TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 240, June 1978

ERICKSON'S CONJECTURE ON THE RATE OF ESCAPE OF d-DIMENSIONAL RANDOM WALK

BY

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ABSTRACT. We prove a strengthened form of a conjecture of Erickson to the effect that any genuinely d-dimensional random walk S_n, $d \geq 3$, goes to infinity at least as fast as a simple random walk or Brownian motion in dimension d. More precisely, if S_n^* is a simple random walk and B_t a Brownian motion in dimension d, and ψ: $[1,\infty) \to (0,\infty)$ a function for which $t^{-1/2}\psi(t)\downarrow 0$, then $\psi(n)^{-1}|S_n^*| \to \infty$ w.p.1, or equivalently, $\psi(t)^{-1}|B_t| \to \infty$ w.p.1, iff $\int_1^\infty \psi(t)^{d-2} t^{-d/2} < \infty$; if this is the case, then also $\psi(n)^{-1}|S_n| \to \infty$ w.p.1 for *any* random walk S_n of dimension d.

1. **Introduction.** Let $X_1, X_2, \ldots$ be independent identically distributed random d-dimensional vectors and $S_n = \sum_{i=1}^n X_i$. Assume throughout that $d \geq 3$ and that the distribution function F of X_1 satisfies

$$\text{supp}(F) \text{ is not contained in any hyperplane.} \tag{1.1}$$

The celebrated Chung-Fuchs recurrence criterion [1, Theorem 6] implies that[1] $|S_n| \to \infty$ w.p.1 irrespective of F. In [4] Erickson made the much stronger conjecture that there should even exist a uniform escape rate for S_n, viz. that $n^{-\alpha}|S_n| \to \infty$ w.p.1 for all $\alpha < \frac{1}{2}$ and all F satisfying (1.1). Erickson proved his conjecture in special cases and proved in all cases that $n^{-\alpha}|S_n| \to \infty$ w.p.1 for $\alpha < 1/2 - 1/d$. Our principal result is the following theorem which contains Erickson's conjecture and which makes precise the intuitive idea that a simple random walk S_n^* (which corresponds to the distribution F^* which puts mass $(2d)^{-1}$ at each of the points $(0, 0, \ldots, \pm 1, 0, \ldots, 0)$) goes to infinity slower than any other random walk.

THEOREM. *Let $d \geq 3$, S_n and S_n^* as above, and let $\{B_t\}_{t \geq 0}$ be a d-dimensional Brownian motion ($EB_t \equiv 0$, $EB_t(i)B_t(j) = t\delta_{i,j}$). Assume that the function ψ: $[1,\infty) \to (0,\infty)$ satisfies*

$$t^{-1/2}\psi(t)\downarrow 0. \tag{1.2}$$

Then $\psi(n)^{-1}|S_n^*| \to \infty$ w.p.1 and $\psi(t)^{-1}|B_t| \to \infty$ w.p.1 are both equivalent to

Received by the editors June 25, 1976.

AMS (MOS) subject classifications (1970). Primary 60J15, 60F20.

Key words and phrases. Random walk, escape rate, concentration functions.

[1]For any vector $v \in \mathbf{R}^d$ we denote its components by $v(1), \ldots, v(d)$, and $|v| = \{\sum_{i=1}^d v^2(i)\}^{1/2}$. "w.p.1" stands for "with probability one".

$$\int_1^{\infty} \psi(t)^{d-2} t^{-d/2} < \infty. \tag{1.3}$$

If (1.2) *and* (1.3) *hold, then also* $\psi(n)^{-1}|S_n| \to \infty$ *w.p.*1 *whenever* F *satisfies* (1.1).

As we shall see this theorem is almost immediate from

PROPOSITION 1. *If* $d \geq 3$ *and* F *satisfies* (1.1) *then there exist constants* $0 < \Gamma_1(F), \Gamma_2(F) < \infty$ *such that for all* $k \geq 1$ *and all* $A > 0$,

$$P\left\{|S_n| \leq A \text{ for some } 2^k \leq n < 2^{k+1}\right\} \leq \Gamma_1(F)\left\{\left((A+1)2^{-k/2}\right)^{d-2} + \exp -\Gamma_2(F)k\right\}. \tag{1.4}$$

It is worthwhile contrasting the theorem with the observation (see [4, end of §5][2]) that even if (1.1)–(1.3) hold, there may exist a deterministic sequence of vectors a_n such that

$$\liminf_{n\to\infty} \psi(n)^{-1}|S_n - a_n| = 0 \quad \text{w.p.1.} \tag{1.5}$$

Thus, the theorem is not merely a matter of estimating concentration functions for S_n. Nevertheless the result is intimately connected with concentration functions. Erickson's proof for $\psi(t) = t^{\alpha}$, $\alpha < 1/2 - 1/d$, uses a concentration function inequality of Esseen [6], and our proof makes heavy use of the following inequalities for concentration functions in dimension two. These estimates too are closely related to those of Esseen [5, Theorem 3], but are more generally applicable. We note that Corollary 1 shows that in the identically distributed case the concentration function decreases like n^{-1}.

PROPOSITION 2. *Let* $Z_1, Z_2, \ldots, Z_n$ *be independent random two-vectors with distribution functions* $G_1, G_2, \ldots, G_n$. *Let* $\rho, \rho_1, \rho_2, \ldots, \rho_n$ *be strictly positive numbers such that* $\rho_i \leq \rho$, *and let* $A_1, A_2, \ldots, A_n$ *be sets in* $\mathbf{R}^2 \times \mathbf{R}^2$ *such that* $A_i \subset \{u \in \mathbf{R}^2\colon |u| > \rho_i\}^2$. *Define the symmetrization* G_i^s *of* G_i *by*[3]

$$G_i^s(B) = \int_{\mathbf{R}^2} G_i(B+u)\, dG_i(u) \tag{1.6}$$

and put[4]

$$q_i = \int_{|u|>\rho_i} dG_i^s(u), \quad \sigma_i^2 = \inf_{|\theta|=1} \int_{|u|\leq\rho_i} \langle\theta,u\rangle^2\, dG_i^s(u),$$

$$C = \sum_{i=1}^{n} \left\{\sigma_i^2 + \frac{\rho_i^2}{q_i} \int_{u,v\in A_i} dG_i^s(u)\, dG_i^s(v)\right\}. \tag{1.7}$$

[2]More details are given in the recent article by K. B. Erickson, *Slowing down d-dimensional random walks*, Ann. Probability 5 (1977), 645–651.

[3]For a distribution function F and Borel set A, $F(A)$ denotes the mass assigned to A by the Borel measure induced by F.

[4]$\langle u,v\rangle$ denotes the usual inner product of two vectors of the same dimension.

Finally, denote the angle between two vectors u and v, determined in such a way that it lies in $[0,\pi]$, *by* $\varphi(u,v)$. *Then there exists a universal constant* $K_1 < \infty$ *such that*

$$\sup_{z\in\mathbf{R}^2} P\left\{\left|\sum_1^n Z_i + z\right| \leqslant \rho\right\}$$
$$\leqslant \frac{K_1\rho^2}{C}\exp\left\{\sum_{i=1}^n \frac{\rho_i^2}{Cq_i}\int_{u,v\in A_i} dG_i^s(u)\, dG_i^s(v)\right.$$
$$\left.\cdot\log\left(1+\frac{\rho_i}{|v|\,|\sin\varphi(u,v)|}\right)\right\}. \tag{1.8}$$

COROLLARY 1. *Let $Z_1, \ldots, Z_n$ be independent random two-vectors, all with the same distribution function G, and define G^s as in* (1.6). *If* $\rho > 0$ *and*

$$q = \int_{|u|>\rho} dG^s(u), \quad \sigma^2 = \inf_{|\theta|=1}\int_{|u|<\rho}\langle\theta,u\rangle^2\, dG^s(u),$$

and if there exist sets $B_1, B_2 \subset \{z \in \mathbf{R}^2\colon |z| > \rho\}$ *and a constant* $c > 0$ *such that*

$$|v|\,|\sin\varphi(u,v)| \geqslant c \quad \textit{whenever } u \in B_1,\ v \in B_2, \tag{1.9}$$

then

$$\sup_{z\in\mathbf{R}^2} P\left\{\left|\sum_1^n Z_i + z\right| \leqslant \rho\right\}$$
$$\leqslant \frac{K_1}{n}\rho^2(1+\rho c^{-1})\left\{\sigma^2 + \frac{\rho^2}{q}\int_{B_1} dG^s(u)\int_{B_2} dG^s(v)\right\}^{-1}. \tag{1.10}$$

ACKNOWLEDGEMENT. We wish to thank Professor Erickson for making his results [4] available to us before publication and for some helpful conversations.

2. **Reduction to a two dimensional problem.** In this section we shall reduce the proof of Proposition 1 to a two dimensional problem. This problem will then be settled together with Proposition 2, by means of estimates on two dimensional concentration functions in §3. Throughout $K_1, K_2, \ldots$ will be strictly positive constants which depend on the dimension d only and whose numerical value is immaterial for our purposes. They will not necessarily have the same value at each appearance. $\Gamma_1, \Gamma_2, \ldots$ and $k_1, k_2, \ldots$ will be constants which depend on F and d or some other parameters; where necessary these parameters will be indicated explicitly, except that we shall not indicate dependence on d. For any random variable Y we write $Y^s = Y - Y'$ where

Y' is independent of Y and has the same distribution as Y. If Y has distribution function G, then Y^s has the distribution function G^s given by

$$G^s(B) = \int G(B+y)\, dG(y).$$

The first reduction shows that we may assume that F has certain smoothness properties. It is of a purely technical nature and has little to do with the basic idea of the proof of Proposition 1. The reader should skip the proof of Lemma 1 at first reading.

LEMMA 1. *It suffices to prove Proposition* 1 *in the case where*

$$F(dx) \geqslant a\, dx \quad on\ C_0 \tag{2.1}$$

for some $a > 0$ *and some closed cube* $C_0 \subset \mathbf{R}^d$ (*which does not reduce to a point*).

PROOF. Let F satisfy (1.1). We shall construct an F_1 which satisfies (2.1) (and a fortiori (1.1)) such that Proposition 1 holds for the original S_n as soon as it holds for a random walk, $S_n(F_1)$ say, whose increments have distribution function F_1. For this purpose we first find a large cube

$$C_1 = \{z \in \mathbf{R}^d : -L \leqslant z(i) \leqslant L, 1 \leqslant i \leqslant d\} \tag{2.2}$$

such that $F(C_1) > 0$ and such that the intersection of supp(F) with C_1 is not contained in any hyperplane. This is possible by (1.1). Define the distribution functions G and H by[5]

$$p_0 = \tfrac{1}{2}F(C_1), \qquad G(S) = (2p_0)^{-1}F(S \cap C_1)$$

and

$$H(S) = (1-p_0)^{-1}\{\tfrac{1}{2}F(S \cap C_1) + F(S \cap C_1^c)\}.$$

Then we can write $F = p_0 G + (1-p_0)H$, and if $I_1, I_2, \ldots, U_1, U_2, \ldots, V_1, V_2, \ldots$ are totally independent random variables with

$$P\{I_j = 0\} = 1 - P\{I_j = 1\} = p_0,$$

$$P\{U_j \in S\} = G(S), \quad P\{V_j \in S\} = H(S),$$

then the joint distribution of $\{X_n\}_{n \geqslant 1}$ is the same as of $\{U_n(1-I_n) + V_n I_n\}_{n \geqslant 1}$ (compare [9, p. 1184]). We shall therefore assume that U, V and I are defined on our original probability space and that $X_n = U_n(1-I_n) + V_n I_n$. Let $\sigma_1 < \sigma_2 < \ldots$ be the successive (random) indices for which $I_n = 1$. Then, with $\sigma_0 = 0$, the $\sigma_{i+1} - \sigma_i$ are independent and all with the distribution

$$P\{\sigma_{i+1} - \sigma_i = r\} = p_0^{r-1}(1-p_0), \qquad r \geqslant 1. \tag{2.3}$$

[5]For any set $S \subset \mathbf{R}^d$, S^c denotes its complement, i.e., $\mathbf{R}^d \setminus S$.

Even more, if $\mathcal{F}_n$ is the σ-field generated by $\{U_j, V_j, I_j\colon 1 \leq j \leq n\}$, then for each $\{\mathcal{F}_n\}$ stopping time T and

$$\sigma^*(T) = \text{smallest } \sigma_i \text{ which exceeds } T,$$

one has

$$P\{\sigma^*(T) = T + r|\mathcal{F}_T\} = p_0^{r-1}(1-p_0), \qquad r \geq 1.$$

It is also not hard to see that the $\{S_{\sigma_{i+1}} - S_{\sigma_i}\}_{i \geq 0}$ are independent and all with the same distribution function[6]

$$F_2 = \sum_{r=1}^{\infty} P\{\sigma_1 = r\} G^{*(r-1)} * H = \sum_{r=1}^{\infty} p_0^{r-1}(1-p_0) G^{*(r-1)} * H,$$

and that also, conditional on $\mathcal{F}_T$, $S_{\sigma^*(T)} - S_T$ has the distribution F_2 on the set $\{T < \infty\}$. Now take

$$T = \inf\{n \in [2^k, 2^{k+1}) \text{ with } |S_n| \leq A\}$$

($= \infty$ if no such T exists). Then, since $|S_T| \leq A$,

$$\begin{aligned}
&P\left\{|S_{\sigma_l}| \leq 2A \text{ for some } l \in \left[\tfrac{1}{2}(1-p_0)2^k, 2(1-p_0)2^{k+2}\right)\right\} \\
&\quad \geq P\left\{T < \infty, |S_{\sigma^*(T)} - S_T| \leq A\right\} \\
&\quad\quad - P\left\{T < \infty, \sigma^*(T) = \sigma_l \text{ with } l \notin \left[\tfrac{1}{2}(1-p_0)2^k, 2(1-p_0)2^{k+2}\right)\right\} \\
&\quad \geq P\{T < \infty\} F_2\left(\{z \in \mathbf{R}^d\colon |z| \leq A\}\right) \\
&\quad\quad - P\left\{T < \infty, \sigma^*(T) - T > 2^k\right\} \\
&\quad\quad - P\left\{\exists l \notin \left[\tfrac{1}{2}(1-p_0)2^k, 2(1-p_0)2^{k+2}\right) \text{ with } \sigma_l \in [2^k, 2^{k+2})\right\}. \qquad (2.4)
\end{aligned}$$

Now for some $A_0 = A_0(F) < \infty$ and all $A \geq A_0$,

$$F_2\left(\{z \in \mathbf{R}^d\colon |z| \leq A\}\right) \geq \tfrac{1}{2}, \quad P\left\{T < \infty, \sigma^*(T) - T > 2^k\right\} \leq p_0^{2^k}.$$

Also, with

$$l_1 = \left[\tfrac{1}{2}(1-p_0)2^k\right], \quad l_2 = \left[2(1-p_0)2^{k+2}\right],$$

the last term in (2.4) is at most

$$P\left\{\sigma_{l_1} \geq 2^k \text{ or } \sigma_{l_2} \leq 2^{k+2}\right\} \leq \Gamma_3 \exp -\Gamma_4 2^k,$$

for some $\Gamma_3, \Gamma_4 < \infty$ depending on p_0 only (by (2.3) and standard exponential

[6] $G * H$ denotes the convolution of G and H, and $G^{*(s)}$ denotes the s-fold convolution of G with itself.

estimates; compare [9, formulae (5.40) – (5.42)]). Thus (2.4) yields for $A > A_0$,

$$P\left\{|S_n| \leq A \text{ for some } 2^k \leq n < 2^{k+1}\right\} = P\{T < \infty\}$$
$$\leq 2P\left\{|S_{\sigma_l}| \leq 2A \text{ for some } l \in \left[\tfrac{1}{2}(1-p_0)2^k, 2(1-p_0)2^{k+2}\right)\right\}$$
$$+ 2p_0^{2^k} + 2\Gamma_3 \exp - \Gamma_4 2^k. \tag{2.5}$$

It is clear from (2.5) that if Proposition 1 holds for the random walk $S_n(F_2) = S_{\sigma_n}$, whose increments $S_{\sigma_{i+1}} - S_{\sigma_i}$ all have the distribution F_2, then Proposition 1 also holds for the original S_n. Actually (2.5) was derived only for $A \geq A_0$. However, for $A < A_0$ or even $A \leq 2^{k/8}$, (1.4) is immediate from Esseen's estimate [6, Corollary to Theorem 6.2]

$$P\left\{|S_n| \leq A\right\} \leq K_2(A+1)^d \sup_z P\left\{|S_n + z| \leq 1\right\}$$
$$\leq \Gamma_5(F)(A+1)^d n^{-d/2}, \tag{2.6}$$

for some $\Gamma_5(F) < \infty$.

F_2 itself does not have to satisfy (2.1). However, set

$$G_2 = \sum_{r=1}^{\infty} p_0^{r-1}(1-p_0)G^{*(r-1)}, \tag{2.7}$$

so that $F_2 = G_2 * H$, and assume that we can find a distribution function G_1 with a continuous density such that

$$\int_{\mathbf{R}^d} x^\nu \, dG_1(x) = \int_{\mathbf{R}^d} x^\nu \, dG_2(x) \quad \text{for } \|\nu\| \leq 16. \tag{2.8}$$

(Here we use the standard multi-index notation; $x^\nu = \prod_{i=1}^d x(i)^{\nu(i)}$, for positive integers $\nu(i)$ and $\|\nu\| = \sum_{i=1}^d \nu(i)$.) We claim that then $F_1 \equiv G_1 * H$ has the required properties. Before constructing G_1 we shall prove this claim. (2.1) is obvious for F_1; indeed G_1 has a continuous density and hence F_1 has a density which is lower semicontinuous. Thus we merely have to prove that the validity of Proposition 1 for $S_n(F_1)$ implies the validity of Proposition 1 for $S_n(F_2)$ (since we already showed that Proposition 1 then also holds for our original S_n). Now fix k and $A \geq 2^{k/8}$; we already saw above that (1.4) follows from (2.6) if $A < 2^{k/8}$ so that these are the only values of interest. Let N, $\chi_1, \chi_2, \ldots$ be independent random d-vectors, also independent of $\{S_{\sigma_l}\}_{l \geq 0}$, and such that N has a normal distribution with mean zero and covariance matrix $k^{-1}A^2$ times the identity matrix, and such that each χ_i has distribution F_1. Then, for any n,

$$P\left\{|S_{\sigma_n}| < 2A\right\} < P\left\{|S_{\sigma_n}(j)| < 2A, 1 < j < d\right\}$$

$$< P\left\{|S_{\sigma_n}(j) + N(j)| < 3A, 1 < j < d\right\} + \sum_{j=1}^{d} P\left\{|N(j)| > A\right\}$$

$$< P\left\{|S_{\sigma_n}(j) + N(j)| < 3A, 1 < j < d\right\} + K_3 \exp - \frac{A^2}{2A^2k^{-1}}$$

$$< P\left\{|S_{\sigma_n}(j) + N(j)| < 3A, 1 < j < d\right\} + K_3 \exp - \frac{k}{2}. \tag{2.9}$$

Similarly

$$P\left\{\left|\sum_{i=1}^{n} \chi_i(j) + N(j)\right| < 3A, 1 < j < d\right\}$$

$$< P\left\{\left|\sum_{1}^{\bar{n}} \chi_i\right| < 4d^{1/2}A\right\} + K_3 \exp - \frac{k}{2}. \tag{2.10}$$

If $\varphi_1, \varphi_2, \psi$ are the characteristic functions of, respectively, G_1, G_2 and H, then the characteristic functions of $S_{\sigma_n} + N$ and $\Sigma_1^n \chi_i + N$ are

$$\exp\left(-\frac{A^2}{2k}|\theta|^2\right)\varphi_2(\theta)^n\psi(\theta)^n \quad \text{and} \quad \exp\left(-\frac{A^2}{2k}|\theta|^2\right)\varphi_1(\theta)^n\psi(\theta)^n.$$

Thus, by the inversion formula,

$$\Bigg| P\left\{|S_{\sigma_n}(j) + N(j)| < 3A,\ 1 < j < d\right\}$$

$$- P\left\{\left|\sum_{i=1}^{n} \chi_i(j) + N(j)\right| < 3A,\ 1 < j < d\right\}\Bigg|$$

$$< \pi^{-d}\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d\theta_1 \cdots d\theta_d \prod_{j=1}^{d} \left|\frac{\sin 3\theta_j A}{\theta_j}\right|$$

$$\cdot |\varphi_2(\theta)^n - \varphi_1(\theta)^n| \exp\left(-\frac{A^2}{2k}|\theta|^2\right)|\psi(\theta)|^n$$

$$< \left(\frac{3A}{\pi}\right)^d n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\varphi_1(\theta) - \varphi_2(\theta)| \exp\left(-\frac{A^2}{2k}|\theta|^2\right) d\theta_1 \cdots d\theta_d.$$

But, by virtue of (2.8),

$$
\begin{aligned}
|\varphi_1(\theta) - \varphi_2(\theta)| &= \left|\int e^{i\langle\theta,x\rangle}\, d(G_1(x) - G_2(x))\right| \\
&= \left|\int \left\{ e^{i\langle\theta,x\rangle} - \sum_{r=0}^{15} \frac{i^r \langle\theta,x\rangle^r}{r!} \right\} d(G_1(x) - G_2(x))\right| \\
&\leqslant \frac{1}{16!}\int \langle\theta,x\rangle^{16}\, d(G_1(x) + G_2(x)) = \frac{2}{16!}\int \langle\theta,x\rangle^{16}\, dG_2(x) \\
&\leqslant \frac{2}{16!}|\theta|^{16} \sum_{r=1}^{\infty} p_0^{r-1}(1-p_0)((r-1)L)^{16}
\end{aligned}
$$

(see (2.7) and recall that G is concentrated on C_1). Consequently, for $A \geqslant 2^{k/8}$ and $n \leqslant 2^{k+2}$,

$$
\begin{aligned}
&\Bigg| P\left\{|S_{\sigma_n}(j) + N(j)| \leqslant 3A,\ 1 \leqslant j \leqslant d\right\} \\
&\qquad - P\left\{\left|\sum_{i=1}^{n} \chi_i(j) + N(j)\right| \leqslant 3A,\ 1 \leqslant j \leqslant d\right\}\Bigg| \\
&\leqslant \Gamma_6(F) n A^d \int \cdots \int |\theta|^{16} \exp\left(-\frac{A^2}{2k}|\theta|^2\right) d\theta_1 \cdots d\theta_d \\
&\leqslant \Gamma_7(F) n A^{d-d-16} k^{(d+16)/2} \leqslant \Gamma_7(F) k^{(d+16)/2} 2^{-k+2}.
\end{aligned}
$$

Combined with (2.9) and (2.10) this yields

$$
P\{|S_{\sigma_n}| \leqslant 2A\} \leqslant P\left\{\left|\sum_1^n \chi_i\right| \leqslant 4d^{1/2}A\right\} + \Gamma_8(F)e^{-k/2}. \tag{2.11}
$$

Interchanging the subscripts 1 and 2 we prove similarly

$$
P\{|S_{\sigma_n}| \leqslant A\} \geqslant P\left\{\left|\sum_1^n \chi_i\right| \leqslant \frac{1}{2} d^{-1/2}A\right\} - \Gamma_8(F)e^{-k/2}, \tag{2.12}
$$

for all $A \geqslant 2^{k/8}$, $n \leqslant 2^{k+2}$. Finally, to prove our claim we appeal to the proof of Theorem 3 in [9]. Just as in the estimates on pp. 1179–1181 of [9],

$$
\begin{aligned}
&P\{|S_{\sigma_n}| \leqslant A \text{ for some } 2^k \leqslant n < 2^{k+1}\} \\
&\qquad \leqslant \left(\sum_{n=0}^{2^k} P\{|S_{\sigma_n}| \leqslant A\}\right)^{-1} \sum_{n=2^k}^{2^{k+2}-1} P\{|S_{\sigma_n}| \leqslant 2A\},
\end{aligned} \tag{2.13}
$$

and also

$$P\left\{\left|\sum_1^n \chi_i\right| \leqslant 4d^{1/2}A \text{ for some } 2^k \leqslant n < 2^{k+2}\right\}$$

$$\geqslant \left[\sum_{n=0}^{2^{k+2}} P\left\{\left|\sum_1^n \chi_i\right| \leqslant 8d^{1/2}A\right\}\right]^{-1} \sum_{n=2^k}^{2^{k+2}-1} P\left\{\left|\sum_1^n \chi_i\right| \leqslant 4d^{1/2}A\right\}$$

$$\geqslant K_4\left[\sum_{n=0}^{2^k} P\left\{\left|\sum_1^n \chi_i\right| \leqslant \tfrac{1}{2}d^{-1/2}A\right\}\right]^{-1} \sum_{n=2^k}^{2^{k+2}-1} P\left\{\left|\sum_1^n \chi_i\right| \leqslant 4d^{1/2}A\right\}. \quad (2.14)$$

Since

$$\sum_{n=0}^{2^k} P\left\{|S_{\sigma_n}| \leqslant A\right\} \to \infty \quad \text{as } k \to \infty \text{ and } A \geqslant 2^{k/8},$$

it follows from (2.11)–(2.14) that for $k \geqslant k_1(F)$,

$$P\left\{|S_{\sigma_n}| \leqslant A \text{ for some } 2^k \leqslant n < 2^{k+1}\right\}$$

$$\leqslant 2K_4^{-1}\left[P\left\{\left|\sum_1^n \chi_i\right| \leqslant 4d^{1/2}A \text{ for some } 2^k \leqslant n < 2^{k+1}\right\}\right.$$

$$\left. + P\left\{\left|\sum_1^n \chi_i\right| \leqslant 4d^{1/2}A \text{ for some } 2^{k+1} \leqslant n < 2^{k+2}\right\}\right]$$

$$+\Gamma_9(F)\exp(-k/2).$$

Since $\Sigma_1^n \chi_i$ can be taken for $S_n(F_1)$ (its increments χ_n have distribution F_1), we see from this that if Proposition 1 holds for $S_n(F_1)$, then it also holds for $S_n(F_2)$ and for the original S_n. (Note that we can always obtain (1.4) for $k < k_1$ by increasing Γ_1.)

To complete the proof of the lemma we show finally that there exists a G_1 with a continuous density and satisfying (2.8). Let there be M positive integer d-vectors ν with $1 \leqslant \|\nu\| \leqslant 16$ and consider the set

$$\mathfrak{M} = \left\{(z_\nu)_{1 \leqslant \|\nu\| \leqslant 16}\colon |z_\nu| \leqslant \Delta_\nu + 1,\ z_\nu = \int x^\nu\, dR \text{ for some distribution function } R \text{ on } \mathbf{R}^d \text{ and all } 1 \leqslant \|\nu\| \leqslant 16\right\},$$

where the constants $\Delta_\nu < \infty$ are chosen large enough such that

$$\mu_\nu \equiv \int x^\nu\, dG_2(x) \in [-\Delta_\nu, \Delta_\nu], \qquad 1 \leqslant \|\nu\| \leqslant 16.$$

Then $\mathfrak{M}$ is a bounded convex subset of $\mathbf{R}^M$ and $\mu = (\mu_\nu)_{1 \leqslant \|\nu\| \leqslant 16} \in \mathfrak{M}$.

Assume first that $\mu \in \partial \mathfrak{M}$. Then there exists a supporting hyperplane of $\mathfrak{M}$ at μ (see [3, Theorem 8]), i.e., there exist constants c_0, c_ν, $1 \leqslant \|\nu\| \leqslant 16$, not all zero, such that

$$\sum_{1 \leqslant \|\nu\| \leqslant 16} c_\nu z_\nu + c_0 \geqslant 0 \quad \text{for all } z \in \mathfrak{M}, \tag{2.15}$$

and

$$\sum_{1 \leqslant \|\nu\| \leqslant 16} c_\nu \mu_\nu + c_0 = 0. \tag{2.16}$$

Then, if

$$P(x) \equiv \sum_{\nu=1}^{M} c_\nu x^\nu + c_0.$$

(2.15) implies that $\int P(x)\, dR(x) \geqslant 0$ for all distribution functions R on $\mathbf{R}^d$ with

$$\left| \int x^\nu \, dR(x) \right| \leqslant \Delta_\nu + 1, \qquad 1 \leqslant \|\nu\| \leqslant 16.$$

It is not hard to see from this and (2.16) that $P(x) = 0$ for $x \in \text{supp}(G_2)$, and, hence, for $x \in \text{supp}(G^{*0}) = \{0\}$ as well as for

$$x \in \text{supp}(G^{*(s)}) = \{x_1 + \cdots + x_s : x_i \in \text{supp } G\} \qquad \text{for some } s \geqslant 1.$$

$P(0) = 0$ shows that $c_0 = 0$. Moreover, if $y_1, \ldots, y_{d+1}$ are $(d+1)$ points in $\text{supp}(G) = \text{supp}(F) \cap C_1$ which do not lie in any hyperplane, then also, for any integers $k_1 \geqslant 0, \ldots, k_{d+1} \geqslant 0$, $\Sigma k_i y_i \in \text{supp}(G_2)$ and hence

$$\sum_{1 \leqslant \|\nu\| \leqslant 16} c_\nu (k_1 + \cdots + k_{d+1})^{-16} (k_1 y_1 + \cdots + k_{d+1} y_{d+1})^\nu = 0.$$

If we let $k_i \to \infty$ such that

$$k_i (k_1 + \cdots + k_{d+1})^{-1} \to \lambda_i \geqslant 0 \quad \text{with } \sum_{i=1}^{d+1} \lambda_i = 1,$$

we obtain

$$\sum_{\|\nu\| = 16} c_\nu (\lambda_1 y_1 + \cdots + \lambda_{d+1} y_{d+1})^\nu = 0$$

for all $\lambda_i \geqslant 0$ with $\Sigma_{i=1}^{d+1} \lambda_i = 1$, i.e.,

$$\sum_{\|\nu\| = 16} c_\nu x^\nu = 0$$

for all x in the closed convex hull of $y_1, \ldots, y_{d+1}$. Since these $(d+1)$ points do not lie in any hyperplane, their convex hull has a nonempty interior (see [3, Theorem 4]) and we conclude $c_\nu = 0$ for $\|\nu\| = 16$. But then also

$$\sum_{1 < \|\nu\| < 15} c_\nu (k_1 + \cdots + k_{d+1})^{-15} (k_1 y_1 + \cdots + k_{d+1} y_{d+1})^\nu = 0,$$

and continuing in this way we find that all $c_\nu = 0$. This was excluded, so that $\mu \in \partial \mathfrak{M}$ is impossible. But then μ lies in the interior of $\mathfrak{M}$ and for each of the 2^M choices of α_ν with $\alpha_\nu = +1$ or -1, we can find a point $z(\alpha) \in \mathfrak{M}$ such that

$$\operatorname{sgn}(z_\nu(\alpha) - \mu_\nu) = \operatorname{sgn} \alpha_\nu \tag{2.17}$$

and

$$z_\nu(\alpha) = \int x^\nu \, dR_\alpha(x)$$

for some R_α with a continuous density. Indeed any point of $\mathfrak{M}$ can be approximated by the moments of continuous densities on $\mathbf{R}^d$. It is a question of simple algebra only to deduce from (2.17) that some convex combination of the $z(\alpha)$ equals μ. Say

$$\mu_\nu = \sum_\alpha \gamma(\alpha) z_\nu(\alpha), \qquad 1 \leq \|\nu\| \leq 16,$$

with $\gamma(\alpha) \geq 0$, $\Sigma_\alpha \gamma(\alpha) = 1$. Then clearly $G_1 \equiv \Sigma_\alpha \gamma(\alpha) R_\alpha$ has the moments μ_ν for $\|\nu\| \leq 16$ and has a continuous density as required. □

From now on we assume that (2.1) *holds*. We introduce the following quantities: A_k will be any fixed positive number not less than $2^{k/8}$. ω will stand for a generic unit vector in $\mathbf{R}^d$ and for any such vector we set

$$t(\omega) = t(\omega,k) = \min\left\{n \colon P\left\{|\langle S_n, \omega\rangle| \geq A_k\right\} \geq (8d+8)^{-1}\right\}.$$

As we shall see in the next lemma $t(\omega)$ is bounded on $|\omega| = 1$, and we can therefore pick an $\omega_d = \omega_d^k$ which maximizes $t(\cdot,k)$, i.e. for which

$$t(\omega_d^k) = \max_{|\omega|=1} t(\omega,k).$$

After that we can successively pick $\omega_{d-1}^k, \ldots, \omega_1^k$ such that $\langle \omega_i^k, \omega_j^k \rangle = \delta_{i,j}$ and

$$t(\omega_{d-l-1}^k) = \max\left\{t(\omega,k) \colon \langle \omega, \omega_j^k \rangle = 0, d - l \leq j \leq d\right\}.$$

We define

$$T = T(k) = t(\omega_2^k, k) \tag{2.18}$$

and note that by our construction $\omega_1, \ldots, \omega_d$ form an orthonormal basis for $\mathbf{R}^d$,

$$T(k) \leq t(\omega_j^k, k), \qquad j \geq 3, \tag{2.19}$$

and if

$$\mathcal{H} = \mathcal{H}^k = \text{span of } \{\omega_1^k, \omega_2^k\} = \{z \colon z = \alpha\omega_1^k + \beta\omega_2^k\},$$

then

$$t(\omega,k) \leqslant T(k) \quad \text{for } \omega \in \mathcal{K}. \tag{2.20}$$

Next we take $M(\omega_2^k,k) = 1$ and for all $\omega \neq \omega_2$ we take $M(\omega) = M(\omega,k)$ as the maximum over $1 \leqslant n \leqslant T(k)$ of any $1-(8d+8)^{-1}$ quantile of $A_k^{-1}|\langle S_n,\omega\rangle|$, i.e.,

$$P\{A_k^{-1}|\langle S_n,\omega\rangle| \leqslant M(\omega)\} \geqslant 1-(8d+8)^{-1}, \qquad 1 \leqslant n \leqslant T(k), \tag{2.21}$$

and for some $1 \leqslant N(\omega) = N(\omega,k) \leqslant T(k)$,

$$P\{A_k^{-1}|\langle S_N,\omega\rangle| \geqslant M(\omega)\} \geqslant (8d+8)^{-1}. \tag{2.22}$$

By definition of T, (2.22) also holds for $\omega = \omega_2$ and $N(\omega_2,k) = T(k)$. Also, because $M(\omega,k)$ must be at least as big as a $1-(8d+8)^{-1}$ quantile of $A_k^{-1}\langle S_{t(\omega,k)},\omega\rangle$ if $t(\omega,k) \leqslant T$, we may and shall choose $M(\omega,k)$ such that

$$M(\omega,k) \geqslant 1 \quad \text{for } \omega \in \mathcal{K}. \tag{2.23}$$

Observe also that $t(\omega_j) \geqslant T(k)$ for $j \geqslant 2$ and therefore

$$\begin{aligned} P\{|\langle S_n,\omega\rangle| \leqslant 2A_k\} &\geqslant P\{|\langle S_{n-1},\omega\rangle| \leqslant A_k\}\, P\{|X_n| \leqslant A_k\} \\ &\geqslant 1-(4d+4)^{-1} \end{aligned} \tag{2.24}$$

whenever

$$k \geqslant k_2(F), \quad n \leqslant T(k) \quad \text{and} \quad \omega = \omega_j \quad \text{with } 2 \leqslant j \leqslant d \tag{2.25}$$

for a suitable $k_2(F) < \infty$.

LEMMA 2. *There exists a $\Gamma_{10} = \Gamma_{10}(F) < \infty$ such that for all $|\omega| = 1, k \geqslant 1$,*

$$t(\omega,k) \leqslant \Gamma_{10}A_k^2. \tag{2.26}$$

Moreover there exist a universal $K_0 < \infty$ and a $k_3 = k_3(F) < \infty$ such that for all $|\omega| = 1, k \geqslant k_3$,

$$T(k)P\{|\langle X_1,\omega\rangle| > M(\omega,k)A_k\} \leqslant K_0 \tag{2.27}$$

and[7]

$$T(k)\sigma^2\{\langle X_1,\omega\rangle I[|\langle X_1,\omega\rangle| \leqslant M(\omega,k)A_k]\} \leqslant K_0\{M(\omega,k)A_k\}^2. \tag{2.28}$$

$$T(k) \to \infty \quad \textit{and} \quad A_k \inf_{|\omega|=1} M(\omega,k) \to \infty \quad \textit{as } k \to \infty, \tag{2.29}$$

and there exists a universal $0 < \eta < 1$ (see (2.44)) such that for all $|\omega| = 1$, $k \geqslant 1$,

$$P\{|\langle S_T,\omega\rangle| \geqslant \tfrac{1}{2}M(\omega,k)A_k\} \geqslant \eta. \tag{2.30}$$

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[7]For a random variable Y, $\sigma^2\{Y\}$ denotes its variance. $I[\ ]$ denotes the indicator function of the event between square brackets.

PROOF. For any real random variable Y its concentration function $Q(Y,\cdot)$ is defined by

$$Q(Y,L) = \sup_y P\{y \leqslant Y \leqslant y + L\}.$$

It is easy to see and well known (see [7, Chapters 1.1 and 2.1]) that this sup is taken on for some y, that for $0 \leqslant L_1 \leqslant L_2$,

$$Q(Y,L_2) \leqslant (L_1^{-1}L_2 + 1)Q(Y,L_1), \tag{2.31}$$

and that for Y_1 and Y_2 independent,

$$Q(Y_1 + Y_2,L) \leqslant \min\{Q(Y_1,L),Q(Y_2,L)\}. \tag{2.32}$$

If $Y_1, Y_2, \ldots$ are independent and identically distributed with distribution function G, then Theorem 3.1 of Esseen [6] gives

$$Q\left(\sum_1^n Y_i,L\right) \leqslant K_5 n^{-1/2}\left\{L^{-2}\int_{|x|<2L} x^2\, dG^s(x) + \int_{|x|>2L} dG^s(x)\right\}^{-1/2} \tag{2.33}$$

for some universal constant $K_5 < \infty$, independent of G, n and L. We shall now apply this with $Y_i = \langle X_i,\omega\rangle$ for an arbitrary unit vector ω. First choose L again such that supp$(F) \cap C_1$ is not contained in any hyperplane, where C_1 is as in (2.2). Then

$$\Gamma_{11}^2 \equiv L^{-2} \inf_{|\omega|=1} \int_{|\langle x,\omega\rangle|<2L} \langle x,\omega\rangle^2\, dF^s(x) > 0,$$

and thus by (2.31) and (2.33),

$$\begin{aligned} P\{|\langle S_n,\omega\rangle| \leqslant A_k\} &\leqslant 2\{L^{-1}A_k + 1\}Q(\langle S_n,\omega\rangle,L) \\ &\leqslant 2\{L^{-1}A_k + 1\}K_5\Gamma_{11}^{-1}n^{-1/2} < 1 - (8d+8)^{-1} \end{aligned}$$

as soon as

$$n > \left(\frac{8d+8}{8d+7}\right)^2\{2L^{-1}A_k + 2\}^2 K_5^2\Gamma_{11}^{-2}.$$

This implies (2.26). The same argument shows for any $L' \geqslant L$,

$$\begin{aligned} P\{|\langle S_n,\omega\rangle| \leqslant \tfrac{1}{2}L'\} &\leqslant Q(\langle S_n,\omega\rangle,L') \\ &\leqslant (L^{-1}L' + 1)K_5\Gamma_{11}^{-1}n^{-1/2} \leqslant 1 - (8d+8)^{-1} \end{aligned}$$

as soon as $n > (2K_5)^2(L^{-1}L' + 1)^2\Gamma_{11}^{-2}$. Thus also $M(\omega)A_k > \frac{1}{2}L'$ whenever $T(k) > (2K_5)^2(L^{-1}L' + 1)^2\Gamma_{11}^{-2}$. However, we must have $T(k) \to \infty$ as $k \to \infty$ because for each fixed t,

$$\sup_\omega P\{|\langle S_t,\omega\rangle| \geqslant A_k\} \leqslant P\{|S_t| \geqslant 2^{k/8}\} \to 0, \qquad k \to \infty.$$

Thus also for each fixed L', $A_k \inf_\omega M(\omega,k) > \frac{1}{2}L'$ eventually, and (2.29) holds.

We turn to (2.27) and (2.28). We have for each ω,

$$P\left\{|\langle S_{T-1},\omega\rangle| \leqslant M(\omega)A_k\right\} \geqslant 1-(8d+8)^{-1} \geqslant \tfrac{1}{2} \tag{2.34}$$

(see (2.21) for $\omega \neq \omega_2$, and the definition of $T(k)$ and $M(\omega_2,k)=1$ for $\omega=\omega_2$). On the other hand, by (2.31) with $L_1 = \frac{1}{8}M(\omega)A_k$ and (2.33), the left-hand side of (2.34) is bounded above by

$$17K_5(T-1)^{-1/2}P\left\{|\langle X_1^s,\omega\rangle| > \tfrac{1}{4}M(\omega)A_k\right\}^{-1/2},$$

so that

$$(T-1)P\left\{|\langle X_1^s,\omega\rangle| > \tfrac{1}{4}M(\omega)A_k\right\} \leqslant 2^{12}K_5^2. \tag{2.35}$$

Now (2.32) for $Y_1 = Y$, $Y_2 = Y'$ shows that

$$Q(Y^s,L) \leqslant Q(Y,L)$$

and this, together with (2.35), yields

$$\begin{aligned} Q\left(\langle X_1,\omega\rangle, \tfrac{1}{2}M(\omega)A_k\right) &\geqslant Q\left(\langle X_1^s,\omega\rangle, \tfrac{1}{2}M(\omega)A_k\right) \\ &\geqslant P\left\{|\langle X_1^s,\omega\rangle| < \tfrac{1}{4}M(\omega)A_k\right\} \geqslant 1-(T-1)^{-1}2^{12}K_5^2. \end{aligned} \tag{2.36}$$

Finally, take $\Gamma_{12} = \Gamma_{12}(F)$ so large that

$$P\left\{|X_1| > \Gamma_{12}\right\} \leqslant \tfrac{1}{2}, \tag{2.37}$$

and $k_3 = k_3(F)$ so large that for all $k \geqslant k_3$,

$$\tfrac{1}{2}A_k \inf_\omega M(\omega,k) > \Gamma_{12}$$

as well as

$$(T(k)-1)^{-1}2^{12}K_5^2 < \tfrac{1}{4} \quad \text{and} \quad T(k) > 2.$$

Then (2.36) shows that there exists some interval $[a,a+\frac{1}{2}M(\omega)A_k]$ which contains $\langle X_1,\omega\rangle$ with a probability at least $1-(T(k)-1)^{-1}2^{12}K_5^2 \geqslant \frac{3}{4}$. By (2.37), for $k \geqslant k_3$ this interval must intersect $[-\Gamma_{12},+\Gamma_{12}] \subset [-\frac{1}{2}M(\omega)A_k, \frac{1}{2}M(\omega)A_k]$. Thus we proved for $k \geqslant k_3$,

$$\begin{aligned} P\left\{|\langle X_1,\omega\rangle| \leqslant M(\omega)A_k\right\} &\geqslant P\left\{\langle X_1,\omega\rangle \in \left[a,a+\tfrac{1}{2}M(\omega)A_k\right]\right\} \\ &\geqslant 1-(T(k)-1)^{-1}2^{12}K_5^2. \end{aligned}$$

This proves (2.27) for $k \geqslant k_3(F)$.

To obtain (2.28) we again apply (2.33) with $Y_i = \langle X_i,\omega\rangle$ and take $L = M(\omega)A_k$. Combined with (2.31) and (2.34) this gives

$$\begin{aligned} 3K_5(T-1)^{-1/2}M(\omega)A_k &\left\{\int_{|\langle x,\omega\rangle| \leqslant 2M(\omega)A_k} \langle x,\omega\rangle^2\, dF^s(x)\right\}^{-1/2} \\ &\geqslant 1-(8d+8)^{-1} \geqslant \tfrac{1}{2}, \end{aligned}$$

and therefore,

$$T(k)\int_{|\langle x,\omega\rangle|<2M(\omega)A_k}\langle x,\omega\rangle^2\,dF^s(x) \le 2^7\{K_5M(\omega)A_k\}^2.$$

Now let Y be any random variable and $c \ge 0$ a constant. Denote the distribution function of Y by G and put

$$d = \left\{\int_{|y|<c} dG(y)\right\}^{-1}\int_{|y|<c} y\,dG(y) = E\{Y|\,|Y|< c\}.$$

Then $|d| \le c$ and

$$\int_{|y|<2c} y^2\,dG^s(y) \ge \iint_{\substack{|y_1|<c\\|y_2|<c}} (y_1-y_2)^2\,dG(y_1)dG(y_2)$$

$$= \iint_{\substack{|y_1|<c\\|y_2|<c}} \left\{(y_1-d)^2 + 2(y_1-d)(y_2-d) + (y_2-d)^2\right\} dG(y_1)\,dG(y_2)$$

$$= 2\int_{|y_2|<c} dG(y_2)\int_{|y_1|<c}(y_1-d)^2\,dG(y_1).$$

Consequently,

$$\sigma^2\{YI[|Y|< c]\} \le E\left\{(YI[|Y|< c]-d)^2\right\}$$

$$= \int_{|y|<c}(y-d)^2\,dG(y) + d^2\int_{|y|>c} dG(y)$$

$$\le \tfrac{1}{2}(P\{|Y|< c\})^{-1}\int_{|y|<2c} y^2\,dG^s(y) + P\{|Y|> c\}c^2.$$

When this is applied to $Y = \langle X_1,\omega\rangle$, $c = M(\omega)A_k$, we obtain from the above inequality:

$$T(k)\sigma^2\{\langle X_1,\omega\rangle I[|\langle X_1,\omega\rangle| \le M(\omega)A_k]\}$$

$$\le \left(2^6K_5^2\left(P\{|\langle X_1,\omega\rangle| \le M(\omega)A_k\}\right)^{-1}\right.$$

$$\left. + T(k)P\{|\langle X_1,\omega\rangle| > M(\omega)A_k\}\right)\{M(\omega)A_k\}^2.$$

Together with (2.27)and (2.29) this implies (2.28).

Lastly we prove (2.30). For $\omega = \omega_2$ (2.30) is immediate since we already observed that (2.22) holds for $\omega = \omega_2$ with N replaced by T. We therefore fix $\omega \ne \omega_2$ for the remainder of the proof. Since, for any $n \le T$, $\langle S_T,\omega\rangle$ is the sum of the independent random variables $\langle S_{[Tn^{-1}]n},\omega\rangle$ and $\langle S_{T-[Tn^{-1}]n},\omega\rangle$, we have

$$\begin{aligned}
&P\{|\langle S_T,\omega\rangle|<\tfrac{1}{2}M(\omega)A_k\}\\
&\quad< 33Q\left(\langle S_T,\omega\rangle,\tfrac{1}{32}M(\omega)A_k\right) \quad \text{(by (2.31))}\\
&\quad< 33Q\left(\langle S_{[Tn^{-1}]n},\omega\rangle,\tfrac{1}{32}M(\omega)A_k\right) \quad \text{(by (2.32))}\\
&\quad< 33K_5[T/n]^{-1/2}\left(P\{|\langle S_n^s,\omega\rangle|>\tfrac{1}{16}M(\omega)A_k\}\right)^{-1/2} \quad \text{(by (2.33))}\\
&\quad< 33K_5[T/n]^{-1/2}\left\{1-Q\left(\langle S_n,\omega\rangle,\tfrac{1}{8}M(\omega)A_k\right)\right\}^{-1/2} \quad \text{(as in (2.36))}. \qquad (2.38)
\end{aligned}$$

Now in order to prove (2.30) we only have to consider those ω for which

$$P\{|\langle S_T,\omega\rangle|\geq \tfrac{1}{2}M(\omega)A_k\} \leq \tfrac{1}{2}.$$

For such an ω (2.38) implies for all $n \leq T$,

$$Q\left(\langle S_n,\omega\rangle,\tfrac{1}{8}M(\omega)A_k\right) \geq 1 - 2^{14}K_5^2 nT^{-1}. \qquad (2.39)$$

Now apply (2.39) with $n = N'$, where

$$N' = N'(\omega,k) = \min\left\{n: P\{|\langle S_n,\omega\rangle| \geq \tfrac{1}{2}M(\omega)A_k\} \geq (8d+8)^{-1}\right\}.$$

Then, by (2.22), $N'(\omega,k) \leq N(\omega,k) \leq T(k)$. Also, by the very definition of N',

$$P\{\langle S_{N'},\omega\rangle \geq \tfrac{1}{2}M(\omega)A_k\} \geq (16)^{-1}(d+1)^{-1} \qquad (2.40)$$

or the inequality holds with $\langle S_{N'},\omega\rangle$ replaced by $-\langle S_{N'},\omega\rangle$. For the sake of definiteness assume (2.40). Then any interval containing $\langle S_{N'},\omega\rangle$ with a probability larger than $(16d+15)(16d+16)^{-1}$ must contain points to the right of $\frac{1}{2}M(\omega)A_k$. By (2.39) there exists such an interval of length $\frac{1}{8}M(\omega)A_k$ which contains $\langle S_{N'},\omega\rangle$ with a probability at least

$$1 - 2^{14}K_5^2 N'T^{-1} > (16d+15)/(16d+16)$$

whenever $T/N' \geq 2^{18}K_5^2(d+1)+4$. In this case, therefore,

$$P\{\tfrac{3}{8}M(\omega)A_k \leq \langle S_{N'},\omega\rangle\} \geq 1 - 2^{14}K_5^2 N'T^{-1} \qquad (2.41)$$

and

$$\begin{aligned}
&P\{\langle S_T,\omega\rangle \geq \tfrac{1}{2}M(\omega)A_k\}\\
&\quad\geq P\Big\{\langle S_{(j+1)N'} - S_{jN'},\omega\rangle \geq \tfrac{3}{8}M(\omega)A_k,\\
&\qquad\qquad 0 \leq j < \left[\frac{T}{N'}\right], \langle S_{T-[T(N')^{-1}]N'},\omega\rangle \geq -\tfrac{1}{2}M(\omega)A_k\Big\}\\
&\quad\geq \left(1 - 2^{14}K_5^2\frac{N'}{T}\right)^{T/N'} \min_{l<N'} P\{|\langle S_l,\omega\rangle| < \tfrac{1}{2}M(\omega)A_k\}\\
&\quad\geq \frac{8d+7}{8d+8}\exp - 2^{15}K_5^2 \quad \text{(use the definition of } N'). \qquad (2.42)
\end{aligned}$$

If $2 \leqslant T/N' < 2^{18}K_5^2(d+1)+4$, we use (2.40) instead of (2.41) and obtain, as in (2.42),

$$P\left\{\langle S_T,\omega\rangle \geqslant \tfrac{1}{2}M(\omega)A_k\right\}$$

$$\geqslant P\left\{\langle S_{(j+1)N'} - S_{jN'},\omega\rangle \geqslant \tfrac{1}{2}M(\omega)A_k,\right.$$

$$\left. 0 \leqslant j < \left[\frac{T}{N'}\right], \langle S_{T-[T(N')^{-1}]N'},\omega\rangle \geqslant \frac{1}{2}M(\omega)A_k\right\}$$

$$\geqslant \left((16d+16)^{-1}\right)^{T/N'}(8d+7)(8d+8)^{-1}$$

$$\geqslant (8d+7)(8d+8)^{-1}\exp - \left(2^{18}K_5^2(d+1)+4\right)\log(16d+16). \quad (2.43)$$

Lastly, if $1 \leqslant T(N')^{-1} < 2$, then $N' \leqslant N \leqslant T < 2N'$ and a fortiori $T - N < N'$. But then

$$P\left\{|\langle S_T,\omega\rangle| \geqslant \tfrac{1}{2}M(\omega)A_k\right\}$$

$$\geqslant P\left\{|\langle S_N,\omega\rangle| \geqslant M(\omega)A_k\right\}P\left\{|\langle S_{T-N},\omega\rangle| \leqslant \tfrac{1}{2}M(\omega)A_k\right\}$$

$$\geqslant \frac{1}{8d+8}\min_{l<N'} P\left\{|\langle S_l,\omega\rangle| \leqslant \tfrac{1}{2}M(\omega)A_k\right\} \quad \text{(by (2.22))}$$

$$\geqslant (8d+7)(8d+8)^{-2} \quad \text{(by definition of } N'\text{)}.$$

Thus in all cases (2.30) holds with

$$\eta = \frac{8d+7}{8d+8}\exp - \left(2^{18}K_5^2(d+1)+4\right)\log(16d+16). \quad \square \quad (2.44)$$

We can now give the reduction of Proposition 1 to a two dimensional problem. For the remainder of the proof we shall use the abbreviation

$$m(k) = M\left(\omega_1^k,k\right). \quad (2.45)$$

Next we define

$$J_n = J_n^k = I\left[|\langle X_n,\omega_1^k\rangle| > m(k)A_k \text{ or } |\langle X_n,\omega_2^k\rangle| > A_k\right],$$

$$\nu_r = \nu_r^k = \inf\left\{n\colon \sum_{l=1}^{n}(1-J_l) = rT(k)\right\},$$

$$Y_r = Y_r^k = \sum_{\nu_{r-1}<n\leqslant\nu_r} X_n. \quad (2.46)$$

Y_r is the sum over a bunch of X_n's, exactly $T(k)$ of which satisfy $|\langle X,\omega_1^k\rangle| \leqslant m(k)A_k$ and $|\langle X,\omega_2^k\rangle| \leqslant A_k$. It is easy to see from this that the Y_r, $r \geqslant 1$, are independent and identically distributed. Moreover, we can write

$$Y_r = \sum_{\nu_{r-1}<n\leqslant\nu_r} X_nJ_n + \sum_{\nu_{r-1}<n\leqslant\nu_r} X_n(1-J_n),$$

and by definition the second term contains exactly $T(k)$ summands with $1 - J_n \neq 0$. Thus if $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are independent, each α_i (β_i) with the conditional distribution of X_1, given $J_1 = 1$ (respectively $J_1 = 0$), then

$$\sum_{\nu_{r-1} < n \leq \nu_r} X_n(1 - J_n) \tag{2.47}$$

has the same distribution as

$$\sum_{n=1}^{T(k)} \beta_n. \tag{2.48}$$

Moreover, the sum (2.47) is independent of $\Sigma_{\nu_{r-1}<n\leq\nu_r} X_n J_n$, which has the same distribution as

$$\sum_{n=1}^{\Lambda} \alpha_n \tag{2.49}$$

where $\Lambda = \Lambda(k)$ is independent of all α_i, β_i and has the negative binomial distribution

$$P\{\Lambda = l\} = P\{\nu_r - \nu_{r-1} = T(k) + l\}$$
$$= \binom{T(k) - 1 + l}{l}(P\{J_1 = 1\})^l(1 - P\{J_1 = 1\})^{T(k)},$$
$$l \geq 0. \tag{2.50}$$

LEMMA 3. *There exist constants* $\Gamma_{13}(F) - \Gamma_{15}(F) < \infty$ *such that for* $2^{k/8} \leq A_k \leq 2^{k/2}$ *and any choice of the unit vector* $\omega_0 = \omega_0^k \in \mathcal{K}^k$ *and* $k \geq 1$ *one has*

$$P\{|S_n| \leq A_k \text{ for some } 2^k \leq n < 2^{k+1}\} \leq \Gamma_{13}\{A_k 2^{-k/2}\}^{d-2}$$
$$\cdot \sum_{2^{k-1} \leq rT(k) \leq (K_0+\Gamma_{10}+2)2^k} P\left\{\left|\left\langle \sum_{n=1}^{r} Y_n^k, \omega_j^k \right\rangle\right| \leq 4M(\omega_j^k, k)A_k \text{ for } 0 \leq j \leq 2\right\}$$
$$+\Gamma_{14} \exp - \Gamma_{15} k. \tag{2.51}$$

PROOF. Let $\mathcal{G}_n$ be the σ-field generated by $X_1, \ldots, X_n$, and consider for some fixed k the $\{\mathcal{G}_n\}$ stopping times

$$\tau = \tau(k) = \min\{n: 2^k \leq n < 2^{k+1}, |S_n| \leq A_k\}$$
$$(= \infty \text{ if no such } n \text{ exists}),$$

$$\sigma = \sigma(k) = \inf\{\nu_r: \nu_r \geq \tau\}.$$

Then the set

$$\{\tau < \infty, J_l = 0 \text{ for exactly } s \text{ indices } l \text{ with}$$
$$l \leq \tau \text{ but } l \text{ greater than the last } \nu_r \leq \tau\} \tag{2.52}$$

belongs to $\mathcal{G}_\tau$. Now fix an $\omega \in \mathcal{H} \cup \{\omega_3, \ldots, \omega_d\}$ and set

$$M^*(\omega,k) = \begin{cases} M(\omega,k) & \text{if } \omega \in \mathcal{H}, \\ 1 & \text{if } \omega \in \{\omega_3, \ldots, \omega_d\}. \end{cases}$$

Then $M^*(\omega,k) \geq 1$ by (2.23), and on the set (2.52),

$$P\{|\langle S_\sigma,\omega\rangle| > 4M^*(\omega,k)A_k|\mathcal{G}_\tau\}$$
$$\leq P\{|\langle S_\sigma - S_\tau,\omega\rangle| > 3M^*(\omega,k)A_k|\mathcal{G}_\tau\}$$
$$= P\{|\langle S_\gamma,\omega\rangle| > 3M^*(\omega,k)A_k\},$$

where

$$\gamma = \gamma(s,k) = \inf\left\{n: \sum_{i=1}^{n}(1 - J_i) = T(k) - s\right\}$$

(note that only $0 \leq s < T(k)$ can occur).

By definition $\gamma(s,k) \geq T(k) - s$, and for each L and $k \geq k_2(F)$,

$$P\{|\langle S_\gamma,\omega\rangle| > 3M^*(\omega,k)A_k\}$$
$$\leq P\{|\langle S_{T(k)-s},\omega\rangle| > 2M^*(\omega,k)A_k\}$$
$$+ P\{|\langle S_\gamma - S_{T(k)-s},\omega\rangle| > M^*(\omega,k)A_k\}$$
$$\leq (4d + 4)^{-1} + P\{\gamma - (T(k) - s) > L\}$$
$$+ P\left\{\max_{j \leq L}|\langle S_{T(k)-s+j} - S_{T(k)-s},\omega\rangle| > M^*(\omega,k)A_k\right\}$$

(see (2.21) and (2.24)).

Now for all $s < T(k)$,

$$P\{\gamma(s) - (T(k) - s) > L\} = P\left\{\sum_{1}^{T(k)-s+L}(1 - J_i) < T(k) - s\right\}$$

$$\leq P\left\{\sum_{1}^{T(k)+L} J_i > L\right\} \leq \frac{T(k) + L}{L} P\{J_i = 1\}.$$

Moreover, by (2.46) and (2.27), for $k \geq k_3(F)$,

$$P\{J_i^k = 1\} \leq P\{|\langle X_1,\omega_i^k\rangle| > M(\omega_i^k,k)A_k \text{ for } i = 1 \text{ or } i = 2\}$$
$$\leq 2K_0T(k)^{-1}. \tag{2.53}$$

Since $T(k) \to \infty$ with k (see (2.29)) we can fix an L, independent of $s < T(k)$, such that for all $k \geqslant k_3$,

$$P\{\gamma(s) - (T(k) - s) > L\} \leqslant 2K_0(T(k) + L)(LT(k))^{-1} \leqslant (8d + 8)^{-1}.$$

With L fixed in this way

$$P\left\{\max_{j \leqslant L} |\langle S_{T(k)-s+j} - S_{T(k)-s}, \omega\rangle| > M^*(\omega,k)A_k\right\}$$

$$\leqslant \sum_{j=0}^{L} P\left\{|S_j| > \inf_{|\omega|=1} M^*(\omega,k)A_k\right\} \leqslant \frac{1}{8d+8}$$

as soon as $\inf_\omega M^*(\omega,k)A_k$ is large enough. Thus by (2.29) we can find $k_4(F) < \infty$, such that for $k \geqslant k_4(F)$ one has on the set (2.52),

$$P\{|\langle S_\sigma, \omega\rangle| > 4M^*(\omega,k)A_k | \mathcal{G}_\tau\} \leqslant (2d+2)^{-1},$$

uniformly in ω and $s < T(k)$. Now with $\omega_j, 1 \leqslant j \leqslant d$, as right after Lemma 1 and $\omega_0 \in \mathcal{K}$ we have

$$P\left\{\tau < \infty, |\langle S_\sigma, \omega_j\rangle| > 4M^*(\omega,k)A_k \text{ for some } 0 \leqslant j \leqslant d\right\}$$

$$\leqslant P\{\tau < \infty\} \sum_{j=0}^{d} P\left\{|\langle S_\sigma, \omega_j\rangle| > 4M^*(\omega,k)A_k | \tau < \infty\right\} \leqslant \tfrac{1}{2} P\{\tau < \infty\}.$$

After taking complements with respect to $\{\tau < \infty\}$ and using the fact that σ equals some ν_r, we obtain

$$\begin{aligned} P\left\{|S_n| \leqslant A_k \text{ for some } 2^k \leqslant n < 2^{k+1}\right\} &= P\{\tau < \infty\} \\ &\leqslant 2P\left\{\tau < \infty, |\langle S_\sigma, \omega_j\rangle| \leqslant 4M^*(\omega_j,k)A_k \text{ for all } 0 \leqslant j \leqslant d\right\} \\ &\leqslant 2P\left\{\exists \nu_r \in [2^k, (K_0 + \Gamma_{10} + 2)2^k) \text{ with } \right. \\ &\qquad \left. |\langle S_{\nu_r}, \omega_j\rangle| \leqslant 4M^*(\omega_j,k)A_k \text{ for all } 0 \leqslant j \leqslant d\right\} \\ &\quad + 2P\left\{\tau < \infty, \sigma \geqslant (K_0 + \Gamma_{10} + 2)2^k\right\}. \end{aligned} \tag{2.54}$$

The last term in (2.54) is easily seen to be exponentially small. Indeed, $\tau < 2^{k+1}$ whenever $\tau < \infty$ so that

$$P\left\{\tau < \infty, \sigma \geqslant (K_0 + \Gamma_{10} + 2)2^k\right\}$$

$$\leqslant E\left\{P\{\sigma - \tau > (K_0 + \Gamma_{10})2^k | \mathcal{G}_\tau\}; \tau < \infty\right\},$$

and on the set (2.52),

$$P\{\sigma - \tau > (K_0 + \Gamma_{10})2^k | \mathcal{G}_\tau\} = P\{\gamma(s,k) > (K_0 + \Gamma_{10})2^k\}$$

$$\leqslant P\{\gamma(s,k) - (T(k) - s) > K_0 2^k\} = P\left\{\sum_{1}^{T(k)-s+K_0 2^k} J_i > K_0 2^k\right\} \tag{2.55}$$

(recall $T(k) - s \leq T(k) \leq \Gamma_{10} A_k^2 \leq \Gamma_{10} 2^k$ by (2.26) and $A_k \leq 2^{k/2}$). Since $T(k) - s \leq \Gamma_{10} 2^k$ and $P\{J_i = 1\} \to 0$ as $k \to \infty$ (see (2.53)), standard exponential estimates show that the last member of (2.55) has an upper bound of the form $\Gamma_{16}(F) \exp - \Gamma_{17}(F) 2^k$ uniformly in $s \leq T(k)$ (see [10, Chapter 9, problems 12–16]).

The first term in the last member of (2.54) is more troublesome. Recall that F is now assumed to satisfy (2.1). By taking C_0 smaller if necessary we may assume that

$$C_0 = \left\{ z \in \mathbf{R}^d : \left| \langle z, \omega_j \rangle - c_j \right| \leq \lambda, 1 \leq j \leq d \right\}$$

for some c_j and $\lambda > 0$. Then we can write $F = a(2\lambda)^d G_3 + (1 - a(2\lambda)^d) H_3$, where G_3 is the uniform distribution on C_0 and H_3 is some other distribution function on $\mathbf{R}^d$. Correspondingly, we may assume that $X_n = E_n \theta_n + (1 - E_n)\psi_n$ where the E_n, θ_i, ψ_j are independent,

$$P\{E_n = 1\} = 1 - P\{E_n = 0\} = a(2\lambda)^d, \qquad n \geq 1,$$

each θ_i has distribution G_3 and each ψ_j has distribution H_3. Now we want to estimate for $3 \leq j \leq d$ the conditional probability of

$$\left| \langle S_{\nu_r}, \omega_j \rangle \right| \leq 4 M^*(\omega_j, k) A_k = 4 A_k \tag{2.56}$$

given $\nu_r = s$, $E_n = 1$ for $n = n_1, n_2, \ldots, n_p$ $(1 \leq n_1 < \cdots < n_p \leq s)$, but not for any other $n \leq s$, and given the values of X_n, $n \notin \{n_1, \ldots, n_p\}$, as well as $\langle X_n, \omega_j \rangle$ for $0 \leq j \leq 2$ and $n \in \{n_1, \ldots, n_p\}$. Given all these data we know that for $j \geq 3$,

$$\langle S_{\nu_r}, \omega_j \rangle = \langle S_s, \omega_j \rangle = \sum_{n \in \{n_1, \ldots, n_p\}} \langle \theta_n, \omega_j \rangle + D_j,$$

where

$$D_j = \sum_{\substack{l \notin \{n_1, \ldots, n_p\} \\ 1 \leq l \leq s}} \langle X_l, \omega_j \rangle$$

is some known constant. Moreover, we know the values of $\langle \theta_n, \omega_j \rangle$ for $n \in \{n_1, \ldots, n_p\}$ and $0 \leq j \leq 2$. (Note that also the event $\{\nu_r = s\}$ is determined only by $J_1, \ldots, J_s$ and hence by $\langle X_l, \omega_1 \rangle$, $\langle X_l, \omega_2 \rangle$, $1 \leq l \leq s$.) However, θ_n has a uniform distribution on C_0, and therefore the conditional distribution of $\langle \theta_n, \omega_3 \rangle, \ldots, \langle \theta_n, \omega_d \rangle$, given $\langle \theta_n, \omega_1 \rangle = x_1$ and $\langle \theta_n \omega_2 \rangle = x_2$, is the same for all x_1, x_2, to wit the uniform distribution on the $(d-2)$ dimensional cube

$$C_2 = \left\{ x = (x(3), \ldots, x(d)) : |x(i) - c_i| \leq \lambda, 3 \leq i \leq d \right\}.$$

This remains true even if we condition on $\langle \theta_n, \omega_0 \rangle$ as well, because $\langle \theta_n, \omega_0 \rangle$ is a function of $\langle \theta_n, \omega_1 \rangle$ and $\langle \theta_n, w_2 \rangle$ (since ω_0 lies in $\mathcal{H}$, the plane spanned by ω_1

and ω_2). Thus, if $\delta_1, \delta_2, \ldots$ are independent random variables, each one uniformly distributed on C_2, then

$$P\left\{ |\langle S_{\nu_r}, \omega_j \rangle| \le 4M^*(\omega_j,k)A_k \text{ for } 0 \le j \le d \,|\, \nu_r = s, \right.$$
$$\left. \sum_{n=1}^{\nu_r} E_n = p, |\langle S_{\nu_r}, \omega_j \rangle| \le 4M^*(\omega_j,k)A_k \text{ for } 0 \le j \le 2 \right\}$$
$$\le \sup_{D_j} P\left\{ \left| \sum_{l=1}^{p} \delta_l(j) + D_j \right| \le 4M^*(\omega_j,k)A_k = 4A_k \text{ for } 3 \le j \le d \right\}. \quad (2.57)$$

Now the corollary to Theorem 6.2 of [6] applies to the independent identically distributed $(d-2)$ vectors $\{\delta_l\}_{l \ge 1}$ whose distribution is not concentrated on any $(d-3)$ dimensional hyperplane and therefore the last member of (2.57) is bounded by

$$\Gamma_{18}(F) p^{-(d-2)/2} \prod_{j=3}^{d} \{4\lambda^{-1}A_k + 1\} \le \Gamma_{19}(F)\{A_k p^{-1/2}\}^{d-2}.$$

It follows that for fixed r,

$$P\left\{ \nu_r \in [2^k, (K_0 + \Gamma_{10} + 2)2^k) \right.$$
$$\left. \text{and } |\langle S_{\nu_r}, \omega_j \rangle| \le 4M^*(\omega_j,k)A_k \text{ for } 0 \le j \le d \right\}$$
$$\le P\left\{ \sum_{n=1}^{2^k} E_n \le \frac{1}{2} 2^k P\{E_1 = 1\} \right\}$$
$$+ \Gamma_{19}(F) A_k^{d-2} \left(2^{k-1} P\{E_1 = 1\}\right)^{-(d-2)/2}$$
$$\cdot P\left\{ \nu_r \in [2^k, (K_0 + \Gamma_{10} + 2)2^k), \sum_{n=1}^{\nu_r} E_n > 2^{k-1} P\{E_1 = 1\}, \right.$$
$$\left. |\langle S_{\nu_r}, \omega_j \rangle| \le 4M^*(\omega_j,k)A_k = 4M(\omega_j,k)A_k \text{ for } 0 \le j \le 2 \right\}. \quad (2.58)$$

Since $P\{E_1 = 1\} = a(2\lambda)^d$ is independent of k, the first term in the right-hand side of (2.58) is at most $\Gamma_{20}(F) \exp - \Gamma_{21}(F) 2^k$ (see [10, Chapter 9, problems 12–16]) and we obtain

$$P\Big\{\exists \nu_r \in \big[2^k, (K_0 + \Gamma_{10} + 2)2^k\big) \text{ with } \big|\langle S_{\nu_r}, \omega_j\rangle\big| < 4M^*(\omega_j,k)A_k \text{ for all } 0 \leqslant j \leqslant d\Big\}$$

$$\leqslant \sum_{2^{k-1} \leqslant rT(k) < (K_0+\Gamma_{10}+2)2^k} \Big[\big(\Gamma_{20} \exp - \Gamma_{21}2^k\big) + \big(\Gamma_{22}(F)\{A_k 2^{-k/2}\}^{d-2}\big) \cdot P\big\{\big|\langle S_{\nu_r}, \omega_j\rangle\big| < 4M(\omega_j,k)A_k \text{ for } 0 \leqslant j \leqslant 2\big\}\Big]$$

$$+ P\Big\{\nu_r \in \big[2^k, (K_0 + \Gamma_{10} + 2)2^k\big) \text{ for some } r < 2^{k-1}(T(k))^{-1} \text{ or } r \geqslant (K_0 + \Gamma_{10} + 2)2^k(T(k))^{-1}\Big\}. \quad (2.59)$$

Since ν_r increases with r and $\nu_r \geqslant rT(k)$ (by definition of ν_r) the last term in the right-hand side of (2.59) is at most

$$P\Big\{\nu_n \geqslant 2^k \text{ for } n = \big[2^{k-1}(T(k))^{-1}\big]\Big\}$$

and we leave it to the reader to show that this term again is bounded by $\Gamma_{23} \cdot \exp - \Gamma_{24}2^k$. Thus, since $S_{\nu_r} = \Sigma_{n=1}^r Y_n$, the last member of (2.54) is indeed bounded by the right-hand side of (2.51) for $k \geqslant \max(k_2,k_3,k_4)$ and suitable $\Gamma_{13} - \Gamma_{15}$. Obviously we can then insure the validity of (2.51) for $k < \max(k_2,k_3,k_4)$ by increasing Γ_{14}. □

3. **Completion of the proof.** Lemma 3 shows that Proposition 1 will follow once we show that there exist $k_5(F) < \infty$ and $K_2 < \infty$ such that for a suitable choice of $\omega_0^k \in \mathcal{H}^k$,

$$\sum_{r=s}^{2s} P\left\{\left|\sum_{n=1}^{r} \langle Y_n^k, \omega_j^k\rangle\right| < 4M(\omega_j^k,k)A_k \text{ for } 0 \leqslant j \leqslant 2\right\} \leqslant K_2, \quad \text{for all } k \geqslant k_5, s \geqslant 1 \text{ and } A_k \in [2^{k/8}, 2^{k/2}]. \quad (3.1)$$

Note that we can always obtain (1.4) for $k < k_5$ by increasing Γ_1; moreover, (1.4) is vacuously true for $A > 2^{k/2}$, and, as observed before, (1.4) follows from (2.6) when $A \leqslant 2^{k/8}$. Note also that the last statement of our theorem will be immediate from Proposition 1, because (1.2)–(1.4) imply for all $b > 0$,

$$P\{\psi(n)^{-1}|S_n| \leqslant b \text{ for some } n \geqslant 2^k\}$$

$$\leqslant \sum_{l=k}^{\infty} P\{|S_n| \leqslant 2^{1/2}b\psi(2^l) \text{ for some } 2^l \leqslant n < 2^{l+1}\}$$

$$\leqslant \Gamma_1(F)\left\{\sum_{l=k}^{\infty} \{2^{1/2}b\psi(2^l) + 1\}^{d-2} 2^{-l(d-2)/2} + \exp - l\Gamma_2(F)\right\}$$

$$\leqslant K_3\Gamma_1(F)b^{d-2}\int_{2^{k-1}}^{\infty} \psi(t)^{d-2} t^{-d/2}\, dt + K_3\Gamma_1(F)2^{-k(d-2)/2}$$

$$+ \Gamma_1(F)\{1 - \exp - \Gamma_2(F)\}^{-1} \exp - k\Gamma_2(F) \to 0 \qquad (k \to \infty).$$

Moreover, it is well known that if (1.2) holds, then $\psi(n)^{-1}|S_n^*| \to \infty$ w.p.1 or $\psi(t)^{-1}|B(t)| \to \infty$ w.p.1 if and only if (1.3) holds (see [2, Theorems 5,6] or [8, §4.2.15]; for S_n^* one can also use the multidimensional analogue of Remark 2, p. 1182 in [9]). Thus to prove our theorem it suffices to prove (1.4), and this in turn has been reduced to proving (3.1).

This will be done by means of certain inequalities on two dimensional concentration functions which are contained in Proposition 2 and some corollaries. We therefore begin with their proofs; the notation is as in Proposition 2.

PROOF OF PROPOSITION 2. By [6, formula (6.9)]

$$\sup_{z\in\mathbf{R}^2} P\left\{\left|\sum_1^n Z_i + z\right| \leqslant \rho\right\}$$

$$\leqslant K_4\rho^2\int_{|\theta|\leqslant\rho^{-1}} d\theta \exp -\left[\frac{1}{2}\sum_{i=1}^n\int_{\mathbf{R}^2}(1-\cos\langle\theta,u\rangle)\,dG_i^s(u)\right]$$

$$\leqslant K_4\rho^2\int_{|\theta|\leqslant\rho^{-1}} d\theta \exp -\left[\frac{1}{2}\sum_{i=1}^n\left\{\int_{|u|\leqslant\rho_i}(1-\cos\langle\theta,u\rangle)\,dG_i^s(u)\right.\right.$$

$$+\frac{1}{2q_i}\int_{|u|,\,|v|>\rho_i}\left[(1-\cos\langle\theta,u\rangle)\right.$$

$$\left.\left.\left.+(1-\cos\langle\theta,v\rangle)\right]dG_i^s(u)\,dG_i^s(v)\right\}\right]$$

$$\leqslant K_4\rho^2\int_{|\theta|\leqslant\rho^{-1}}\exp\left\{-K_5\sum_{i=1}^n\int_{|u|\leqslant\rho_i}\langle\theta,u\rangle^2\,dG_i^s(u)-\sum_{i=1}^n\frac{1}{4q_i}\right.$$

$$\cdot\int_{u,v\in A_i}\left[(1-\cos\langle\theta,u\rangle)+(1-\cos\langle\theta,v\rangle)\right]$$

$$\left. dG_i^s(u)\,dG_i^s(v)\right\}d\theta. \tag{3.2}$$

Now take C as in (1.7) and apply Hölder's inequality, as in [5, p. 212] or [6, p. 294]. It then follows that the last member of (3.2) is bounded by

$$K_4\rho^2\left\{\int_{|\theta|\leqslant\rho^{-1}} d\theta\exp - K_5C|\theta|^2\right\}^{C^{-1}\Sigma_{i=1}^n\sigma_i^2}\cdot\exp\sum_i\frac{\rho_i^2}{Cq_i}\int_{u,v\in A_i}dG_i^s(u)\,dG_i^s(v)$$

$$\cdot\log\int_{|\theta|\leqslant\rho^{-1}}d\theta\exp-\frac{C}{4\rho_i^2}(2-\cos\langle\theta,u\rangle-\cos\langle\theta,v\rangle).$$

Clearly

$$\int_{|\theta|<\rho^{-1}} d\theta \exp - K_5C|\theta|^2 \leqslant \int_{\mathbf{R}^2} d\theta \exp - K_5C|\theta|^2 = \frac{\pi}{K_5C}\cdot$$

It is also clear that

$$\int_{|\theta|<\rho^{-1}} d\theta \exp - \frac{C}{4\rho_i^2}(2 - \cos\langle\theta,u\rangle - \cos\langle\theta,v\rangle)$$

is invariant under a rotation of the pair of vectors u,v and therefore equals ($\varphi = \varphi(u,v)$; recall also $|u|, |v| > \rho \geqslant \rho_i$ for $u, v \in A_i$)

$$\int_{|\theta|<\rho^{-1}} d\theta \exp - \frac{C}{4\rho_i^2}\Big[(1 - \cos\theta_i|u|) + \big(1 - \cos(\theta_1|v|\cos\varphi + \theta_2|v|\sin\varphi)\big)\Big]$$

$$\leqslant \int_{|\theta_1|<\rho^{-1}} d\theta_1 \exp - \frac{C}{4\rho_i^2}\left[1 - \cos\theta_1|u|\right]$$

$$\cdot \sup_{b\in\mathbf{R}} \int_{|\theta_2|<\rho^{-1}} d\theta_2 \exp - \frac{C}{4\rho_i^2}\left[1 - \cos(b + \theta_2|v|\sin\varphi)\right]$$

$$\leqslant K_6 \frac{\rho_i}{|u|C^{1/2}}\left(|u|\rho^{-1} + 1\right)\cdot \frac{\rho_i}{|v|\,|\sin\varphi|C^{1/2}}\left(|v|\,|\sin\varphi|\rho^{-1} + 1\right)$$

(compare [5, p. 213])

$$\leqslant \frac{2}{C}K_6\left(1 + \frac{\rho_i}{|v|\,|\sin\varphi(u,v)|}\right).$$

(1.8) now follows by putting these estimates together. □

Corollary 1 is immediate from Proposition 1 by specialization (take all $\rho_i = \rho$ and $A_i = B_1 \times B_2$). More important for our proof, though, is the more technical

COROLLARY 2. *Let* $Z_1, Z_2, \ldots$ *be independent identically distributed two-vectors and assume that there exist constants* $d_1 > 0$, $-\infty < d_2 < \infty$, $d_3 > 0$ *such that*

$$p_1 \equiv P\{Z_1^s(2) > d_1, Z_1^s(1) \leqslant d_2Z_1^s(2)\} > 0, \tag{3.3}$$

as well as

$$p_2 \equiv P\{Z_1^s(2) > d_1, Z_1^s(1) \geqslant d_2Z_1^s(2) + d_3\} > 0. \tag{3.4}$$

Then for all $\rho \geqslant 0$ *and all* $n \geqslant 1$,

$$\sup_z P\left\{\left|\sum_1^n Z_i + z\right| \leqslant \rho\right\} \leqslant \Gamma_3 \frac{1}{n}\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\}, \tag{3.5}$$

for some constant $\Gamma_3 = \Gamma_3(d_i, p_i) < \infty$ *depending on* $p_1, p_2, d_1 - d_3$ *only. In fact* $\Gamma_3 \leqslant \Gamma_4(D) < \infty$ *for all* p_i, d_i *and* $D < \infty$ *with*

$$|d_2| \leqslant D, \quad d_1 d_3^{-1} \leqslant D, \quad p_1 \geqslant D^{-1}, \quad p_2 \geqslant D^{-1}. \tag{3.6}$$

Moreover, for all $s \geqslant 1$ *and* $\rho \geqslant 0$,[8]

$$E\#\left\{r \in [s,2s]: \left|\sum_1^r Z_i\right| \leqslant \rho\right\} \leqslant \Gamma_3\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\}(1 + \log 2). \tag{3.7}$$

If instead of (3.3) *and* (3.4) *we assume that for some* $d_1 > 0$, $-\infty < d_2 < \infty$, *and* $d_3 > 0$,

$$p_3 \equiv P\{Z_1(2) > d_1, Z_1(1) \leqslant d_2 Z_1(2)\} > 0 \tag{3.8}$$

as well as

$$p_4 \equiv P\{Z_1(2) > d_1, Z_1(1) \geqslant d_2 Z_1(2) + d_3\} > 0, \tag{3.9}$$

then for each $s \geqslant 1$, $\rho \geqslant 0$,

$$E\#\left\{r \in [s,2s]: \left|\sum_1^r Z_n\right| \leqslant \rho\right\} \leqslant \Gamma_5\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\}, \tag{3.10}$$

now with $\Gamma_5 = \Gamma_5(p_i, d_i) < \infty$ *depending only on* p_3, p_4 *and* $d_1 - d_3$. *Moreover,* $\Gamma_5 \leqslant \Gamma_6(D) < \infty$ *for all* p_i, d_i *and* $D < \infty$ *with*

$$|d_2| \leqslant D, \quad d_1 d_3^{-1} \leqslant D, \quad p_3 \geqslant D^{-1}, \quad p_4 \geqslant D^{-1}. \tag{3.11}$$

Lastly, assume that (3.8) *and* (3.9) *hold and that* $W_1, W_2, \ldots$ *are independent random vectors each with the distribution of* $\Sigma_{i=1}^{\Lambda} Z_i$, *where* Λ *is independent of the* Z_i, *has the distribution given by* (2.50), *and that for some* d_4, d_5,

$$0 < d_4 \leqslant T(k)P\{J_1 = 1\} \leqslant d_5 < \infty, \quad P\{J_1 = 1\} \leqslant \tfrac{1}{2}. \tag{3.12}$$

Under these conditions one has for all $\rho \geqslant 0$,

$$\sup_z P\left\{\left|\sum_{r=1}^n W_r + z\right| \leqslant \rho\right\} \leqslant \Gamma_7 \frac{1}{n}\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\} \tag{3.13}$$

for some $\Gamma_7 = \Gamma_7(p_i, d_i)$ *depending on* p_3, p_4 *and* $d_1 - d_5$ *only. Moreover,* $\Gamma_7 \leqslant \Gamma_8(D) < \infty$ *for all* p_i, d_i *and* $D < \infty$ *which satisfy* (3.11) *and* $d_4 \geqslant D^{-1}$, $d_5 \leqslant D$. *Estimates* (3.10) *and* (3.13) *remain valid if* (3.8) *and* (3.9) *hold with* Z_1 *replaced by* $-Z_1$ *in one or both of these formulae.*

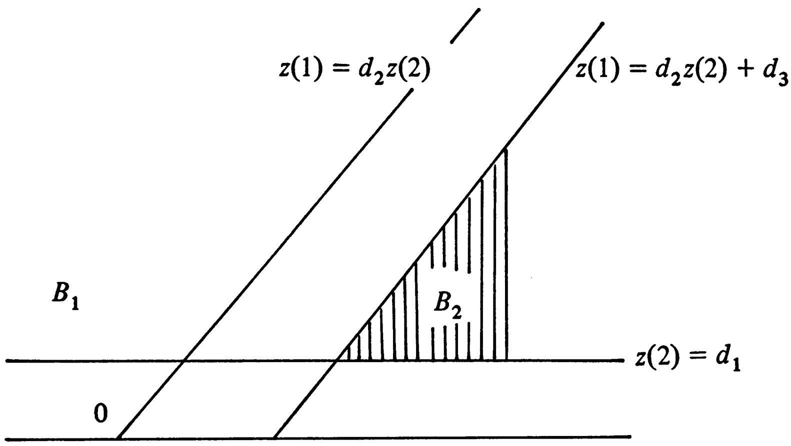
PROOF. We first prove (3.5) with $\rho = d_1$ from (3.3) and (3.4). This is an immediate application of Corollary 1 with

[8] #{ } denotes the number of elements in the set between braces.

$$B_1 = \{z \in \mathbf{R}^2 \colon z(2) > d_1, z(1) \leqslant d_2 z(2)\}$$

and

$$B_2 = \{z \in \mathbf{R}^2 \colon z(2) > d_1, z(1) \geqslant d_2 z(2) + d_3\} \quad \text{(see figure).}$$



Clearly, $B_1, B_2 \subset \{z \in \mathbf{R}^2 \colon |z| > d_1\}$ and

$$\int_{B_i} dG^s(u) = p_i \leqslant q = \int_{|u|>d_1} dG^s(u), \qquad i = 1, 2.$$

Moreover, for $u \in B_1$, $v \in B_2$,

$$|\sin \varphi(u,v)| = (|u|\,|v|)^{-1} |u(2)v(1) - u(1)v(2)|,$$

and, consequently,

$$|v|\,|\sin \varphi(u,v)| \geqslant \frac{u(2)}{|u(1)|+u(2)} \left| v(1) - \frac{u(1)}{u(2)} v(2) \right|$$

$$\geqslant \frac{u(2)}{|u(1)|+u(2)} \left\{ d_2 v(2) + d_3 - \frac{u(1)}{u(2)} v(2) \right\}. \qquad (3.14)$$

Now $u(1) \leqslant d_2 u(2)$ for $u \in B_1$ so that the last member of (3.14) is positive. If $|u(1)| \leqslant (1 + 2|d_2|)u(2)$, then it is even bounded below by $\frac{1}{2}(1 + |d_2|)^{-1} d_3$; and if $|u(1)| > (1 + 2|d_2|)u(2)$, then necessarily $u(1) < -2|d_2|u(2)$ and the last member of (3.14) is bounded below by

$$\frac{u(2)}{|u(1)|+u(2)} \, \frac{1}{2} \, \frac{|u(1)|}{u(2)} v(2) \geqslant \frac{1}{4} \, \frac{1}{1+|d_2|} d_1.$$

Thus in all cases

$$|v|\,|\sin \varphi(u,v)| \geqslant c \equiv \frac{1}{4} \, \frac{1}{1+|d_2|} \min\{d_1, d_3\}, \qquad (3.15)$$

and (3.5) follows from (1.10) with

$$\Gamma_3 \equiv K_1(1 + d_1c^{-1})q(p_1p_2)^{-1} \leq K_1(p_1p_2)^{-1}\{1 + 4(1 + |d_2|)(1 + d_1d_3^{-1})\}. \quad (3.16)$$

Recall that we took $\rho = d_1$ so far. However, once we have (3.5) for $\rho = d_1$ it follows for each $\rho > 0$ because any disc of radius ρ can be covered by at most $K_7(\rho^2 d_1^{-2} + 1)$ discs of radius d_1. It is immediate from (3.16) that Γ_3 is bounded above whenever p_i, d_i satisfy (3.6). Also (3.7) is immediate from (3.5) because $\sum_{r=s}^{2s} r^{-1} \leq 1 + \log 2$.

Next we prove (3.13) by showing that (3.3) and (3.4) hold for suitable values of p_1, p_2 and Z replaced by W. For this purpose take B_1 as above. Assume further that $z_1, z_2, z_1' \in B_1$ satisfy

$$z_1(1) - d_2 z_1(2) \leq \min\{z_2(1) - d_2 z_2(2), z_1'(1) - d_2 z_1'(2)\} \quad (3.17)$$

as well as

$$z_2(2) \geq z_1'(2). \quad (3.18)$$

Then

$$z_1(1) + z_2(1) - z_1'(1) - d_2(z_1(2) + z_2(2) - z_1'(2)) \leq z_2(1) - d_2 z_2(2) \leq 0, \quad (3.19)$$

and

$$z_1(2) + z_2(2) - z_1'(2) \geq z_1(2) > d_1. \quad (3.20)$$

(3.19) and (3.20) just say that $z_1 + z_2 - z_1' \in B_1$, so that $z_1 + z_2 - z_1' \in B_1$ as soon as (3.17) and (3.18) hold. In agreement with the notational convention from before Lemma 1 we now take Λ' an independent copy of Λ and put

$$W_1 = \sum_{i=1}^{\Lambda} Z_i, \quad W_1' = \sum_{i=1}^{\Lambda'} Z_i', \quad W_1^s = W_1 - W_1'.$$

By comparing the different orders of the quantities in (3.17) and (3.18) we now obtain

$$\begin{aligned} P\{W_1^s \in B_1\} &\geq P\{\Lambda = 2, \Lambda' = 1\} \cdot \int_{\substack{z_1, z_2, z_1' \in B_1 \\ (3.17) \text{ and } (3.18) \text{ hold}}} P\{Z_1 \in dz_1, Z_2 \in dz_2, Z_1' \in dz_1'\} \\ &\geq \frac{1}{6} P\{\Lambda = 2\} P\{\Lambda' = 1\} \cdot \int_{z_1, z_2, z_1' \in B_1} P\{Z_1 \in dz_1\} P\{Z_2 \in dz_2\} P\{Z_1' \in dz_1'\} \\ &= \frac{1}{6} P\{\Lambda = 2\} P\{\Lambda' = 1\} (P\{Z_1 \in B_1\})^3. \end{aligned} \quad (3.21)$$

Now, by (2.50) and (3.12),

$$P\{\Lambda = 2\} \geq \frac{1}{2!}\left(T(k)P\{J_1 = 1\}\right)^2 \exp - 2d_5 \geq \tfrac{1}{2}d_4^2 \exp - 2d_5.$$

Similarly,

$$P\{\Lambda = 1\} \geq d_4 \exp - 2d_5,$$

and finally, $P\{Z_1 \in B_1\} = p_3$ (by (3.8)). Thus

$$\begin{aligned} P\{W_1^s(2) > d_1, W_1^s(1) < d_2 W_1^s(2)\} \\ = P\{W_1^s \in B_1\} \geq K_8 d_4^3 p_3^3 \exp - 4d_5. \end{aligned} \tag{3.22}$$

Next consider $P\{W_1^s \in B_2\}$. Assume $z_1, z_2, z_1' \in B_2$ are such that

$$z_1(1) - d_2 z_1(2) \geq \max\{z_2(1) - d_2 z_2(2), z_1'(1) - d_2 z_1'(2)\}, \tag{3.23}$$

and (3.18) holds; then we get

$$\begin{aligned} z_1(1) + z_2(1) - z_1'(1) - d_2\left(z_1(2) + z_2(2) - z_1'(2)\right) \\ \geq z_2(1) - d_2 z_2(2) \geq d_3, \end{aligned}$$

and as before, $z_1(2) + z_2(2) - z_1'(2) > d_1$. Thus, as in (3.21) and (3.22),

$$\begin{aligned} P\{W_1^s \in B_2\} &\geq P\{\Lambda = 2, \Lambda' = 1\} \\ &\quad \cdot \int_{\substack{z_1, z_2, z_1' \in B_2 \\ (3.18) \text{ and } (3.23) \text{ hold}}} P\{Z_1 \in dz_1, Z_2 \in dz_2, Z_1' \in dz_1'\} \\ &\geq K_8 d_4^3 \exp - 4d_5 \left(P\{Z_1 \in B_2\}\right)^3 \\ &\geq K_8 d_4^3 p_4^3 \exp - 4d_5. \end{aligned} \tag{3.24}$$

(3.22) and (3.24) show that (3.3) and (3.4) hold for W_1^s instead of Z_1^s and

$$p_1 = K_8 d_4^3 p_3^3 \exp - 4d_5, \quad p_2 = K_8 d_4^3 p_4^3 \exp - 4d_5. \tag{3.25}$$

(3.13) is therefore immediate from (3.5). Also the bound $\Gamma_7 \leq \Gamma_8(D)$, whenever (3.11) holds together with $d_4 \geq D^{-1}$, $d_5 \leq D$, is immediate from (3.25) and the fact that $\Gamma_3 \leq \Gamma_4(D)$ on (3.6). Since we may everywhere in this argument replace Z_1 by $-Z_1$ without influencing the distribution of W_1^s, no change is needed if Z_1 is replaced by $-Z_1$ in (3.8) or (3.9).

Lastly, we prove (3.10). For this purpose let $\Lambda_1, \Lambda_2, \ldots$ be a sequence of independent random variables, also independent of the Z_i, and

$$P\{\Lambda_i = l\} = \left(\tfrac{1}{2}\right)^{l+1}, \qquad l \geq 0. \tag{3.26}$$

(3.26) corresponds to $T(k) = 1$, $P\{J_1 = 1\} = \frac{1}{2}$ in (2.50). Thus (3.12) is satisfied with $d_4 = d_5 = \frac{1}{2}$ in this case. Set $\tau_0 = 0$, and for $r \geq 1$,

$$\tau_r = \sum_{i=1}^{r} \Lambda_i, \quad W_r = \sum_{i=\tau_{r-1}+1}^{\tau_r} Z_i. \tag{3.27}$$

Then the W_r are independent, identically distributed and by what we just proved (3.13) holds with Γ_7 depending only on $p_3, p_4, d_1 - d_3$ (since now $d_4 = d_5 = \frac{1}{2}$) and $\Gamma_7 \leq \Gamma_8(D)$ whenever (3.11) holds. Moreover,

$$E\#\left\{r \in [s,2s]: \left|\sum_1^r Z_i\right| \leq \rho\right\} \leq (s+1)P\{\tau_{[s/2]} > s \quad \text{or} \quad \tau_{4s} \leq 2s\}$$
$$+ E\#\left\{r \in [\tau_{[s/2]}, \tau_{4s}): \left|\sum_1^r Z_i\right| \leq \rho\right\}. \tag{3.28}$$

By (3.26),

$$E\Lambda_i = 1, \tag{3.29}$$

and by standard exponential bounds for the geometric distribution (compare [9, formulae (5.40)–(5.42)]),

$$P\{|\tau_n - n| \geq \tfrac{1}{2}n\} = P\left\{\sum_1^n \Lambda_i \leq \frac{1}{2}n \text{ or } \sum_1^n \Lambda_i \geq \frac{3}{2}n\right\}$$
$$\leq K_9 \exp - K_{10}n. \tag{3.30}$$

Thus, the first term in the right-hand side of (3.28) is at most

$$2(s+1)K_9 \exp - K_{10}\frac{s}{2} \leq K_{11}.$$

The second term in the right-hand side of (3.28) equals

$$\sum_{[s/2] \leq j < 4s} E\#\left\{r \in [\tau_j, \tau_{j+1}): \left|\sum_{i=1}^r Z_i\right| \leq \rho\right\}$$
$$= \sum_{[s/2] \leq j < 4s} \sum_{m=0}^{\infty} P\left\{\left|\sum_1^{\tau_j + m} Z_i\right| \leq \rho, \tau_{j+1} - \tau_j > m\right\}$$
$$= \sum_{[s/2] \leq j < 4s} \sum_{m=0}^{\infty} \int_{\mathbf{R}^2} P\left\{\tau_{j+1} - \tau_j > m, \sum_{l=\tau_j+1}^{\tau_j+m} Z_l \in dz\right\}$$
$$\cdot P\left\{\left|\sum_{i=1}^{\tau_j} Z_i + z\right| \leq \rho\right\}$$
$$\leq \sum_{[s/2] \leq j < 4s} \sum_{m=0}^{\infty} P\{\tau_{j+1} - \tau_j > m\} \sup_z P\left\{\left|\sum_1^j W_i + z\right| \leq \rho\right\}$$
$$\leq \sum_{[s/2] \leq j < 4s} E\{\tau_{j+1} - \tau_j\}\Gamma_7 \frac{1}{j}\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\} \quad \text{(by (3.13))}$$
$$\leq \Gamma_7 K_{12}\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\} \quad \text{(by (3.29))}. \tag{3.31}$$

The last estimate works only for $s \geqslant 2$, but for $s = 1$ (3.10) needs no proof anyway. From (3.28), (3.31) and the estimate for the first term in the right-hand side of (3.28), we finally see that (3.28) is, for $s \geqslant 2$, at most

$$K_{11} + \Gamma_7 K_{12}\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\} \leqslant K_{13}(\Gamma_7 + 1)\left\{\left(\frac{\rho}{d_1}\right)^2 + 1\right\}. \quad \square \tag{3.32}$$

We now return to the proof of the bound in (3.1). The notation will be as in § 2 for the remainder of this section. We distinguish two cases:

$$T(k)P\{J_1^k = 1\} \leqslant 2^{-5}\eta \tag{3.33}$$

and

$$2^{-5}\eta < T(k)P\{J_1^k = 1\} \leqslant 2K_0. \tag{3.34}$$

(η and K_0 are as in Lemma 2. (3.33) and (3.34) are the only possibilities as we saw in (2.53).) For the remainder of the proof we are only interested in ω components with $\omega \in \mathcal{H}$, the plane spanned by ω_1,ω_2. We shall therefore think of all random variables as being two dimensional and specified by their ω_1 and ω_2 component. We remind the reader of the notation $m(k) = M(\omega_1^k,k)$ and of the choice $M(\omega_2^k,k) = 1$. For any random vector X we define another random vector $\tilde{X} \in \mathcal{H}$ by scaling of the ω_1 and ω_2 component as follows:

$$\tilde{X} = \tilde{X}^k = A_k^{-1}\left\{\frac{1}{m(k)}\langle X,\omega_1^k\rangle\omega_1^k + \langle X,\omega_2^k\rangle\omega_2^k\right\}. \tag{3.35}$$

For most quantities we shall no longer indicate its dependence on k explicitly.

Lemma 4. *There exist $k_6 = k_6(F) < \infty$ and $K_{14} < \infty$ such that the left-hand side of* (3.1) *is bounded by K_{14} for all $s \geqslant 1$ and all $k \geqslant k_6(F)$ for which* (3.33) *holds.*

Proof. By dropping the condition on the ω_0 component and using the above notation we see that the left-hand side of (3.1) is bounded above by

$$\sum_{r=s}^{2s} P\left\{\left|\sum_{n=1}^{r}\langle\tilde{Y}_n,\omega_j\rangle\right| \leqslant 4 \text{ for } j = 1,2\right\}. \tag{3.36}$$

Of course,

$$\tilde{Y}_r = \sum_{n=\nu_{r-1}+1}^{\nu_r}\{\langle\tilde{X}_n(1-J_n),\omega_1\rangle\omega_1 + \langle\tilde{X}_n(1-J_n),\omega_2\rangle\omega_2\}$$
$$+ \sum_{n=\nu_{r-1}+1}^{\nu_r}\{\langle\tilde{X}_nJ_n,\omega_1\rangle\omega_1 + \langle\tilde{X}_nJ_n,\omega_2\rangle\omega_2\}. \tag{3.37}$$

By the observations made immediately following the introduction of Y_r in §2 (see (2.47)) the two sums in the right-hand side of (3.37) are independent, and if we call the first sum $\tilde{Z}_r$, i.e.,

$$\tilde{Z}_r = \sum_{n=\nu_{r-1}+1}^{\nu_r} \{\langle \tilde{X}_n(1-J_n),\omega_1\rangle\omega_1 + \langle \tilde{X}_n(1-J_n),\omega_2\rangle\omega_2\},$$

then the two dimensional analogue of (2.32) gives

$$\sup_z P\left\{\left|\sum_{n=1}^{r} \langle \tilde{Y}_n,\omega_j\rangle + z_j\right| < 4, j = 1, 2\right\}$$
$$< \sup_z P\left\{\left|\sum_{n=1}^{r} \langle \tilde{Z}_n,\omega_j\rangle + z_j\right| < 4, j = 1, 2\right\}. \tag{3.38}$$

Moreover, as observed in (2.47) and (2.48), each $\tilde{Z}_l$ has the distribution of $\Sigma_{n=1}^{T(k)} \tilde{\beta}_n$, where the β_n are independent, and each with the distribution of X_1, given $J_1 = 0$. Thus the variance of each $\langle \tilde{Z}_l,\omega\rangle$ equals

$$\sigma^2(\omega,k) \equiv T(k)\sigma^2\{\langle \tilde{X}_1,\omega\rangle | J_1 = 0\}.$$

Note also that

$$|\tilde{X}_1| < A_k^{-1}(m_k^{-1}|\langle X_1,\omega_1\rangle| + |\langle X_2,\omega_2\rangle|) < 2$$

whenever $J_1 = 0$. Put

$$\Delta_k = \inf\{\sigma^2(\omega,k): |\omega| = 1, \omega \in \mathcal{K}\}$$

and apply the corollary to Theorem 6.2 of Esseen [6] to the independent and identically distributed random two-vectors $\tilde{\beta}_1, \ldots, \tilde{\beta}_{rT(k)}$. Esseen's $\chi_1(4)$ can now be taken at least

$$2 \inf_{|\omega|=1} \sigma^2\{\langle \tilde{\beta}_1,\omega\rangle\} = 2T(k)^{-1}\Delta_k,$$

and therefore the right-hand side of (3.38) is bounded by

$$K_{15}(rT(k)T(k)^{-1}\Delta_k)^{-1} = K_{15}(r\Delta_k)^{-1}.$$

Consequently, the left-hand side of (3.1) is at most

$$\sum_{r=s}^{2s} K_{15}(r\Delta_k)^{-1} < \frac{K_{15}}{\Delta_k}(1 + \log 2).$$

Thus we have a bound of the desired form for (3.1) as soon as Δ_k exceeds some strictly positive ε_1^2. We choose ε_1 as follows:

$$K_{16} = 32K_0^{1/2}\eta^{-1/2} + 4, \quad K_{17} = \frac{1}{16}(10K_0^{1/2}\eta^{-1/2} + 2)^{-1}, \tag{3.39}$$

$$\varepsilon_1 = \frac{1}{32}\eta^{1/2}\frac{K_{17}^2}{K_{17} + 8K_{16}}. \tag{3.40}$$

(K_0 and η are as in Lemma 2.) For the remainder of the lemma we may now assume

$$\Delta_k < \varepsilon_1^2. \tag{3.41}$$

By definition of Δ_k this means that there exists a unit vector $\Omega_1 \in \mathcal{K}$ with $\sigma^2(\Omega_1,k) < 2\varepsilon_1^2$. By Chebyshev's inequality we then have, with $\mu = E\langle \tilde{Z}_1,\Omega_1\rangle$ and $T = T(k)$,

$$\begin{aligned} P&\left\{\left|\left\langle \tilde{S}_T,\Omega_1\right\rangle - \mu\right| \geq 8\varepsilon_1\eta^{-1/2}\right\} \\ &\leq P\left\{\tilde{S}_T \neq \tilde{Z}_1\right\} + P\left\{\left|\langle \tilde{Z}_1,\Omega_1\rangle - \mu\right| \geq 8\varepsilon_1\eta^{-1/2}\right\} \\ &\leq P\left\{J_n = 1 \text{ for some } n \leq T\right\} + \sigma^2(\Omega_1,k)/64\varepsilon_1^2\eta^{-1} \\ &\leq \eta/32 + \eta/32 \quad (\text{by (3.33) and (3.41)}) = \eta/16. \end{aligned} \tag{3.42}$$

We first show that (3.42) necessitates

$$|\mu| \geq K_{17} - 8\varepsilon_1\eta^{-1/2} \geq \tfrac{1}{2}K_{17} \tag{3.43}$$

for $k \geq k_2(F) + k_3(F)$. Indeed if (3.43) fails, then (3.42) shows

$$P\left\{\left|\left\langle \tilde{S}_T,\Omega_1\right\rangle\right| \geq K_{17}\right\} \leq \eta/2. \tag{3.44}$$

But if

$$\Omega_1 = (\cos\varphi_0)\omega_1 + (\sin\varphi_0)\omega_2,$$

and we define

$$\Omega_2 = \left\{m(k)^{-2}\cos^2\varphi_0 + \sin^2\varphi_0\right\}^{-1/2}\left(\frac{\cos\varphi_0}{m(k)}\,\omega_1 + (\sin\varphi_0)\omega_2\right),$$

then (3.44) says

$$P\left\{|\langle S_T,\Omega_2\rangle| \geq K_{17}\left\{m(k)^{-2}\cos^2\varphi_0 + \sin^2\varphi_0\right\}^{-1/2}A_k\right\} \leq \eta/2. \tag{3.45}$$

At the same time Ω_2 is a unit vector in $\mathcal{K}$, and thus by (2.23), $M(\Omega_2,k) \geq 1$, and by (2.30),

$$P\left\{|\langle S_T,\Omega_2\rangle| \geq \tfrac{1}{2}A_k\right\} \geq P\left\{|\langle S_T,\Omega_2\rangle| \geq \tfrac{1}{2}M(\Omega_2)A_k\right\} \geq \eta. \tag{3.46}$$

(3.45) and (3.46) together imply

$$K_{17}\left\{m(k)^{-2}\cos^2\varphi_0 + \sin^2\varphi_0\right\}^{-1/2} \geq \tfrac{1}{2}$$

and

$$\sin^2\varphi_0 \leq 4K_{17}^2 \leq \tfrac{3}{4}, \quad |\cos\varphi_0| \geq \tfrac{1}{2}. \tag{3.47}$$

However, we also have from (2.30):

$$P\left\{|\langle S_T,\omega_1\rangle| \geq \tfrac{1}{2}m(k)A_k\right\} \geq \eta; \tag{3.48}$$

and from (2.24):

$$P\{|\langle S_T,\omega_2\rangle| < 2A_k\} \geq \frac{3}{4} \quad \text{for } k \geq k_2(F). \tag{3.49}$$

(3.44) and (3.48) together give

$$\begin{aligned}\frac{\eta}{2} &< P\{|\langle S_T,\omega_1\rangle| \geq \tfrac{1}{2}m(k)A_k \text{ and } |\langle \tilde{S}_T,\Omega_1\rangle| < K_{17}\} \\ &= P\{|\langle S_T,\omega_1\rangle| \geq \tfrac{1}{2}m(k)A_k \text{ and} \\ &\qquad |\langle S_T,\omega_1\rangle \cos\varphi_0 + \langle S_T,\omega_2\rangle \sin\varphi_0 m(k)| < K_{17}m(k)A_k\} \\ &< P\left\{|\langle S_T,\omega_2\rangle \sin\varphi_0 m(k)| \geq m(k)A_k\left(\tfrac{1}{2}|\cos\varphi_0| - K_{17}\right) \geq \frac{1}{8}m(k)A_k\right\} \\ &< P\{|\langle S_T,\omega_2\rangle| \geq (16K_{17})^{-1}A_k\} \quad (\text{by } (3.47)).\end{aligned} \tag{3.50}$$

(3.49) and (3.50) together show that any interval which contains $\langle S_T,\omega_2\rangle$ with a probability $(1 - \frac{1}{4}\eta)$ must contain both $2A_k$ and $(16K_{17})^{-1}A_k$ or both $-2A_k$ and $-(16K_{17})^{-1}A_k$. Thus any such interval must have length at least

$$\{(16K_{17})^{-1} - 2\}A_k \geq 10K_0^{1/2}\eta^{-1/2}A_k \quad (\text{see } (3.39)).$$

For $k \geq k_3(F)$, however, this contradicts

$$\begin{aligned}&P\left\{\left|\langle S_T,\omega_2\rangle - \sum_{n=1}^{T(k)} E\{\langle X_n,\omega_2\rangle I[|\langle X_n,\omega_2\rangle| \leq A_k]\}\right| \geq 4K_0^{1/2}\eta^{-1/2}A_k\right\} \\ &\qquad \leq \sum_{n=1}^{T(k)} P\{|\langle X_n,\omega_2\rangle| > A_k\} \\ &\qquad\quad + \frac{\eta}{16}\frac{T(k)}{K_0A_k^2}\sigma^2\{\langle X_1,\omega_2\rangle I[|\langle X_n,\omega_2\rangle| \leq A_k]\} \\ &\qquad \leq T(k)P\{J_1 = 1\} + \frac{K_0}{K_0}\frac{\eta}{16} \quad (\text{by } (2.28) \text{ and } M(\omega_2) = 1) \\ &\qquad \leq \eta/8 \quad (\text{by } (3.33)).\end{aligned} \tag{3.51}$$

Thus, the assumption that (3.43) fails indeed leads to a contradiction when $k \geq k_2 + k_3$, and for the remainder we may use (3.43) when $k \geq k_6(F) \equiv k_2(F) + k_3(F)$.

Now bring in the unit vector

$$\Omega_3 = (-\sin\varphi_0)\omega_1 + (\cos\varphi_0)\omega_2$$

orthogonal to Ω_1 in $\mathcal{H}$. If

$$Q(\langle \tilde{S}_T,\Omega_3\rangle,K_{17}) \geq 1 - \eta/4, \tag{3.52}$$

then there exists a constant f such that

$$P\left\{\left|\langle \tilde{S}_T,\Omega_3\rangle - f\right| < \tfrac{1}{2}K_{17}\right\} \geq 1 - \eta/4,$$

which together with (3.42) and (3.40) would imply

$$P\left\{\left|\tilde{S}_T - \mu\Omega_1 - f\Omega_3\right| \leq \tfrac{1}{2}K_{17} + 8\varepsilon_1\eta^{-1/2} < K_{17}\right\} \geq 1 - \eta/2,$$

and a fortiori for Ω_4 a unit vector in $\mathcal{H}$ orthogonal to $\mu\Omega_1 + f\Omega_3$,

$$P\left\{\left|\langle \tilde{S}_T,\Omega_4\rangle - \langle \mu\Omega_1 + f\Omega_3,\Omega_4\rangle\right| = \left|\langle \tilde{S}_T,\Omega_4\rangle\right| < K_{17}\right\} \geq 1 - \eta/2.$$

Apart from the change from Ω_1 to Ω_4 this again is (3.44), which we showed to be impossible. Thus (3.52) must fail and no interval of length K_{17} can contain $\langle \tilde{S}_T,\Omega_3\rangle$ with probability $1 - \eta/4$. Thus, if ν is a median of $\langle \tilde{S}_T,\Omega_3\rangle$, then

$$P\left\{\left|\langle \tilde{S}_T,\Omega_3\rangle - \nu\right| > \tfrac{1}{2}K_{17}\right\} \geq \eta/4,$$

and at least one of the inequalities

$$P\left\{\langle \tilde{S}_T,\Omega_3\rangle > \nu + \tfrac{1}{2}K_{17}\right\} \geq \eta/8 \tag{3.53}$$

or

$$P\left\{\langle \tilde{S}_T,\Omega_3\rangle < \nu - \tfrac{1}{2}K_{17}\right\} \geq \eta/8 \tag{3.54}$$

must hold. For the sake of definiteness assume (3.53) holds. Since ν is a median of $\langle \tilde{S}_T,\Omega_3\rangle$ we also have

$$P\left\{\langle \tilde{S}_T,\Omega_3\rangle \leq \nu\right\} \geq \tfrac{1}{2}. \tag{3.55}$$

We are almost ready to apply Corollary 2; we merely need an estimate on ν, or, more generally, the distribution of $|\tilde{S}_T|$. Just as in (3.51) we have for any $x \geq 0$ and $k \geq k_6(F)$,

$$\begin{aligned}
&P\left\{\left|\langle \tilde{S}_T,\omega_1\rangle - T(k)E\left\{\langle \tilde{X}_1,\omega_1\rangle I\left[|\langle X_1,\omega_1\rangle| \leq m(k)A_k\right]\right\}\right| \geq x\right\} \\
&\quad \leq T(k)P\left\{|\langle X_1,\omega_1\rangle| > m(k)A_k\right\} \\
&\qquad + \frac{T(k)}{x^2}\sigma^2\left\{\langle \tilde{X}_1,\omega_1\rangle I\left[|\langle X_1,\omega_1\rangle| \leq m(k)A_k\right]\right\} \\
&\quad \leq T(k)P\{J_1 = 1\} \\
&\qquad + \frac{T(k)}{x^2}\{m(k)A_k\}^{-2}\sigma^2\left\{\langle X_1,\omega_1\rangle I\left[|\langle X_1,\omega_1\rangle| \leq m(k)A_k\right]\right\} \\
&\quad \leq \eta/32 + x^{-2}K_0 \quad \text{(see (3.33) and (2.28))}.
\end{aligned} \tag{3.56}$$

But also by (2.21),

$$P\left\{\left|\langle \tilde{S}_T,\omega_1\rangle\right| < 2\right\} = P\left\{\left|\langle S_T,\omega_1\rangle\right| < 2m(k)A_k\right\}$$
$$\geq 1-(8d+8)^{-1} \geq \tfrac{3}{4}. \tag{3.57}$$

Together with (3.56) this shows (take $x = 8K_0^{1/2}\eta^{-1/2}$ in (3.56))

$$\left|T(k)E\left\{\langle \tilde{X}_1,\omega_1\rangle I\left[\left|\langle X_1,\omega_1\rangle\right| \leq m(k)A_k\right]\right\}\right| \leq 8K_0^{1/2}\eta^{-1/2}+2,$$

and therefore (again take $x = 8K_0^{1/2}\eta^{-1/2}$ in (3.56))

$$P\left\{\left|\langle \tilde{S}_T,\omega_1\rangle\right| \geq 16K_0^{1/2}\eta^{-1/2}+2\right\} \leq \eta/16.$$

The same inequality holds when ω_1 is replaced by ω_2 (use (3.49) to get (3.57)) and, therefore,

$$P\left\{\left|\langle \tilde{S}_T,\Omega_3\rangle\right| \geq 32K_0^{1/2}\eta^{-1/2}+4\right\}$$
$$\leq P\left\{\left|\tilde{S}_T\right| \geq 32K_0^{1/2}\eta^{-1/2}+4\right\} \leq \eta/8 < \tfrac{1}{2}.$$

In particular, we must have

$$|\nu| \leq K_{16} = \left(32K_0^{1/2}\eta^{-1/2}+4\right) \quad \text{(see (3.39))}. \tag{3.58}$$

We can now complete the proof of the lemma by means of an application of Corollary 2. By (3.43) either $\mu > 0$ or $\mu < 0$. For the sake of argument take $\mu > 0$. Then (3.42) and (3.43) give

$$P\left\{K_{17}/4 < \mu/2 \leq \mu - 8\varepsilon_1\eta^{-1/2} \leq \langle \tilde{S}_T,\Omega_1\rangle \leq \mu + 8\varepsilon_1\eta^{-1/2}\right\}$$
$$\geq 1-\eta/16. \tag{3.59}$$

(3.59) and (3.55) yield

$$P\left\{\langle \tilde{S}_T,\Omega_1\rangle > \tfrac{1}{4}K_{17}, \langle \tilde{S}_T,\Omega_3\rangle \leq \left(\nu+\tfrac{1}{8}K_{17}\right)\mu^{-1}\langle \tilde{S}_T,\Omega_1\rangle\right\}$$
$$\geq P\left\{\mu - 8\varepsilon_1\eta^{-1/2} \leq \langle \tilde{S}_T,\Omega_1\rangle \leq \mu + 8\varepsilon_1\eta^{-1/2}, \langle \tilde{S}_T,\Omega_3\rangle \leq \nu\right\}$$
$$\geq P\left\{\langle \tilde{S}_T,\Omega_3\rangle \leq \nu\right\} - \frac{\eta}{16} \geq \frac{1}{2} - \frac{\eta}{16} \geq \frac{1}{4}. \tag{3.60}$$

In exactly the same way one obtains from (3.59) and (3.53):

$$P\left\{\langle \tilde{S}_T,\Omega_1\rangle \geq \tfrac{1}{4}K_{17}, \langle \tilde{S}_T,\Omega_3\rangle \geq \left(\nu+\tfrac{1}{8}K_{17}\right)\mu^{-1}\langle \tilde{S}_T,\Omega_1\rangle + \tfrac{1}{8}K_{17}\right\}$$
$$\geq P\left\{\mu - 8\varepsilon_1\eta^{-1/2} \leq \langle \tilde{S}_T,\Omega_1\rangle \leq \mu + 8\varepsilon_1\eta^{-1/2}, \langle \tilde{S}_T,\Omega_3\rangle \geq \nu + \tfrac{1}{2}K_{17}\right\}$$
$$\geq \eta/8 - \eta/16 = \eta/16. \tag{3.61}$$

Lastly, we observe that

$$P\{\tilde{Y}_1 \neq \tilde{S}_T\} = P\left\{\sum_1^{\nu_1} X_n \neq \sum_1^{T} X_n\right\} = P\{\nu_1 \neq T\}$$
$$\leq P\{J_n = 1 \text{ for some } n \leq T\} \leq \tfrac{1}{32}\eta \quad \text{(by (3.33))},$$

so that with

$$d_1 = \tfrac{1}{4}K_{17}, \quad d_2 = \left(\nu + \tfrac{1}{8}K_{17}\right)\mu^{-1}, \quad d_3 = \tfrac{1}{8}K_{17},$$

(3.60) and (3.61) give

$$P\left\{\langle \tilde{Y}_1,\Omega_1\rangle > d_1, \langle \tilde{Y}_1,\Omega_3\rangle \leq d_2\langle \tilde{Y}_1,\Omega_1\rangle\right\} \geq 1/4 - \eta/32 \geq 1/8,$$

and

$$P\left\{\langle \tilde{Y}_1,\Omega_1\rangle > d_1, \langle \tilde{Y}_1,\Omega_3\rangle \geq d_2\langle \tilde{Y}_1,\Omega_1\rangle + d_3\right\} \geq \eta/16 - \eta/32 = 2^{-5}\eta.$$

These inequalities are just (3.8) and (3.9) for $Z_1(1) = \langle \tilde{Y}_1,\Omega_3\rangle$, $Z_1(2) = \langle \tilde{Y}_1,\Omega_1\rangle$ and $p_3 = \frac{1}{8}$, $p_4 = 2^{-5}\eta$. Moreover, $d_3d_1^{-1}$, p_3, p_4 all have strictly positive lower bounds which are independent of F, whereas (see (3.43) and (3.58))

$$|d_2| \leq \left(K_{16} + \frac{1}{8}K_{17}\right)|\mu|^{-1} \leq 2\left(K_{16} + \frac{1}{8}K_{17}\right)K_{17}^{-1}.$$

It follows there from (3.10) and (3.11) that for some $K_{18} < \infty$ and all $s \geq 1$, (3.36) is bounded by

$$E\#\left\{r \in [s,2s]: \left|\sum_{n=1}^{r}\{\langle \tilde{Y}_n,\Omega_1\rangle\Omega_1 + \langle \tilde{Y}_n,\Omega_3\rangle\Omega_3\}\right| \leq 8\right\}$$

(because Ω_1,Ω_3 is an orthonormal basis of $\mathcal{K}$)

$$\leq K_{18}\left\{\left(32K_{17}^{-1}\right)^2 + 1\right\}.$$

This gives the desired bound for (3.36) and (3.1) under (3.33).

LEMMA 5. *There exist a* $k_7 = k_7(F) < \infty$ *and* $K_{19} < \infty$ *such that the left-hand side of* (3.1) *is bounded by* K_{19} *for all* $k \geq k_7(F)$ *for which* (3.34) *holds.*

PROOF. Set

$$\tilde{W}_r = \sum_{n=\nu_{r-1}+1}^{\nu_r}\{\langle \tilde{X}_nJ_n,\omega_1\rangle\omega_1 + \langle \tilde{X}_nJ_n,\omega_2\rangle\omega_2\}. \tag{3.62}$$

As mentioned in (3.37) and the lines following it, $\tilde{Y}_r = \tilde{Z}_r + \tilde{W}_r$ and all $\tilde{Z}_r, \tilde{W}_s$, $r \geq 1$, $s \geq 1$, are independent. Just as in (3.38) the two dimensional analogue of (2.32) therefore gives for any unit vectors $\Omega_i \in \mathcal{K}$ and $x_i \geq 0$, $i \in I$ (some finite set of positive integers)

$$\sup_z P\left\{\left|\left\langle \sum_{n=1}^{r}\tilde{Y}_n,\Omega_i\right\rangle + z_i\right| \leq x_i, i \in I\right\}$$

$$\leq \sup_z P\left\{\left|\left\langle \sum_{n=1}^{r}\tilde{W}_n,\Omega_i\right\rangle + z_i\right| \leq x_i, i \in I\right\}. \tag{3.63}$$

The distribution of

$$W_r = \sum_{n=\nu_{r-1}+1}^{\nu_r} X_n J_n$$

was given in (2.49) and (2.50). As there we take $\alpha_1,\alpha_2,\ldots$ independent and each with the conditional distribution of X_1 given $J_1 = 1$. Then $\tilde{\alpha}_i$ has the conditional distribution of $\tilde{X}_1$ given $J_1 = 1$ and $\tilde{W}_r$ the distribution of $\Sigma_{i=1}^{\Lambda}\tilde{\alpha}_i$. Moreover, by (2.46),

$$P\{|\langle\tilde{\alpha}_i,\omega_1\rangle| > 1 \text{ or } |\langle\tilde{\alpha}_1,\omega_2\rangle| > 1\}$$
$$= P\{|\langle X_1,\omega_1\rangle| > m(k)A_k \text{ or } |\langle X_1,\omega_2\rangle| > A_k | J_1 = 1\} = 1,$$

and a fortiori

$$P\{|\tilde{\alpha}_i| > 1\} = 1. \tag{3.64}$$

Now consider the unit vectors

$$u_l = \left(\cos(2l+1)\frac{\pi}{16}\right)\omega_1 + \left(\sin(2l+1)\frac{\pi}{16}\right)\omega_2, \qquad l = 0,\ldots,7,$$

all of which lie in $\mathcal{H} \cap \{z \in \mathbf{R}^2: z(2) > 0\}$, and are such that every vector in $\mathcal{H}$ makes an angle at most $\pi/16$ with some u_l or $-u_l$. Thus one can choose a $0 \leq l \leq 7$ such that

$$P\left\{\varphi(\tilde{\alpha}_1,u_l) \leq \frac{\pi}{16} \text{ or } \varphi(\tilde{\alpha}_1,-u_l) \leq \frac{\pi}{16}\right\} \geq \frac{1}{8}. \tag{3.65}$$

With l fixed in this way one can also find two orthogonal unit vectors Ω_1,Ω_2 in $\mathcal{H}$ such that $\varphi(\Omega_1,u_l) \leq \pi/16$,

$$P\{\varphi(\tilde{\alpha}_1,\Omega_1) \leq \pi/8 \text{ or } \varphi(\tilde{\alpha}_1,-\Omega_1) \leq \pi/8,$$
$$\langle\tilde{\alpha}_1,\Omega_1\rangle \cdot \langle\tilde{\alpha}_1,\Omega_2\rangle \geq 0\} \geq 1/16 \tag{3.66}$$

as well as

$$P\{\varphi(\tilde{\alpha}_1,\Omega_1) \leq \pi/8 \text{ or } \varphi(\tilde{\alpha}_1,-\Omega_1) \leq \pi/8,$$
$$\langle\tilde{\alpha}_1,\Omega_1\rangle\langle\tilde{\alpha}_1,\Omega_2\rangle \leq 0\} \geq 1/16. \tag{3.67}$$

Note that (3.66) merely says that there is a probability at least $\frac{1}{16}$ for $\tilde{\alpha}_1$ to lie in the first or third quadrant with respect to the Ω_1, Ω_2 axes and even within $\pi/8$ from the positive or negative Ω_1 axis. (3.67) says the same thing with first and third quadrant replaced by second and fourth quadrant; such Ω_1,Ω_2 are easily obtained by continuity considerations, for if one chooses Ω_1 first along one boundary line of the set in braces in (3.65) then there is probability at least $\frac{1}{8}$ that $\tilde{\alpha}_1$ lies in the first or third quadrant, and when Ω_1 is rotated to the other boundary line then one ends up with probability at least $\frac{1}{8}$ that $\tilde{\alpha}_1$ lies in the second or fourth quadrant. Note that Ω_1,Ω_2 really depend on k, since $\tilde{\alpha}_1$ does. The same holds for φ_1 and ω_0 below, but this will not influence the sequel. Assume

$$\Omega_1 = (-\sin \varphi_1)\omega_1 + (\cos \varphi_1)\omega_2, \quad \Omega_2 \text{ or } -\Omega_2 = (\cos \varphi_1)\omega_1 + (\sin \varphi_1)\omega_2$$

and fix ω_0, a unit vector in $\mathcal{H}$, $K_{20} < \infty$ and $\varepsilon_2 > 0$ as follows:

$$\omega_0 = \{m(k)^{-2}\cos^2\varphi_1 + \sin^2\varphi_1\}^{-1/2} \cdot \{m(k)^{-1}(\cos \varphi_1)\omega_1 + (\sin \varphi_1)\omega_2\}, \tag{3.68}$$

$$L_0 = [32K_0\eta^{-1}] + 1, \quad x_0 = 24K_0\eta^{-1} + 1, \tag{3.69}$$

$$K_{20} = \min\{(64L_0)^{-1}, 2^{-7}(12x_0 + 3)^{-1}\}, \quad \varepsilon_2 = (32L_0)^{-1}\eta. \tag{3.70}$$

For brevity we shall write

$$\Theta = \Theta(k) = \{m(k)^{-2}\cos^2\varphi_1 + \sin^2\varphi_1\}^{1/2}.$$

We consider three separate cases now. Which case we are in depends on which of the inequalities (3.71)–(3.73) holds:

$$P\{|\langle\tilde{\alpha}_1,\Omega_2\rangle| > \tfrac{1}{2}\} \geqslant \varepsilon_2, \tag{3.71}$$

$$P\{K_{20}M(\omega_0,k)\Theta(k) < |\langle\tilde{\alpha}_1,\Omega_2\rangle| \leqslant \tfrac{1}{2}\} \geqslant \varepsilon_2, \tag{3.72}$$

$$P\{|\langle\tilde{\alpha}_1,\Omega_2\rangle| > K_{20}M(\omega_0,k)\Theta(k)\} \leqslant 2\varepsilon_2. \tag{3.73}$$

First assume (3.71); without loss of generality we may then assume (if necessary replace Ω_2 by $-\Omega_2$ and/or Ω_1 by $-\Omega_1$) that

$$P\{\langle\tilde{\alpha}_1,\Omega_2\rangle > \tfrac{1}{2}, \langle\tilde{\alpha}_1,\Omega_1\rangle \geqslant 0\} \geqslant \tfrac{1}{4}\varepsilon_2. \tag{3.74}$$

In this case we take

$$\Omega_3 = \frac{1}{\sqrt{2}}\{-\Omega_1 + \Omega_2\}, \quad \Omega_4 = \frac{1}{\sqrt{2}}\{\Omega_1 + \Omega_2\}.$$

Ω_3 and Ω_4 are orthogonal unit vectors, bisecting the second (resp. first) quadrant with respect to Ω_1, Ω_2. Obviously $\langle\tilde{\alpha}_1,\Omega_2\rangle > \frac{1}{2}$, $\langle\tilde{\alpha}_1,\Omega_1\rangle \geqslant 0$ entails

$$\langle\tilde{\alpha}_1,\Omega_3\rangle = -\langle\tilde{\alpha}_1,\Omega_4\rangle + \sqrt{2}\,\langle\tilde{\alpha}_1,\Omega_2\rangle \geqslant -\langle\tilde{\alpha}_1,\Omega_4\rangle + 1/\sqrt{2}$$

as well as

$$\langle\tilde{\alpha}_1,\Omega_4\rangle > 2^{-3/2}.$$

Thus (3.74) implies

$$P\{\langle\tilde{\alpha}_1,\Omega_4\rangle > 2^{-3/2}, \langle\tilde{\alpha}_1,\Omega_3\rangle \geqslant -\langle\tilde{\alpha}_1,\Omega_4\rangle + 2^{-1/2}\} \geqslant \frac{1}{4}\varepsilon_2. \tag{3.75}$$

But also, from (3.64) and (3.67), one has

$$P\left\{\langle\tilde{\alpha}_1,\Omega_4\rangle \geqslant \cos\frac{3\pi}{8}, \langle\tilde{\alpha}_1,\Omega_3\rangle \leqslant -\langle\tilde{\alpha}_1,\Omega_4\rangle\right\} \geqslant \frac{1}{32}$$

or the same inequality with $\tilde{\alpha}_1$ replaced by $-\tilde{\alpha}_1$. Thus if we put

$$d_1 = \tfrac{1}{3}, \quad d_2 = -1, \quad d_3 = 2^{-1/2}, \quad p_3 = \tfrac{1}{32}, \quad p_4 = \tfrac{1}{4}\varepsilon_2,$$

then (3.9) holds with $Z_1(1) = \langle \tilde{\alpha}_1, \Omega_3 \rangle$, $Z_1(2) = \langle \tilde{\alpha}_1, \Omega_4 \rangle$, and (3.8) holds with the same replacement or with $-Z_1(1) = \langle \tilde{\alpha}_1, \Omega_3 \rangle$, $-Z_1(2) = \langle \tilde{\alpha}_1, \Omega_4 \rangle$. Moreover, by (3.34) and (2.29) we also have (3.12) with $d_4 = 2^{-5}\eta$, $d_5 = 2K_0$ as soon as $k > k_7$ for suitable $k_7 = k_7(F) < \infty$. Thus, by (3.63) and (3.13),

$$\sup_z P\left\{\left|\sum_{n=1}^{r} \langle \tilde{Y}_n, \omega_j \rangle + z_j\right| \leq 4, j = 1,2\right\}$$
$$\leq \sup_z P\left\{\left|\sum_{n=1}^{r} \tilde{W}_n + z\right| \leq 8\right\} \leq K_{21} r^{-1}$$

for some $K_{21} < \infty$ depending on ε_2, η and K_0 only. Thus K_{21} does not depend on F, and in the case where (3.71) holds we obtain the bound

$$\sum_{r=s}^{2s} K_{21} r^{-1} \leq K_{21}(1 + \log 2)$$

for (3.36) and the left-hand side of (3.1).

Next we consider the case where (3.72) holds. Again we assume that the signs of Ω_1, Ω_2 have been chosen such that

$$P\left\{K_{20} M(\omega_0, k)\Theta(k) < \langle \tilde{\alpha}_1, \Omega_2 \rangle \leq \tfrac{1}{2}, \langle \tilde{\alpha}_1, \Omega_1 \rangle \geq 0\right\} \geq \tfrac{1}{4}\varepsilon_2 \tag{3.76}$$

(compare (3.74)). Since $|\tilde{\alpha}_1| > 1$ w.p.1 (see (3.64)), $|\langle \tilde{\alpha}_1, \Omega_2 \rangle| \leq \tfrac{1}{2}$ implies $|\langle \tilde{\alpha}_1, \Omega_1 \rangle| > \tfrac{1}{2}$. Thus, if we define for any random vector X,

$$\bar{X} = \langle \tilde{X}, \Omega_1 \rangle \Omega_1 + \{M(\omega_0, k)\Theta(k)\}^{-1} \langle \tilde{X}, \Omega_2 \rangle \Omega_2$$
$$= \langle \tilde{X}, \Omega_1 \rangle \Omega_1 + M(\omega_0, k)^{-1}\{m(k)^{-2}\cos^2\varphi_1 + \sin^2\varphi_1\}^{-1/2} \langle \tilde{X}, \Omega_2 \rangle \Omega_2,$$

then (3.76) implies

$$P\left\{\langle \bar{\alpha}_1, \Omega_1 \rangle = \langle \tilde{\alpha}_1, \Omega_1 \rangle > \tfrac{1}{2}, \langle \bar{\alpha}_1, \Omega_2 \rangle > K_{20}\right\} \geq \tfrac{1}{4}\varepsilon_2. \tag{3.77}$$

Also, (3.67) together with (3.64) gives

$$P\left\{|\langle \bar{\alpha}_1, \Omega_1 \rangle| \geq \cos\frac{\pi}{8}, \langle \bar{\alpha}_1, \Omega_1 \rangle \langle \bar{\alpha}_1, \Omega_2 \rangle \leq 0\right\}$$
$$\geq P\{|\tilde{\alpha}_1| > 1, \varphi(\tilde{\alpha}_1, \Omega_1) \leq \pi/8$$
$$\text{or } \varphi(\tilde{\alpha}_1, -\Omega_1) \leq \pi/8, \langle \tilde{\alpha}_1, \Omega_1 \rangle \langle \tilde{\alpha}, \Omega_2 \rangle \leq 0\} \geq 1/16,$$

so that

$$P\left\{\langle \bar{\alpha}_1, \Omega_1 \rangle \geq \cos\frac{\pi}{8}, \langle \bar{\alpha}_1, \Omega_2 \rangle \leq 0\right\} \geq \frac{1}{32} \tag{3.78}$$

or

$$P\left\{\langle -\bar{\alpha}_1,\Omega_1\rangle > \cos\frac{\pi}{8}, \langle -\bar{\alpha}_1,\Omega_2\rangle < 0\right\} > \frac{1}{32}. \tag{3.79}$$

If we take

$$d_1 = \tfrac{1}{2},\quad d_2 = 0,\quad d_3 = K_{20},\quad p_3 = \frac{1}{32},\quad p_4 = \frac{1}{4}\varepsilon_2,$$

then (3.77) is again (3.9) with $Z_1(1) = \langle\bar{\alpha}_1,\Omega_2\rangle, Z_1(2) = \langle\bar{\alpha}_1,\Omega_1\rangle$; (3.78) becomes (3.8) and (3.79) is (3.8) with Z replaced by $-Z$. As before we also have (3.12) with $d_4 = 2^{-5}\eta$ and $d_5 = 2K_0$ for $k > k_7$ from (3.34) and (2.29). Thus, by (3.13) and (3.63),

$$\sup_z P\left\{\left|\sum_{n=1}^{r}\langle\bar{Y}_n,\Omega_j\rangle + z_j\right| < 8, j = 1,2\right\}$$
$$\leq \sup_z P\left\{\left|\sum_{n=1}^{r}\langle\overline{W}_n,\Omega_j\rangle + z_j\right| \leq 8, j = 1,2\right\}$$
$$\leq \sup_z P\left\{\left|\sum_{n=1}^{r}\overline{W}_n + z\right| \leq 16\right\} \leq K_{22}r^{-1}. \tag{3.80}$$

Again $K_{22} < \infty$ depends neither on F nor k. However,

$$\sum_{n=1}^{r}\langle\bar{Y}_n,\Omega_1\rangle = \sum_{n=1}^{r}\langle\tilde{Y}_n,\Omega_1\rangle$$
$$= A_k^{-1}\left\{-\sum_{n=1}^{r}\langle Y_n,\omega_1\rangle\frac{1}{m(k)}\sin\varphi_1 + \sum_{n=1}^{r}\langle Y_n,\omega_2\rangle\cos\varphi_1\right\} \tag{3.81}$$

and

$$\left|\sum_{n=1}^{r}\langle\bar{Y}_n,\Omega_2\rangle\right| = \{M(\omega_0,k)\Theta(k)\}^{-1}\left|\sum_{n=1}^{r}\langle\tilde{Y}_n,\Omega_2\rangle\right|$$
$$= \{A_kM(\omega_0,k)\Theta(k)\}^{-1}$$
$$\cdot\left|\sum_{n=1}^{r}\langle Y_n,\omega_1\rangle\frac{1}{m(k)}\cos\varphi_1 + \sum_{n=1}^{r}\langle Y_n,\omega_2\rangle\sin\varphi_1\right|$$
$$= A_k^{-1}M(\omega_0,k)^{-1}\left|\sum_{n=1}^{r}\langle Y_n,\omega_0\rangle\right|. \tag{3.82}$$

It follows that the condition

$$\left|\sum_{n=1}^{r}\langle Y_n,\omega_j\rangle\right| \leq 4M(\omega_j,k)A_k \tag{3.83}$$

for $j = 0$ is equivalent to

$$\left|\sum_{n=1}^{r}\left\langle \bar{Y}_n,\Omega_2\right\rangle\right| < 4,$$

and condition (3.83) for $j = 1$ and 2 implies

$$\left|\sum_{n=1}^{r}\left\langle \bar{Y}_n,\Omega_1\right\rangle\right| \leq 4|\sin\varphi_1| + 4|\cos\varphi_1| \leq 8$$

(recall $M(\omega_1,k) = m(k)$, $M(\omega_2,k) = 1$). Thus, (3.1) is bounded by

$$\sum_{r=s}^{2s}\sup_z P\left\{\left|\sum_{n=1}^{r}\left\langle \bar{Y}_n,\Omega_j\right\rangle + z_j\right| \leq 8, j = 1,2\right\}$$

$$\leq \sum_{r=s}^{2s} K_{22}r^{-1} \quad \text{(see (3.80))} \leq K_{22}(1 + \log 2).$$

This settles the case where (3.72) holds and we are left with the case where (3.73) holds. Without loss of generality we may also assume that (3.71) fails. We begin with an estimate on the distribution of $\langle \tilde{S}_T,\Omega_2\rangle$ in this case. Analogously to (3.82),

$$\left|\left\langle \tilde{S}_T,\Omega_2\right\rangle\right| = \Theta(k)A_k^{-1}\left|\left\langle S_T,\omega_0\right\rangle\right|,$$

so that by (2.30),

$$P\left\{\left|\left\langle \tilde{S}_T,\Omega_2\right\rangle\right| \geq \tfrac{1}{2}\Theta(k)M(\omega_0,k)\right\} \geq \eta. \tag{3.84}$$

Obviously,

$$\tilde{S}_T = \sum_{n=1}^{T}\tilde{X}_nJ_n + \sum_{n=1}^{T}\tilde{X}_n(1 - J_n).$$

Moreover, for any $L \geq 1$,

$$P\left\{\left|\sum_{n=1}^{T}\langle \tilde{X}_nJ_n,\Omega_2\rangle\right| \geq \frac{1}{4}\Theta(k)M(\omega_0,k)\right\}$$

$$\leq P\left\{\sum_{n=1}^{T}J_n > L\right\} + P\left\{\sum_{i=1}^{L}\left|\langle\tilde{\alpha}_i,\Omega_2\rangle\right| \geq \frac{1}{4}\Theta(k)M(\omega_0,k)\right\}; \tag{3.85}$$

indeed once we know that $J_n = 1$ exactly for $n \in \{n_1, \ldots, n_L\}$ and no other $n \in [1,T]$, the conditional distribution of

$$\sum_{n=1}^{T}\tilde{X}_nJ_n \tag{3.86}$$

is simply the distribution of

$$\sum_{i=1}^{L}\tilde{\alpha}_i \tag{3.87}$$

(compare (2.49) and (2.50)). The right-hand side of (3.85) is bounded by

$$\frac{T}{L} EJ_1 + LP\left\{|\langle \tilde{\alpha}_1, \Omega_2\rangle| \geq \frac{1}{4L}\Theta(k)M(\omega_0,k)\right\}.$$

Now take $L = L_0$ (see (3.69)). Then it follows from (2.53), (3.70) and (3.73) that (3.85) is at most

$$(T/L_0)P\{J_1 = 1\} + L_0 P\{|\langle \tilde{\alpha}_1,\Omega_2\rangle| > K_{20}\Theta M(\omega_0,k)\}$$
$$\leq (2/L_0)K_0 + L_0 2\varepsilon_2 \leq \eta/8.$$

When this bound for (3.85) is combined with (3.84) we find

$$P\left\{\left|\sum_{n=1}^{T} \langle \tilde{X}_n(1 - J_n),\Omega_2\rangle\right| \geq \frac{1}{4}\Theta(k)M(\omega_0,k)\right\} \geq \frac{7}{8}\eta. \tag{3.88}$$

Next we need an estimate for the distribution of

$$\sum_{n=1}^{T} \langle \tilde{X}_n(1 - J_n),\omega_j\rangle, \qquad j = 1,2,$$

which is a slight variation on (3.56). E.g., take $j = 1$ and write temporarily

$$I_n = I\big[|\langle X_n,\omega_1\rangle| \leq m(k)A_k\big] = I\big[|\langle \tilde{X}_n,\omega_1\rangle| \leq 1\big].$$

Then, as in (3.56), for $k \geq k_3(F)$,

$$P\left\{\left|\sum_{n=1}^{T} \langle \tilde{X}_n I_n,\omega_1\rangle - TE\langle \tilde{X}_1,\omega_1\rangle I_1\right| \geq x\right\}$$
$$\leq (T(k)/x^2)\sigma^2\{\langle \tilde{X}_1,\omega_1\rangle I[|\langle X_1,\omega_1\rangle| \leq m(k)A_k]\} \leq K_0/x^2. \tag{3.89}$$

But also

$$I_n - (1 - J_n) = I\big[|\langle \tilde{X}_n,\omega_1\rangle| \leq 1, |\langle \tilde{X}_n,\omega_2\rangle| > 1\big],$$

so that

$$\left|\sum_{n=1}^{T} \langle \tilde{X}_n(1 - J_n),\omega_1\rangle - \sum_{n=1}^{T} \langle \tilde{X}_n I_n,\omega_1\rangle\right|$$
$$\leq \sum_{n=1}^{T} |\langle \tilde{X}_n,\omega_1\rangle|\,|1 - J_n - I_n|$$
$$\leq \#\{n \leq T: |\langle \tilde{X}_n,\omega_1\rangle| \leq 1, |\langle \tilde{X}_n,\omega_2\rangle| > 1\}$$
$$\leq \sum_{n=1}^{T} J_n.$$

It follows that

$$P\left\{\left|\sum_{n=1}^{T}\langle\tilde{X}_n(1-J_n),\omega_1\rangle-\sum_{n=1}^{T}\langle\tilde{X}_nI_n,\omega_1\rangle\right|>x\right\}$$
$$<x^{-1}T(k)EJ_1<2x^{-1}K_0\quad(\text{by }(2.53)).$$

Combined with (3.89) this estimate yields

$$P\left\{\left|\sum_{n=1}^{T}\langle\tilde{X}_n(1-J_n),\omega_1\rangle-TE\langle\tilde{X}_1,\omega_1\rangle I\left[|\langle\tilde{X}_1,\omega_1\rangle|<1\right]\right|>2x\right\}$$
$$<K_0(2x^{-1}+x^{-2}).$$

This inequality remains valid when ω_1 is replaced by ω_2 throughout and thus, if we define

$$\rho=\rho(k)=-E\langle\tilde{X}_1,\omega_1\rangle I\left[|\langle\tilde{X}_1,\omega_1\rangle|<1\right]\sin\varphi_1$$
$$+E\langle\tilde{X}_1,\omega_2\rangle I\left[|\langle\tilde{X}_1,\omega_2\rangle|<1\right]\cos\varphi_1,$$

then

$$P\left\{\left|\sum_{n=1}^{T}\langle\tilde{X}_n(1-J_n),\Omega_1\rangle-T(k)\rho(k)\right|>4x\right\}<2K_0(2x^{-1}+x^{-2}).\quad(3.90)$$

Now observe that by (2.21) and (2.24),

$$P\left\{|\langle\tilde{S}_T,\Omega_1\rangle|<3\right\}>P\left\{|\langle\tilde{S}_T,\omega_1\rangle|<1,|\langle\tilde{S}_T,\omega_2\rangle|<2\right\}$$
$$>1-P\left\{|\langle S_T,\omega_1\rangle|>m(k)A_k\right\}-P\left\{|\langle S_T,\omega_2\rangle|>2A_k\right\}$$
$$>1-3(8d+8)^{-1}\geq\frac{1}{2}.$$

Thus, for x_0 as in (3.69) we have

$$2K_0\left(2x_0^{-1}+x_0^{-2}\right)<\eta/4<1/4,$$

and

$$P\left\{\left|\sum_{n=1}^{T}\langle\tilde{X}_nJ_n,\Omega_1\rangle+T(k)\rho(k)\right|<4x_0+3,\right.$$
$$\left.\left|\sum_{n=1}^{T}\langle\tilde{X}_n(1-J_n),\Omega_1\rangle-T(k)\rho(k)\right|<4x_0\right\}$$
$$\geq P\left\{|\langle\tilde{S}_T,\Omega_1\rangle|<3,\left|\sum_{n=1}^{T}\langle\tilde{X}_n(1-J_n),\Omega_1\rangle-T(k)\rho(k)\right|<4x_0\right\}\geq 1/4.$$

Using the relation between the distributions of (3.86) and (3.87) we obtain

$$\frac{1}{4} < P\left\{\left|\sum_{n=1}^{T}\langle \tilde{X}_n J_n, \Omega_1\rangle + T(k)\rho(k)\right| < 4x_0 + 3\right\}$$

$$< P\left\{\sum_{1}^{T} J_n > L_0\right\}$$

$$+ \sum_{l=0}^{L_0} P\left\{\sum_{1}^{T} J_n = l\right\} P\left\{\left|\sum_{1}^{l}\langle \tilde{\alpha}_i, \Omega_1\rangle + T(k)\rho(k)\right| < 4x_0 + 3\right\}$$

$$< \frac{\eta}{8} + \max_{l < L_0} P\left\{\left|\sum_{1}^{l}\langle \tilde{\alpha}_i, \Omega_1\rangle + T(k)\rho(k)\right| < 4x_0 + 3\right\}$$

(compare the estimates for (3.85)). Thus, for some $0 < l_0 < L_0$,

$$P\left\{\left|\sum_{1}^{l_0}\langle \tilde{\alpha}_i, \Omega_1\rangle + T(k)\rho(k)\right| < 4x_0 + 3\right\} > \frac{1}{8}. \tag{3.91}$$

Similarly we get from (3.88) and (3.90),

$$\frac{5}{8}\eta < P\left\{\left|\sum_{n=1}^{T}\langle \tilde{X}_n(1 - J_n), \Omega_2\rangle\right| > \frac{1}{4}\Theta(k)M(\omega_0, k),\right.$$
$$\left.\left|\sum_{n=1}^{T}\langle \tilde{X}_n(1 - J_n), \Omega_1\rangle - T(k)\rho(k)\right| < 4x_0\right\}$$

$$< P\left\{\sum_{1}^{T} J_n > L_0\right\}$$

$$+ \max_{l < L_0} P\left\{\left|\sum_{n=1}^{T-l}\langle \tilde{\beta}_n, \Omega_2\rangle\right| > \frac{1}{4}\Theta(k)M(\omega_0, k),\right.$$
$$\left.\left|\sum_{n=1}^{T-l}\langle \tilde{\beta}_n, \Omega_1\rangle - T(k)\rho(k)\right| < 4x_0\right\},$$

where we used the same notation as in (2.47) and (2.48). Again there exists a $0 < l_1 < L_0$ such that

$$P\left\{\left|\sum_{n=1}^{T-l_1}\langle \tilde{\beta}_n, \Omega_2\rangle\right| > \frac{1}{4}\Theta(k)M(\omega_0, k),\right.$$
$$\left.\left|\sum_{n=1}^{T-l_1}\langle \tilde{\beta}_n, \Omega_1\rangle - T(k)\rho(k)\right| < 4x_0\right\} > \frac{\eta}{2}. \tag{3.92}$$

But, by (2.29) and (2.53) we can choose $k_7(F) < \infty$ such that for $k \geqslant k_7$,

$$P\left\{\left|\sum_{T-l_1+1}^{T} \langle \tilde{\beta}_n, \Omega_2 \rangle\right| \geqslant \frac{1}{8}\Theta(k)M(\omega_0,k)\right\}$$

$$= P\left\{\left|\sum_{j=1}^{l_1} \langle \beta_j, \omega_0 \rangle\right| \geqslant \frac{1}{8}M(\omega_0,k)A_k\right\}$$

$$\leqslant l_1 P\left\{|X_1| \geqslant (8l_1)^{-1}M(\omega_0,k)A_k | J_1 = 0\right\}$$

$$\leqslant l_1 (P\{J_1 = 0\})^{-1} P\left\{|X_1| > (8l_1)^{-1}M(\omega_0,k)A_k\right\} \leqslant \eta/8.$$

A similar estimate gives for $k \geqslant k_7$,

$$P\left\{\left|\sum_{T-l_1+1}^{T} \langle \tilde{\beta}_n, \Omega_1 \rangle\right| \geqslant 4x_0\right\} \leqslant \frac{\eta}{8},$$

and we therefore conclude from (3.92) that

$$P\left\{\left|\sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_2 \rangle\right| \geqslant \frac{1}{8}\Theta(k)M(\omega_0,k),\right.$$

$$\left.\left|\sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_1 \rangle - T(k)\rho(k)\right| \leqslant 8x_0\right\} \geqslant \frac{\eta}{4}. \tag{3.93}$$

However, with α_i, β_j and Λ independent, Y_i has the distribution of $\Sigma_{n=1}^{T}\beta_n + \Sigma_{i=1}^{\Lambda}\alpha_i$ (see (2.46)–(2.50)); (3.91) and (3.93) together therefore give

$$P\left\{|\langle \tilde{Y}_1, \Omega_2 \rangle| \geqslant \tfrac{1}{16}\Theta(k)M(\omega_0,k), |\langle \tilde{Y}_1, \Omega_1 \rangle| \leqslant 12x_0 + 3\right\}$$

$$\geqslant P\{\Lambda = l_0\} P\left\{\left|\sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_2 \rangle\right| \geqslant \frac{1}{8}\Theta(k)M(\omega_0,k),\right.$$

$$\left.\left|\left\langle \sum_{n=1}^{T} \tilde{\beta}_n, \Omega_1 \right\rangle - T(k)\rho(k)\right| \leqslant 8x_0\right\}$$

$$\cdot P\left\{\left|\sum_{i=1}^{l_0} \langle \tilde{\alpha}_i, \Omega_2 \rangle\right| \leqslant \frac{1}{16}\Theta(k)M(\omega_0,k),\right.$$

$$\left.\left|\sum_{i=1}^{l_0} \langle \tilde{\alpha}_i, \Omega_1 \rangle + T(k)\rho(k)\right| \leqslant 4x_0 + 3\right\}$$

$$\geqslant P\{\Lambda = l_0\} \cdot \frac{\eta}{4}\left\{\frac{1}{8} - l_0 P\left\{|\langle \tilde{\alpha}_1, \Omega_2 \rangle| > K_{20}\Theta(k)M(\omega_0,k)\right\}\right\}$$

$$\geqslant \tfrac{1}{64}\eta P\{\Lambda = l_0\} \quad \text{(see (3.70) and (3.73)).} \tag{3.94}$$

Moreover, by (2.50) and (3.34) for any l,

$$P\{\Lambda = l\} \geq (l!)^{-1}(TP\{J_1 = 1\})^l(1 - 2K_0T^{-1})^T$$
$$\geq (l!)^{-1}(2^{-5}\eta)^l \exp - 4K_0,$$

as soon as $K_0T(k)^{-1} \leq \frac{1}{4}$, i.e., for $k \geq k_7(F)$ if $k_7(F)$ is chosen large enough (see (2.29)). Thus, in terms of $\overline{Y}_1$, (3.94) yields for $k \geq k_7$ and some $K_{23} > 0$, independent of F and k,

$$P\left\{\nu_1 = T + l_0, \left|\left\langle \overline{Y}_1,\Omega_2\right\rangle\right| \geq \tfrac{1}{16},\right.$$
$$\left.\left|\left\langle \overline{Y}_1,\Omega_1\right\rangle\right| \leq 16(12x_0 + 3)\left|\left\langle \overline{Y}_1,\Omega_2\right\rangle\right|\right\} \geq K_{23}.$$

We can therefore find a number $x_1 = x_1(F,k)$, $|x_1| \leq 16(12x_0 + 3)$ such that

$$P\left\{\nu_1 = T + l_0, \left\langle \overline{Y}_1,\Omega_2\right\rangle \geq \tfrac{1}{16}, \left\langle \overline{Y}_1,\Omega_1\right\rangle \leq x_1\left\langle \overline{Y}_1,\Omega_2\right\rangle\right\} \geq \tfrac{1}{4}K_{23}, \quad (3.95)$$

as well as

$$P\left\{\nu_1 = T + l_0, \left\langle \overline{Y}_1,\Omega_2\right\rangle \geq \tfrac{1}{16},\right.$$
$$\left.\left\langle \overline{Y}_1,\Omega_1\right\rangle \geq x_1 \left\langle \overline{Y}_1,\Omega_2\right\rangle\right\} \geq \tfrac{1}{4}K_{23}, \quad (3.96)$$

or both of these inequalities hold with $\langle \overline{Y}_1,\Omega_2\rangle$ replaced by $-\langle \overline{Y}_1,\Omega_2\rangle$. By changing the sign of Ω_2, if necessary, we may restrict ourselves to the case where (3.95) and (3.96) hold. Lastly, we observe that (3.64), (3.70), the inequality $|x_1|K_{20} \leq 16(12x_0 + 3)K_{20} \leq \frac{1}{4}$, (3.73) and the fact that (3.71) fails imply

$$P\left\{\left|\left\langle \bar{\alpha}_{l_0+1}, \Omega_2\right\rangle\right| \leq K_{20}, \left|\left\langle \bar{\alpha}_{l_0+1}, \Omega_1\right\rangle\right| \geq \left|x_1\left\langle \bar{\alpha}_{l_0+1}, \Omega_2\right\rangle\right| + \tfrac{1}{4}\right\}$$
$$\geq P\left\{\left|\left\langle \tilde{\alpha}_{l_0+1},\Omega_2\right\rangle\right| \leq \min\left\{\tfrac{1}{2}, K_{20}\Theta(k)M(\omega_0,k)\right\}, \left|\left\langle \tilde{\alpha}_{l_0+1},\Omega_1\right\rangle\right| \geq \tfrac{1}{2}\right\}$$
$$\geq P\left\{\left|\left\langle \tilde{\alpha}_{l_0+1},\Omega_2\right\rangle\right| \leq \min\left\{\tfrac{1}{2}, K_{20}\Theta(k)M(\omega_0,k)\right\}, |\tilde{\alpha}_{l_0+1}| > 1\right\}$$
$$\geq 1 - \varepsilon_2 - 2\varepsilon_2 \geq \tfrac{1}{2}.$$

Let us assume that

$$P\left\{\left|\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle\right| \leq K_{20}, \left\langle \bar{\alpha}_{l_0+1},\Omega_1\right\rangle \geq \left|x_1\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle\right|\right.$$
$$\left.+ \tfrac{1}{4} \geq x_1\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle + \tfrac{1}{4}\right\} \geq \tfrac{1}{4}.$$

(If this fails then we have

$$P\left\{\left|\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle\right| \leq K_{20}, \left\langle \bar{\alpha}_{l_0+1},\Omega_1\right\rangle \leq -\left|x_1\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle\right|\right.$$
$$\left.- \tfrac{1}{4} \leq x_1\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle - \tfrac{1}{4}\right\} \geq \tfrac{1}{4};$$

this can be treated analogously.) Then we have from (2.50), (3.96) and (3.34),

$$P\left\{\left\langle \bar{Y}_1,\Omega_2\right\rangle \geq \tfrac{1}{16} - K_{20} \geq \tfrac{1}{32}, \left\langle \bar{Y}_1,\Omega_1\right\rangle \geq x_1\left\langle \bar{Y}_1,\Omega_2\right\rangle + \tfrac{1}{4}\right\}$$

$$\geq P\{\Lambda = l_0 + 1\}$$

$$\cdot P\left\{\left\langle \sum_{1\leq i\leq l_0} \bar{\alpha}_i + \sum_{n=1}^{T} \bar{\beta}_n,\Omega_2\right\rangle \geq \tfrac{1}{16},\right.$$

$$\left\langle \sum_{1\leq i\leq l_0} \bar{\alpha}_i + \sum_{n=1}^{T} \bar{\beta}_n,\Omega_1\right\rangle \geq x_1\left\langle \sum_{1\leq i\leq l_0} \bar{\alpha}_i + \sum_{n=1}^{T} \beta_n,\Omega_2\right\rangle,$$

$$\left.\left|\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle\right| \leq K_{20}, \left\langle \bar{\alpha}_{l_0+1},\Omega_1\right\rangle \geq x_1\left\langle \bar{\alpha}_{l_0+1},\Omega_2\right\rangle + \tfrac{1}{4}\right\}$$

$$\geq \tfrac{1}{4}\,\frac{P\{\Lambda = l_0 + 1\}}{P\{\Lambda = l_0\}}\,P\left\{\nu_1 = T + l_0, \left\langle \bar{Y}_1,\Omega_2\right\rangle \geq \tfrac{1}{16},\right.$$

$$\left.\left\langle \bar{Y}_1,\Omega_1\right\rangle \geq x_1\left\langle \bar{Y}_1,\Omega_2\right\rangle\right\}$$

$$\geq \tfrac{1}{16} K_{23} T(k) P\{J_1 = 1\}(l_0 + 1)^{-1} \geq 2^{-9}\eta K_{23}(L_0 + 1)^{-1} = K_{24}, \qquad (3.97)$$

say, with $K_{24} > 0$ independent of F and k. We can now complete the proof by a final application of Corollary 2 as before. (3.95) and (3.97) again are versions of (3.8) (resp. (3.9)) with $Z_1(1) = \langle \bar{Y}_1,\Omega_1\rangle$, $Z_1(2) = \langle \bar{Y}_1,\Omega_2\rangle$, $d_1 = 1/32$, $d_2 = x_1$, $d_3 = \frac{1}{4}$, $p_3 = \frac{1}{4}K_{23}$, $p_4 = K_{24}$. Thus, by (3.10) we have for some $K_{25} < \infty$,

$$\sum_{r=s}^{2s} P\left\{\left|\sum_1^r \bar{Y}_n\right| \leq 12\right\} \leq K_{25}, \qquad s \geq 1. \qquad (3.98)$$

As in (3.81)–(3.83) the left-hand side of (3.98) is, however, an upper bound for the left-hand side of (3.1). Thus (3.1) holds also in this last case. □

Lemmas 4 and 5 prove (3.1) with $k_5 = \max(k_6,k_7)$, $K_2 = \max(K_{14}, K_{19})$. As observed in the beginning of this section, this proves Proposition 1 and our theorem.

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