac{\partial}{\partial z_a} - z_{k+a}\frac{\partial}{\partial z_{k+b}} + \delta_{ab} \text{ gives } \psi_{n'+\varepsilon_b-\varepsilon_a} \quad (1 \leqslant a, b \leqslant k, a \neq b).$$

Also,

$$z_{2k+c}\frac{\partial}{\partial z_{2k+d}} \text{ gives } \psi_{n'+\varepsilon_{2k+c}-\varepsilon_{2k+d}} \quad (1 \leqslant c, d \leqslant l-k),$$

$$\frac{\partial^2}{\partial z_a \partial z_{2k+c}} - z_{k+a}\frac{\partial}{\partial z_{2k+c}} \text{ gives } \psi_{n'+\varepsilon_{k+a}-\varepsilon_{2k+c}} \quad (1 \leqslant a \leqslant k, 1 \leqslant c \leqslant l-k),$$

$$z_{2k+c}\left(z_a - \frac{\partial}{\partial z_{k+a}}\right) \text{ gives } \psi_{n'-\varepsilon_{k+a}+\varepsilon_{2k+c}} \quad (1 \leqslant a \leqslant k, 1 \leqslant c \leqslant l-k).$$

Thus one $\psi_{n'} \in \mathcal{H}^0_{r+t,s+t}$ gives every $\psi_{n''} \in \mathcal{H}^0_{r+t,s+t}$. Q.E.D.

Now we combine our three lemmas to prove the algebraic analog of Theorem 4.27:

A.7. THEOREM. *Let $\mathcal{V}$ be a nonzero subspace of $\mathcal{H}^0_d$ that is stable under $d\nu(\mathfrak{p}'_E)$ if $k > 1$, under $d\nu(\mathfrak{p}_E)$ if $k = 1$. Then $\mathcal{V}$ has finite codimension in $\mathcal{H}^0_d$.*

PROOF. Let $0 \neq f \in \mathcal{V}$. Then Lemma A.5 gives us some $\psi_n \in \mathcal{V}$ of the form $d\nu(\Xi)\cdot f$, $\Xi \in \mathfrak{U}(\mathfrak{p}_E^{(1)})$. Lemma A.6 says that $d\nu(\mathfrak{U}(\mathfrak{p}_E^{(1)}))\cdot\psi_n$ has finite codimension in $\mathcal{H}^0_d$. As $d\nu(\mathfrak{U}(\mathfrak{p}_E^{(1)}))\cdot\psi_n \subset \mathcal{V}$, the theorem is proved. Q.E.D.

REFERENCES

1. R. L. Anderson, J. Fischer and R. Raczka, *Coupling problem for $U(p, q)$ ladder representations*, Proc. Roy. Soc. Ser. A **302** (1968), 491–500. MR **36** #2738.

2. V. Bargmann, *Group representations on Hilbert spaces of analytic functions*, Analytic Methods in Mathematical Physics, Gordon and Breach, 1970, pp. 27–63. MR **50** #3787.

3. Carey and Hannabuss, Preprint.

4. Corwin-Ne'eman-Sternberg, *Graded Lie Algebras in mathematics and physics*, Rev. Modern Phys. **47** (1975), 573–604.

5. D. Ž. Djoković and G. Hochschild, *Semi-simplicity of 2-graded Lie algebras*. II, Illinois J. Math. **20** (1976), 134–143.

6. S. Gelbart, *Harmonics on Stiefel manifolds and generalized Hankel transforms*, Bull. Amer. Math. Soc. **78** (1972), 451–455.

———, *Holomorphic discrete series for the real symplectic group*, Invent. Math. **19** (1973), 49–58. MR **47** #8770.

7. K. I. Gross and R. A. Kunze, *Bessel functions and representation theory*. II, *Holomorphic discrete series and metaplectic representations*, J. Functional Analysis **25** (1977), 1–49.

8. Harish-Chandra, *Representations of semi-simple Lie grouss*. VI, Amer. J. Math. **78** (1956), 564–628. MR **18**, 490.

9. C. Itzykson, *Remarks on boson commutation rules*, Comm. Math. Phys. **4** (1967), 92–122. MR **35** #2569.

10. A. Joseph, *The minimal orbit in a simple Lie algebra and its associated maximal ideal*, Ann. Sci. École Norm. Sup. **9** (1976), 1–29.

11. V. Kac, *Lie superalgebras* (to appear).

12. M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials* (preprint).

13. D. Kazhdan, *Some applications of the Weil representation* (preprint).

14. S. Kobayashi, *On automorphism groups of homogeneous complex manifolds*, Proc. Amer. Math. Soc. **12** (1961), 359–361. MR **24** #A3664.

15. ______, *Irreducibility of certain unitary representations*, J. Math. Soc. Japan **20** (1968), 638–642. MR **37** #6968.

16. G. Mack and I. Todorov, *Irreducibility of the ladder representations of* $U(2, 2)$ *when restricted to the Poincaré subgroup*, J. Mathematical Phys. **10** (1969), 2078–2085. MR **40** #3816.

17. G. W. Mackey, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311. MR **20** #4789.

18. S. Murakami, *On automorphisms of Siegel domains*, Lecture Notes in Math., vol. 286, Springer-Verlag, Berlin and New York, 1972. MR **51** #944.

19. H. Ozeki and M. Wakimoto, *On polarizations of certain homogeneous spaces*, Hiroshima Math. J. **2** (1972), 445–482. MR **49** #5236a.

20. I. I. Pjateckiĭ-Šapiro, *Automorphic functions and the geometry of classical domains*, Fizmatgiz, Moscow, 1961; English transl., Gordon and Breach, New York, 1969. MR **25** #231; **40** #5908.

21. L. Pukánszky, *On the unitary representations of exponential groups*, J. Functional Analysis **2** (1968), 73–113. MR **37** #4205.

22. ______, *On the theory of exponential groups*, Trans. Amer. Math. Soc. **126** (1967), 487–507. MR **35** #301.

23. J. H. Rawnsley, *On the cohomology groups of a polarization and diagonal quantization* (preprint).

24. ______, *Representations of a semi-direct product by quantization*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 345–350.

25. H. Rossi and M. Vergne, *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math. **136** (1976), 1–59.

26. ______, *Continuation analytique de la série discrete holomorphic*, Noncommutative Harmonic Analysis, Lecture Notes in Math., vol. 466, Springer-Verlag, Berlin and New York, 1975.

27. L. P. Rothschild and J. A. Wolf, *Representations of semisimple groups associated to nilpotent orbits*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 155–174. MR **50** #10158.

28. W. Schmidt, *Some properties of square-integrable representations of semi-simple Lie groups*, Ann. of Math.(2) **102** (1975), 535–564.

29. D. Shale, *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. **103** (1962), 149–167. MR **25** #956.

30. M. Takeuchi, *Homogeneous Siegel domains*, Publ. Study Group of Geometry, **7**, Tokyo, 1973.

31. J. A. Tirao, *Square integrable representations of semisimple Lie groups*, Trans. Amer. Math. Soc. **190** (1974), 57–75. MR **49** #3036.

32. I. T. Todorov, *Discrete series of hermitian representations of the Lie algebra of* $U(p, q)$, Lecture Notes, ICTP, Trieste, 1975.

33. M. Vergne, *Étudie de certains représentations induites d'un groupe de Lie résoluble*

exponentiel, Ann. Sci. Ecole Norm. Sup. (4) **3** (1970), 353–384. MR **43** #4966.

34. N. Wallach, *Analytic continuation of the discrete series*. I, Trans. Amer. Math. Soc. (to appear).

35. ______, *Analytic continuation of the discrete series*. II, Trans. Amer. Math. Soc. (to appear).

36. ______, *On the unitarizability of representations with highest weights*, Noncommutative Harmonic Analysis, Lecture Notes in Math., vol. 466, Springer-Verlag, Berlin and New York, 1975.

37. A. Weil, *Sur certains groupes d'operateurs unitaires*, Acta Math. **111** (1964), 143–211. MR **29** #2324.

38. J. A. Wolf, *Unitary representations of maximal parabolic subgroups of the classical groups*, Mem. Amer. Math. Soc. No. 180 (1976).

39. ______, *Representations of certain semi-direct product groups*, J. Functional Analysis **19** (1975), 339–372.

40. ______, *Unitary representations of partially holomorphic cohomology spaces*, Mem. Amer. Math. Soc. No. 138 (1974).

41. ______, *Remark on nilpotent orbits*, Proc. Amer. Math. Soc. **51** (1975), 213–216.

Department of Astronomy and Physics, Tel Aviv University, Tel Aviv, Israel

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel

Department of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

Current address (Shlomo Sternberg): Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

Current address (J. A. Wolf): Department of Mathematics, University of California, Berkeley, California 94720