TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 240, June 1978

OPERATORS WITH SMALL SELF-COMMUTATORS

BY

J. W. DEL VALLE

ABSTRACT. Let A be a bounded operator on a Hilbert space H. The self-commutator of A, denoted $[A]$, is $A^*A - AA^*$. An operator is of commutator rank n if the rank of $[A]$ is n. In this paper operators of commutator rank one are studied. Two particular subclasses are investigated in detail. First, completely nonnormal operators of commutator rank one for which A^*A and AA^* commute are completely characterized. They are shown to be special types of simple weighted shifts. Next, operators of commutator rank one for which $\{A^n e\}_{n=0}^{\infty}$ is an orthogonal sequence (where e is a generator of the range of $[A]$) are characterized as a type of weighted operator shift.

1. Introduction. Let H be a Hilbert space. An *operator* from H to a Hilbert space K is understood to be a bounded linear transformation from H to K. If $H = K$, the operator is said to be *on* H. If A is an operator on H, the *self-commutator* of A, denoted $[A]$, is $A^*A - AA^*$. A is *completely nonnormal*, or *abnormal*, if A does not possess a nonzero reducing subspace M such that $A|M$ is normal, and A is of *commutator rank n* if the rank of $[A]$ is n.

Let $\mathfrak{B}(H)$ denote the set of all operators on H, and for each nonnegative integer n, let

$$\mathfrak{D}_n(H) = \{A: A \in \mathfrak{B}(H) \text{ and Rank } [A] = n\}, \quad \text{and}$$

$$\mathfrak{S}_n(H) = \{A: A \in \mathfrak{D}_n(H) \text{ and } A^*A \text{ and } AA^* \text{ commute}\}.$$

These last two sets will often be written respectively as $\mathfrak{D}_n$ and $\mathfrak{S}_n$ when their application to the underlying space H is not to be emphasized. It is immediate that the classes $\mathfrak{D}_n$ and $\mathfrak{S}_n$ consist entirely of normal operators if and only if $n = 0$. Probably the simplest examples of $\mathfrak{S}_n$ operators are weighted unilateral and bilateral shifts: $Ae_k = s_k e_{k+1}$, where $\{e_k\}$ is an orthonormal basis on H. Of course, to be in $\mathfrak{S}_n$, the sequence of absolute values of the weights in the shift must have exactly n jumps or changes.

If A is an operator in $\mathfrak{D}_1$, then by multiplying A by a nonzero real constant and, if necessary, replacing A by its adjoint, it may be assumed, without loss of generality, that $[A] = P$, where P is a one-dimensional selfadjoint projection. With this in mind, let

Received by the editors February 29, 1976 and, in revised form, November 10, 1976.
AMS (MOS) subject classifications (1970). Primary 47A65, 47B20, 47B47; Secondary 49G20.
Key words and phrases. Operator, normal, hyponormal, commutator.

$$\mathfrak{D}_1'(H) = \{A: A \in \mathfrak{D}_1(H) \text{ and } [A] = P\}, \quad \text{and}$$
$$\mathfrak{S}_1'(H) = \mathfrak{S}_1(H) \cap \mathfrak{D}_1'(H),$$

where P is as above. For convenience, several of the theorems of the paper refer to the "normalized" hyponormal operators in $\mathfrak{D}_1'$ and $\mathfrak{S}_1'$, rather than $\mathfrak{D}_1$ and $\mathfrak{S}_1$, but they have obvious trivial extensions to their respective larger classes.

From the preliminary definitions above it follows that the self-commutator, $[A]$, of a normal operator A is zero. A natural attempt to extend the known structure of normal operators to larger classes is to consider transformations A whose self-commutator is small. For example, it might be assumed that the range of $[A]$ is finite dimensional. However, even if the rank of $[A]$ is one, the structure of A is evidently very complicated (see, for example, **[5]**, **[7]**, **[10]**, and **[11]**), and there appears to be no useful generalization of the spectral theorem that can be used to analyze such operators.

The purpose of this paper is to study operators A for which $[A]$ has rank one. Some of the theorems stated require the additional condition that A^*A and AA^* commute. Naturally, this extra hypothesis makes the operator behave in a more nearly normal fashion, since all normal operators satisfy the condition. §2 presents a structure theory for the operators in $\mathfrak{S}_1$. In Theorem 2.4 they are essentially characterized up to unitary equivalence. In §3 a special subclass of $\mathfrak{D}_1$ is studied. This subclass consists of the operators A in $\mathfrak{D}_1$ with the property that the sequence $\{A^n e\}_{n=0}^{\infty}$ is orthogonal, where e is any generator of the (one-dimensional) range of $[A]$. Theorem 3.7 is the principal result. A forthcoming paper **[1]** treats the class $\mathfrak{S}_2$, which contains a considerably larger and richer variety of operators than are in $\mathfrak{S}_1$. The methods and techniques employed in this and the forthcoming paper are principally algebraic, and are to be contrasted with those of the papers referred to in the above paragraph, where the approach is largely in terms of singular integrals. In particular, constant use is made of the standard polar factorization of an operator A as $U\sqrt{A^*A}$, where U is a partial isometry with the same kernel as A^*A. Details of this factorization are given in **[2, pp. 68 and 263]**. Whenever a radical sign is applied to an operator, as in this instance, it is assumed that the operator is nonnegative and selfadjoint, for which there is only one (nonnegative selfadjoint) square root.

Before initiating the study of $\mathfrak{D}_1$ and $\mathfrak{S}_1$, two fundamental theorems should be noted. The first states that an operator can always be decomposed into a normal part and a completely nonnormal part.

THEOREM 1.1. *If A is an operator on the Hilbert space H, then there is a reducing subspace M of H (possibly trivial) such that $A|M$ is normal and $A|M^{\perp}$ is completely nonnormal. Furthermore, the decomposition is unique, and*

$$M = \bigcap_{m,n=0}^{\infty} \operatorname{Ker}(A^n A^{*m} - A^{*m} A^n).$$

A proof of this result appears in [6, p. 498].

There is a variation of Theorem 1.1 whose application is useful in the following sections.

THEOREM 1.2. *Let A be an operator on H. If K is the smallest reducing subspace of A containing the range of $[A]$, then $A|K$ is the completely nonnormal component of A.*

PROOF. Let K be as defined in the statement of this theorem. From Theorem 1.1, $A = A_1 \oplus A_2$ on $M \oplus M^{\perp}$, where A_1 is completely nonnormal and A_2 is normal. Since $[A] = [A_1] \oplus [A_2]$, $[A_2] = 0$, and so it is clear that $K \subseteq M$, because M is a reducing subspace of A containing the range of $[A]$. If this containment were proper, then A_1 itself could be further reduced into $A_{11} \oplus A_{12}$ on $K \oplus K^{\perp}$, where $K^{\perp}$ is the orthogonal complement of K in M. But the definition of K implies that $[A_{12}] = 0$, which contradicts the fact that A_1 is completely nonnormal. Therefore, $K = M$, and the theorem is proved.

If the structure of the nonnormal component of the operator A can be determined, then the known theory of normal operators can be applied to the other component. Since it is the nonnormal behavior of nearly-normal operators which is presently under investigation, it is evident that there is no loss of generality in assuming that A is completely nonnormal, and some of the results in this paper are stated in that context. This is, of course, no tangible restriction, but merely a convenience which streamlines the study and simplifies the statements of some of the results.

2. The class $\mathfrak{S}_1$. The purpose of the present section is to characterize the operators in $\mathfrak{S}_1$. Reference is made quite often to a particular type of weighted shift, W, defined on the orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$, by $We_n = te_{n+1}$ if $n < 0$, and $We_n = se_{n+1}$ if $n \geqslant 0$, where s and t are nonnegative real numbers such that $s^2 = t^2 + 1$. Its associated matrix is written below and thereafter referred to only by the symbol "$\mathfrak{M}$".

$$\mathfrak{M} = \begin{bmatrix} \ddots & & & & & & & \\ \ddots & t & 0 & & & & 0 & \\ & & \ddots & \ddots & & & & \\ & & & t & 0 & & & \\ & & & & s & 0 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & s & 0 \\ & & & & & & & \ddots \ \ddots \end{bmatrix}.$$

A special case of matrix $\mathfrak{M}$ deserves mention. If $t = 0$ and $s = 1$, then the operator A reduces into $U \oplus 0$ on $H_1 \oplus H_2$, where $H_1 = \operatorname{Span}\{e_n\}_{n=0}^{\infty}$, and $H_2 = \operatorname{Span}\{e_{-n}\}_{n=1}^{\infty}$, and where U is the unilateral shift. On the other hand, if $t \neq 0$, then A has no reducing subspaces. This fact follows immediately from Theorem 11 in [4], which states that a bilateral weighted shift has a reducing subspace if and only if the absolute values of its sequence of weights form a periodic sequence.

The following simple fact is needed for Theorem 2.2.

LEMMA 2.1. *Let $T \in \mathfrak{B}(H)$, and suppose that P is a projection of rank one that commutes with T. Then the range of P is an eigenspace of T.*

PROOF. The proof is trivial and well known. Let e be a nonzero vector in Range(P). Since $TP = PT$, then $TPe = PTe$, so $Te = PTe$. But $P(Te) \in$ Range$(P) = \operatorname{Span}\{e\}$, so there is a scalar α such that $PTe = \alpha e$, and so $Te = \alpha e$.

THEOREM 2.2. *Let $A \in \mathfrak{B}(H)$ and suppose that A has the factorization $A = U\sqrt{T}$, where U is unitary and $T = A^*A$. (It is not being assumed that* Ker(T) = Ker(U); *that is, it is not necessary that $U\sqrt{T}$ be the canonical polar factorization of A.) Suppose $A \in \mathfrak{S}_1'$. Then there is a reducing subspace M of A such that $A|M$ has matrix $\mathfrak{M}$ and $A|M^{\perp}$ is normal.*

PROOF. Let $[A] = P$, a projection of rank one. Then $A^*A - AA^* = P$, so $\sqrt{T}U^*U\sqrt{T} - U\sqrt{T}\sqrt{T}U^* = P$, which implies that

$$T - UTU^* = P, \tag{2.1}$$

since U is unitary. Since $A \in \mathfrak{S}_1'$, A^*A and AA^* commute, so that P commutes with both A^*A and AA^*. Let e be a unit vector in Range(P). By Lemma 2.1, e is an eigenvector for both A^*A and AA^*. Since both of these operators are nonnegative, there exist nonnegative real scalars a and b such that

$$\begin{aligned} A^*Ae &= ae, \quad \text{and} \\ AA^*e &= be, \text{ with } a = b + 1, \end{aligned} \tag{2.2}$$

since $(A^*A - AA^*)e = Pe$ implies that $(a - b)e = e$. From (2.1) follow the identities

$$U(A^*A)U^* = AA^* \quad \text{and} \quad U^*(AA^*)U = A^*A. \tag{2.3}$$

It is now shown, as illustrated in the diagram below, that the vector in the left-hand column is an eigenvector for both A^*A and AA^*, with respective eigenvalue (a or b) as indicated.

$$\begin{array}{c|cc|c} & A^*A & AA^* & \\ \hline (U^*)^n e & b & b & n > 0 \\ e & a & b & \\ (U^n)e & a & a & n > 0 \\ \hline \end{array} \tag{2.4}$$

The proof is by induction on n for $n \geqslant 0$. The case $n = 0$ (the middle row above) has been established in (2.2). Thus suppose $n \geqslant 0$ and suppose the existence of the specified eigenvalues for this n. Then by (2.3) and (2.4),

$$\begin{aligned}(AA^*)(U^{n+1}e) &= (UU^*)(AA^*U)(U^ne)\\ &= U(U^*AA^*U)U^ne = U(A^*A)U^ne\\ &= U(aU^ne) = aU^{n+1}e, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}(A^*A)(U^{*n+1}e) &= (U^*U)(A^*AU^*)(U^{*n}e)\\ &= U^*(UA^*AU^*)(U^{*n}e) = U^*(AA^*)(U^{*n}e)\\ &= U^*(bU^{*n}e) = bU^{*n+1}e. \end{aligned} \tag{2.6}$$

There exist constants c and d such that $PU^{n+1}e = ce$ and $PU^{*n+1}e = de$. Thus, by (2.5),

$$\begin{aligned}A^*A(U^{n+1}e) &= (AA^* + P)(U^{n+1}e)\\ &= aU^{n+1}e + PU^{n+1}e = aU^{n+1}e + ce. \end{aligned} \tag{2.7}$$

Since P and A^*A commute, $P(A^*A)U^{n+1}e = (A^*A)PU^{n+1}e$, so

$$P(aU^{n+1}e + ce) = (A^*A)(ce),$$

$$a(PU^{n+1}e) + P(ce) = c(A^*A)e, \quad ace + ce = cae.$$

Thus $c = 0$, so that from (2.7),

$$(A^*A)(U^{n+1}e) = aU^{n+1}e. \tag{2.8}$$

Observe here that $c = 0$ also implies that $PU^{n+1}e = 0$; that is, $PU^me = 0$ for all $m > 0$. Similarly, by (2.6),

$$\begin{aligned}AA^*(U^{*n+1}e) &= (A^*A - P)(U^{*n+1}e)\\ &= bU^{*n+1}e - PU^{*n+1}e = bU^{*n+1}e - de, \end{aligned} \tag{2.9}$$

and therefore,

$$P(AA^*)U^{*n+1}e = (AA^*)PU^{*n+1}e,$$

$$P(bU^{*n+1}e - de) = (AA^*)(de), \quad bde - de = dbe,$$

so that $d = 0$, and (2.9) implies that

$$(AA^*)(U^{*n+1}e) = bU^{*n+1}e \tag{2.10}$$

The induction is now complete from (2.5), (2.6), (2.8), and (2.10).

Let $e_n = U^ne$ for each integer n. In particular $e_0 = e$. Let $M_1 = \operatorname{Span}\{e_n\}_{n=0}^{\infty}$, and $M_2 = \operatorname{Span}\{e_{-n}\}_{n=1}^{\infty}$. From (2.4) it is clear that A^*A is reduced by each of M_1 and M_2. Specifically, $A^*A|M_1 = aI_1$ and $A^*A|M_2 = bI_2$, where I_1 and I_2 are the respective identities of M_1 and M_2. Since eigenvectors corresponding to distinct eigenvalues of a selfadjoint operator are orthogonal, M_1 and M_2 are orthogonal. Let $M = M_1 \oplus M_2$. From (2.4),

$$\sqrt{T}\,|M = \sqrt{A^*A}\,|M = \sqrt{a}\, I_1 \oplus \sqrt{b}\, I_2. \tag{2.11}$$

$$\text{If } n > 0 \text{ then } A^n e = \sqrt{a}^{\,n}(U^n e). \tag{2.12}$$

The proof of (2.12) follows by induction. For if $n = 0$, then $A^0 e = e = \sqrt{a}^{\,0}(U^0 e)$, and if (2.12) holds for an integer $n \geqslant 0$, then by (2.11),

$$\begin{aligned}(A^{n+1})e &= (U\sqrt{T}\,)^{n+1} e = U\sqrt{T}\,(A)^n e\\ &= U\sqrt{T}\,(\sqrt{a}^{\,n} U^n e) = \sqrt{a}^{\,n} U(\sqrt{T}\, U^n e)\\ &= \sqrt{a}^{\,n} U(\sqrt{a}\, U^n e) = \sqrt{a}^{\,n+1}(U^{n+1}e).\end{aligned}$$

Using (2.4) and (2.11), an analogous inductive argument (omitted) shows that

$$\text{If } n \geqslant 0, \text{ then } (A^*)^n e = \sqrt{b}^{\,n}(U^{*n}e). \tag{2.13}$$

If $m > 0$, it follows from the comment below (2.8) that $(U^m e, e) = (U^m e, Pe) = (PU^m e, e) = 0$. Therefore, since U is unitary, $\{U^n e\}_{n=-\infty}^{\infty}$ is an orthonormal sequence; that is, $U|M$ is a bilateral shift of multiplicity one. The theorem now follows immediately from (2.11), Theorem 1.2, and the fact that $A = U\sqrt{T}$.

An additional fact is needed for the proof of Theorem 2.4.

LEMMA 2.3. *Let $A \in \mathfrak{N}_1$ and suppose that A is completely nonnormal. Then A has no nontrivial reducing subspaces.*

PROOF. Suppose to the contrary. Let $H = H_1 \oplus H_2$ be a nontrivial decomposition of H (that is, $H_1 \neq \{0\} \neq H_2$), such that H_1 and H_2 reduce A. Let $A_1 = A|H_1$ and $A_2 = A|H_2$.

Let e be a unit vector in the range of $[A]$, with respective components e_1 and e_2, such that $e = e_1 \oplus e_2$ in $H_1 \oplus H_2$. Suppose that $e_1 \neq 0$ and $e_2 \neq 0$. Choose $f \in H_1$ such that $[A_1]f \neq 0$. This choice is possible since A is completely nonnormal, so that A_1 cannot be normal. Then there is a nonzero scalar λ such that $[A](f \oplus 0) = \lambda e = \lambda(e_1 \oplus e_2) = \lambda e_1 \oplus \lambda e_2$. But $[A](f \oplus 0) = [A_1]f \oplus 0$, so that $\lambda e_2 = 0$, which contradicts the fact that neither λ nor e_2 is zero.

Thus it is impossible for both e_1 and e_2 to be nonzero, so one of them, say e_2, is zero, and $e = e_1 \oplus 0$.

Suppose $h \in H_2$. Then for some scalar μ, $[A](0 \oplus h) = \mu(e_1 \oplus 0) = \mu e_1 \oplus 0$. And also, $[A](0 \oplus h) = [A_1](0) \oplus [A_2](h) = 0 \oplus [A_2](h)$. Thus $[A_2]h = 0$. Since this is true for all $h \in H_2$, $[A_2] = 0$, which implies that A is normal on H_2, contrary to hypothesis.

Therefore the assumed decomposition does not exist, and A is irreducible, as required.

THEOREM 2.4. *Suppose A is a completely nonnormal operator in $\mathfrak{S}_1'(H)$. Then A is either a unilateral shift or else a weighted bilateral shift with matrix $\mathfrak{M}$. Conversely every such shift is in $\mathfrak{S}_1'$.*

PROOF. Let $A = U\sqrt{A^*A}$ be the polar factorization of A. In general U is a partial isometry. But suppose $\operatorname{Ker}(U) \neq \{0\}$. Then $\operatorname{Ker}(\sqrt{A^*A}) \neq \{0\}$, so there is a nonzero vector f in H for which $\sqrt{A^*A}\, f = 0$. Thus $A^*Af = 0$, which implies that $(A^*Af, f) = 0$, or $Af = 0$; that is, f is an eigenvector for A. But $A^*A - AA^* = [A] \geqslant 0$ by hypothesis, so A is hyponormal, and in [2] the solution to problem 163 shows that the span of eigenvectors of a hyponormal operator reduces A. However, since A is completely nonnormal, A can have no nontrivial reducing subspaces, by Lemma 2.3. Hence $\operatorname{Ker}(U) = \{0\}$, so U is an isometry.

Problem 118 in [2] shows that either U is unitary, or else U is a direct sum of (one or more) copies of the unilateral shift, plus (possibly) a unitary operator. If U is unitary, then the hypotheses of Theorem 2.2 above are satisfied for A, and the result follows immediately from this theorem, since the normal part is absent.

On the other hand suppose U is not unitary. Then $U = \Sigma_{n=0} \oplus\ U_n$ on a decomposition $\Sigma_{n=0} \oplus\ H_n$ of H, such that $U|H_n = U_n$, where U_0 is unitary and U_n is a unilateral shift for $n \geqslant 1$, it being understood that the U_0 and H_0 summands are absent if U has no unitary component, and that the sum otherwise extends over all occurrences of unilateral shifts in U. U will now be extended to a unitary operator in the following manner. For $n \geqslant 1$, let $\hat{H}_n$ be a Hilbert space so that the unilateral shift U_n extends in the obvious way to a (unitary) bilateral shift $\hat{U}_n$ on $H_n \oplus \hat{H}_n$. Let $H^\perp = \Sigma_{n=1} \oplus\ \hat{H}_n$ and $K = H \oplus H^\perp$, and let $V = U \oplus \Sigma_{n=1} \oplus\ \hat{U}_n$ be the required unitary extension of U to K.

Let $T = A^*A$ on H and $\hat{T} = T \oplus 0$ on K. Since $T \geqslant 0$, so is $\hat{T}$. Let $\hat{A} = A \oplus 0$ on K. It is clear that H reduces $\hat{A}$, and that $[\hat{A}] = [A] \oplus 0$, so $[\hat{A}]$ is of rank one. Suppose $f \in H$. Then $\hat{A}(f \oplus 0) = Af \oplus 0 = (U\sqrt{T}f) \oplus 0$, and $V\sqrt{\hat{T}}\,(f \oplus 0) = V(\sqrt{T}f \oplus 0) = (U\sqrt{T}f) \oplus 0$. And if $g \in H^\perp$, then

$$\hat{A}(0 \oplus g) = A(0) \oplus O(g) = 0,$$

and $V\sqrt{\hat{T}}\,(0 \oplus g) = V(\sqrt{T}(0) \oplus O(g)) = V(0) = 0$. Since $\hat{A}$ and $V\sqrt{\hat{T}}$ agree on each of H and $H^\perp$, $\hat{A} = V\sqrt{\hat{T}} = V\sqrt{A^*A \oplus 0} = V\sqrt{\hat{A}^*\hat{A}}$.

Thus $\hat{A} \in \mathfrak{B}(K)$ satisfies the hypotheses of Theorem 2.2, so there is a reducing subspace $M \subseteq K$ for $\hat{A}$ such that $\hat{A}|M$ is a bilateral shift with matrix $\mathfrak{M}$. Let $M = M_1 \oplus M_2$, where $M_1 = M \cap H$ and $M_2 = M \cap H^\perp$. Then $\hat{A}$ is reduced by each of M, H, and $H^\perp$, so M_1 and M_2 also reduce $\hat{A}$, and $\hat{A}|M = \hat{A}|M_1 \oplus \hat{A}|M_2 = \hat{A}|M_1 \oplus 0$, since $\hat{A}|H^\perp = 0$ implies that $\hat{A}|M_2 = 0$. Hence the nonnormal component of $\hat{A}|M$ is completely contained in H, and since A is completely nonnormal, A has no nontrivial reducing subspaces, by Lemma 2.3. Thus $M_1 = H$; for otherwise $\hat{A}|M_1 = A|M_1$ would be a nonzero proper component of A.

If $t \neq 0$ in matrix $\mathfrak{M}$, then $\hat{A}|M$ $(= \hat{A}|M_1 \oplus \hat{A}|M_2)$ is irreducible (see the remark in the second paragraph of this section), so one of M_1 or M_2 must be $\{0\}$. But $M_1 = H$, so $M_2 = \{0\}$; that is, $M = M_1 = H$, so that $A = \hat{A}|H = \hat{A}|M$, an operator with matrix $\mathfrak{M}$, as required.

If $t = 0$, it is clear from matrix $\mathfrak{M}$ that $\hat{A}|M = W \oplus 0$, where W is a unilateral shift on M_1 and 0 is the zero operator on M_2. Hence $A = \hat{A}|M_1 = W$, a unilateral shift.

The converse is obvious.

Recall that the hypothesis that $s^2 = t^2 + 1$ is necessary for the operator to be in $\mathcal{E}_1'$. By relaxing this condition and requiring instead only that s and t be any two distinct nonnegative real numbers the general completely nonnormal operator in $\mathcal{E}_1$ can be represented. In this manner the pairs (s, t) can be used as unitary invariants for such operators. The details are obvious and are omitted.

3. Weighted operator shifts in $\mathfrak{D}_1$. Suppose e is a generator of the one-dimensional range of an operator A in $\mathfrak{D}_1$. This section investigates the conditions under which the sequence $\{A^n e\}_{n=0}^{\infty}$ is orthogonal. Theorem 3.7 shows that this orthogonality hypothesis is equivalent to the condition that A be a weighted operator shift.

LEMMA 3.1. *Let $\{H_n\}_{n=-\infty}^{\infty}$ be a sequence of Hilbert spaces and let $H = \sum_{n=-\infty}^{\infty} \oplus H_n$ be their direct sum. Suppose P is a selfadjoint projection on H of rank one. Then P has the operator matrix form $[P_{ij}]$, where P_{ij}: $H_j \to H_i$ is defined by $P_{ij} = (\ ,f_j)f_i$ for $f_i \in H_i$ and $f_j \in H_j$, and $\sum_{n=-\infty}^{\infty}\|f_n\|^2 = 1$. Conversely, every such operator P is a one-dimensional projection.*

This is a generalization of Theorem 1 on p. 172 of [3]. The proof is straightforward, but tedious, and is therefore omitted.

LEMMA 3.2. *Suppose $A \in \mathfrak{D}_1'$ and is a bilateral weighted operator shift on the Hilbert space $H = \sum_{-\infty}^{\infty} \oplus H_n$. There exists an m such that if g is any element in the range of $[A]$, then $g \in H_m$.*

PROOF. Since A is an operator shift, it has a matrix representation, with respect to the basis $\{H_n\}$, whose $(n + 1, n)$ entry is A_n: $H_n \to H_{n+1}$, and whose other entries are zeroes. A simple calculation shows that $[A]$ is the diagonal operator matrix whose (n, n) entry is $A_n^* A_n - A_{n-1}A_{n-1}^*$.

It is assumed that $[A] = P$, a one-dimensional selfadjoint projection, and it follows from Lemma 3.1 that there is a unit vector $f = (\ldots, f_{-1}, f_0, f_1, \ldots)$, in Range$(P)$, with $f_n \in H_n$, such that P has the matrix $[P_{mn}]$, where $P_{mn} =$

$(\ ,f_n)f_m\colon H_n \to H_m$. If there were an m and n, with $m \neq n$, such that $f_m \neq 0$ and $f_n \neq 0$, then P_{mn} would be nonzero. But since P_{mn} is the (m, n) entry of $[A]$, this would imply that $[A]$ had a nonzero entry off the main diagonal. Therefore, there can exist at most one m for which $f_m \neq 0$, and since $\Sigma_{-\infty}^{\infty}\|f_n\|^2 = 1$, there is exactly one m such that $f_m \neq 0$. Since any two nonzero points in the range of P are linearly dependent, the result follows.

COROLLARY 3.3. *If A is a weighted operator shift in $\mathcal{D}_1'$ and e is in the range of $[A]$, then $\{A^{*m}e, A^n e\}_{m=1,}^{\infty}\,{}_{n=0}^{\infty}$ is an orthogonal sequence.*

PROOF. This follows immediately from Lemma 3.2, since if $f \in H_k$, then $Af \in H_{k+1}$ and $A^*f \in H_{k-1}$.

The following definition and lemma allow a converse of Corollary 3.3 to be established.

DEFINITION 3.4. Let $A \in \mathcal{B}(H)$. A *chain* $\mathcal{C}$ of A is either the identity operator I on H, or else a finite product $B_1B_2B_3 \cdots B_n$, where each B_k is either A or A^*, for $1 \leqslant k \leqslant n$. Suppose that there are r occurrences of A and s occurrences of A^* in $\mathcal{C}$. Let $\mathcal{L}(\mathcal{C}) = r + s$ denote the *length* of $\mathcal{C}$, and let $\mathcal{J}(\mathcal{C}) = r - s$, the *index* of $\mathcal{C}$. (This index, defined on the chain $\mathcal{C}$, is in no way related to the Fredholm index, defined for the single operator A.) Also define the "partial indices" $\mathcal{J}_1$ and $\mathcal{J}_2$ by $\mathcal{J}_1(\mathcal{C}) = r$ and $\mathcal{J}_2(\mathcal{C}) = s$.

LEMMA 3.5. *Let $A \in \mathcal{D}_1'(H)$ and let the unit vector e generate the range of $[A]$. Let $\mathcal{C}_1 = B_1B_2 \cdots B_n$ and $\mathcal{C}_2 = C_1C_2 \cdots C_m$ be chains of A (either of which could have length zero). For convenience let $\mathcal{J}_1(\mathcal{C}_1) = r$, $\mathcal{J}_2(\mathcal{C}_1) = s$, $\mathcal{J}_1(\mathcal{C}_2) = u$, and $\mathcal{J}_2(\mathcal{C}_2) = v$. Suppose that $\mathcal{J}(\mathcal{C}_1) \neq \mathcal{J}(\mathcal{C}_2)$; that is, $r - s \neq u - v$. If $\{A^n e\}_{n=0}^{\infty}$ is an orthogonal sequence, then $(\mathcal{C}_1 e, \mathcal{C}_2 e) = (A^{r+v}e, A^{s+u}e) = 0$.*

The gist of the proof is that, under the given hypotheses, the factors A and A^*, applied to e, can be "commuted within the inner product." For example $(A^*A^4e, AA^*e) = (A^2A^*A^2e, A^*Ae)$.

PROOF. The proof is by induction on the sum $r + s + u + v = m + n$, the combined lengths of $\mathcal{C}_1$ and $\mathcal{C}_2$. Let this sum equal p. If $p = 0$, then $\mathcal{C}_1 = \mathcal{C}_2 = I$, the identity, and so the statement of the theorem is true vacuously, since then, $\mathcal{J}(\mathcal{C}_1) = \mathcal{J}(\mathcal{C}_2)$, contrary to hypothesis.

Suppose the theorem holds for all chains $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ for which $\mathcal{L}(\hat{\mathcal{C}}_1) + \mathcal{L}(\hat{\mathcal{C}}_2) < p$, where p is a fixed positive integer. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be chains of respective lengths n and m, such that $m + n = p$.

Then $(\mathcal{C}_1 e, \mathcal{C}_2 e) = (B_1B_2 \cdots B_n e, C_1C_2 \cdots C_m e)$. If no A precedes (is to the left of) an A^* in $\mathcal{C}_1$, then this inner product has the form

$$(A^{*s}A^r e, C_1C_2 \cdots C_m e).$$

Otherwise the product has the form

$$(A_1A_2 \cdots A_kA^*B_{k+2} \cdots B_ne, C_1C_2 \cdots C_me),$$

where each A_i is A (the subscripts just serve to count the occurrences of A). In this case

$$\begin{aligned}(\mathcal{C}_1e, \mathcal{C}_2e) &= (A_1A_2 \cdots A_{k-1}(AA^*)B_{k+2} \cdots B_ne, \mathcal{C}_2e)\\ &= (A_1 \cdots A_{k-1}(A^*A - P)B_{k+2} \cdots B_ne, \mathcal{C}_2e)\\ &= (A_1 \cdots A_{k-1}A^*AB_{k+2} \cdots B_ne, \mathcal{C}_2e),\\ &\quad - (A_1 \cdots A_{k-1}PB_{k+2} \cdots B_ne, \mathcal{C}_2e) \qquad (3.1)\end{aligned}$$

where, of course, $P = [A] = A^*A - AA^*$. Since for any $f \in H$, Pf is a scalar times e, due to the fact that the range of P is one dimensional, it follows that the last term in (3.1) can be written as a scalar times $(A_1 \cdots A_{k-1}e, \mathcal{C}_2e)$. The length of the chain $A_1 \cdots A_{k-1} = A^{k-1}$ is $k - 1$, and the length of $\mathcal{C}_2$ is m. There are two cases:

(i) $\mathcal{J}(B_{k+2} \cdots B_n) = 0$, or

(ii) $\mathcal{J}(B_{k+2} \cdots B_n) \neq 0$.

If $\mathcal{J}(B_{k+2} \cdots B_n) = 0$, then

$$\begin{aligned}\mathcal{J}(A_1 \cdots A_{k-1}B_{k+2} \cdots B_n) &= \mathcal{J}(A_1 \cdots A_{k-1}) + \mathcal{J}(B_{k+2} \cdots B_n)\\ &= (k-1) + 0 = k - 1,\end{aligned}$$

so

$$\begin{aligned}&= \mathcal{J}(\mathcal{C}_1) = \mathcal{J}(B_1 \cdots B_n)\\ &= \mathcal{J}(B_1 \cdots B_{k-1}(B_kB_{k+1})B_{k+2} \cdots B_n)\\ &= \mathcal{J}(A_1 \cdots A_{k-1}(A^*A)B_{k+2} \cdots B_n)\\ &= \mathcal{J}(A_1 \cdots A_{k-1}B_{k+2} \cdots B_n)\\ &= k - 1,\end{aligned}$$

so that $\mathcal{J}(\mathcal{C}_2) \neq k - 1$, because $\mathcal{J}(\mathcal{C}_1) \neq \mathcal{J}(\mathcal{C}_2)$ by hypothesis. Thus for some scalar λ,

$$(A_1 \cdots A_{k-1}PB_{k+2} \cdots B_ne, \mathcal{C}_2e) = \lambda(A_1 \cdots A_{k-1}e, \mathcal{C}_2e) = 0$$

by the inductive hypothesis. On the other hand, if case (ii) holds, that is, if $\mathcal{J}(B_{k+2} \cdots B_n) \neq 0$, then

$$(B_{k+2} \cdots B_ne, e) = (B_{k+2} \cdots B_ne, Ie) = 0,$$

again by the inductive hypothesis, since $\mathcal{L}(B_{k+2} \cdots B_n) < n < p$. Hence $B_{k+2} \cdots B_ne$ and e are orthogonal, so that $(PB_{k+2} \cdots B_n)e = 0$, and therefore $(A_1 \cdots A_{k-1}PB_{k+2} \cdots B_ne, \mathcal{C}_2e) = 0$.

So no matter whether case (i) or case (ii) holds, the last term of (3.1) vanishes, and the corresponding equation can be simplified to

$$(\mathcal{C}_1e, \mathcal{C}_2e) = (A_1 \cdots A_{k-1}A^*AB_{k+2} \cdots B_ne, \mathcal{C}_2e). \qquad (3.2)$$

The effect of the preceding has been to shift the leftmost occurrence of A^* in $\mathcal{C}_1$ one position further to the left, if this term were not already to the extreme left.

This process can be repeated as often as necessary to yield

$$(\mathcal{C}_1 e, \mathcal{C}_2 e) = (A^{*s} A^r e, \mathcal{C}_2 e) = (A^r e, A^s \mathcal{C}_2 e). \tag{3.3}$$

The technique employed above can now be applied to the inner product $(A^s \mathcal{C}_2 e, A^r e)$ to move the v occurrences of A^* in $\mathcal{C}_2$ to the left, so that this product is equal to $(A^{*v} A^s A^u e, A^r e)$. Thus, from (3.3),

$$\begin{aligned}(\mathcal{C}_1 e, \mathcal{C}_2 e) &= \overline{(A^s \mathcal{C}_2 e, A^r e)} \\ &= \overline{(A^{*v} A^s A^u e, A^r e)} = \overline{(A^s A^u e, A^v A^r e)} = (A^{r+v} e, A^{u+s} e). \end{aligned} \tag{3.4}$$

The given condition that $\mathcal{J}(\mathcal{C}_1) \neq \mathcal{J}(\mathcal{C}_2)$ implies that $r - s \neq u - v$, so $r + v \neq u + s$, so that $(A^{r+v} e, A^{s+u} e) = 0$ in (3.4), using the fact that $\{A^n e\}_{n=0}^\infty$ was given to be an orthogonal sequence.

This terminates the induction argument and thus establishes the lemma.

THEOREM 3.6. *Let $A \in \mathfrak{D}_1'(H)$ and assume that A is completely nonnormal. Let e be a unit vector in the range of $[A]$ and suppose that $\{A^n e\}_{n=0}^\infty$ is an orthogonal sequence in H. Then A is a weighted operator shift on H, with decomposition $H = \sum_{n=-\infty}^\infty \oplus H_n$. The H_n, as defined below in the proof, are uniquely determined up to the labeling of the indices.*

PROOF. For each integer n, let

$$\Lambda_n = \{\mathcal{C}: \mathcal{C} \text{ is a chain of } A \text{ of index } n\},$$

and define H_n by $H_n = \operatorname{Span}\{\mathcal{C}e: \mathcal{C} \in \Lambda_n\}$. By Lemma 3.5 the H_n are pairwise orthogonal, for if $\mathcal{C}_\alpha e \in H_n$ and $\mathcal{C}_\beta e \in H_m$, where $m \neq n$, then $\mathcal{J}(\mathcal{C}_\alpha) = n \neq m = \mathcal{J}(\mathcal{C}_\beta)$, so $(\mathcal{C}_\alpha e, \mathcal{C}_\beta e) = 0$, and since H_n and H_m are generated by chains with the respective indices of $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$, it follows, by taking limits, that H_n and H_m are orthogonal.

Let $K = \sum_{n=-\infty}^\infty \oplus H_n$. If $g \in H_n$, then $A(\mathcal{C}g) = (A\mathcal{C})g \in H_{n+1}$ by definition of H_{n+1}, since $\mathcal{J}(A\mathcal{C}) = \mathcal{J}(A) + \mathcal{J}(\mathcal{C}) = 1 + n$. Similarly, $A^*(\mathcal{C}g) = (A^*\mathcal{C})g \in H_{n-1}$, since $\mathcal{J}(A^*\mathcal{C}) = \mathcal{J}(A^*) + \mathcal{J}(\mathcal{C}) = -1 + n$. Using the continuity of both A and A^* and the definition of H_n, it follows that if $f \in H_n$, then $Af \in H_{n+1}$ and $A^*f \in H_{n-1}$. This fact simultaneously shows that

(i) Each of A and A^* leaves K invariant, so that K reduces A, and

(ii) A is a weighted operator shift on K: $A(H_n) \subseteq H_{n+1}$.

By Lemma 2.3, A has no nontrivial reducing subspaces, so $H = K$. That is, $H = \sum_{-\infty}^\infty \oplus H_n$.

Finally, to show the essential uniqueness of the decomposition, suppose that $\sum_{-\infty}^\infty \oplus \hat{H}_n$ is a second direct sum of H, with respect to which A is also a weighted operator shift. Lemma 3.2 shows that if f is a unit vector in the range of $[A]$, then $f \in \hat{H}_m$ for some m, and relabeling the indices of the $\hat{H}_n$ if necessary, it may be assumed that $f \in \hat{H}_0$. In particular, for the vector e introduced earlier in this proof, $e \in \hat{H}_0$. Then the fact that A is a shift makes it clear that if n is an integer and $\mathcal{C}$ is a chain of index n, then $\mathcal{C}e \in \hat{H}_n$. This implies that $H_n \subseteq \hat{H}_n$ for all n. If there were an n such that $H_n \neq \hat{H}_n$, then $\Sigma \oplus H_n$ would be properly contained in $\Sigma \oplus \hat{H}_n$, a contradiction, since both of these sums are equal to H. Thus $H_n = \hat{H}_n$ for all n.

Of course in the more general case where the point e is in $\hat{H}_m$ for some m, then all that can be concluded is that $H_n = \hat{H}_{m+n}$ for all n.

THEOREM 3.7. *Let $A \in \mathfrak{D}_1$. Then A is a weighted operator shift if and only if $\{A^n e\}_{n=0}^{\infty}$ is an orthogonal sequence (where e is a nonzero vector in the range of $[A]$), and in this case, it is also true that $\{A^{*m}e, A^n e\}_{m=1,}^{\infty}{}_{n=0}^{\infty}$ is an orthogonal sequence.*

PROOF. This is an immediate consequence of Corollary 3.3 and Theorem 3.6.

It is possible to present a structure theory for the completely nonnormal operators considered in this section, on the basis of which all of these transformations may explicitly be synthesized or decomposed. The "generators" of each such operator A are A^*A and AA^*, which may be chosen arbitrarily as any two positive selfadjoint operators subject only to the condition that their difference be of rank one. On this basis it is easy to specify unitary invariants.

BIBLIOGRAPHY

1. J. W. Del Valle, *A structure theory for operators of commutator rank two* (to appear).

2. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

3. ______, *Finite-dimensional vector spaces*, 2nd ed., Van Nostrand, Princeton, N.J., 1958. MR **19**, 725.

4. R. L. Kelley, *Weighted shifts on Hilbert space*, Ph.D. Thesis, Univ. of Michigan, 1966.

5. W. Koppelman, *Spectral multiplicity theory for a class of singular integral operators*, Trans. Amer. Math. Soc. **113** (1964), 87–100. MR **29** #1555a.

6. B. B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J. **23** (1973/74), 497–511. MR **49** #7823.

7. J. D. Pincus, *Commutators, generalized eigenfunction expansions and singular integral operators*, Trans. Amer. Math. Soc. **121** (1966), 358–377. MR **32** #6228.

8. C. R. Putnam, *Commutation properties of Hilbert space operators and related topics*, Springer-Verlag, New York, 1967. MR **36** #707.

9. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Ungar, New York, 1955. MR **17**, 175.

10. M. Rosenblum, *A spectral theory for self-adjoint singular integral operators*, Amer. J. Math. **88** (1966), 314–328. MR **33** #6453.

11. Dao-xeng Xa (Tao-hsing Hsia), *On non-normal operators*. I, Acta Math. Sinica **12** (1962), 216–228 = Chinese Math. **3** (1963), 232–246. MR **26** #6773.

Current address: Department of Mathematics, University of Georgia, Athens, Georgia 30601