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CONDITIONALLY COMPACT SEMITOPOLOGICAL ONE-PARAMETER INVERSE SEMIGROUPS OF PARTIAL ISOMETRIES

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ABSTRACT. The algebraic structure of one-parameter inverse semigroups has been completely described. Furthermore, if B is the bicyclic semigroup and if B is contained in any semitopological semigroup, the relative topology on B is discrete. We show that if F is an inverse semigroup generated by an element and its inverse, and F is contained in a compact semitopological semigroup, then the relative topology is discrete; in fact, if F is any one-parameter inverse semigroup contained in a compact semitopological semigroup, then the multiplication on F is jointly continuous if and only if the inversion is continuous on F, and we describe $\bar{F}$ in that case. We also show that if $\{J_t\}$ is a one-parameter semigroup of bounded linear operators on a (separable) Hilbert space, then $\{J_t\} \cup \{J_t^*\}$ generates a one-parameter inverse semigroup T with $J_t^{-1} = J_t^*$ if and only if $\{J_t\}$ is a one-parameter semigroup of partial isometries, and we describe the weak operator closure of T in that case.

1. Introduction. An *inverse semigroup* is a semigroup in which each element x has a unique inverse x^{-1} with the properties that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If G is any subgroup of the positive real numbers under ordinary multiplication, let $P = G \cap [1, \infty)$. Let $F_P = \{(x, y, z) \in P^3\colon\ y \geqslant x$ and $y \geqslant z\}$ together with the following multiplication:

$$(x, y, z)(r, s, t) = \left(\frac{xzr}{y \wedge zr}, \frac{yzrs}{(y \wedge zr)(zr \wedge s)}, \frac{zrt}{zr \wedge s} \right)$$

where $x \wedge y = \min(x, y)$. Any inverse semigroup generated by a homomorphic image of P is a homomorphic image of F_P [6] and is called a *one-parameter inverse semigroup*. We mention at this point one homomorphic image of F_P. Let $B_P = P \times P$ together with this multiplication:

$$(x, y)(z, w) = \left(\frac{xz}{y \wedge z}, \frac{yw}{y \wedge z} \right).$$

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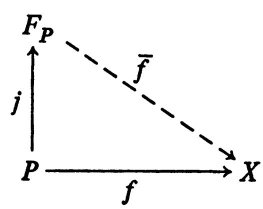
If $P = \{1, x, x^2, \dots\}$ where $x > 1$, then B_P is the *bicyclic semigroup* whose properties are discussed in [4].

Suppose that Y is a semigroup with a Hausdorff topology in which the maps $x \to xy$ and $x \to yx$ of $Y \to Y$ are continuous for each y in Y. Then the multiplication on Y is said to be *separately continuous* and Y is called a *semitopological semigroup*. If the map $(x, y) \to xy$ of $Y \times Y \to Y$ is continuous, then the multiplication on Y is *jointly continuous* (or just *continuous*) and Y is called a *topological* semigroup.

Let X be a one-parameter inverse semigroup generated by the image of P under a continuous homomorphism f: $P \to X$. Let j: $P \to F_P$ be the map $x \to (x, x, x)$. Then j is an isomorphism of P into F_P and we can identify P with its image in F_P. Then $x^{-1} = (1, x, 1)$ and hence $(x, y, z) = xy^{-1}z$. Furthermore, if $\bar{f}$: $F_P \to X$ is defined by

$$\bar{f}(xy^{-1}z) = f(x)f(y)^{-1}f(z),$$

then $\bar{f}$ is a homomorphism and the following diagram commutes [6], [1]:



If F_P is endowed with the relative product topology which it inherits from R^3, then the multiplication on F_P is jointly continuous. We shall show that if X is embedded densely in a compact semitopological semigroup and that if the inversion on $f(P) \cup f(P)^{-1}$ is continuous, then in fact $\bar{f}$ is a *continuous* homomorphism of F_P onto X.

One-parameter inverse semigroups occur naturally in $B(H)$, the semigroup of bounded linear operators on Hilbert space. In the weak operator topology, the closed unit ball is a compact semitopological semigroup, and the map $T \to T^*$ is continuous where T^* is the adjoint of the operator T. A partial isometry is an operator which is isometric on the orthogonal complement of its kernel; it is known [8, p. 99] that T is a partial isometry if and only if $TT^*T = T$. We will show that if $J = \{J_t: t \in [0, \infty)\}$ is a one-parameter semigroup of partial isometries, then $J \cup J^*$ generates a one-parameter inverse semigroup.

This example shows that it is not unreasonable to ask that the map $x \to x^{-1}$ be continuous on X, and in that case we are able to give a description of X.

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1.1. REMARK. The closure of an inverse semigroup in a compact topological semigroup is an inverse semigroup, but Brown and Moran [3, Proposition 1] give an example of a unitary group whose weak closure is not an inverse semigroup.

2. The one-parameter inverse semigroups. In this section we summarize the possible congruence relations on F_P (and hence its homomorphic images), as given in [6, 3.10]. We then point out that every one-parameter semigroup J of partial isometries generates a one-parameter inverse semigroup X, and that each of the possibilities for X can be realized as an inverse semigroup generated by a partial isometry.

If $1 < t \in P$, let $I_t = \{xy^{-1}z \colon y > t\} \subseteq Fp$ and $I_t^0 = \{xy^{-1}z \in F_P; y > t\}$. I_t and I_t^0 are the only ideals of F_P. Let α and β be the congruences on F_P defined so that $xy^{-1}z \,\alpha\, rs^{-1}t$ if and only if $x = r$ and $yt = sz$ and $xy^{-1}z \,\beta\, rs^{-1}t$ if and only if $z = t$ and $yr = xs$. Then $F_P/\alpha \approx Bp \approx F_P/\beta$. For each subgroup N of G, define σ_N to be the group congruence on F_P defined so that $xy^{-1}z \,\sigma_N\, rs^{-1}t$ if and only if $xzs/rty \in N$. Then if I is an ideal of F_P and $\delta = \alpha, \beta$, or σ_N, define the relation $I\delta$ by $xy^{-1}z \, I\delta \, rs^{-1}t$ if and only if either

(a) $xy^{-1}z$ and $rs^{-1}t$ are in I and $xy^{-1}z \,\delta\, rs^{-1}t$ or

(b) $xy^{-1}z = rs^{-1}t \notin I$.

2.1. PROPOSITION (EBERHART AND SELDEN). *Every congruence on F_P is $I\delta$ for some ideal I and congruence $\delta = \alpha, \beta$, or σ_N. Hence the only possible homomorphic images of F_P are $(F_P \setminus I) \cup B_P$, $(F_P \setminus I) \cup (G/N)$ or F_P* [6, *Theorem* 3.10].

2.2. PROPOSITION. *Let T be a semigroup with an involution $*$, i.e., a mapping $u \to u^*$ such that $(uv)^* = v^*u^*$ and $(u^*)^* = u$, and let $R = \{R_t \colon t \in P\}$ be a one-parameter subsemigroup of T. Then $\langle R, R^*\rangle$, the semigroup generated by $R \cup R^*$, is an inverse semigroup if and only if*

(i) *$R_tR_t^*R_t = R_t$ for each t in P and*

(ii) *$R_tR_t^*R_s^*R_s = R_s^*R_sR_tR_t^*$ for each s and t in P.*

PROOF. The conditions are clearly necessary. Now if R is as above, and $s > t$, we have

$$\begin{aligned} R_sR_t^* &= R_{s-t}R_tR_t^* = R_{s-t}R_{s-t}^*R_{s-t}R_tR_t^* \\ &= R_{s-t}R_tR_t^*R_{s-t}^*R_{s-t} = R_sR_s^*R_{s-t}. \end{aligned}$$

Similarly, $R_t^*R_s = R_{s-t}R_s^*R_s$, and it follows that any "word" from $\langle R, R^*\rangle$ can be expressed as a triple $R_xR_y^*R_z$ where $y > x$ and $y > z$. The arguments of [6, pp. 56–57] can be used to show that the map $h: F_P \to \langle R, R^*\rangle$ defined by $h(xy^{-1}z) = R_xR_y^*R_z$ is a homomorphism, and since F_P is an inverse semigroup, so is its image $\langle R, R^*\rangle$.

Now let $J = \{J_t\}_{t=0}^{\infty}$ be a one-parameter semigroup of partial isometries on a (separable) Hilbert space. By [7, Theorem B],

$$J_t = U_t \oplus K_t \oplus S_t \oplus T_t,$$

where U_t is unitary, K_t is purely isometric, S_t is purely coisometric, and T_t is a nilpotent and hence, by [12], the direct integral of truncated shifts. Recall that a *truncated shift* is an operator which is unitarily equivalent to R_t, where for f in $\mathfrak{L}^2[K, a]$ (if K is a separable Hilbert space and $a > 0$, $\mathfrak{L}^2[K, a]$ is the Hilbert space of measurable K-valued functions on $[0, a]$ with square-integrable K-norm):

$$R_tf(x) = \begin{cases} 0 & \text{if } x < t \\ f(x - t) & \text{if } t \leqslant x \leqslant a \end{cases}$$

and $R_t = 0$ if $t \geqslant a$. This a is called the *index* of R_t. Since

$$R_t^*f(x) = \begin{cases} 0 & \text{if } a - t < x \leqslant a \\ f(x + t) & \text{if } 0 \leqslant x \leqslant a - t \end{cases}$$

and $R_t^* = 0$ if $t \geqslant a$, we have the following.

2.3. PROPOSITION. *If $R = \{R_t\}$ is a one-parameter semigroup of truncated shifts of index a, then $\langle R, R^*\rangle$ is isomorphic to F_P/I_a.*

PROOF. Since, for x, y, and z in P, with $y < a$,

$$R_xR_y^*R_zf(t) = \begin{cases} 0 & \text{if } t < x \text{ or } t > a - y + x \\ f(t - x + y - z) & \text{otherwise,} \end{cases}$$

it is easy to see that R satisfies the conditions of 2.2 and hence $\langle R, R^*\rangle$ is an inverse semigroup. Furthermore, one sees that $R_xR_y^*R_z = 0$ if and only if $y \geqslant a$, and hence by 2.1, $\langle R, R^*\rangle \approx F_P/I_a$.

It now follows that if $T = \{T_t\}$ is a one-parameter semigroup of nilpotent partial isometries of index a (and hence a direct integral of truncated shifts of index less than or equal to a), then $\langle T, T^*\rangle$ is isomorphic to $(F_P \setminus I_a) \cup \{0\}$.

Proposition 2.2 also gives the following.

2.4. PROPOSITION. *If $J = \{J_t\} = \{U_t \oplus K_t \oplus S_t \oplus T_t\}$ is a one-parameter semigroup of partial isometries, then $\langle J, J^*\rangle$ is a one-parameter inverse semigroup. Furthermore, if $\{U_t\} \neq I$ is a semigroup of unitary operators, $\{K_t\}$ of nonunitary isometries, $\{S_t\}$ of nonunitary coisometries, and $\{T_t\}$ of nilpotents of index a, then*

(1) $\langle\{U_t \oplus S_t\}, \{U_t^* \oplus S_t^*\}\rangle \approx (F_P \setminus I_a) \cup R^+$;

(2) $\langle\{K_t \oplus T_t\}, \{K_t^* \oplus T_t^*\}\rangle \approx (F_P \setminus I_a) \cup B_P$;

(3) $\langle\{K_t \oplus S_t\}, \{K_t^* \oplus S_t^*\}\rangle \approx F_P$;
(4) $\langle\{K_t\}, \{K_t^*\}\rangle \approx B_P$.

3. Conditionally compact one-parameter inverse semigroups. We now assume that X is a one-parameter inverse semigroup densely embedded in a compact semitopological semigroup Y. In case $P = \{1, x, x^2, \ldots\}$ where $x > 1$, we show that X has the discrete topology as a subspace of Y. Furthermore, if P is dense in $[1, \infty)$, and the inversion $x \to x^{-1}$ on $P \cup P^{-1}$ is continuous, then the multiplication is continuous on X. We then assume, from 3.2.7 on, that inversion is continuous on X, and describe $\overline{X}$.

Since semigroups of the type B_P are somewhat more accessible than those of the more general type, and some information is known about them in case $P = \{1, x, x^2, \ldots\}$ for $x > 1$, we devote a short section to these semigroups.

3.1. Bicyclic Semigroups. As we have mentioned before, if $P = \{1, x, x^2, \ldots\}$ where $x > 1$, then B_P is the familiar bicyclic semigroup, B, so-called because it is generated by two mutually inverse elements p and q subject to the generating relation $pq = 1$; we summarize the results of [2] on this subject:

(a) B is a discrete subspace of $\overline{B}$ in the relative topology. (Originally proved in [5], as was (b).)

(b) $\overline{B} \setminus B$ is a closed two-sided ideal of $\overline{B}$.

(c) The minimal idempotent e of the monothetic semigroup $\Gamma = \{q, q^2, \ldots\}$ is the minimal idempotent of $\overline{B}$, and the minimal ideal K of Γ, which is a compact topological group, with identity e, in the relative topology from $\overline{B}$, is the minimal ideal of $\overline{B}$. Furthermore $K = e\overline{B} = \overline{B}e$.

(d) The decreasing sequence $q^n p^n$ of idempotents converges to an idempotent w; wB is a cyclic group with identity $w\overline{B} = \overline{wB} = \overline{B} \setminus B$.

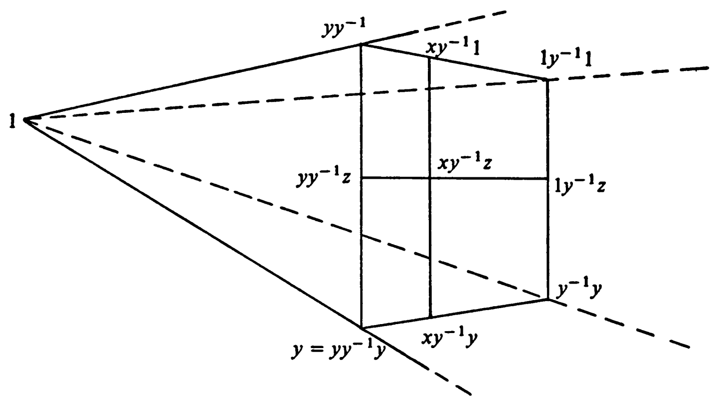
(e) If the map $x \to x^*$ is a continuous involution, then e and w are selfadjoint, and for each $x \in \overline{B}$, $(ex)^* = ex^* = (ex)^{-1}$ in K.

(f) If $B = \langle T, T^*\rangle$ where T is an isometry, then w is the Wold idempotent which decomposes H into $H_1 \oplus H_2$ where $T|H_1$ is unitary and $T|H_2$ is a unilateral shift. The minimal idempotent e decomposes H into $H_3 \oplus H_4$, where $T|H_3$ is unitary, and H_4 is the closed linear span of the eigenvalues of T (and T^*) corresponding to its unimodular eigenvalues.

These results show that we have no more (nor less) information about the closure of B than we have about the closure of a group, which examples in [3] and [12] show may be arbitrarily pathological.

If P is a dense subsemigroup of $[1, \infty)$, the existence of the idempotent w follows exactly as in [2], as does the fact that wB_P is a group. That $\overline{B}_P \setminus B_P = \overline{wB_P}$ turns out to be equivalent to the continuity of the inversion on B_P as we see for F_P in 3.2.6, and from 3.2.7 on, we shall assume this condition.

3.2. F_P. Suppose $X = F_P$. Then X may be pictured as below:



The idempotents of F_P are precisely those of the form $x(xy)^{-1}y$ where x and y are elements of P. [6, 2.13]. Now the idempotents $\{xx^{-1}: x \in P\}$ and $\{x^{-1}x: x \in P\}$ form two decreasing nets as $x \to \infty$, and hence there exist two idempotents E and F such that $xx^{-1} \to E$ and $x^{-1}x \to F$ as $x \to \infty$. Since xx^{-1} is decreasing, $E \neq xx^{-1}$ for any x in P, and similarly, $F \neq x^{-1}x$ for any x in P. Furthermore, $xx^{-1}E = E = Exx^{-1}$ for every x in P and $x^{-1}xF = F = Fx^{-1}x$ for every x in P.

Proposition 3.2.1. *E and F commute with every element $xy^{-1}z$ of F_P, and hence with every element of $\overline{F_P}$.*

Proof. As $x \to \infty$, $xx^{-1} \to E$, so if $r \in P$, $rxx^{-1} = r(r^{-1}r)(xx^{-1}) = r(xx^{-1})(r^{-1}r)$ (since idempotents commute) $= (rx)(rx)^{-1}r \to Er$; but $rxx^{-1} \to rE$, so $Er = rE$. On the other hand, for each r in P, $xx^{-1}r^{-1} = xx^{-1}r^{-1}rr^{-1} = r^{-1}rxx^{-1}r^{-1} = r^{-1}(rx)(rx)^{-1} \to r^{-1}E$; but $xx^{-1}r^{-1} \to Er^{-1}$, so $Er^{-1} = r^{-1}E$ for each r in P. Hence if $xy^{-1}z \in F_P$, $Exy^{-1}z = xEy^{-1}z = xy^{-1}Ez = xy^{-1}zE$. Similarly, F commutes with each element of F_P and hence with each element of $\overline{F_P}$.

Proposition 3.2.2. *If $\{z_\alpha\}$ and $\{y_\alpha\}$ are unbounded increasing nets from P, then $\{x_\alpha x_\alpha^{-1}y_\alpha^{-1}y_\alpha\}$ converges to EF.*

Proof. Let $\{x_{\alpha_\delta}x_{\alpha_\delta}^{-1}y_{\alpha_\delta}^{-1}y_{\alpha_\delta}\}$ be a subnet which converges, say to Z. Then

$$EFx_{\alpha_\delta}x_{\alpha_\delta}^{-1}y_{\alpha_\delta}^{-1}y_{\alpha_\delta} = Ex_{\alpha_\delta}x_{\alpha_\delta}^{-1}Fy_{\alpha_\delta}^{-1}y_{\alpha_\delta} = EF,$$

but this net converges to EFZ, so $EF = EFZ$. Now let $x \in P$, and assume $x_{\alpha_\delta} > x$. Then

$$xx^{-1}x_{\alpha_\delta}x_{\alpha_\delta}^{-1}y_{\alpha_\delta}^{-1}y_{\alpha_\delta} = x_{\alpha_\delta}x_{\alpha_\delta}^{-1}y_{\alpha_\delta}^{-1}y_{\alpha_\delta} \to Z,$$

so $xx^{-1}Z = Z$ for each x in P. Similarly $Zx^{-1}x = Z$ for each x in P, and hence, letting x approach ∞, $EZ = Z = ZF$, so $EFZ = EZF = EZ = Z$, so $EF = EFZ = Z$. It now follows that $x_\alpha x_\alpha^{-1} y_\alpha^{-1} y_\alpha \to EF$.

PROPOSITION 3.2.3. *$EF_P \cap F_P = \varnothing$; $FF_P \cap F_P = \varnothing$ and $EFF_P \cap F_P = \varnothing$.*

PROOF. Suppose $E \in F_P$; then $E = xx^{-1}y^{-1}y$ for some x and y in P. But if $k > x$, $E = kk^{-1}E = kk^{-1}xx^{-1}y^{-1}y = kk^{-1}y^{-1}y \neq xx^{-1}y^{-1}y = E$. Thus E is not in F_P, and a similar maneuver shows that F and EF are not in F_P. Now we show that $EF_P \cap F_P = \varnothing$. Suppose that there exists x in P such that $Ex^{-1} \in F_P$. Then $E = Exx^{-1} = xEx^{-1} \in F_P$. So there is no such x. Now if $xy^{-1}z$ is any element of F_P such that $Exy^{-1}z \in F_P$, then

$$\begin{aligned} E(y/x)^{-1} &= Exx^{-1}(y/x)^{-1} = Exy^{-1} \\ &= xy^{-1}Ezz^{-1} = (Exy^{-1}z)z^{-1} \in F_P, \end{aligned}$$

and this contradiction proves that there is no such $xy^{-1}z$. The proof for F and EF is similar.

PROPOSITION 3.2.4. *If $\{x_\alpha y_\alpha^{-1} z_\alpha\}$ is a net from F_P such that $\{y_\alpha\}$ is unbounded, then if $x_\alpha y_\alpha^{-1} z_\alpha \to u$, $u \notin F_P$. In fact, $u \in \overline{EF_P} \cup \overline{FF_P}$.*

PROOF. Note that if e is any idempotent and x any element of $\overline{F}_P$, then $ex = x$ if and only if $x \in \overline{eF_P}$. Now if $\{x_\alpha\}$ is unbounded, for any k in P, x_α is eventually larger than k, so $kk^{-1}x_\alpha y_\alpha^{-1} z_\alpha = x_\alpha y_\alpha^{-1} z_\alpha \to u$, so $kk^{-1}u = u$, and hence $Eu = u$. If u were in F_P, u would be in EF_P, so $u \notin F_P$. Similarly, if $\{z_\alpha\}$ is unbounded, u would be in$FF_P \setminus F_P$. If $\{x_\alpha\}$ and $\{z_\alpha\}$ are both bounded nets and $\{y_\alpha\}$ is unbounded, then for every $x > \sup\{x_\alpha\}$,

$$\begin{aligned} x^{-1}xx_\alpha y_\alpha^{-1} z_\alpha &= x^{-1}xx_\alpha x_\alpha^{-1}(y_\alpha/x_\alpha)^{-1} z_\alpha \\ &= x_\alpha x_\alpha^{-1}x^{-1}x(y_\alpha/x_\alpha)^{-1} z_\alpha \\ &= x_\alpha x_\alpha^{-1}(y_\alpha/x_\alpha)^{-1} z_\alpha = x_\alpha y_\alpha^{-1} z_\alpha \to u. \end{aligned}$$

But $x^{-1}xx_\alpha y_\alpha^{-1} z_\alpha \to x^{-1}xu$, so $x^{-1}xu = u$, and again $u \in FF_P \setminus F_P$.

PROPOSITION 3.2.5. *If $P = \{1, x, x^2, x^3, \ldots\}$, then F_P has the discrete topology as a subspace of any compact semitopological semigroup.*

PROOF. Suppose $x_\alpha y_\alpha^{-1} z_\alpha \to xy^{-1}z$. Then by 3.2.4, $\{y_\alpha\}$, $\{x_\alpha\}$ and $\{z_\alpha\}$ are bounded nets in P. Hence we can find x_0, y_0, z_0 and subnets $x_{\alpha_\delta}, y_{\alpha_\delta}, z_{\alpha_\delta}$ such that $x_{\alpha_\delta} \to x_0$, $y_{\alpha_\delta} \to y_0$, and $z_{\alpha_\delta} \to z_0$. Thus eventually $x_{\alpha_\delta} = x_0$, $y_{\alpha_\delta} = y_0$, and $z_{\alpha_\delta} = z_0$, and hence eventually $x_{\alpha_\delta} y_{\alpha_\delta}^{-1} z_{\alpha_\delta} = x_0 y_0^{-1} z_0$. Thus since

$$x_{\alpha_\delta} y_{\alpha_\delta}^{-1} z_{\alpha_\delta} \to xy^{-1}z, \qquad x_0 y_0^{-1} z_0 = xy^{-1}z.$$

This shows that the only convergent nets are eventually constant, and so the relative topology on F_P is the discrete.

Proposition 3.2.5 assures us that if P is $\{1, x, x^2, \ldots\}$, then both the multiplication on F_P and the inversion map on F_P are continuous functions. Moreover, since every convergent net which is bounded in its second coordinate converges to a point of F_P, and every convergent net which is unbounded in its second coordinate converges to a point of $\overline{EP_P}$ or $\overline{FF_P}$, we see that $\overline{EF_P} \cup \overline{FF_P}$ is an ideal of $\overline{F_P}$ and $\overline{F_P} \setminus F_P = \overline{EF_P} \cup \overline{FF_P}$. We next show that all these related conditions are equivalent.

PROPOSITION 3.2.6. *If F_P is contained densely in a compact semitopological semigroup, and $P \subseteq F_P$ is a continuous embedding, then the following are equivalent*:

(a) *Inversion is continuous on $P \cup P^{-1}$*;

(b) *$P^{-1} \subseteq F_P$ is a continuous embedding*;

(c) *The topology on F_P is the product topology*;

that is, $x_\alpha y_\alpha^{-1} z_\alpha \to xy^{-1}z$ if and only if $x_\alpha \to x, y_\alpha \to y$, and $z_\alpha \to z$;

(d) *If $\{y_\alpha\}$ is a bounded net from P, and if $x_\alpha y_\alpha^{-1} z_\alpha \to u$, $u \in F_P$*;

(e) $\overline{F_P} \setminus F_P = \overline{EF_P} \cup \overline{FF_P}$.

(f) *$\overline{F_P} \setminus F_P$ is an ideal of $\overline{F_P}$, if $\overline{F_P} \setminus F_P \neq \varnothing$.*

PROOF. The implications (a) $\Leftrightarrow$ (b) and (c) $\Rightarrow$ (d) are easy. We show (a) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a), and (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (d).

(a) $\Rightarrow$ (c). Suppose inversion is continuous in $P \cup P^{-1}$. We note that if $x_\alpha y_\alpha^{-1} z_\alpha \to u$ where $x_\alpha \to x$ and $z_\alpha \to z$, and if $x > 1$, and $1 < r < x$, then since $r < x_\alpha$ eventually, $rr^{-1}x_\alpha y_\alpha^{-1} z_\alpha = x_\alpha y_\alpha^{-1} z_\alpha$ eventually, so $rr^{-1}u = u$. (Similarly $ut^{-1}t = u$ if $t < z$.) Now if $x_\alpha \to x$, we show that $x_\alpha x_\alpha^{-1} \to xx^{-1}$. We can assume that $x_\alpha x_\alpha^{-1} \to u$ for some $u \in \overline{F_P}$. If $r > x$, then eventually $r > x_\alpha$, so $r^{-1}x_\alpha x_\alpha^{-1} = r^{-1}$, and so $r^{-1} = r^{-1}u$; letting $r \to x$, $r^{-1} \to x^{-1}$ by assumption and thus $x^{-1} = x^{-1}u$, and $xx^{-1} = xx^{-1}u$. If $x_\alpha \not< x$ frequently, then

$$xx^{-1}x_\alpha x_\alpha^{-1} = x_\alpha x_\alpha^{-1}$$

frequently, so $xx^{-1}u = u$. On the other hand, if $x_\alpha < x$ eventually, then since $xx_\alpha^{-1} \to xx^{-1}$ we have

$$xx_\alpha^{-1}u = (x/x_\alpha)x_\alpha x_\alpha^{-1}u = x/x_\alpha u$$

by the above, so $xx^{-1}u = u$, so in either case, $u = xx^{-1}u = xx^{-1}$. Similarly $x_\alpha^{-1}x_\alpha \to x^{-1}x$. Now if $x_\alpha \to x$, $y_\alpha \to y$, $z_\alpha \to z$, we can assume $x_\alpha y_\alpha^{-1} z_\alpha \to u$. Let $r > x$ and $t > z$; then

$$rr^{-1}x_\alpha y_\alpha^{-1} z_\alpha t^{-1}t = r\left(\frac{ry_\alpha t}{x_\alpha z_\alpha}\right)^{-1} t,$$

so

$$rr^{-1}ut^{-1}t = r\left(\frac{ryt}{xz}\right)^{-1} t = rr^{-1}(yt/xz)t.$$

Letting $r \to x$, we have $xx^{-1}ut^{-1}t = xx^{-1}(yt/xz)^{-1}z = x(yt/z)^{-1}t = x(y/z)^{-1}t^{-1}t$, and letting $t \to z$,

$$u = xx^{-1}uz^{-1}z = x(y/z)^{-1}z^{-1}z = xy^{-1}z.$$

It follows that $x_\alpha y_\alpha^{-1}z_\alpha \to xy^{-1}z$ if and only if $x_\alpha \to x, y_\alpha \to y$, and $z_\alpha \to z$.

(d) $\Rightarrow$ (a). Let $x_\alpha \to x$ in P, and suppose $x_\alpha^{-1} \to rs^{-1}t$ in F_P. We show $r \leqslant x$, and $t \leqslant x$. For suppose $r > x$. Then $r > x_\alpha$ eventually, so since

$$rr^{-1}\left(\frac{r}{x_\alpha}\right) = rx_\alpha^{-1} \to rrs^{-1}t = r^2\left(\frac{r^2s}{r^2 \wedge s}\right)\frac{r^2t}{r^2t}$$

we have

$$rr^{-1}r/x = r^2\left(\frac{r^2s}{r^2 \wedge s}\right)^{-1}\frac{r^2t}{r^2 \wedge s}$$

and hence $r = 1$, and this contradiction shows that $r \leqslant x$; similarly $t \leqslant x$. Now suppose that $s < x$. Then since $s < x_\alpha$ eventually, and $x_\alpha^{-1} = x_\alpha^{-1}ss^{-1} \to rs^{-1}tss^{-1} = rt(st)^{-1}t$, we have $t = 1$. Similarly, we can show $r = 1$. Thus $x_\alpha^{-1} \to s^{-1}$ where $s < x$. Now if $x_\alpha < x$ frequently, the fact that

$$(x/x)x^{-1} = x_\alpha^{-1}xx^{-1} \to s^{-1}xx^{-1} = (x/s)x^{-1}$$

implies that $x = s$. If $x_\alpha > x$ frequently, then

$$s^{-1} = \lim_\alpha x_\alpha^{-1} = \lim_\alpha\left(x_\alpha^{-1}xx^{-1}\right) = s^{-1}xx^{-1} = (x/s)x^{-1},$$

so $x = s$. Hence if $x_\alpha^{-1} \to rs^{-1}t$, $r \leqslant x$, $t \leqslant x$, and $s \geqslant x$. Suppose $s > x$. Then $s > x_\alpha$ eventually, so since $(s/x_\alpha)s^{-1} = x_\alpha^{-1}ss^{-1} \to rs^{-1}tss^{-1} = rt(st)^{-1}t$, $t = 1$, and $s/x = r$ Similarly, $r = 1$, so $s = x$. Therefore, if $x_\alpha \to x$, then $x_\alpha^{-1} \to x^{-1}$.

(d) $\Rightarrow$ (e). Let $u \in \bar{F}_P \setminus F_P$. Then $u = \lim_\alpha x_\alpha y_\alpha^{-1}z_\alpha$, where $\{y_\alpha\}$ is unbounded. As in 3.2.4, either $Eu = u$ or $Fu = u$. Hence $u \in \overline{EF_P} \cup \overline{FF_P}$. If $v \in \overline{EF_P} \cup \overline{FF_P}$, then $Ev = v$ or $Fv = v$, so by 3.2.3, $v \in \bar{F}_P \setminus F_P$.

(e) $\Rightarrow$ (f). Suppose $\bar{F}_P \setminus F_P = \overline{EF_P} \cup \overline{FF_P}$. If $u \in \bar{F}_P \setminus F_P$, then either $Eu = u$ or $Fu = u$, so if $v \in \bar{F}_P$, $uv = Euv \in \overline{EF_P}$, or else $uv = Fuv \in \overline{FF_P}$, so $uv \in \bar{F}_P \setminus F_P$.

(f) $\Rightarrow$ (d). We first show that if $\{y_\alpha\}$ is bounded and $y_\alpha^{-1} \to u$, then $u \in F_P$. Let $r > y_\alpha$ for each α. We can assume y_α converges, say to y in P, and hence $r^{-1}ru = r^{-1}r/y$, and since $r^{-1}ru \in F_P$, $u \in F_P$. Now let $x_\alpha y_\alpha^{-1}z_\alpha \to v$. Let $r > x_\alpha$ and $t > z_\alpha$ for each α. Then

$$rr^{-1}x_\alpha y_\alpha^{-1}z_\alpha t^{-1}t = r\left(\frac{ry_\alpha t}{x_\alpha z_\alpha}\right)^{-1}t;$$

we can assume $ry_\alpha t^{-1}/x_\alpha z_\alpha$ converges to q, hence $q \in F_P$; so $rr^{-1}vt^{-1}t = rqt \in F_P$ and hence $v \in F_P$.

3.3. For the remainder of this paper, we will assume X is a homomorphic image of F_P embedded densely in a compact semitopological semigroup, and that the inversion on X is a continuous mapping.

PROPOSITION 3.3.1. *If $X = F_P/\delta$ and if $\delta^\natural$ is the natural map of F_P to X, then, under the assumption that inversion is continuous on X, $\delta^\natural$ is a continuous homomorphism from F_P with the product topology onto X.*

PROOF. It follows as in 3.2.6 that $\delta^\natural$ is continuous on $F_P \setminus I$ where $\delta = I\sigma$. Now since $\delta^\natural$ must be continuous on P, we know $I = I_a$ for some $a > 1$ and if $\sigma = \sigma_N$ for some subgroup N of G, N must be a closed subgroup of G. We can assume P is dense in $[1, \infty)$. We prove the proposition for $F_P/I_a\alpha$ and point out that the proofs for the remaining cases are similar.

Suppose $X = F_P/I_a\alpha$. Then we note that $X \approx (F_P \setminus I_a) \cup B_P$, where the multiplication is as follows: If both u and v are in F_P then uv is as usual. If either u or v is in B_P, then $uv = \alpha^\natural(u)\alpha^\natural(v)$. For the sake of simplifying the notation, we equate $xy^{-1}z$ with $(x, y/z)$ if $y > a$.

Let $x_\alpha \to a$ from $[1, a)$, and suppose $x_\alpha x_\alpha^{-1} \to e$. Since for each $r < a$, $x_\alpha x_\alpha^{-1} rr^{-1} = x_\alpha x_\alpha^{-1}$ eventually, $err^{-1} = e$. Now $x_\alpha x_\alpha^{-1}(a, a) = (a, a)$, so $e(a, a) = (a, a)$. But $e(a/x_\alpha)^{-1} \to e$ as $x_\alpha \to a$, and

$$e(a/x_\alpha)^{-1} = ex_\alpha x_\alpha^{-1}(a/x_\alpha)^{-1} = e(x_\alpha, a) \to e(a, a) = (a, a),$$

so $e = (a, a)$. Similarly, if $x_\alpha^{-1}x_\alpha \to f, f(1, 1) = (1, 1)$, so since

$$f(a/x_\alpha) = fx_\alpha^{-1}x_\alpha(a/x_\alpha) = fx_\alpha^{-1}(a, 1) = f(a/x_\alpha, 1) \to f(1, 1)$$

and $f(a/x_\alpha) \to f$, we have $f = f(1, 1) = (1, 1)$. Now suppose that $x_\alpha \to x$, and $y_\alpha \to b > a$; we assume $x_\alpha y_\alpha^{-1} \to u$. Then $y_\alpha > a$ eventually, so $x_\alpha y^{-1} = (x_\alpha, y_\alpha)$ eventually, and $x_\alpha y_\alpha^{-1} \to (x, b)$. If $y_\alpha \to a$, we can assume $y_\alpha < a$ eventually. As above, if $r < a$, $x_\alpha y_\alpha^{-1} = x_\alpha y_\alpha^{-1} rr^{-1} \to urr^{-1}$ so $u = urr^{-1}$ and hence, letting $r \to a$, $u = u(a, a)$. But $x_\alpha y_\alpha^{-1}(a, a) \to u(a, a)$ and $x_\alpha y_\alpha^{-1}(a, a) = (x_\alpha a/y_\alpha, a) \to (x, a)$ so $u = (x, a)$. If $z_\alpha \to z$, and $y_\alpha \to b > a$, then since $y_\alpha^{-1}z_\alpha = (1, y_\alpha/z_\alpha)$ eventually, $y_\alpha^{-1}z_\alpha \to (1, b/z)$. If $y_\alpha \to a$, assume $y_\alpha^{-1}z_\alpha \to u$, and $y_\alpha < a$ eventually. Then if $r < a$, $r^{-1}ru = u$, and, letting $r \to a$, $(1, 1)u = u$ by the above; but $(1, 1)(y_\alpha^{-1}z_\alpha) = (1, y_\alpha/z_\alpha) \to (1, a/z)$ so $u = (1, a/z)$. Now we show in general that if $x_\alpha \to x$, $y_\alpha \to y$, and $z_\alpha \to z$, then $x_\alpha y_\alpha^{-1}z_\alpha \to xy^{-1}z$. We can assume $x_\alpha y_\alpha^{-1}z_\alpha \to u$. If $y > a$, then $x_\alpha y_\alpha^{-1}z_\alpha = (x_\alpha, y_\alpha/z_\alpha)$ eventually and we are done. If $y = a$ and $x > 1$, suppose $x < a$, and let $x < r < a$. Then eventually $rr^{-1}x_\alpha y_\alpha^{-1}z_\alpha = r(ry_\alpha/x_\alpha)^{-1}z_\alpha \to (r, ry/xz)$ so $rr^{-1}u = (r, ry/xz)$, and, letting $r \to x$, $xx^{-1}u = (x, y/z)$ but as before $xx^{-1}u = u$, so $u = (x, y/z)$. Finally, if $x = a$, then $u = (a, a)u = (a, a/z)$. Thus if $\delta^\natural(x_\alpha) \to \delta^\natural(x)$, $\delta^\natural(y_\alpha) \to \delta^\natural(y)$, and $\delta^\natural(z_\alpha) \to \delta^\natural(z)$, then $\delta^\natural(x_\alpha y_\alpha^{-1}z_\alpha) \to \delta^\natural(xy^{-1}z)$, and it follows that $\delta^\natural$ is a continuous homomorphism of F_P onto X. The proof is exactly analogous if $X = F_P/I_a\sigma_N$.

Now recall that for each X, the decreasing nets $\{\delta^{\natural}(x)\delta^{\natural}(x)^{-1}: x \in P\}$ and $\{\delta^{\natural}(x)^{-1}\delta^{\natural}(x): x \in P\}$ converge to idempotents E and F respectively as $x \to \infty$. Furthermore, in each case, E and F commute with X, and 3.3.1 allows us to conclude that in fact, $\overline{X} = X \cup \overline{EX} \cup \overline{FX}$. We point out that if $X = F_P/I_a\sigma_N$, $E = F = \delta^{\natural}(aa^{-1})$ is the identity element of the group which is the minimal ideal of X. If $X = F_P/I_a\alpha$, $F = (1, 1) = \delta^{\natural}(a^{-1}a)$ but $E \notin X$. In this case, $\overline{X} \setminus X = \overline{EX}$. Similarly, if $X = F_P/I_a\beta$, then $\overline{X} \setminus X = \overline{Fx}$.

Proposition 3.3.3. *If X is any homomorphic image of F_P, then $EX \approx B_P$ or EX is a commutative group; EX is a group if and only if $E = EF$, and these remarks are true if E and F are interchanged. Furthermore, $\overline{X} = X \cup \overline{EX} \cup \overline{FX}$.*

Proof. Since E commutes with each element of X, the map $u \to Eu$ is a homomorphism of X to EX. Hence $EX = F_P/I_b\sigma$ for some $b \in P$ and congruence σ on FP. Since $Exx^{-1} = E$ for each x in $(I_b\sigma)^{\natural}(P)$, b = 1, and hence $EX = F_P/\beta$ or F_P/σ_N. In the first case, $EX \approx B_P$ with E as identity and

$$E\delta^{\natural}(xy^{-1}z) = E(y/x)^{-1}z \approx (y/x, z),$$

and the idempotents $Ex^{-1}x$ converge to EF as $x \to \infty$. It is easy to see that EFX is a commutative group in this case. In the second case, EX is a group and since $Ex^{-1}x = E$, $E = EF$. The discussion in the previous paragraph together with 3.2.6 implies that $\overline{X} = X \cup \overline{EX} \cup \overline{FX}$.

The idempotents E and F are thus maximal among idempotents of $\overline{X} \setminus X$, if $\overline{X} \setminus X \neq \varnothing$. We now turn to a discussion of minimal idempotents of $\overline{X}$. The one-parameter semigroups $\overline{\delta^{\natural}(P)}$ and $\overline{\delta^{\natural}(P^{-1})}$ possess as kernels (minimal ideals) commutative compact topological groups K and K', respectively, with identities e_0 and f_0. If $X = B_P$, it follows as in [2] that $e_0 = f_0$ and hence $K = K'$ is the minimal ideal for $\overline{X}$; furthermore $K \subseteq \overline{EX}$ and $K = \overline{EX}$ if and only if $e_0 = E = w$. These facts are easy to prove if X is any of the other images of F_P; we discuss the case where $X = F_P$.

If $\{x_\alpha\}$ is a net from P which converges to e_0, by 3.2.6, $\{x_\alpha\}$ is unbounded. If $x \in P$, $x < x_\alpha$ frequently, so $xx^{-1}x_\alpha = x_\alpha$ frequently and hence $xx^{-1}e_0 = e_0 = Ee_0 = e_0E$. Similarly $e_0F = e_0 = Fe_0$. Thus $EFe_0 = e_0$ so $e_0 \in \overline{EFX}$, and hence e_0 commutes with each element of $\overline{X}$. Now if $y \in P$, $x_\alpha y^{-1} \to e_0y^{-1}$. But since $x_\alpha > y$ frequently, $x_\alpha y^{-1} = x_\alpha x_\alpha^{-1}x_\alpha/y$, and $e_0x_\alpha y^{-1} \to e_0y^{-1}$. Thus since $e_0x_\alpha y^{-1} = e_0x_\alpha x_\alpha^{-1}x_\alpha/y = e_0x_\alpha/y$, e_0y^{-1} is in $\overline{K} = K$. Similarly $y^{-1}e_0$ is in K. Thus $e_0X = Xe_0 \subseteq K$, and $e_0\overline{X} = \overline{X}e_0 \subseteq K$. Now if e is any idempotent of $\overline{X}$, $(e_0e)(e_0e) = e_0e_0ee = e_0e$, and since $e_0e \in K$, $e_0e = e$. Hence e_0 is the minimal idempotent for $\overline{X}$. It follows that $e_0 = f_0$ and as in [2, Proposition 2] that K is the kernel for $\overline{X}$ and that $K = e_0\overline{X}$. We summarize this discussion in the following proposition.

PROPOSITION 3.3.4. *If $X = F_P/\delta$, then the kernel of $\overline{X}$ is K, the compact topological group which is the kernel of $\overline{\delta^{\natural}(P)}$. If e_0 is the identity element of K, then $K = e_0\overline{X}$.*

We now derive some corollaries and give some examples from the space of bounded linear operators on complex Hilbert space. This one is proved as in [**2**, Proposition 5].

COROLLARY 3.3.5. *If $\overline{X}$ is a compact semitopological semigroup with a continuous involution $t \to t^*$, all idempotents of X, EX and FX are selfadjoint, as are EF and e_0. If $x \in K$, then $x^* \in K$, and furthermore, $x^*x = e = xx^*$ for every x in K.*

The following corollary was proved in [**1**, Theorem 4].

COROLLARY 3.3.6. *If $\overline{X}$ is a compact topological semigroup, then since $EX \neq B_P \neq FX$, $E = F$ and hence $\overline{X} = X \cup \overline{EX}$.*

EXAMPLE 3.3.7. If U is unitary, and T and S are unilateral shifts, let $J = U \oplus T \oplus S^*$. Then $J^n = U^n \oplus T^n \oplus (S^*)^n$; let $X = \langle J, J^* \rangle$. We have $E = I_1 \oplus O_2 \oplus I_3$, $F = I_1 \oplus I_2 \oplus O_3$, and $EF = I_1 \oplus O_2 \oplus O_3$. EX is the bicyclic semigroup generated by $U \oplus O_2 \oplus S^* = p$ and $U^* \oplus O_2 \oplus S = q$ with $1 = pq = E$, and FX is the bicyclic semigroup $\langle U \oplus T \oplus O_3, U^* \oplus T^* \oplus O_3 \rangle$. It is easy to see that the structure of $\overline{EFX}$ is totally dependent on U.

REMARK 3.3.9. If $\{J_t\} = J$ is a one-parameter semigroup of partial isometries on a finite-dimensional complex space, then $J_t = U_t \oplus T_t$, where U_t is unitary and T_t is a truncated shift.

PROOF. The uniform and weak topologies coincide, so $\overline{X} = \langle J \cup J^* \rangle$ is a compact topological semigroup. If $J_t = U_t \oplus S_t \oplus C_t^* \oplus T_t$ where either S_t or C_t is a nonunitary isometry, then either EX or FX would be bicyclic. Since this cannot happen in a compact topological semigroup, S_t and C_t^* are both unitary.

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