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CLASSIFYING OPEN PRINCIPAL FIBRATIONS

BY

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ABSTRACT. Let G be a compact metric group. We shall construct classifying spaces for open principal G-fibrations over compact metric spaces.

1. Introduction. Let G be a compact metric group. J. Milnor [11] gave the first functorial construction of a universal G-bundle (classifying space) $EG \to BG$. R. Milgram [10] gave a later construction with better properties, some of which were developed by N. E. Steenrod [16]. If G is a Lie group, every principal fibration with completely regular total space and with fibre G is a fibre-bundle (A. Gleason [4]) (hence also open), so that BG classifies certain (open) principal G-fibrations. We shall extend this result to arbitrary compact metric groups G and compact metric spaces X. J. Cohen [1] extensively studied open principal fibrations, and proved, in particular, that every open principal fibration is an inverse limit of fibre-bundles (see §2). We shall use Cohen's result and some properties of Milgram's resolution (§4) to construct a universal open principal G-fibration

$$\hat{\xi}_G\colon G \to \hat{E}G \to \hat{B}G. \tag{1.1}$$

We describe $\hat{\xi}_G$ in §2. In §§5–6 we describe a natural isomorphism

$$\alpha\colon [X, \hat{B}G] \xrightarrow{\cong} k_G(X), \tag{1.2}$$

where $k_G(X)$ denotes the class of isomorphism classes of open principal G-fibrations over the compact metric space X.

(1.3) REMARKS. (a) J. P. May [9, see especially §§4–6] classified principal fibrations up to weak equivalence, a coarser equivalence relation. Compare weak homotopy equivalence versus homotopy equivalence for compact metric spaces.

(b) In §6 we shall see that an open principal G-fibration over a CW complex is a fibre-bundle, and hence that there is a natural *weak* homotopy equivalence $BG \to \hat{B}G$.

(c) This paper is an outgrowth of the authors' work [3] on strong pro-homotopy theory (Steenrod homotopy theory). See also Remark (6.6(b)).

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2. Open principal G-fibrations. We recall some of the theory of open principal G-fibrations due to J. Cohen [1]. Let G be a compact metric group. Suppose G acts principally (on the right) on a compact metric space E; i.e., for any point y in E the orbit $\{yg \mid g \in G\}$ is homeomorphic to G. If the quotient map $p: E \to E/G$ is open and is a Hurewicz fibration (has the covering homotopy property), p is called an *open principal G-fibration.*

Open principal G-fibrations arise naturally as the limits of bundle maps. More precisely, let $\{G_n\}$ be a *Lie series* for G (L. Pontryagin [13, §46]), i.e., $\{G_n\}$ is a tower of Lie groups bonded by surjections π_n and $\lim\{G_n\} = G$. (We may use a tower for the Lie series of G because G is a compact metric group.) Assume, without loss of generality, that $G_0 = \{e\}$.

Cohen associates to an open principal G-fibration $G \to E \to X$ the tower

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \| \\
G_{n+1} & \longrightarrow & E_{n+1} \equiv E \times_G G_{n+1} & \xrightarrow{p_{n+1}} & X \\
\downarrow{\scriptstyle \pi_{n+1}} & & \downarrow{\scriptstyle \pi'_{n+1}} & & \| \\
G_n & \longrightarrow & E_n \equiv E \times_G G_n & \xrightarrow{p_n} & X \\
\downarrow & & \downarrow & & \| \\
\vdots & & \vdots & & \vdots
\end{array}$$

of G_n-bundles over X. We shall frequently call such a tower a *principal G_n-fibration* over X and use the notation

$$\{\xi_n: G_n \to E_n \to X\}. \tag{2.2}$$

(2.3) REMARKS. The maps π_{n+1} and π'_{n+1} above are principal bundle maps with fibre $K_{n+1} \equiv \ker(\pi_{n+1}: G_{n+1} \to G_n)$.

Conversely, Cohen shows that applying the inverse limit functor lim to the principal $\{G_n\}$-fibration (2.2) yields an open principal G-fibration where $G = \lim\{G_n\}$.

We shall call principal $\{G_n\}$-fibrations over a compact metric space X *isomorphic* if the diagrams (2.1) are level-wise isomorphic over id_X. Let $k_{\{G_n\}}(X)$ denote the *class of isomorphism classes of principal $\{G_n\}$-fibrations* over X and $k_G(X)$ the *class of isomorphism classes of open principal G-fibrations* over X. Summarizing, we have the following

(2.4) PROPOSITION (J. COHEN [1]). *Let G be a compact metric group with Lie series $\{G_n\}$ and let X be a compact metric space. Then there are natural isomorphisms*

$$k_G(X) \underset{\lim}{\overset{\{\cdot \times_G G_n\}}{\rightleftarrows}} k_{\{G_n\}}(X).$$

Therefore, (1.2) will follow if we can show

$$[X, \hat{B}G] \cong k_{\{G_n\}}(X). \tag{2.5}$$

We shall need the following notation. If H is a topological group, let

$$\xi_H \colon H \to EH \to BH$$

be Milgram's resolution (classifying space) [10] applied to H. See §4.

Let G be a compact metric group with Lie series $\{G_n\}$. Let

$\xi_{\{G_n\}} \equiv \{G_n \to EG_n \to BG_n\}$,

$\hat{E}G \equiv \lim\{EG_n\}$, $\hat{B}G \equiv \lim\{BG_n\}$,

$\hat{\xi}_G \equiv G \to \hat{E}G_n \to \hat{B}G_n$, and

$\hat{\xi}_{\{G_n\}} \equiv \{G_n \to E'G_n \to \hat{B}G_n\}$

(take pullbacks of EG_n over $\hat{B}G$). (Note that $\lim\{E'G\} = \hat{E}G$.)

We shall call $\hat{\xi}_G$ the *standard universal principal open G-fibration*, see (1.2), (6.4).

There is a natural map $BG \to \hat{B}G$. In §6 we shall see that this map is a weak homotopy equivalence.

(2.6) PROPOSITION. *For any compact metric space X, any levelwise map of principal $\{G_n\}$-fibrations, or any map of open principal G-fibrations over* id_X *is an isomorphism.*

PROOF. The first result follows easily from the analogous result in bundle theory; see, e.g., [6, p. 42]. The second result then follows by the *proof* of Proposition (2.4). □

3. Compatible local sections. We shall need suitable versions of local sections and associated partitions of unity for principal $\{G_n\}$-fibrations $\{G_n \to E_n \to X\}$ over compact metric spaces. An easy inductive argument, using local sections and associated partitions of unity for the principal bundle maps π'_{n+1}; $E_{n+1} \to E_n$ associated with a principal $\{G_n\}$-fibration (see (2.1)–(2.3)), yields the following.

(3.1) PROPOSITION. *Let $\{G_n \to E_n \to X\}$ be a principal $\{G_n\}$-fibration over a compact metric space X. Then, for each $n \geqslant 0$, there is a finite open cover*

$$\mathcal{U}_n = \{U_{(i_0,i_1,\dots,i_n)} \mid i_k \in I_k\},$$

a family of local sections.

$$\mathcal{S}_n = \{s_{(i_0,i_1,\dots,i_n)} \colon U_{(i_0,i_1,\dots,i_n)} \to E_n\},$$

and a partition of unity

$$\mathcal{H}_n = \{h_{(i_0,i_1,\dots,i_n)} \colon X \to [0,1]\}$$

subordinate to $\mathcal{U}_n$, which satisfy the following compatibility conditions.

$(\mathfrak{U}_n)$ $U_{(i_0,i_1,\ldots,i_n,i_{n+1})} \subset U_{(i_0,i_1,\ldots,i_n)}$.
$(\mathfrak{S}_n)$ $\pi'_{n+1} \circ s_{(i_0,i_1,\ldots,i_n,i_{n+1})} = s_{(i_0,i_1,\ldots,i_n)}|: U_{(i_0,i_1,\ldots,i_n,i_{n+1})} \to E_n$.
$(\mathfrak{H}_n)$ $h_{(i_0,i_1,\ldots,i_n)} = \Sigma_{i_{n+1}} h_{(i_0,i_1,\ldots,i_n,i_{n+1})}$. □

4. Milgram's construction. Following N. E. Steenrod [16], G. Segal [17], S. Mac Lane [7], and [5], we describe the main properties of Milgram's resolution needed later. S. Mac Lane [7] and the second-named author [5] described Milgram's resolution $G \to EG \to BG$ of a topological group as the geometric realization (J. Milnor [12]) of a simplicial resolution $G \to \mathfrak{E}G \to \mathfrak{B}G$.

(4.1) *Standard simplicial resolutions*; *see, e.g.*, [7]. Given a topological group G, define a simplicial G-space (simplicial object over the category of right G-spaces) $\{\mathfrak{E}G_n, d_i: \mathfrak{E}G_n \to \mathfrak{E}G_{n-1}, s_i: \mathfrak{E}G_n \to \mathfrak{E}G_{n+1} | n > 0, 0 < i < n\}$ by

$$\mathfrak{E}G_n = G^{n+1} = \{(g_0, g_1, \ldots, g_n)\},$$
$$d_i(g_0, g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$
$$s_i(g_0, g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n).$$

Regard G as a constant simplicial object, and let $\mathfrak{B}G \equiv \mathfrak{E}G \times_G *$ be the quotient simplicial space. The sequence $G \to \mathfrak{E}G \to \mathfrak{B}G$ forms the *standard simplicial resolution* of G.

(4.2) *Milgram's resolution*. Following G. Segal [17] and others, let R: simplicial spaces $\to CG$ (compactly generated spaces) be the extension of J. Milnor's geometric realization functor [12]. Then [16], [17], [7], [5], the sequence

$$(G \to EG \to BG) \equiv (G = RG \to R\mathfrak{E}G \to R\mathfrak{B}G)$$

is Milgram's resolution (classifying space) for G. The following results show the usefulness of Milgram's construction.

(4.3) PROPOSITION (STEENROD [16]). *Let G and H be topological groups. Then $E(G \times H) \cong EG \times EH$ as $(G \times H)$-spaces, hence $B(G \times H) \cong BG \times BH$.* □

(4.4) PROPOSITION. *Let p: $G \to H$ be a surjection of Lie groups with kernel K. Then the induced map Bp: $BG \to BH$ is a bundle map with fibre BK.*

PROOF (outlined). By L. Pontryagin [13, §44], G is locally isomorphic to $K \times H$ *as sets* under a *local* multiplication $K \times H \to G$. By the proof of (4.3) (in terms of simplicial G-spaces), BG is locally isomorphic to $BK \times BH$ near the basepoint. But the map Bp is homogeneous because the map Ep: $EG \to EH$ is a map of topological groups [16]. The conclusion follows. □

We shall need the following formulas for classifying maps and homotopies between classifying maps. Compare, e.g., [6, pp. 54–57].

(4.5) *Alternate coordinates for Milgram's resolution.* If G is a topological group, the map $G^{n+1} \to G^{n+1}$ given by

$$(g_0, g_1, \ldots, g_n) \mapsto (g_0 g_1 \cdots g_n, g_1 g_2 \cdots g_n, \ldots, g_n)$$

yields an isomorphism of G-spaces, where G acts on the rightmost factor of the "domain" G^{n+1}, and G acts diagonally (on the right) on the "range" G^{n+1}. This yields "*delete-repeat*" coordinates for EG and, hence, EG with these coordinates:

$\mathfrak{S}G = G^{n+1} = \{(g_0, g_1, \ldots, g_n)\}$,
$d_i(g_0, g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n)$, and
$s_i(g_0, g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_{i-1}, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_n)$.

Thus [16] each point in EG may be represented uniquely as $((g_0, g_1, \ldots, g_n), (t_0, t_1, \ldots, t_n))$ where $g_i \neq g_{i+1}$ and $t_i \neq 0$ for all i.

(4.6) *Classifying maps.* As in [6, pp. 54–57], given a principal G-bundle ξ: $G \to E \to X$ over a compact metric space X, a finite open cover $\mathfrak{U} = \{U_i\}$ of X, local sections s_i: $U_i \to E$, and a partition of unity $\{h_i\}$ subordinate to $\mathfrak{U}$, there is a canonical classifying map $X \to BG$ for ξ given by an explicit formula. We omit the details.

(4.7) *Linear homotopies.* Compare [6, pp. 54–57]. The maps $f_{m,n}(G^{m+1} \times \Delta^m) * (G^{n+1} \times \Delta^n) \to EG$ defined by

$$\begin{aligned} f_{m,n}\big(((g_0, g_1, \ldots, g_m), (t_0, t_1, \ldots, t_m)), s, \\ (g_0', g_1', \ldots, g_n'), (t_0', t_1', \ldots, t_m')\big) \\ = \big((g_0, g_1, \ldots, g_m, g_0', g_1', \ldots, g_m'), \\ ((1-s)(t_0, t_1, \ldots, t_m), s(t_0', t_1', \ldots, t_n'))\big) \end{aligned}$$

define an equivariant map μ: $EG * EG$ (diagonal action on the join) $\to EG$ whose restriction to each factor is the identity.

Now suppose that f, f': $X \rightrightarrows BG$ classify $G \to E \to X$. Let $\tilde{f}, \tilde{f}'$: $E \rightrightarrows EG$ be the associated maps on total spaces. Define an equivariant homotopy H: $E \times I \to EG$ by the formula $H(y, s) = \mu(f(y), s, f'(y))$. Then the induced map H: $X \times I \to BG$ provides a canonical "*linear homotopy*" from f to f'.

5. The transformation α. Let f: $X \to \hat{B}G$ be a continuous map. We may associate to f the principal $\{G_n\}$-fibration $f^*\xi_{\{G_n\}}$ over X where $\xi_{\{G_n\}}$: $\{G_n\} \to \{E'G_n\} \to \hat{B}G$ is the universal $\{G_n\}$-fibration over BG; see §2. This yields a function from the set of continuous maps $X \to \hat{B}G$ to $k_{\{G_n\}}(X)$. In this section we shall prove that homotopic maps $X \to \hat{B}G$ induce isomorphism principal $\{G_n\}$-fibrations over X, and thus define a natural transformation of functors

$$\alpha\colon [\ , \hat{B}G] \to k_{\{G_n\}}(\,), \quad \alpha_X\colon [c, \hat{B}G] \to k_{\{G_n\}}(X)$$

from compact metric spaces to pointed sets.

(5.1) PROPOSITION. *Let X be a compact metric space and let* $\xi = \{\xi_n$: $G_n \to$

$E_x \to X \times I\}$ be a principal $\{G_n\}$-fibration over $X \times I$. Let $\xi^0 = \{\xi_n^0\}$ and $\xi^1 = \{\xi_n^1\}$ be the restrictions of ξ to $X \times 0$ and $X \times 1$, respectively. Then there is an isomorphism $\xi^0 \to \xi^1$.

PROOF (outlined). Use Proposition (3.1) and the usual proof that the restrictions of an ordinary bundle over $X \times I$ to $X \times 0$ and $X \times 1$ are isomorphic (e.g., Husemoller [6, pp. 54–57]) to inductively define compatible bundle maps f_n: $\xi_n \to \xi_n^1 \subset \xi_n$. The required isomorphism is given by $\{f_n|: \xi_n^0 \to \xi_n^1\}$. □

(5.2) COROLLARY. *Let f, g: $X \rightrightarrows \hat{B}G$ be homotopic maps. Then f and g induce isomorphism principal $\{G_n\}$-fibrations over X.*

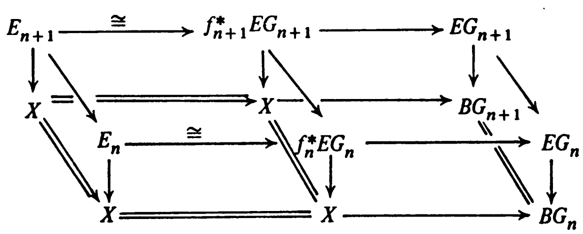
PROOF. Let H: $X \times I \to \hat{B}G$ be a homotopy from f to g. Then $f^*\xi = (H^*\xi)^0 \cong (H^*\xi)^1$ (by (5.1)) $= g^*\xi$. □

This yields a well-defined natural transformation α: $[\ ,\hat{B}G] \to k_{\{G_n\}}(\ )$.

6. Proof of (1.2). We shall prove our main result, that the natural transformation α: $[X, \hat{B}G] \to k_{\{G_n\}}(X)$ is an isomorphism for compact metric X, by constructing classifying maps for principal $\{G_n\}$-fibrations.

(6.1) *Construction of classifying maps.* Let $\xi = \{\xi_n: G_n \to E_n \to X\}$ be a principal $\{G_n\}$-fibration over a compact metric space X.

Use Proposition (3.1) to obtain compatible finite open covers $\mathcal{U}_n$, families of local sections $\mathcal{S}_n$, and associated partitions of unity $\mathcal{K}_n$. Use (4.6) to obtain classifying maps f_n: $X \to BG_n$ for each bundle ξ_n: $G_n \to E_n \to X$ from this data. A lengthy but straightforward computation shows that the diagrams



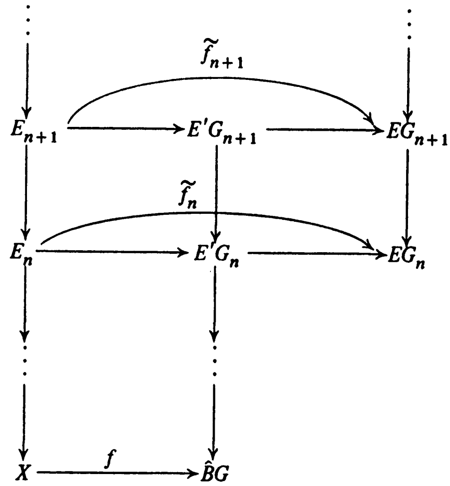
commute. Our construction thus yields the following.

(6.2) PROPOSITION. *Let $\xi = \{\xi_n: G_n \to E_n \to X\}$ be a principal $\{G_n\}$-fibration over a compact metric space X. Then there is a classifying map f: $X \to \hat{B}G$ with $\xi \cong f^*\xi_{\{G_n\}}$.*

PROOF. Let $f = \lim_n\{f_n\}$: $X \to \hat{B}G \equiv \lim_n\{BG_n\}$. The conclusion follows. □

(6.3) PROPOSITION. *Isomorphic principal $\{G_n\}$-fibrations over a compact metric space have homotopic classifying maps.*

PROOF. Let $\xi = \{\xi_n: G_n \to E_n \to X\}$ and $\xi' = \{\xi'_n: G_n \to E'_n \to X\}$ be isomorphic principal $\{G_n\}$-fibrations over X "classified" by f and mapping X to $\hat{B}G$, and $\xi \cong f^*\hat{\xi}_G$, and $\xi' \cong f'^*\hat{\xi}_G$. Because $\xi \cong \xi'$, $f'(= f' \circ \mathrm{id}_X)$ also classifies ξ. We obtain a commutative diagram



in which $\{\tilde{f}_n: E_n \to EG_n\}$ is a compatible family of equivariant maps and a similar diagram involving $f': X \to \hat{B}G$ and $\{\tilde{f}'_n: E_n \to EG_n\}$. Now use Proposition (4.8) to obtain compatible, equivariant "linear" homotopies $\{\tilde{H}_n: E_n \times I \to EG_n\}$ with $H_n|_0 = f_n$ and $H_n|_1 = f'_n$. Passing to quotient spaces yields compatible homotopies $\{H_n: X \times I \to BG_n\}$. Finally, the required homotopy is given by $H = \lim\{H_n\}: X \times I \to \hat{B}G$ $(H|_0 = f$ and $H|_1 = f')$. □

(6.4) *Proof of* (1.2). Propositions (6.2)–(6.3) yield a well-defined function $\beta: k_G(X) \to [X, \hat{B}G]$, where β associates to a bundle ξ its classifying map $\beta(\xi) \in [X, \hat{B}G]$. By construction, $\xi \cong \alpha(\beta(\xi)) \equiv \beta(\xi)^*\hat{\xi}_G$. Also, if $[f] \in [X, \hat{B}G]$, $\beta(\alpha(f))$ and f both classify $f^*\hat{\xi}_G$, so $[\beta(\alpha(f))] = [f]$ by (6.7). Therefore $\alpha: [X, \hat{B}G] \to k_G(X)$ and β are inverse isomorphisms. □

(6.5) PROPOSITION. (a) *Every open principal G-fibration over a CW complex is a principal G-bundle.* (b) *The natural map $BG \to \hat{B}G$ is a weak homotopy equivalence.*

PROOF. By (1.2) or a direct argument similar to (5.1), k_G is a homotopy functor, so each open principal G-fibration over a contractible space is a principal G-bundle. (a) now follows. (b) now follows from studying open principal G-fibrations over spheres. □

(6.6) REMARKS. (a) The above classification results can be extended to open principal G-fibrations over paracompact spaces.

(b) The classifying space $\hat{B}G = \lim\{BG_n\}$ can be interpreted as the homotopy inverse limit [**6**, §4], $\operatorname{holim}\{BG_n\}$, of the tower of fibrations $\{BG_n\}$; see (4.6). In this sense $\hat{B}G$ is a kind of completion of BG; consider, for example, profinite groups G.

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