TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 240, June 1978

STRONG DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

BY

C. J. NEUGEBAUER

ABSTRACT. Let F be a differentiation basis in R^n, i.e., a family of measurable sets S contracting to 0 such that $\|M_F f\|_p \leqslant A_p \|f\|_p$, where M_F is the Hardy-Littlewood maximal operator. For $f \in \Lambda_\alpha^{pq}$, we let $E_F(f)$ be the complement of the Lebesgue set of f relative to F, and we show that E_F has L_α^{pq}-capacity 0, where L_α^{pq} is a capacity associated with Λ_α^{pq} in much the same way as the Bessel capacity $B_{\alpha p}$ is associated with L_α^p.

1. With the Bessel potential space $L_\alpha^p(R^n)$ there is associated the Bessel capacity $B_{\alpha p}$ which is defined for $E \subset R^n$ by

$$B_{\alpha p}(E) = \inf\{\|g\|_p^p: g \in L^p, g \geqslant 0, G_\alpha * g \geqslant 1 \text{ on } E\},$$

where G_α is the Bessel kernel given by $\widehat{G_\alpha}(x) = (1 + 4\pi^2|x|^2)^{-\alpha/2}$ [5, p. 132]. The capacity $B_{\alpha p}$ is an outer measure on R^n and its relation to H^r, Hausdorff measure of dimension r, is given by the following [4]. If $p > 1$, $\alpha p < n$, then $H^{n-\alpha p}(E) = 0$ implies $B_{\alpha p}(E) = 0$, and $B_{\alpha p}(E) = 0$ implies $H^{n-\alpha p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$.

Let F be a family of measurable sets $S \subset R^n$ with $0 < |S| < \infty$, and let $S \to 0$ stand for the generic notation for the limit as $j \to \infty$ of any sequence $\{S_j\} \subset F$ such that, for any $\varepsilon > 0$, $S_j \subset \{|x| < \varepsilon\}$, $j \geqslant j(\varepsilon)$. With such a family we associate the Hardy-Littlewood maximal operator

$$M_F f(x) = \sup\left\{ \frac{1}{|S|} \int_{S+x} |f(y)|\, dy: S \in F \right\}.$$

The behavior of $M_F f$ is decisive in the study of the differentiability of the integral. If, for example, $\|M_F f\|_p \leqslant A_p \|f\|_p$ (there are many interesting families F with this property [3]), then the set

$$E_F(f) = \left\{ x: \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|\, dy \not\to 0 \text{ as } S \to 0 \right\}$$

has measure 0 for $f \in L^p$. If $f \in L_\alpha^p$, then as shown in [2] the exceptional set $E_F(f)$ has even $B_{\alpha p}$-capacity 0.

The purpose of this paper is to prove an analogous result for the Lipschitz spaces $\Lambda_\alpha^{pq}(R^n)$ (see [5], [6]), using instead of $B_{\alpha p}$ a "Lipschitz" capacity L_α^{pq}.

Received by the editors February 20, 1976.

AMS (MOS) subject classifications (1970). Primary 26A54, 26A16; Secondary 46E35.

Key words and phrases. Lipschitz spaces, Lipschitz capacity, differentiation.

In view of the inclusion relations

$$L_\alpha^p \subset \Lambda_\alpha^{pr} \subset \Lambda_\alpha^{pq} \subset L_\beta^p, \quad p > 1,$$

$r = \max(p, 2)$, $q \geq r$, $\beta < \alpha$ [6, pp. 441, 452] our differentiability result lies then intermediate to those of [2].

We recall that, if F is the family of all oriented rectangles, a nonregular family, then $\|M_F f\|_p \leq A_p \|f\|_p$, $1 < p \leq \infty$ [5]. For regular families F, i.e., those for which there is $c > 0$ such that every $E \in F$ lies in a sphere S with $|S| < c|E|$, the size of $E_F(f)$ for $f \in \Lambda_\alpha^{pq}$ can be studied with the techniques developed in [8], especially the mean value property $\int_{|y|<r} G_\alpha(x-y)\,dy \leq cG_\alpha(x)r^n$. One sees easily that this tool is no longer available for nonregular families, and its substitute $\|M_F f\|_p \leq A_p\|f\|_p$, which is not a condition on the kernel G_α, requires one to study the interplay between Λ_α^{pq} and M_F.

2. Let F be a family as in §1, and let $tF = \{tS\colon S \in F\}$, $0 < t \leq 1$, where $tS = \{ts\colon s \in S\}$. Let

$$M_t f(x) = \sup\left\{ \frac{1}{|tS|} \int_{tS+x} |f(y)|\,dy\colon S \in F \right\}.$$

LEMMA 1. *If* $\|M_F f\|_p \leq A_p\|f\|_p$, $f \in L^p$, *then* $\|M_t f\|_p \leq A_p\|f\|_p$, $0 < t \leq 1$.

PROOF. $|tS|^{-1}\int_{tS+x}|f(u)|\,du = |s|^{-1}\int_{S+x/t}|f(tv)|\,dv \leq M_F(\delta_t f)(x/t)$, where $(\delta_t f)(x) = f(tx)$. Hence

$$\begin{aligned}\|M_t f\|_p &\leq \left\{ \int_{R^n} M_F(\delta_t f)\left(\frac{x}{t}\right)^p dx \right\}^{1/p} \\ &= \left\{ t^n \int_{R^n} M_F(\delta_t f)(u)^p\,du \right\}^{1/p} \leq t^{n/p} A_p \left\{ \int_{R^n} |(\delta_t f)(x)|^p\,dx \right\}^{1/p} \\ &= A_p\|f\|_p.\end{aligned}$$

We note that A_p is independent of t.

For $0 < \alpha \leq 1$, the Lipschitz space $\Lambda_\alpha^{pq}(R^n)$ consists of all functions $f \in L^p(R^n)$ for which the norm

$$\text{(i)} \qquad \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) + f(x-t) - 2f(x)\|_p^q}{|t|^{n+\alpha q}}\,dt \right\}^{1/q} = \|f\|_{\Lambda_\alpha^{pq}}$$

is finite [5, p. 151]; here $1 \leq p, q \leq \infty$. If $0 < \alpha < 1$, the above norm is equivalent to

$$\text{(ii)} \qquad \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) - f(x)\|_p^q}{|t|^{n+\alpha q}}\,dt \right\}^{1/q}$$

[5, p. 153]. If $\alpha > 1$, Λ_α^{pq} is the collection of $f \in L^p(R^n)$ for which the norm $\|f\|_p + \sum_{j=1}^n \|\partial f/\partial x_j\|_{\Lambda_{\alpha-1}^{pq}} < \infty$ [5, p. 153], where $\partial f/\partial x_j$ is taken in the sense of distribution.

LEMMA 2. *Let* $0 < \alpha < 1$ *and* $\|M_F f\|_p \leq A_p \|f\|_p, f \in L^p$. *Then* $\|M_t f\|_{\Lambda_\alpha^{pq}} \leq A_p \|f\|_{\Lambda_\alpha^{pq}}$.

PROOF. By Lemma 1 we only need to show that $\|M_F f\|_{\Lambda_\alpha^{pq}} \leq A_p \|f\|_{\Lambda_\alpha^{pq}}$. If $f_t(x) = f(x + t)$, one easily verifies that $|M_F f(x + t) - M_F f(x)| \leq M_F(f_t - f)(x)$, and the result follows from (ii).

3. If G_α is the Bessel potential of order $\alpha > 0$, and if $J_\alpha f = G_\alpha * f$, then J_α: $\Lambda_\beta^{pq} \to \Lambda_{\alpha+\beta}^{pq}$ is an isomorphism and the norms $\|f\|_{\Lambda_\beta^{pq}}$, $\|J_\alpha f\|_{\Lambda_{\alpha+\beta}^{pq}}$ are equivalent [5, Chapter 5]. This result will be used frequently in the sequel.

For $0 < \alpha < \infty$, $1 \leq p, q \leq \infty$, we define, for $E \subset R^n$,

$$L_\alpha^{pq}(E) = \inf\{\|g\|_{\Lambda_\alpha^{pq}} : g \geq 0 \text{ and } g \geq 1 \text{ on } E\}.$$

For $0 < \gamma < \alpha$ we define, for $E \subset R^n$,

$$B_\gamma^{pq}(E) = \inf\{\|g\|_{\Lambda_\gamma^{pq}} : g \geq 0 \text{ and } G_{\alpha-\gamma} * g \geq 1 \text{ on } E\}.$$

It is easily verified that L_α^{pq}, B_γ^{pq} are capacities, i.e., they are monotone, countably subadditive, and assign 0 to $E = \varnothing$ (see [4, p. 251]).

The usefulness of B_γ^{pq} is exhibited in the following lemma.

LEMMA 3. $L_\alpha^{pq}(E) = 0$ *if and only if there exists* $0 < \gamma < \alpha$ *such that* $B_\gamma^{pq}(E) = 0$.

PROOF. ($\rightarrow$) Let $0 < \gamma < \alpha$ and let $\varepsilon > 0$. Choose $g \in \Lambda_\alpha^{pq}$ so that $g \geq 0$, $g \geq 1$ on E, and $\|g\|_{\Lambda_\alpha^{pq}} \leq \varepsilon$. If $g = G_{\alpha-\gamma} * \psi$, $\psi \in \Lambda_\gamma^{pq}$, then $\||\psi|\|_{\Lambda_\gamma^{pq}} < K\varepsilon$, and $B_\gamma^{pq}(E) = 0$.

($\leftarrow$) If $B_\gamma^{pq}(E) = 0$, then there is $g \geq 0$ in Λ_γ^{pq} with $G_{\alpha-\gamma} * g \geq 1$ on E and $\|g\|_{\Lambda_\gamma^{pq}} \leq \varepsilon$. As before $\|G_{\alpha-\gamma} * g\|_{\Lambda_\alpha^{pq}} < K\varepsilon$, and $L_\alpha^{pq}(E) = 0$.

LEMMA 4. *Let* $1 < p \leq \infty$. *The relation between the Bessel capacity* $B_{\alpha p}$ *and the Lipschitz capacity* L_α^{pq} *is given by*

(i) $B_{\alpha p}(E) = 0$ *implies* $L_\alpha^{pq}(E) = 0$ $q \geq \max(p, 2)$.

(ii) $L_\alpha^{pq}(E) = 0$ *implies* $B_{\gamma p}(E) = 0, 0 < \gamma < \alpha$.

PROOF. (i) We use Lemma 3 and verify that $B_\gamma^{pq}(E) = 0$ for $0 < \gamma < \alpha$. Since $B_{\alpha p}(E) = 0$, there is $g \geq 0$ in L^p such that $\|g\|_p \leq \varepsilon$ and $G_\alpha * g(x) \geq 1$, $x \in E$. If $\psi = G_\gamma * g$, then $\|\psi\|_{\Lambda_\gamma^p} \leq M\|g\|_p \leq M \cdot \varepsilon$ [6, p. 452].

(ii) If $0 < \gamma < \alpha$, we have $B_{\alpha-\gamma}^{pq}(E) = 0$, and hence there is $g \geq 0$ with $\|g\|_{\Lambda_{\alpha-\gamma}^{pq}} \leq \varepsilon$ and $G_\gamma * g(x) \geq 1$, $x \in E$. Since $\|g\|_p \leq \varepsilon$, we get $B_{\gamma p}(E) = 0$.

COROLLARY. *If* $L_\alpha^{pq}(E) = 0$ *and* $\alpha p > 1$, *then* $H^{n-1}(E) = 0$.

PROOF. Let $0 < \gamma < \alpha$ with $\gamma p > 1$. Since $B_{\gamma p}(E) = 0$, we see from [4, Theorem 22] that $H^{n-1}(E) = 0$.

The next lemma shows that $f \in \Lambda_\alpha^{pq}$ can be defined absolutely modulo sets of L_α^{pq}-capacity 0.

LEMMA 5. *Let $f \in \Lambda_\alpha^{pq}$, $0 < \gamma < \alpha$, and $f = G_{\alpha-\gamma} * \psi$, $\psi \in \Lambda_\gamma^{pq}$. Then $G_{\alpha-\gamma} * |\psi|(x) < \infty$ for L_α^{pq}-a.e. x.*

PROOF. Let $E = \{x: G_{\alpha-\gamma} * |\psi|(x) = \infty\}$. Then

$$B_\gamma^{pq}(E) \leq B_\gamma^{pq}\{x: G_{\alpha-\gamma} * |\psi|(x) \geq k\}$$
$$= B_\gamma^{pq}\left\{x: G_{\alpha-\gamma} * \frac{|\psi|}{k}(x) \geq 1\right\} \leq \frac{1}{k}\|\psi\|_{\Lambda_\gamma^{pq}} \to 0 \quad \text{as } k \to \infty.$$

The result now follows from Lemma 3.

LEMMA 6. *Let $\psi_j \to f(\Lambda_\alpha^{pq})$. Then there exists a subsequence $\{j_i\}$ such that $\psi_{j_i} \to f$ for L_α^{pq}-a.e. x.*

PROOF. The proof is standard and we give it for the sake of completeness. Let $0 < \gamma < \alpha, f = G_{\alpha-\gamma} * g, \psi_j = G_{\alpha-\gamma} * \phi_j, g, \phi_j \in \Lambda_\gamma^{pq}$. Then

$$B_\gamma^{pq}\{x: |\psi_j - f|(x) \geq \varepsilon\} \leq B_\gamma^{pq}\{x: G_{\alpha-\gamma} * (|\phi_j - g|/\varepsilon)(x) \geq 1\}$$
$$\leq \|\phi_j - g\|_{\Lambda_\gamma^{pq}}/\varepsilon \to 0 \quad \text{as } j \to \infty.$$

If we select now $\{j_i\}$ such that for $A_i = \{x: |\psi_{j_i} - f|(x) \geq 1/2^i\}$ we have $B_\gamma^{pq}(A_i) \leq 1/2^i$, then on $A = \bigcap_{k \geq 1} \bigcup_{i \geq k} A_i$, $\psi_{j_i} \to f$ and $B_\gamma^{pq}(A) = 0$.

LEMMA 7. *If $\|M_F f\|_p \leq A_p\|f\|_p$ and $0 < \alpha < 1$, then $\|\int_0^1 M_t g(x)\,dt\|_{\Lambda_\alpha^{pq}} \leq A_p\|g\|_{\Lambda_\alpha^{pq}}$ where M_F and M_t are the maximal operators associated with F and tF.*

PROOF. Let $\psi(x) = \int_0^1 M_t g(x)\,dt$. Then $\|\psi\|_p \leq A_p\|g\|_p$ (Lemma 1), and

$$\|\psi(x+\tau) - \psi(x)\|_p \leq \int_0^1 \|M_t(g_\tau - g)\|_p\,dt$$
$$\leq A_p\|g(x+\tau) - g(x)\|_p,$$

where $g_\tau(x) = g(x+\tau)$.

From Lemma 5 we deduce that for $g \in \Lambda_\alpha^{pq}$, $M_t g(x) \in L^1(dt, [0, 1])$ for $L_\alpha^{pq} =$ a.e. x.

4. We are now ready to state and prove our differentiability results. We let F be a family of sets as in §1, and we let $tF = \{tS: S \in F\}, 0 < t \leq 1$.

THEOREM 1. *Assume that for $f \in L^p(R^n)$, $\|M_F f\|_p \leq A_p\|f\|_p$. For $f \in \Lambda_\alpha^{pq}(R^n)$, let*

$$E_t(f) = \left\{ x:\ \limsup_{S \to 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)|\, dy > 0 \right\}.$$

Then $L_\alpha^{pq}(E_t) = 0$.

PROOF. We have to verify that, for $\sigma > 0$, $L_\alpha^{pq}(E_\sigma) = 0$, where

$$E_\sigma = \left\{ x \in E_t:\ \limsup_{S \to 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)|\, dy > \sigma \right\}.$$

Let $0 < \beta < \min(1, \alpha)$. By Lemma 3 we need to show that $B_\beta^{pq}(E) = 0$. Let $\eta > 0$ be given.

Let $C_0^\infty(R^n)$ be the space of infinitely differentiable functions with compact support. By [6, p. 444] there is a sequence $\{\psi_j\} \subset C_0^\infty$ such that $\psi_j \to f(\Lambda_\alpha^{pq})$. Choose now $\psi_j^\beta, f^\beta \in \Lambda_\beta^{pq}$ with $G_{\alpha-\beta} * \psi_j^\beta = \psi_j$ and $G_{\alpha-\beta} * f^\beta = f$. Then $\psi_j^\beta \to f^\beta(\Lambda_\beta^{pq})$, and from this we get that

$$B_\beta^{pq}\{x: |\psi - f|(x) \geq \delta\} \leq \|\psi^\beta - f^\beta\|_{\Lambda_\beta^{pq}}/\delta = o(1) \quad \text{as } j \to \infty.$$

If $\Sigma = tS$, we have

$$\begin{aligned} \frac{1}{|\Sigma|} \int_{\Sigma+x} |f(y) - f(x)|\, dy &\leq \frac{1}{|\Sigma|} \int_{\Sigma+x} |f(y) - \psi_j(y)|\, dy \\ &\quad + \frac{1}{|\Sigma|} \int_{\Sigma+x} |\psi_j(y) - \psi_j(x)|\, dy \\ &\quad + |\psi_j(x) - f(x)| \\ &= A_{ij}(x) + A_{2j}(x) + A_{3j}(x). \end{aligned}$$

(i) Since $\{x:\ M_t(f - \psi_j)(x) \geq \sigma/3\} \subset \{x:\ G_{\alpha-\beta} * M_t(f^\beta - \psi_j^\beta)(x) > \sigma/3\}$, we see that

$$\begin{aligned} B_\beta^{pq}\{x: M_t(f - \psi_j)(x) \geq \sigma/3\} &\leq (3/\sigma)\left\| M_t\left(f^\beta - \psi_j^\beta\right)\right\|_{\Lambda_\beta^{pq}} \\ &\leq (3/\sigma) A_p \left\| f^\beta - \psi_j^\beta \right\|_{\Lambda_\beta^{pq}} \leq \eta/2, \quad \text{if } j \geq j_1(\eta). \end{aligned}$$

The next to the last inequality follows from Lemma 2 (applicable since $\beta < 1$). Since $A_{1j}(x) \leq M_t(f - \psi_j)(x)$, there is a set E_{1j} such that $B_\beta^{pq}(E_{1j}) \leq \eta/3$, $j \geq j_1(\eta)$, and $x \notin E_{1j}$ implies $A_{1j}(x) < \sigma/3$.

(ii) Since $B_\beta^{pq}\{x: A_{3j}(x) \geq \sigma/3\} \leq (3/\sigma)\|\psi_j^\beta - f^\beta\|_{\Lambda_\beta^{pq}}$, there is $j_3(\eta)$ such that for $j \geq j_3(\eta)$, $A_{3j}(x) < \sigma/3$ except for $x \in E_{3j}$ with $B_\beta^{pq}(E_{3j}) \leq \eta/2$.

(iii) Let now $j_0 \geq \max[j_1(\eta), j_3(\eta)]$. Then there is $\tau(\sigma)$ such that $S \subset B(0, \tau)$ implies $|\psi_{j_0}(y) - \psi_{j_0}(x)| \leq \sigma/3$, $y \in tS + x$. If then $x \notin E_{1j_0} \cup E_{3j_0}$ and $S \subset B(0, \tau)$, we get $|\Sigma|^{-1}\int_{\Sigma+x}|f(y) - f(x)|\, dy \leq \sigma$, $\Sigma = tS$, and hence $E_\sigma \subset E_{1j_0} \cup E_{3j_0}$. Then $B_\beta^{pq}(E_\sigma) < \eta$, and the proof is complete.

THEOREM 2. *Under the same hypothesis as for Theorem* 1, *we have for* L_α^{pq}-*a.e.* x,

(i) $$\lim_{S\to 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)|\, dy = 0, \ \ a.e.\ t,\ 0 < t < 1.$$

(ii) $$\lim_{S\to 0} \int_0^1 \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)|\, dy\, dt = 0.$$

PROOF. (i) Let

$$E = \left\{ (t, x) \colon \limsup_{S\to 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)|\, dy > 0 \right\},$$

and let $E_x = \{t\colon (t, x) \in E\}$. We must show that $|E_x| = 0$ for L^{pq}_α-a.e. x.

Let $A_\sigma = \{x\colon |E_x| > \sigma\}$. Let $0 < \beta < \min(1, \alpha)$ and choose $\{\psi_j\} \subset C_0^\infty$ so that $\psi_j \to f(\Lambda^{pq}_\alpha)$. By Lemma 6 we may assume that $\psi_j(x) \to f(x)$ for L^{pq}_α-a.e. x. Let E_1 be the exceptional set, i.e., $L^{pq}_\alpha(E_1) = 0$ and $x \notin E_1$ implies $\psi_j(x) \to f(x)$.

We need another exceptional set E_2. Since $M_t(f - \psi_j)(x) \leq G_{\alpha-\beta} * M_t(f^\beta - \psi_j^\beta)(x)$ (notation as in the proof of Theorem 1), and $\psi_j^\beta \to f^\beta(\Lambda^{pq}_\beta)$, we see that

$$\int_0^1 M_t(f - \psi_j)(x)\, dt \leq G_{\alpha-\beta} * \int_0^1 M_t\left(f^\beta - \psi_j^\beta\right)(x)\, dt \equiv \Psi_j(x).$$

By Lemma 7, $\int_0^1 M_t(f^\beta - \psi_j^\beta)(x)\, dt \in \Lambda^{pq}_\beta$, and hence $\Psi_j \in \Lambda^{pq}_\alpha$.

Next

$$\left\{ x\colon \int_0^1 M_t(f - \psi_j)(x)\, dt > \varepsilon \right\} \subset \left\{ x\colon G_{\alpha-\beta} * \frac{1}{\varepsilon} \int_0^1 M_t\left(f^\beta - \psi_j^\beta\right)(x)\, dt > 1 \right\},$$

and hence

$$B^{pq}_\beta \left\{ x\colon \int_0^1 M_t(f - \psi_j)(x)\, dt > \varepsilon \right\} \leq \frac{1}{\varepsilon} \left\| \int_0^1 M_t\left(f^\beta - \psi_j^\beta\right)(x)\, dt \right\|_{\Lambda^{pq}_\beta}$$

$$\leq \frac{A_p}{\varepsilon} \left\| f^\beta - \psi_j^\beta \right\|_{\Lambda^{pq}_\beta} = o(1) \quad \text{as } j \to \infty.$$

We can now choose a subsequence $\{j_i\}$ so that $\int_0^1 M_t(f - \psi_{j_i})(x)\, dt \to 0$ for L^{pq}_α-a.e. x (see e.g. the proof of Lemma 6). We may assume that $\{j_i\} = \{j\}$. Let

$$E_2 = \left\{ x\colon \int_0^1 M_t(f - \psi_j)(x)\, dt \nrightarrow 0 \right\}.$$

Then $L^{pq}_\alpha(E_2) = 0$.

We return now to the set A_σ introduced at the beginning of the proof, and we claim that, for every $\sigma > 0$, $A_\sigma \subset E_1 \cup E_2$.

If we deny this, then there is $x_0 \in A_\sigma$ and $x_0 \notin E_1 \cup E_2$. Then $|E_{x_0}| > \sigma$. If

$$E_{x_0\lambda} = \left\{ t\colon \limsup_{S\to 0} \frac{1}{|tS|} \int_{tS+x_0} |f(y) - f(x_0)|\, dy > \lambda \right\},$$

then there is $\lambda > 0$ with $|E_{x_0\lambda}| > \sigma$. We choose now j_0 so large that

(i) $\int_0^1 M_t(f - \psi_{j_0})(x_0)\, dt < \sigma\lambda/3$,

(ii) $|\psi_{j_0}(x_0) - f(x_0)| < \lambda/3$,

and we choose $\tau_0 > 0$ such that $S \subset B(0, \tau_0)$ implies

(iii) $|\psi_{j_0}(y) - \psi_{j_0}(x_0)| < \lambda/3, y \in tS + x_0, 0 < t \leq 1$.

Since for $t \in E_{x_0\lambda}$,

$$\lambda \leq M_t(f - \psi_{j_0})(x_0) + \frac{2\lambda}{3}, \quad S \subset B(0, \tau_0),$$

we see that $E_{x_0\lambda} \subset \{t\colon\ M_t(f - \psi_{j_0})(x_0) > \lambda/3\}$. From this $|E_{x_0\lambda}| \leq (3/\lambda)\int_0^1 M_t(f - \psi_{j_0})(x_0)\, dt < \sigma$, a contradiction.

To prove (ii), we first observe that $\int_0^1 M_t f^\beta(x)\, dt \in \Lambda_\beta^{pq}$ (Lemma 7), and hence $G_{\alpha-\beta} * \int_0^1 M_t f^\beta(x)\, dt < \infty$ for L_α^{pq}-a.e. x (Lemma 5). Since $M_t f(x) \leq G_{\alpha-\beta} * M_t f^\beta(x)$, we see that $M_t f(x) \in L^1(dt, [0, 1])$ for L_α^{pq}-a.e. x. Finally,

$$|f(x)| + M_t f(x) \geq \frac{1}{|tS|}\int_{tS+x}|f(y) - f(x)|\, dt \to 0 \quad \text{as } S \to 0,$$

and we only need to apply the Lebesgue dominated convergence theorem to obtain (ii).

COROLLARY. *Under the same hypothesis,*

$$\lim_{S\to 0}\int_0^1 \cdots \int_0^1 \frac{1}{|t_1 t_2 \cdots t_k S|}\int_{t_1 t_2 \cdots t_k S + x}|f(y) - f(x)|\, dy\, dt_1 \cdots dt_k = 0$$

for L_α^{pq}-a.e. x.

5. In this section we will study higher order differentiability. Let F be a differentiation basis as in §1, and let $\delta(S)$ denote the diameter of S. We say that a function $f \in L^p(R^n)$ is in $t_k^p(x_0)$ with respect to F if there exists a polynomial $\Pi_{x_0}(y)$ of degree $\leq k$ such that

$$\left\{\frac{1}{|S|}\int_{S+x_0}|f(y) - \Pi_{x_0}(y)|^p\, dy\right\}^{1/p} = o\big(\delta(S)^k\big), \quad \text{as } S \to 0.$$

If F is the family of all balls centered at the origin, this notion is due to Calderón and Zygmund [1], and for a general F was introduced in [2].

THEOREM 3. *Let $\|M_F f\|_p \leq A_p\|f\|_p$. If $f \in \Lambda_\alpha^{pq}(R^n)$, and k is a nonnegative integer $< \alpha$, then $f \in t_k^1(x)$ with respect to F for $L_{\alpha-k}^{pq}$-a.e. x.*

Of course, the special case $k = 0$ is Theorem 1. The proof of Theorem 3 proceeds along the lines of the corresponding theorem for L_α^p [2]. We shall see that with the help of Theorem 2 it is possible to omit the hypothesis $tF \subset F$, $0 < t \leq 1$, made in [2]. We need

LEMMA 8. *Let $f \in \Lambda_\alpha^{pq}(R^n)$, $1 < \alpha < \infty$. Then for L_α^{pq}-a.e. x, f is absolutely* *continuous on H^{n-1}-a.e. ray from x, and on such a ray fix $f(x+z)-f(x)=\int_0^1 \nabla f(x+tz)\cdot z\,dt$.*

PROOF. The proof is precisely the same as the proof of the corresponding lemma for L_α^p [2] as soon as we establish the existence of a sequence $\{f_j\} \subset C_0^\infty$ such that for L_α^{pq}-a.e. x,

$$G_1 * |\nabla f_j - \nabla f|(x) \to 0 \quad \text{as} j \to \infty.$$

Let $0<\gamma<\min(1, \alpha-1)$, and choose $g \in \Lambda_\gamma^{pq}$ such that $f = G_{\alpha-\gamma} * g$. Let $\{f_i\} \subset C_0^\infty$ such that $f_i \to f(\Lambda_\alpha^{pq})$, and let $g_i \in \Lambda_\gamma^{pq}$ with $f_i = G_{\alpha-\gamma} * g_i$. Then $g_i \to g(\Lambda_\gamma^{pq})$. By Lemma 6, $f_k(x) \to f(x)$ for L_α^{pq}-a.e. x, for some subsequence $\{f_k\}$. Since $\alpha > 1$, $\nabla f_k \to \nabla f\ (\Lambda_{\alpha-1}^{pq})$, and hence $\nabla f_k - \nabla f = G_{\alpha-\gamma-1} * h_k$, $h_k \to 0\ (\Lambda_\gamma^{pq})$. Since $\gamma < 1$, we easily see that $|h_k| \to 0\ (\Lambda_\gamma^{pq})$, and hence for some subsequence $\{h_j\}$, $G_{\alpha-\gamma} * |h_j|(x) \to 0$ for L_α^{pq}-a.e. x. Since $G_1 * G_{\alpha-\gamma-1} * |h_j|(x) = G_{\alpha-\gamma} * |h_j|(x)$, the sequence $\{f_j\}$ has the desired property.

PROOF OF THEOREM 3. Since $\alpha \leq 1$ is Theorem 1, we may assume that $1<\alpha$. Let $f \in \Lambda_\alpha^{pq}$, and let $\Pi_x(y) = \sum_{0<|\beta|<k}(D^\beta f(x)/\beta!)(y-x)^\beta$; we observe that $D^\beta f \in \Lambda_{\alpha-|\beta|}^{pq}$.

Set $R_x(y) = f(y) - \Pi_x(y)$, and let $|\gamma| = k$. Then $D^\gamma R_x(y) = D^\gamma f(y) - D^\gamma f(x)$. By Theorem 1, for $L_{\alpha-k}^{pq}$ – a.e. x,

$$\frac{1}{|S|}\int_{S+x} |D^\gamma R_x(y)|\,dy \to 0 \quad \text{as } S \to 0,$$

and by Theorem 2,

$$\int_0^1 \frac{1}{|tS|}\int_{tS+x} |D^\gamma R_x(y)|\,dy\,dt \to 0.$$

Let now $|\nu| = k-1$. For $L_{\alpha-k}^{pq}$-a.e. x, $D^\nu f$ is AC on H^{n-1}-a.e. ray from x, and on such a ray $D^\nu R_x(y)$ is also AC from which

$$|D^\nu R(x+z) - D^\nu R_x(x)| \leq \int_0^1 |\nabla(D^\nu R_x)(x+tz)\cdot z|\,dt.$$

We note that $D^\nu R_x(x) = 0$. As in [2] for any S (not necessarily in F)

$$\text{(i)} \qquad \frac{1}{\delta(S)}\frac{1}{|S|}\int_{S+x}|D^\nu R_x(z)|\,dz \leq \int_0^1 \frac{1}{|tS|}\int_{tS+x}|\nabla(D^\nu R_x)(z)|\,dz\,dt.$$

If $S \in F$ and $S \to 0$, the integral goes to 0 for $L_{\alpha-k}^{pq}$-a.e. x.

For the next step of the proof we need

$$\text{(ii)} \qquad \int_0^1 \frac{1}{\delta(\tau S)}\frac{1}{|\tau S|}\int_{\tau S+x}|D^\nu R_x(z)|\,dz\,d\tau \to 0$$

as $S \to 0$, $S \in F$, for $L_{\alpha-k}^{pq}$-a.e. x. From (i), replacing S by τS, we obtain

$$\int_0^1 \frac{1}{\delta(\tau S)} \frac{1}{|\tau S|} \int_{\tau S+x} |D^\nu R_x(z)|\, dz\, d\tau$$

$$\leqslant \int_0^1 \int_0^1 \frac{1}{|t\tau S|} \int_{t\tau S+x} |\nabla (D^\nu R_x)(z)|\, dz\, dt\, d\tau$$

and the corollary to Theorem 2 proves (ii).

If now η is a mutli-index, $|\eta| = k - 2$, we get as before

$$|D^\eta R_x(x + z)| \leqslant \int_0^1 |\nabla (D^\eta R_x)(x + tz) \cdot z|\, dt$$

and hence for any S,

$$\frac{1}{\delta(S)} \frac{1}{|S|} \int_{S+x} |D^\eta R_x(z)|\, dz \leqslant \int_0^1 \frac{1}{|tS|} \int_{tS+x} |\nabla (D^\eta R_x)(z)|\, dz\, dt.$$

If we divide this by $\delta(S) \geqslant \delta(tS)$, we finally obtain

$$\frac{1}{\delta(S)^2} \frac{1}{|S|} \int_{S+x} |D^\eta R_x(z)|\, dz$$

$$\leqslant \int_0^1 \frac{1}{\delta(tS)} \frac{1}{|tS|} \int_{tS+x} |\nabla (D^\eta R_x)(z)|\, dz.$$

We apply now (ii).

For the step treating a multi-index of length $k - 3$, we need to know that for $L^{pq}_{\alpha-k}$-a.e. x

$$\int_0^1 \frac{1}{\delta(\sigma S)^2} \frac{1}{|\sigma S|} \int_{\sigma S+x} |D^\eta R_x(z)|\, dz\, d\sigma \to 0$$

as $S \to 0$, $S \in F$. This integral is majorized by

$$\int_0^1 \int_0^1 \frac{1}{\delta(t\sigma S)} \frac{1}{|t\sigma S|} \int_{t\sigma S+x} |\nabla (D^\eta R_x)(z)|\, dz\, dt\, d\sigma$$

$$\leqslant \sum \int_0^1 \int_0^1 \int_0^1 \frac{1}{|\tau t\sigma S|} \int_{\tau t\sigma S+x} |\nabla (D^\rho R_x)(z)|\, dz\, dt\, d\sigma\, d\tau,$$

where Σ extends over all ρ with $|\rho| = k - 1$. By the corollary to Theorem 2 this is $o(1)$ as $S \to 0$, $S \in F$, for $L^{pq}_{\alpha-k}$-a.e. x. The proof of Theorem 3 is now complete.

6. We let F be a family as in §1 and we shall assume now that

(i) $$\|M_F f\|_p \leqslant A_p \|f\|_p, \qquad 1 < p \leqslant \infty.$$

The family of all oriented rectangles is an example satisfying (i). If (i) holds, then for a.e. x,

$$\lim_{S\to 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r\, dy = 0,$$

$f \in L^p$ and $1 \leq r < p$. This is no longer true if $r = p$ as the example due to Saks [7] shows.

If $f \in \Lambda_\alpha^{pq}$, then $f \in L_\beta^p$, $\alpha > \beta$, and hence by Sobolov's theorem $f \in L^r$, where $1/r = 1/p - \beta/n > 1/p - \alpha/n$. Here, of course, we assume that $\alpha p < n$. As in Theorem 1, we wish to study the size of the exceptional set (for which (ii) does not hold) under the hypothesis that $f \in \Lambda_\alpha^{pq}$. For this purpose we need a lemma.

LEMMA 9. *Let* $f \in \Lambda_\alpha^{pq}$, $0 < \alpha < 1$, *and let* $1 \leq r \leq p$. *Then*

$$\||f|^r\|_{\Lambda_\alpha^{p/rq}} \leq A_r \|f\|_p^{r-1} \|f\|_{\Lambda_\alpha^{pq}}.$$

PROOF. Recall that

$$\|f\|_{\Lambda_\alpha^{pq}} = \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) - f(x)\|_p^q}{|t|^{n+\alpha q}}\, dt \right\}^{1/q}.$$

Now $|f(x+t)|^r - |f(x)|^r = (|f(x+t)| - |f(x)|)r \cdot \chi^{r-1}$, where χ is between $|f(x+t)|$, $|f(x)|$. Hence, if $\psi(x,t) = \max(|f(x+t)|, |f(x)|)$, we get

$$\big|\, |f(x+t)|^r - |f(x)|^r \big| \leq r|f(x+t) - f(x)|\psi(x,t)^{r-1}.$$

For $\beta \leq p$ we have

$$\big\| |f(x+t)|^r - |f(x)|^r \big\|_\beta \leq r \left\{ \int |f(x+t) - f(x)|^\beta \psi(x,t)^{\beta(r-1)}\, dx \right\}^{1/\beta}.$$

If we let $s = p/\beta$, $1/t = 1 - 1/s = (p-\beta)/p$, and apply Hölder's inequality, we get

$$\begin{aligned} &\||f(x+t)|^r - |f(x)|^r\|_\beta \\ &\quad \leq r\left[\left\{ \int |f(x+t) - f(x)|^p\, dx \right\}^{\beta/p} \left\{ \int \psi(x,t)^{\beta(r-1)\cdot p/(p-\beta)} \right\}^{(p-\beta)/p} \right]^{1/\beta}. \end{aligned}$$

We let now $\beta = p/r$. Then $\beta(r-1)p/(p-\beta) = p$, and $p \cdot (p-\beta)/p\beta = r - 1$, and hence

$$\||f(x+t)|^r - |f(x)|^r\|_{p/r} \leq r\|f(x+t) - f(x)\|_p\, \|\psi(x,t)\|_p^{r-1}.$$

Since $\psi(x,t) \leq |f(x)| + |f(x+t)|$, we see that $\|\psi(x,t)\|_p \leq 2\|f\|_p$. Since, clearly, $\||f|^r\|_{p/r} = \|f\|_p \|f\|_p^{r-1}$, the proof of the lemma is complete.

Let C_1, C_2 be two capacities on R^n, and let $[C_1, C_2](E) = \inf\{C_1(E') + C_2(E'')\}$, where the inf is extended over all E', E'' with $E = E' \cup E''$. It is easy to check that this is a capacity, and that $[C_1, C_2](E) \leq \min(C_1(E), C_2(E))$. If $\{C_\alpha\}$ is a collection of capacities on R^n, then $C(E) = \sup_\alpha C_\alpha(E)$ is also a capacity on R^n.

Let $f \in \Lambda_\alpha^{pq}$, $\alpha p < n$. As we have seen, $f \in L^r$ for $1/r > 1/p - \alpha/n$. If $p \leq r < s$, $1/s = 1/p - \alpha/n$, let for $r < t < s$, $\alpha_t = \alpha - n/p + n/t$. We define now the capacity $C_{r\alpha}$ by

$$C_{r\alpha}(E)=\begin{cases}\left[L_\alpha^{pq}, L_\alpha^{p/r\,q}\right](E), & 1 \leqslant r<p,\\ \sup\limits_{t>r}\left[L_{\alpha_t}^{tq}, L_\alpha^{t/r\,q}\right](E), & p \leqslant r<s.\end{cases}$$

THEOREM 4. *Assume that* $\|M_F f\|_p \leqslant A_p\|f\|_p$, $1<p\leqslant\infty$. *Let* $f\in\Lambda_\alpha^{pq}$, $0<\alpha<1$, $\alpha p<n$, *and let* $1/r>1/p-\alpha/n$. *Then for* $C_{r\alpha}$*-a.e.* x,

$$\frac{1}{|S|}\int_{S+x}|f(y)-f(x)|^r\,dy\to 0 \quad \text{as } S\to 0.$$

PROOF. We assume first that $1\leqslant r<p$. Let

$$E=\left\{x\colon \limsup_{S\to 0}\frac{1}{|S|}\int_{S+x}|f(y)-f(x)|^r\,dy>0\right\},$$

and let $0<\beta<\alpha$. By Lemma 3 it suffices to show that there exists a decomposition $E=E'\cup E''$ such that $B_\beta^{pq}(E')=0$ and $B_\beta^{p/r\,q}(E'')=0$.

Let $\{\psi_j\}\subset C_0^\infty$ such that $\psi_j\to f(\Lambda_\alpha^{pq})$. Then for each j

$$\left\{\frac{1}{|S|}\int_{S+x}|f(y)-f(x)|^r\,dy\right\}^{1/r}\leqslant\left\{M_F(|f-\psi_j|^r)(x)\right\}^{1/r}$$
$$+\left\{\frac{1}{|S|}\int_{S+x}|\psi_j(y)-\psi_j(x)|^r\,dy\right\}^{1/r}+|\psi_j(x)-f(x)|$$
$$=A_{1j}(x)+A_{2j}(x)+A_{3j}(x).$$

We claim that there is $j_1<j_2<\cdots$ such that $M_F(|f-\psi_{j_i}|^r)(x)\to 0$ as $i\to\infty$ for $B_\beta^{p/r\,q}$-a.e. x.

By Lemma 9 $|f-\psi_i|^r\in\Lambda_\alpha^{p/r\,q}$, and hence $|f-\psi_j|^r=G_{\alpha-\beta}*g_j$, $g_j\in\Lambda_\beta^{p/r\,q}$. Since

$$\{x\colon A_{1j}(x)\geqslant\varepsilon^r\}\subset\{x\colon G_{\alpha-\beta}*M_F g_j(x)\geqslant\varepsilon^r\},$$

we obtain

$$B_\beta^{p/r\,q}\{x\colon A_{1j}(x)\geqslant\varepsilon^r\}\leqslant\frac{1}{\varepsilon^r}\|M_F g_j\|_{\Lambda_\beta^{p/r\,q}}\leqslant\frac{A}{\varepsilon^r}\|g_j\|_{\Lambda_\beta^{p/r\,q}}\quad\text{(Lemma 2).}$$

By Lemma 9, $|f-\psi_j|^r\to 0\ (\Lambda_\alpha^{p/r\,q})$, and hence $g_j\to 0\ (\Lambda_\beta^{p/r\,q})$. The claim now follows (see Lemma 6).

Let $E''=\{x\colon \limsup_{i\to\infty}M_F(|f-\psi_{j_i}|^r)(x)>0\}$. Then $B_\beta^{p/r\,q}(E'')=0$. We may assume that $\psi_{j_i}(x)\to f(x)$ for L_α^{pq}-a.e. x, and if E' is the exceptional set, we have $B_\beta^{pq}(E')=0$. We show that $E\subset E'\cup E''$.

Let $x\notin E'\cup E''$, and let $\varepsilon>0$. Choose j_0 such that $A_{1j_0}(x)\leqslant\varepsilon/3$, $A_{3j_0}(x)\leqslant\varepsilon/3$, and then choose τ_0 such that $S\subset B(0,\tau_0)$ implies $A_{2j_0}(x)\leqslant\varepsilon/3$. It follows then that $x\notin E$.

If $p\leqslant r<s$, $1/s=1/p-\alpha/n$, let $r<t<s$, and let $\alpha_t=\alpha-n/p+$

n/t. By [6, p. 441], $f \in \Lambda_{\alpha_t}^{tq}$. Consequently by the first part of the proof, $[L_{\alpha_t}^{tq}, L_{\alpha_t}^{t/r\,q}](E) = 0$, and the proof of the theorem is complete.

REFERENCES

1. A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171–225. MR **25** #310.

2. D. J. Deignan and W. P. Ziemer, *Strong differential properties of Bessel potentials*, Trans. Amer. Math. Soc. **225** (1977), 113–122.

3. M. deGuzmán, *Differentiation of integrals in R^n*, Lecture Notes in Math., no. 481, Springer-Verlag, Berlin and New York, 1975.

4. N. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26** (1970), 255–292. MR **43** #3474.

5. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970. MR **44** #7280.

6. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space*, J. Math. Mech. **13** (1964), 407–479. MR **29** #462.

7. S. Saks, *Remark on the differentiability of the Lebesgue indefinite integral*, Fund. Math. **22** (1934), 257–261.

8. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials*. I, Ann. Inst. Fourier **11** (1961), 385–475. MR **26** #1485.

MATHEMATICAL SCIENCES DIVISION, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907