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ON THE SEIFERT MANIFOLD OF A 2-KNOT

BY

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ABSTRACT. From geometric facts about embeddings $S^2 \to S^4$ we study the relationship between the smallest number of normal generators (weight) of a group and its preabelian presentations.

Introduction. Let f: $S^2 \to S^4$ be an embedding or 2-knot: f always [**9**, Lemma 1] extends to an embedding $\bar{f}$: $S^2 \times D^2 \to S^4$. Let X be the closure of $S^4 - \operatorname{Im} f$, a compact manifold called the *complement* of f. $\partial X = S^2 \times S^1$ and $\pi_1(X)$ is a finitely presented group G. Such G are called knot groups.

If $z_0 \in S^2$ and $z_0' \in S^1$ we choose $(z_0, z_0') \in \partial X$ as our basepoint. Let x be the element of G represented by $z_0 \times S^1$. Then [**5**, Theorem 1]

(1) $H_1(G) = G/G' = Z$ with generator x (a meridian for f).

(2) $H_2(G) = 0$.

(3) x and all its conjugates generate G.

(G' is the commutator subgroup $[G, G]$ of G.)

In general if $S \subseteq G$, we write $N_G S$ for $\bigcap N$, where $N \lhd G$ and $N \supseteq S$. Then (3) is $N_G\{x\} = G$, and $G' = N_G\{[g, h]: g, h \in G\}$.

Condition (2) is hard to handle geometrically in S^4. We choose instead a somewhat stronger statement:

(2′) G has a finite presentation with $r + 1$ generators and r relators.

Statement (2) is strictly stronger than (2′) and knots exist whose group does not satisfy (2′) [**5**, pp. 106–107]. The purpose of this note is to show:

THEOREM. *Let G be a finitely presented group satisfying* (1), (2′) *and* (3); *then*

(i) *There exists a knot f: $S^2 \to S^4$ with group G.*

(ii) *There exists a submanifold $V \subset S^4$ of the form $(S^1 \times S^2 \# \cdots \# S^1 \times S^2)_0$ (that is, $S^1 \times S^2 \# \cdots \# S^1 \times S^2$ minus an open disk) with $\partial V = f(S^2)$, $\pi_1(V)$ free of rank r.*

(iii) *$\pi_1(V) = \pi_1(S^4 - V)$ is a free group Φ of rank r; there exist monomorphisms ν_0, ν_1: $\Phi \to \Phi$ such that G has a presentation*

$$\langle x, \Phi: \nu_0(\varphi) = x\nu_1(\varphi)x^{-1}, \varphi \in \Phi \rangle.$$

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As a corollary we show that no such group G can be of the form $A * B$, $A \neq 1$, $B \neq 1$ (cf. [3]). We also obtain useful information about the cohomology of G.

1. Seifert manifolds of knots. Suppose $f\colon S^2 \to S^4$ is a knot. A Seifert manifold for f is any compact, framed 3-manifold $V \subseteq S^4$ with boundary $f(S^2)$.

If the group associated to f satisfies (2′) then it has a presentation of the form

$$\langle x, b_1, \ldots, b_r\colon b_1B_1, \ldots, b_rB_r\rangle \qquad (B_1, \ldots, B_r \in F'), \tag{1}$$

where we write F for the free group $\langle x, b_1, \ldots, b_r\colon \rangle$, J for $\langle x\colon \rangle$ and L for $\langle b_1, \ldots, b_r\colon \rangle$ so that $F = J * L$. Let $\lambda\colon F \to L$ be the map defined by $\lambda(x) = 1, \lambda(b_j) = b_j, j = 1, \ldots, r$.

Clearly any group with presentation (1) satisfies conditions (1) and (2′). For it to satisfy condition (3) it is necessary and sufficient that

$$L = N_L\{\lambda(b_1B_1), \ldots, \lambda(b_rB_r)\}. \tag{2}$$

Conversely for any group with a presentation (1) and satisfying condition (2) we can find an embedding $f\colon S^2 \to \Sigma^4$, where Σ is a homotopy sphere and $G = \pi_1(\Sigma^4 - f(S^2))$. In [2] we prove we may assume $\Sigma = S^4$ but the paper contains some gaps. We wish to present that proof here with more detail.

LEMMA (1). *Let G be a finitely presented group satisfying conditions* (1), (2′) *and* (3). *Then we may assume G has a presentation* (1) *such that $\lambda(b_1B_1), \ldots, \lambda(b_rB_r)$ actually generate L.*

PROOF. I. Let $\iota_0\colon D^3 \to D^5$ be the standard embedding of the 3-ball in D^5 and let $W_0 = D^5 - L(D^3)$. The boundary U_0 of W_0 is the complement of the trivial knot $f_0 = \iota_0|S^2$, with group J. Let $\psi_j\colon S^0 \times D^4 \to U_0$ be mutually disjoint embeddings ($j = 1, \ldots, r$). We attach 1-handles $h_j^1 \approx D^1 \times D^4$ to W_0 *via* ψ_j. Let ι' be the embedding $D^3 \to D^5 + \Sigma h_j^1$ and W' the complement of $\iota'(D^3)$. The fundamental group of $U' = \partial W'$ is now $F = J * L$, where the b_j correspond to the h_j^1. Let now $\bar{\psi}_j\colon S^1 \times D^3 \to U'$ be mutually disjoint embeddings representing $b_jB_j \in \pi_1(U')$, that is, $\bar{\psi}_j(S^1 \times 0)$ is a loop in U' representing the word b_jB_j. Attaching 2-handles $h_j^2 \approx D^2 \times D^3$ to W' *via* the ψ_j we obtain an embedding $\iota''\colon D^3 \to T$, where $T = D^5 + \Sigma h_j^1 + \Sigma h_j^2$ is a contractible space with boundary Σ, a homotopy 4-sphere. Embedding $f = \iota''|S^2$ is a knot whose group is presented by (1).

II. Recall now f_0 extends to an embedding $d_0\colon \Delta \to S^4$, where Δ is the 3-disk. Let $d'\colon \mathring{\Delta} \to U'$ be the composition $\mathring{\Delta} \to U_0 \cup \varphi_j(S^0 \times D^4) \subset U'$ which we may assume to be transversal to the $\bar{\psi}_j$. Let $\sigma\colon J * L \to J$ be the map defined by $\sigma x = x$, $\sigma b_j = 1$ its kernel is a free product $\Pi^* L_n$ ($n \in Z$) where $L_n = \langle b_{1n}, \ldots, b_{rn}\colon \rangle$, $b_{jn} = x^n b_j x^{-n}$.

We observe now that $d'\mathring{\Delta}$ has a framing $\bar{\nu}'(z, t) \in U'$ $(z \in \mathring{\Delta}, -1 < t < 1)$ such that $\bar{\nu}'(z, 0) = d'z$. Define ν'_ε: $\mathring{\Delta} \to U' - d'\mathring{\Delta}$ by $z \mapsto \bar{\nu}'(z, \varepsilon)$, $\varepsilon = \pm 1$. If $z_0 \in \mathring{\Delta}$ is the basepoint, we adopt $u_0 = \nu'_1 z_0$ as the basepoint for $U' - d'\mathring{\Delta}$. Let γ be a fixed arc from u_0 to $\nu'_{-1}(z_0)$ in $U' - d'\mathring{\Delta}$. We are going to alter $\mathring{\Delta}$ but we may assume γ remains in the complement of it throughout.

Suppose $j = 1$ for the time being, and let $b_1 B_1 = b_1 w_0 x^{\varepsilon_1} \cdots x^{\varepsilon_s} w_s$ where, if $1 < a < s$, $w_\alpha \in L$, $\Sigma \varepsilon_\alpha = 0$ and so $b_1 B_1 = b_{10} \Pi_\alpha w_{\alpha n_\alpha} (w_{\alpha n_\alpha} = x^{n_\alpha} w_\alpha x^{-n_\alpha})$ for certain $n_\alpha \in Z$. Let $M = \max\{n_\alpha: n_\alpha > 0\}$, $\mu = \min\{n_\alpha: n_\alpha < 0\}$. Define H_M^0 (resp. H_μ^0) to be the subgroup of L generated by the w_α for which $n_\alpha = M$ (resp. $n_\alpha = \mu$). Both subgroups are free of finite rank generated by, say, $h_1^0, \ldots, h_{t_0}^0$ and $g_1^0, \ldots, g_{u_0}^0$. It is understood that each h_p^0 (g_q^0) is one of the w_α $(n_\alpha = M)$ or w_α $(n_\alpha = \mu)$ for $p = 1, \ldots, t_0$; $q = 1, \ldots, u_0$.

With u_0 as basepoint, define maps $\bar{h}_p, \bar{g}_q$: $S^1 \times D^3 \to U' - d'\mathring{\Delta}$ representing h_p^0, g_q^0 respectively. Then we may define new embeddings $\bar{h}'_p, \bar{g}'_q$: $S^1 \times D^3 \to U'$ representing $x h_p^0 x^{-1}$ and $x^{-1} g_q^0 x$ which are transversal to d' and where $\bar{h}'_p(S^1 \times D^3) \cap d'\mathring{\Delta} = S^0 \times D^3$ for each p and a similar relation for $\bar{g}'_q$. The points in $(\bar{h}'_p)^{-1}$ $(\bar{h}'_p(S^1 \times 0) \cap d'\mathring{\Delta})$ partition S^1 in two intervals I_p, I'_p and if we join the endpoints of $\bar{h}'_p(I_p)$ and $\bar{h}'_p(I'_p)$ in $d'\mathring{\Delta}$ one of the resulting loops, say the one coming from I_p, represents h_p^0 in U'. Similarly, let J_q be the corresponding interval for $\bar{g}'_q$. Write

$$\Delta' = \Delta - \bigcup_{p,q} \left\{\bar{h}'_p(\mathring{I}_p \times \mathring{D}^3) \cup \bar{g}'_q(\mathring{J}_q \times \mathring{D}^3)\right\},$$

$$\Delta_1 = \Delta' \cup \bigcup_{p,q} \left\{\bar{h}'_p(I_p \times \dot{D}^3) \cup \bar{g}'_q(J_q \times \dot{D}^3)\right\}.$$

Then d' can be extended to a (framed) embedding d'_1: $\Delta_1 \to U'$ with framing $\bar{\nu}_1$ and we may assume $z_0 \in \Delta_1$, $\gamma \subset U' - d'_1\mathring{\Delta}_1$. The intersection points in $P_0 = \Delta \cap \bar{\psi}_1(S^1 \times 0)$ are in 1-1 correspondence with $\{1, \ldots, s\} \times S^0$, whereas $P_1 = \Delta_1 \cap \bar{\psi}_1(S^1 \times 0)$ is in 1-1 correspondence with $\{\alpha: \mu \neq n_\alpha \neq M\} \times S^0$. Algebraically, this means the loop $\bar{\psi}_1(S^1 \times 0)$ represents a word $b_{10} w'_0 x^{\varepsilon'_1} \cdots x^{\varepsilon'_t} w'_t$ $(t < s)$, $w'_t \in \pi_1(U' - d_1\Delta_1) = L * H_\mu^0 * H_M^0$ and so we can write $[\bar{\psi}_1(S^1 \times 0)] = b_{10} \Pi_\beta w'_{\beta n_\beta}$ where if $1 < \beta < t$, $\mu + 1 < n_\beta < M - 1$.

III. We now repeat the procedure in II on the words w'_β with $n_\beta = M - 1$ or $\mu + 1$ provided $M - 1 > 0$, $\mu + 1 < 0$. If, say, $\mu + 1 = 0$ we work with $\{w'_\beta | n_\beta = M - 1\}$ only. Let H^1_{M-1} and $H^1_{\mu+1}$ be the free group generated by $\{w'_\beta | n_\beta = M - 1\}$ and $\{w'_\beta | n_\beta = \mu + 1\}$ respectively and $\{h_1^1, \ldots, h_{t_1}^1\}$, $\{g_1^1, \ldots, g_{u_1}^1\}$ subsets of the above generating sets that freely generate the groups. Define $h_p^{(1)}, g_q^{(1)}$: $S^1 \times D^3 \to U' - d_1(\mathring{\Delta}_1)$ representing the h_p^1, g_q^1 respectively and alter them to embeddings $\bar{h}_p^{(1)}, \bar{g}_q^{(1)}$: $S^1 \times D^3 \to U'$ representing $x h_p^1 x^{-1}$ and $x^{-1} g_q^1 x$, transversal to d'_1 and intersecting $d_1(\Delta_1)$ in a copy of $S^0 \times D^3$. Again $(\bar{h}_p^{(1)})^{-1}$ $(\bar{h}_p^{(1)}(S^1 \times 0) \cap d_1(\Delta_1))$ partitions S^1 in two intervals

$I_p^{(1)}$ and $I_p^{(1)\prime}$ and we choose, as in II, $I_p^{(1)}$, that one that represents $h_p^{(1)}$. We may alter Δ_1 to Δ_2 and find a framed embedding d_2: $\mathring{\Delta}_2 \to U'$ (with framing $\bar{\nu}_2$) so that $\gamma \subset U' - d_2(\Delta_2)$ and where $P_2 = d_2\Delta_2 \cap \psi_1(S^1 \times 0)$ is in 1-1 correspondence with $\{\alpha: \mu + 2 < n_\alpha < M - 2\}$. After $m = \max\{|\mu|, M\}$ we obtain a framed embedding d_m: $\Delta_m \to U'$ disjoint from $\psi_1(S^1 \times D^3)$ and Δ_m is obtained from the connected sum of S copies of $S^1 \times S^2$ by removing an open disk (or Δ_m is obtained from Δ by attaching S handles $D^1 \times S^2$). We may assume $z_0 \in \Delta_m$, $\gamma \subset U' - d_m\mathring{\Delta}_m$.

IV. The fundamental group $\pi_1(\Delta_1)$ is free and its free generators are in 1-1 correspondence with h_p^0 and g_q^0. Similarly $\pi_1(\Delta_2)$ is free in a set in 1-1 correspondence with h_p^η, g_q^θ, $0 \leqslant \eta, \theta \leqslant 1$. In general $\pi_1(\Delta_m)$ has free generators which we may label $h_p^{(\eta)}$, for $p = 1, \dots, t_\eta$, $1 < \eta < M$ and $g_q^{(\theta)}$, $q = 1, \dots, u_\theta$, $1 \leqslant \theta \leqslant |\mu|$.

On the other hand, $\pi_1(U' - d_m\mathring{\Delta}_m)$ is also free generated by

$$\{b_j: 1 < j < r\} \cup \left\{\left[\bar{h}_p^\eta(S^1 \times 0)\right]; p = 1, \dots, t_\eta, 1 < \eta < M\right\}$$
$$\cup \left\{\left[\bar{g}_q^\theta(S^1 \times 0)\right]; q = 1, \dots, u_\theta, 1 \leqslant \theta \leqslant |\mu|\right\}$$

where, as usual, $[x]$ is the homotopy class of any loop x based on $u_0 = \gamma(0) \in U'$.

The framing $\bar{\nu}_m$ permits us to find [4, p. 573] a compact manifold Y which is a deformation retraction of $U' - d_m(\mathring{\Delta}_m)$ and with boundary $V_0 \cup V_1$, where $V_t \approx \Delta_m$, and where map f_t: $\Delta_m \hookrightarrow V_t \hookrightarrow Y$ is homotopic to ν_{-1} if $t = 0$ and to ν_1 if $t = 1$. If we assume $\gamma \subset Y$ then $z_0 \in V_1$ and f_t defines homomorphisms f_t: $\pi_1(V) \to \pi_1(Y)$ by

$$f_{1*}(h_p^\eta) = w_{1p}, \qquad f_{0*}(h_p^\eta) = \left[\gamma\bar{h}_p^\eta(S^1 \times 0)\gamma^{-1}\right],$$
$$f_{1*}\left(g_q^\theta\right) = \left[\bar{g}_q^\theta(S^1 \times 0)\right], \quad f_{0*}\left(g_q^\theta\right) = w'_{\theta q}, \tag{3}$$

where we adopt the following conventions: $w_{\eta p}$ (resp. $w'_{\theta q}$) is a word in the free group $\pi_1(U' - d_\eta(\mathring{\Delta}_\eta))$ (resp. $\pi_1(U' - d_\theta(\mathring{\Delta}_\theta))$) which is part of a set of free generators for $H_{M-\eta}$ (resp. $H_{\mu+\theta}$). Thus, for example w'_{1q} is one of the w_α ($n_\alpha = \mu$) of II and w_{2p} is one of the w_β ($n_\beta = M - 1$) of III. Since these $w_{\eta p}$, $w'_{\theta q}$ are free generators of $H_{M-\eta}$, $H_{\mu+\theta}$, the maps f_{t*} are monomorphisms. Furthermore if $x = h_p^\eta$ or g_q^θ, the $[f_{1*}(x)]^{-1}f_{0*}(x)$ generate a free factor Φ in $\pi_1(Y)$ since they contain letter x only once. Thus $\pi_1(Y) = L * \Phi$.

V. To work with all r words b_jB_j simultaneously, we repeat the above procedure on Δ_m using an embedding $S^1 \times D^3 \to U' - d_m(\mathring{\Delta}_m)$ representing b_2B_2, etc.

VI. Summarizing: we have found a 3-manifold V obtained from Δ by attaching handles, that is V is of the form $S^1 \times S^2 \# \cdots \# S^1 \times S^2 - \mathring{D}^3$ and a framed embedding d: $\mathring{V} \to U'$. The embeddings $\bar{\psi}_j$: $S^1 \times D^3 \to U'$ are disjoint from $d(\mathring{V})$. If $\Phi = \pi_1(V)$, $\pi_1(U' - d(V)) = \pi_1(Y)$ has the form $L * \Phi$ and we have maps f_1, f_0: $\Phi \to L * \Phi$ where the $y_j = [f_1(x_j)]^{-1} f_0(x_j)$ generate the free factor Φ if the x_j do. Recall $U' = \partial W'$ (see I) and so we can find an embedding ι': $S^2 \to \partial(D^5 + \Sigma h_j^1)$ such that U' is the complement of $\iota'(S^2)$; d: $\mathring{V} \to U'$ can be extended to an embedding $V \to U' \cup \iota(S^2)$. The $\bar{\psi}_j$ represent in $U' - d_m(\mathring{V})$ elements of the form $b_j w_{j0} v_{j1} \cdots v_{js} w_{js}$ ($w_{jl} \in L$, $v_{jl} \in \Phi$) where the w_{jl} correspond to those factors $w_{\alpha n_\alpha}$ with $n_\alpha = 0$. Fix j, by increasing the rank of Φ by 2 we may assume the ψ_j represent words $b_j v_j$ where v_j is in $\Phi_1 = \Phi * \langle c, d: \ \rangle$ where $L * \Phi_1 = \pi_1(U' - d'(V'))$, V' is obtained from V by taking connected sum with $S^1 \times S^2 \# S^1 \times S^2$. In fact, write $xb_j B_j x^{-1} = x\bar{b}_j x^{-1} \cdot xB_j x^{-1}$. To distinguish those four factors we write $x_1 b_j x_2 x_3 B_j x_4$. Deform $\bar{\psi}_j$ to $\bar{\psi}_j'$ so that it represents $xb_j B_j x^{-1}$. Thus $\bar{\psi}_j'(S^1 \times D^3)$ intersects $d_m(V)$ and $\bar{\psi}_j'^{-1}(\bar{\psi}_j'(S^1 \times D^3) \cap d_m\mathring{V}) = \bigcup_{i=1}^4 p_i \times D^3$ where $p_i \in S^1$ corresponds to factor x_i ($i = 1, 2, 3, 4$). The p_i partition S^1 in four intervals, I_1, I_2, I_3, I_4, with $\dot{I}_1 = \{p_4, p_1\}$, $\dot{I}_2 = \{p_1, p_2\}$, etc. Finally, let $\frac{1}{2}D^2 = \{z \in R^4: |z| < \frac{1}{2}\}$. Define

$$V_0' = V - \left\{\bar{\psi}_j'\left(\mathring{I}_1 \times \mathring{D}^3\right) \cup \bar{\psi}_j\left(\mathring{I}_3 \times \tfrac{1}{2}\mathring{D}^3\right)\right\},$$

$$V' = V_0' \cup \left\{\bar{\psi}_j'\left(I_1 \times \dot{D}^3\right) \cup \bar{\psi}_j'\left((S^1 - \mathring{I}^3) \times \tfrac{1}{2}\dot{D}^3\right)\right\}.$$

This V' satisfies the desired condition. By doing this for all j, we obtain a manifold $V'' \subset U'$ of the form $\#(S^1 \times S^2) - \mathring{D}^3$ such that if Y_1 is a deformation retraction of $U' - V''$ which is a compact manifold with $\partial Y_1 = V_0'' \cup V_1''$, then

(a) $\pi_1(V'') = \Phi$ is free; let $\{x_1, \ldots, x_n\}$ be a set of generators.

(b) $\pi_1(Y_1) = L * \Phi$ and $V_t'' \subset Y_1$ define two monomorphisms f_t: $\pi_1(V'') \to \pi_1(Y_1)$ such that $y_i = f_1(x_i) f_0(x_i^{-1})$ are free generators of free factor Φ in $\pi_1(Y_1)$.

(c) The $\bar{\psi}_j$ represent words $b_j v_j$, $v_j \in \Phi$ in $\pi_1(Y_1)$.

Attaching the two handles h_j^2 to U' along the $\bar{\psi}_j$ adds the relations $b_j = \bar{v}_j^1$. Let $Y'' = \partial T - (Y_1 \cap T)$; then $\pi_1(Y'') = \Phi$.

VII. We have found an embedding d'': $V'' \to \partial T = \Sigma$ such that both V'' and Y'' ($\sim \Sigma - d''(V'')$) have free fundamental group Φ with generators $\{x_i\}$ and $\{y_i = f_1(x_i^{-1}) f_0(x_i)\}$ respectively. We can present our group G by

$$\left\langle x, y_1, \ldots, y_n: f_1(x_i^{-1}) x f_0(x_i) x^{-1}, i = 1, \ldots, n \right\rangle. \tag{4}$$

In fact $\Sigma - d''(S^2)$ has fundamental group G. On the other hand $\Sigma - d''(S^2)$ is obtained from Y'' by identifying $\mathring{V}_0''$ to $\mathring{V}_1''$ and so by the van Kampen theorem, G has presentation (4). Clearly presentation (4) satisfies condition (2).

The result in the introduction follows from this proof.

THEOREM (2). *Let G be a group satisfying conditions* (1), (2′) *and* (3); *then there exists a knot f: $S^2 \to S^4$ with associated group G.*

PROOF. Repeat part I of the proof of Lemma (1) using a presentation that satisfies condition (2). Using the same notation we may assume that in T the attaching maps $\bar{\psi}_j$ isotope to maps so that $\bar{\psi}_j(S^1 \times 0)$ intersects the transversal disk of h_k^1 in one point if $j = k$ or it is empty if $j \neq k$. This *is* the geometric meaning of condition (2). In particular ∂T is the standard sphere.

2. Commutator subgroup. In view of our above results we may write a structure theorem for $G' = [G, G]$.

Let Φ_m be a group isomorphic to Φ ($m \in Z$) and a_m: $\Phi \to \Phi_m$ an explicit isomorphism. Consider $P = \Pi_m^* \Phi_m$, an infinite free product. Write

$$\cdots \underset{\Phi}{*} \Phi_{-1} \underset{\Phi}{*} \Phi_0 \underset{\Phi}{*} \Phi_1 \underset{\Phi}{*} \cdots \tag{5}$$

for the quotient $P/N_P\{a_m f_1(x_i)[a_{m+1} f_0(x_i)]^{-1}: m \in Z, i = 1, \ldots, n\}$, that is the infinite free product of copies of Φ with amalgamations by f_0 and f_1 ([**6**], as in [**4**] and [**7**]).

PROPOSITION (3). *If G satisfies conditions* (1), (2′) *and* (3), *G is isomorphic to the group presented by* (5), *where Φ, f_0 and f_1 are as in Lemma* (1).

For the proof of this proposition construct $\tilde{X}$, the universal abelian (infinite cyclic) covering space of X by taking copies Y_m'' of Y'' (cf. Lemma 1) with boundary $V_0''(m) \cup V_1''(m)$. Then $\tilde{X}$ is obtained by identifying $V_1''(m)$ to $V_0''(m+1)$ for all m. An application of the van Kampen theorem and a direct limit yield the desired result since $\pi_1(Y_m'') = \Phi_m$.

COROLLARY (4). *G' has cohomological dimension $\leqslant 3$. If G' is finitely presented then G' is free.*

THEOREM (5). *G' has cohomological dimension $\leqslant 2$.*

PROOF. Let M be a (left) G'-module. Then (5) defines structures of Φ_m-module for M. If G is presented by (4) $x\Phi_m x^{-1} = w\Phi_{m+1}w^{-1}$ for some $w \in G'$ and so $H^*(\Phi_m; M)$ is canonically isomorphic to $H^*(\Phi_{m+1}; M)$ and so $H^*(\Phi; M)$ is independent of m ([**1**, p. 197], [**3**, Lemma 2]). Write J for the free group with generator x. As in [**4**], $H^*(G'; M)$ fits in a Mayer-Vietoris sequence of J-modules, where differentials d_0 and d_1 are both $f_1^* \otimes 1 - f_0^* \otimes x$:

$$0 \to H^0(G'; M) \to H^0(\Phi; M) \otimes ZJ \xrightarrow{d_0} H^0(\Phi; M) \otimes ZJ$$

$$\to H^1(G'; M) \to H^1(\Phi; M) \otimes ZJ \xrightarrow{d_1} H^1(\Phi; M) \otimes ZJ$$

$$\to H^2(G'; M) \to 0.$$

I. For $m \in H^0(\Phi; M) = M^\Phi$, $d_0(m \otimes 1) = m \otimes (1 - x)$ and so in coker d_0 we equate the actions of $f_0(x_i)$ and $f_1(x_i)$ (x_i generate Φ). Since $y_i = f_1(x_i^{-1})f_0(x_i)$ also generate Φ, coker $d_0 = M^\Phi$, and our sequence reduces to

$$0 \to M^\Phi \to H^1(G'; M) \to M_\Phi \otimes ZJ \xrightarrow{d_1} M_\Phi \otimes ZJ \to H^2(G'; M) \to 0 \qquad (6)$$

since $H^1(\Phi; M) = M_\Phi = M \otimes_\Phi Z$ (cf. [1, p. 197]). Tensoring (6) over J with Z we get

$$M_\Phi \xrightarrow{\bar{d}_1} M_\Phi \to H^2(G'; M)_J \to 0, \qquad (7)$$

an exact sequence of abelian groups.

II. To calculate $\bar{d}_1$, let $R = Z\Phi$ and $I = \ker(R \to Z)$ [1, X.4]. If $\xi_i = x_i - 1$ and $\eta_i = y_i - 1$, I is a free R-module in the ξ_i or the η_i ($i = 1, \dots, n$). For a left Φ-module M define ι^*: $M \to \mathrm{Hom}_\Phi(I, M)$ by $\iota^*(m)(w - 1) = wm - m$. Then (loc. cit.) $H^1(\Phi; M) = \operatorname{coker} \iota^*$.

Let $g \in \mathrm{Hom}_\Phi(I; M)$. Then

$$\begin{aligned}(f_1^* - f_0^*)g(\xi_i) &= g(f_1(x_i) - f_0(x_i)) \\ &= f_1(x_i)g\big(1 - f_1(x_i^{-1})f_0(x_i)\big) = -f_1(x_i)g(\eta_i)\end{aligned}$$

and since $f_1(x_i)$ is a *unit* in R, $f_1^* - f_0^* = \bar{d}_1$ is an isomorphism $\mathrm{Hom}_\Phi(I; M) \to \mathrm{Hom}_\Phi(I; M)$ which induces an isomorphism $M_\Phi \to M_\Phi$ and so, by the exactness of (7)

$$H_2(G'; M)_J = 0. \qquad (8)$$

III. By the Lyndon spectral sequence [1, XVI. 7],

$$H^3(G; M) = H^1\big(J; H^2(G'; M)\big) = H^2(G'; M)_J = 0$$

by (8). Q.E.D.

COROLLARY (6). *For a group G satisfying conditions* (1), (2′), (3) *and for all G'-modules M, $H_2(G'; M)_J = 0$.*

PROPOSITION (7). *A group satisfying conditions* (1), (2′) *and* (3) *is never a nontrivial free product*. (*Compare with* [3].)

PROOF. If it is, we may assume $G = H * J$. Then $G' = \amalg_m^* H_m$ where $H_m \approx H$. For all H-modules M, $H^2(H; M) = H^2(G'; M)_J = 0$ by Corollary (6), so that c.d. $H \leqslant 1$, that is, H is free [8, 0.3]. By condition (1), $H/H' = 0$ so H itself must be trivial.

3. Generalizations. Let mS^2 be the disjoint union of m copies of S^2: $mS^2 = S_1^2 + \cdots + S_m^2$. An m-link is an embedding f: $mS^2 \to S^4$. If $G = \pi_1(S^4 - \operatorname{Im} f)$ then [5, Theorem 3]

(4) $H_1(G; Z)$ is free of rank m with generators $x_1, \dots, x_m$,

(5) $H_2(G; Z) = 0$,

(6) $N_G\{x_1, \ldots, x_m\} = G$,

where the x_i are meridians for $f|S_i^2$.

Again if we change (5) by

(5′) G has a presentation with $r + m$ generators and r relations,

we obtain

THEOREM (8). *For any group G satisfying conditions* (4), (5′) *and* (6) *there exists an embedding f: $mS^2 \to S^4$ with associated group G; the commutator subgroup G^1 has $c \cdot d \leq 2$ and G is never a nontrivial free product with $m + 1$ factors.*

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