TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 240, June 1978

PARACOMPACTNESS OF BOX PRODUCTS OF COMPACT SPACES[1]

BY

KENNETH KUNEN[2]

ABSTRACT. We consider box products of countably many compact Hausdorff spaces. Under the continuum hypothesis, the product is paracompact iff its Lindelöf degree is no more than the continuum; in particular, the product is paracompact if each space has weight continuum or less, or if each space is dispersed. Some partial results are proved under Martin's axiom.

0. Introduction. All our spaces are T_3 (regular Hausdorff). The box topology on $\Pi_n X_n$ has as a basis arbitrary products of open sets. Such a product of compact spaces is compact only in trivial cases, but it may be paracompact. By van Douwen **[vD 2]** there are compact X_n of weight ω_2 such that $\Pi_n X_n$ is not even normal. However, Rudin has shown that the continuum hypothesis (CH) implies that $\Pi_n X_n$ is paracompact if each X_n is compact metric **[R 1]** or if each X_n is a compact ordinal **[R 2]**.

In this paper, we obtain easier proofs of some stronger positive results. Under CH, when each X_n is compact, $\Pi_n X_n$ is paracompact iff it is ω_1-Lindelöf (see 3.2). In particular if each X_n has weight $\leqslant \omega_1$, $\Pi_n X_n$ is paracompact under CH (4.1); under Martin's axiom, we can modify our arguments to establish this if we assume also that the X_n are first countable (4.4). In §5, we show that if each X_n is a compact dispersed space, $\Pi_n X_n$ is c-Lindelöf and hence, under CH, paracompact. §1 reviews some general results on paracompact spaces, and §2 applies them to box products. §6 extends the results of §4 and §5 to products of locally compact paracompact spaces.

1. Paracompactness. We collect here some known results and easy consequences thereof.

1.1. THEOREM (MICHAEL **[M 1]**). *The following are equivalent*:

(a) *X is paracompact.*

Received by the editors October 25, 1976.

AMS (MOS) subject classifications (1970). Primary 54B10; Secondary 54D30, 54G10.

Key words and phrases. Box product, paracompact space, dispersed space, Lindelöf degree.

[1]This research was supported by NSF Grant GP-27633.

[2]The author is grateful to the referee for a number of very helpful comments.

(b) *Every open cover of X has a σ-locally finite open refinement.*

(c) *Every open cover of X has a σ-discrete open refinement.*

We remind the reader that all our spaces are T_3, an assumption needed in the theorem. A corollary of the fact that (b) implies (a) is that Lindelöf spaces are paracompact. We shall need a generalization of this, which we state after recalling some definitions.

1.2. DEFINITION. Let κ be an infinite cardinal.

(1) X is κ-Lindelöf iff every open cover of X has a subcover of cardinality $\leqslant \kappa$.

(2) X is κ-open iff the intersection of less than κ open sets in X is open.

Thus, ω-Lindelöf = Lindelöf, and all spaces are ω-open. The fact that Lindelöf spaces are paracompact is the following lemma for $\kappa = \omega$.

1.3. LEMMA. *If X is κ-open and κ-Lindelöf, then X is paracompact.*

PROOF. If $\kappa > \omega$, then X is zero dimensional, since it is regular and ω_1-open. Let $\mathfrak{U}$ be an open cover of X. Then $\mathfrak{U}$ has a refinement $\{V_\alpha: \alpha < \kappa\}$, where each V_α is clopen. Let

$$W_\alpha = V_\alpha - \bigcup \{V_\beta: \beta < \alpha\}.$$

Then $\{W_\alpha: \alpha < \kappa\}$ is a disjoint clopen refinement of $\mathfrak{U}$.

We shall also need the following result on preservation of paracompactness under closed maps.

1.4. LEMMA. *Let F be a closed continuous map from X onto Y. Then*

(1) *If X is paracompact, so is Y.*

(2) *If Y is paracompact and $F^{-1}\{y\}$ is Lindelöf for each $y \in Y$, then X is paracompact.*

PROOF. For (1), see Michael [M 2]. For (2), let $\mathfrak{U}$ be an open cover of X. For each $y \in Y$, choose a countable subfamily of $\mathfrak{U}$, $\{U_n^y: n \in \omega\}$, which covers $F^{-1}\{y\}$. Let

$$V^y = \left\{ z \in Y: F^{-1}\{z\} \subseteq \bigcup_{n \in \omega} U_n^y \right\}.$$

Then $y \in V^y$ and V^y is open since F is closed. Let $\{W^y: y \in Y\}$ be a locally finite open refinement of $\{V^y: y \in Y\}$ which is precise (i.e., $W^y \subseteq V^y$). Let $\mathfrak{R}_n$ be the family $\{U_n^y \cap F^{-1}(W^y): y \in Y\}$. Then $\{\mathfrak{R}_n: n \in \omega\}$ forms a σ-locally finite open refinement of $\mathfrak{U}$. Thus, X is paracompact by Theorem 1.1.

We remark that (2) is well known in the case that each $F^{-1}\{y\}$ is compact, and the proof above is an easy modification of the standard proof for that case.

Lindelöf degree has the same preservation property, with a similar, but easier proof.

1.5. LEMMA. *Let F be a closed continuous map from X onto Y, such that $F^{-1}\{y\}$ is Lindelöf for each $y \in Y$. Then for any infinite κ, Y is κ-Lindelöf iff X is.*

2. The reduced box product. We shall define a certain reduced product, $\Pi_n X_n/\sim$. This will be easier to deal with than $\Pi_n X_n$, as it is ω_1-open. However, for compact X_n, $\Pi_n X_n$ and $\Pi_n X_n/\sim$ will have similar paracompactness and Lindelöf properties. The reduced product was first defined by Knight **[Kn]**, and was applied by Rudin **[R 2]** to obtain information about box products.

$\Pi_n X_n/\sim$ is defined as follows. If $p, q \in \Pi_n X_n$, we say p is equivalent to q ($p \sim q$) iff $p(n) = q(n)$ for all but finitely many n. Let $E(p) = \{q: q \sim p\}$, and, for $A \subseteq \Pi_n X_n$, $E(A) = \bigcup\{E(p): p \in A\}$. We give the quotient space $\Pi_n X_n/\sim = \{E(p): p \in \Pi_n X_n\}$ the usual quotient topology and let $\sigma: \Pi_n X_n \to \Pi_n X_n/\sim$ be the usual projection. So $\sigma(p) = E(p)$; we use $\sigma(p)$ when we are thinking of a point in $\Pi_n X_n/\sim$ and $E(p)$ when we are thinking of a subset of $\Pi_n X_n$.

It is easy to see that each $E(p)$ is closed. Also, the map σ is open, so that the $\sigma(U)$ for U basic in $\Pi_n X_n$ form a base for $\Pi_n X_n/\sim$. We shall rely heavily on the following

2.1. THEOREM (RUDIN). *$\Pi_n X_n/\sim$ is ω_1-open.*

2.2. THEOREM. *If each X_n is compact, the map σ is closed.*

Before proving these, we remark that the interest of 2.2 is in its corollary,

2.3. COROLLARY. *If each X_n is compact, then*
(a) *$\Pi_n X_n$ is paracompact iff $\Pi_n X_n/\sim$ is.*
(b) *For any infinite κ, $\Pi_n X_n$ is κ-Lindelöf iff $\Pi_n X_n/\sim$ is.*

PROOF. Point inverses under σ are σ-compact, and hence Lindelöf, so 1.4 and 1.5 apply.

We turn now to the proofs of Theorems 2.1 and 2.2. 2.1 is essentially proved in **[R 2]**, but we include a proof to show that 2.1 and 2.2 actually both follow from the same Lemma 2.4. For each k, let τ_k be the canonical projection from $\Pi_n X_n = \Pi_{n<k} X_n \times \Pi_{n \geqslant k} X_n$ onto $\Pi_{n \geqslant k} X_n$.

2.4. LEMMA. *If, for each k, U^k is an open subset of $\Pi_{n \geqslant k} X_n$, then $\bigcap\{\tau_k^{-1}(U^k): k < \omega\}$ is open in $\Pi_n X_n$.*

PROOF. Fix $p \in \bigcap\{\tau_k^{-1}(U^k): k < \omega\}$. Then for each k, there is a neighborhood V^k of p contained in $\tau_k^{-1}(U^k)$, which may be taken to be of the form

$$V^k = X_0 \times X_1 \times \cdots \times X_{k-1} \times V_k^k \times V_{k+1}^k \times V_{k+2}^k \times \ldots,$$

each V_n^k $(n \geq k)$ being open in X_n. Let $W_n = \bigcap\{V_n^k\colon k \leq n\}$, and $W = \Pi_n W_n$. Then $W = \bigcap\{V^k\colon k < \omega\}$ is a neighborhood of p contained in each $\tau_k^{-1}(U^k)$.

To deduce 2.1 from 2.4, let σ_k be the map from $\Pi_{n \geq k} X_n$ onto $\Pi_n X_n/\sim$ such that $\sigma = \sigma_k \circ \tau_k$. If V^k is open in $\Pi_n X_n/\sim$ for each $k < \omega$, then $\sigma_k^{-1}(V^k)$ is open in $\Pi_{n \geq k} X_n$, so $\sigma^{-1}(\bigcap\{V^k\colon k < \omega\})$, which is equal to $\bigcap\{\tau_k^{-1}(\sigma_k^{-1}(V^k))\colon k < \omega\}$, is open in $\Pi_n X_n$, so $\bigcap\{V^k\colon k < \omega\}$ is open in $\Pi_n X_n/\sim$.

To deduce 2.2, note that showing σ is closed is equivalent to showing that whenever K is closed in $\Pi_n X_n$, so is $\sigma^{-1}\sigma K$. But this set is just $\bigcup\{\tau_k^{-1}\tau_k K\colon k < \omega\}$. Hence, if each $\tau_k K$ is closed in $\Pi_{n \geq k} X_n$, then by 2.4, $\sigma^{-1}\sigma K$ is closed. But each $\tau_k K$ is closed, since projection from a compact factor is a closed map.

3. On paracompactness and the Lindelöf degree. Under CH, our main tool for proving paracompactness is

3.1. THEOREM (CH). *If each X_n is compact, then $\Pi_n X_n$ is paracompact iff it is ω_1-Lindelöf.*

PROOF. If $\Pi_n X_n$ is ω_1-Lindelöf, then so is $\Pi_n X_n/\sim$ by 2.3. Since $\Pi_n X_n/\sim$ is also ω_1-open by 2.1, it is paracompact by 1.3, so $\Pi_n X_n$ is paracompact by 2.3.

The other direction is more difficult and is not needed for the rest of this paper. We shall in fact establish the following, without CH.

3.2. THEOREM. *If each X_n is compact and $\Pi_n X_n$ is paracompact, then it is c-Lindelöf.*

We remark that we know of no example, under any set-theoretic axioms, of a box product of compact spaces which is normal but not paracompact.

To prove 3.2, we need the following unpublished result of Arhangel'skiĭ:

3.3. THEOREM. *If Y is compact and $\mathfrak{F}$ is a cover of Y by closed G_δ sets and $\mathfrak{F}$ satisfies*

$$\forall H \in \mathfrak{F}(|\{K \in \mathfrak{F}\colon H \cap K \neq 0\}| \leq c), \qquad (*)$$

then $|\mathfrak{F}| \leq c$.

Of course, if the elements of $\mathfrak{F}$ are points, 3.3 is just the well-known result of [A], a simple proof of which is due to Pol [P]. Pol's proof can easily be modified to yield 3.3.

We wish to apply 3.3 with Y the product of the spaces X_n under the usual (Tychonov) topology. To do this we need

3.4. LEMMA. *Let Z be paracompact and $\mathfrak{B}$ any base for Z. Then every open* *cover* $\mathfrak{U}$ *for* Z *has a refinement to a cover* $\mathfrak{F}$ *such that*

(1) *Elements of* $\mathfrak{F}$ *are of the form* $\bigcap_n B_n$, *where each* $B_n \in \mathfrak{B}$, *and* $B_0 \supseteq \mathrm{cl}(B_1) \supseteq B_1 \supseteq \mathrm{cl}(B_2) \supseteq \ldots$.

(2) $\mathfrak{F}$ *satisfies* (*).

We apply 3.4 with Z the box product $\Pi_n X_n$ and $\mathfrak{B}$ the usual base. Then elements of $\mathfrak{F}$ will be closed G_δ sets in Y, so $|\mathfrak{F}| \leqslant c$ by 3.3. Thus, if Z is paracompact, it is c-Lindelöf.

To prove 3.4, we apply paracompactness ω times to produce a sequence of covers $\mathcal{V}_n$ $(n \in \omega)$ of Z with $\mathcal{V}_0 = \mathfrak{U}$ and

(a) Each $\mathcal{V}_{n+1}$ is a locally finite open refinement of $\mathcal{V}_n$.

(b) If $V_{n+1} \in \mathcal{V}_{n+1}$, V_{n+1} intersects only finitely many members of $\mathcal{V}_n$.

(c) If $V_{n+1} \in \mathcal{V}_{n+1}$, there is a $B \in \mathfrak{B}$ and $V_n \in \mathcal{V}_n$ with $V_{n+1} \subseteq B \subseteq \mathrm{cl}(B) \subseteq V_n$.

Let $\mathfrak{F}$ be the set of all sets of the form $\bigcap_n V_n = \bigcap_n B_n$, where each $V_n \in \mathcal{V}_n$, $B_n \in \mathfrak{B}$, and $V_0 \supseteq \mathrm{cl}(B_0) \supseteq B_0 \supseteq V_1 \supseteq \ldots$. $\mathfrak{F}$ satisfies (*) by conditon (b). To see that $\mathfrak{F}$ is a cover, fix $p \in X$. Let T be the tree of all finite sequences $\langle V_1 \ldots V_k \rangle$ such that $p \in V_k$, each $V_n \in \mathcal{V}_n$, and for some $B_1 \ldots B_{k-1} \in \mathfrak{B}$, $V_1 \supseteq \mathrm{cl}(B_1) \supseteq B_1 \supseteq V_2 \supseteq \cdots \supseteq B_{k-1} \supseteq V_k$. Each level of T is finite, since the covers are point finite, but T is infinite, so by König's lemma, there is a path through T. The intersection of this path is a set in $\mathfrak{F}$ containing p.

4. Trivial applications of §§1–3. Theorem 3.1 yields immediately:

4.1. THEOREM (CH). *If each* X_n *is compact and of weight* $\leqslant \omega_1$, *then* $\Pi_n X_n$ *is paracompact.*

We would call an application of §§1–3 nontrivial if the Lindelöf degree of $\Pi_n X_n/\!\sim$ is established by some bound other than its weight. For example, in §5 we show that if each X_n is a compact dispersed space, $\Pi_n X_n/\!\sim$ is c-Lindelöf.

We look now at the situation under Martin's axiom (MA). See Jech [J] for a complete discussion of MA, including a proof of its consistency with $\neg$ CH. We need here only the well-known

4.2. LEMMA. *Assume* MA. *If* $\kappa < c$ *and* $f_\alpha(\alpha < \kappa)$ *are functions from* ω *into* ω, *then there is a* $g\colon \omega \to \omega$ *such that for each* α, $g(n) > f_\alpha(n)$ *for all but finitely many* n.

This gives us, in analogy with Theorem 2.1,

4.3. THEOREM. *If* MA *and each* X_n *is first countable, then* $\Pi_n X_n/\!\sim$ *is c-open.*

PROOF. Fix $\sigma(p) \in \Pi_n X_n/\!\sim$ and let U^α $(\alpha < \kappa)$ be neighborhoods of $\sigma(p)$, where $\kappa < c$. We show that $\bigcap \{U_\alpha\colon \alpha < \kappa\}$ is a neighborhood of $\sigma(p)$. For each n, let $\{V_n^k\colon k < \omega\}$ be a base at $p(n)$ in X_n. For each α, let $f_\alpha\colon \omega \to \omega$ be such that

$$\sigma\Big(\prod_n V_n^{f_\alpha(n)}\Big) \subseteq U_\alpha.$$

Then if g is as in 4.2, $\sigma(\Pi_n V_n^{g(n)})$ is an open neighborhood of $\sigma(p)$ contained in $\bigcap\{U_\alpha\colon \alpha < \kappa\}$.

Then, analogously to 4.1, we have the following, whose proof is obtained essentially by replacing ω_1 by c in the proof of the easy direction of 3.1.

4.4. THEOREM (MA). *If each X_n is compact and first countable, then $\Pi_n X_n$ is paracompact.*

PROOF. By Arhangel'skiĭ [A], each X_n has cardinality $\leqslant c$, and thus weight $\leqslant c$, so $\Pi_n X_n/\sim$ has weight $\leqslant c$, so by 4.3, $\Pi_n X_n/\sim$, and hence also $\Pi_n X_n$ is paracompact.

We do not know whether CH is required for Theorem 4.1, or MA for 4.4. van Douwen [vD 3] has shown that in 4.4, if each X_n is compact metric, then MA may be weakened to the existence of a κ-scale in ω^ω for some κ. The proof shows that in this case, $\Pi_n X_n/\sim$ is κ-metrizable.

Under $\neg$CH, there is no criterion like Theorem 3.1 for paracompactness in terms of Lindelöf degree. By van Douwen [vD 2], there are always compact X_n of weight ω_2 such that $\Pi_n X_n$ is not even normal; under $\neg$CH, this box product would have weight c and hence Lindelöf degree c (since no nontrivial box product has Lindelöf degree $< c$). But the box product of 2-point spaces has the same Lindelöf degree c and is paracompact (in fact discrete).

5. Products of compact dispersed spaces. Rudin [R 2] shows that under CH, if each X_n is a compact ordinal (i.e., a successor ordinal with the order topology), then $\Pi_n X_n$ is paracompact. An easier proof of this may be obtained by first establishing without any set-theoretic assumptions, that $\Pi_n X_n$ is c-Lindelöf, and then quoting 3.1. Following a suggestion of Scott Williams, we can also generalize this to dispersed spaces:

5.1. THEOREM. *If each X_n is a compact dispersed space, then $\Pi_n X_n$ is c-Lindelöf.*

Here, X is dispersed iff each subspace Y contains an isolated (in Y) point. Ordinals are dispersed, since the first point in any subspace of an ordinal is isolated.

For any X, one can form the Cantor-Bendixon sequence $X^{(\alpha)}$, where $X^{(0)} = X$, $X^{(\alpha+1)} = \{x \in X^{(\alpha)}\colon x \text{ is not isolated in } X^{(\alpha)}\}$, and $X^{(\gamma)} = \bigcap\{X^{(\alpha)}\colon \alpha < \gamma\}$ for γ a limit. Then each $X^{(\alpha)}$ is closed in X, the $X^{(\alpha)}$ are nonincreasing, and, if $Y \subseteq X$, $Y^{(\alpha)} \subseteq X^{(\alpha)}$ for all α. X is dispersed iff $X^{(\alpha)} = 0$ for some α. If X is compact dispersed, then the first α such that

$X^{(\alpha)} = 0$ is a successor ordinal, $\beta + 1$, and $X^{(\beta)}$ is finite; we call β the rank of X.

To prove Theorem 5.1, let $\mathcal{U}$ be an open cover of $\Pi_n X_n$. Following [**R 2**], we use a tree to index our attempts to refine $\mathcal{U}$, but we look for a closed rather than an open refinement.

Let $T = c^{<\omega_1} = \bigcup\{c^\xi\colon \xi < \omega_1\}$. Elements $s \in T$ are viewed as c-ary sequences of countable ordinal length, so T is the complete c-ary tree of height ω_1. We write lh(s) for the length (domain) of s, and, if $\eta < \text{lh}(s)$, $s \upharpoonright \eta$ is the sequence of length η formed by restricting s to η. If $s \in T$ and $\mu \in c$, then $s\mu$ is the sequence of length lh$(s) + 1$ which begins with s and ends with the element μ. 0 is the empty sequence, of length 0.

$K \lhd \mathcal{U}$ means $\exists U \in \mathcal{U}$ $(K \subseteq U)$. We shall define, by induction on lh(s), $K(s) = \Pi_n K_n(s)$, a closed box in $\Pi_n X_n$. We shall then check that $\forall p \in \Pi_n X_n$ $\exists s \in T\,(p \in K(s) \lhd \mathcal{U})$, which, since T has cardinality c, will imply that $\mathcal{U}$ has a subcover of cardinality c.

$K(s)$ will be defined so that $K(0) = \Pi_n X_n$, and, for each $s \in T$,

(1) $K(s) = \bigcup\{K(s\mu)\colon \mu < c\}$.

(2) If lh(s) is a limit, $K(s) = \bigcap\{K(s \upharpoonright \xi)\colon \xi < \text{lh}(s)\}$.

(3) For each $\mu < c$, either $K(s\mu) \lhd \mathcal{U}$ or $\exists n[\text{rank}(K_n(s\mu)) < \text{rank}(K_n(s))]$.

By (1) and (2), if $p \in \Pi_n X_n$, there is an $f \in c^{\omega_1}$ with $p \in K(f \upharpoonright \xi)$ for all $\xi < \omega_1$. Then, for each n, the ranks of the $K_n(f \upharpoonright \xi)$ are nonincreasing and hence eventually constant. If they are constant past ξ for each n, then (3) insures that $K(f \upharpoonright \xi) \lhd \mathcal{U}$. Hence, we shall be done if (1)–(3) can be accomplished.

To define $K(s)$, we proceed by induction, taking intersections at limits, so that (2) holds by definition. For a successor stage, fix s. Let $\beta_n = \text{rank}(K_n(s))$ and $Y_n = (K_n(s))^{(\beta_n)}$. Then Y_n is finite. Let $\mathcal{V}$ be a family of open boxes of cardinality c such that $\mathcal{V}$ covers $\Pi_n Y_n$ and cl$(V) \lhd \mathcal{U}$ for all $V \in \mathcal{V}$. This is possible since $|\Pi_n Y_n| \leq c$. Let $K(s\mu)$, for $\mu < c$, enumerate all the c boxes $K = \Pi_n K_n$ such that either

(a) $K = \text{cl}(V) \cap K(s)$ for some $V \in \mathcal{V}$ or

(b) for some n,

(i) $K_n = K_n(s) - (V_n^1 \cup \cdots \cup V_n^j)$ for some $V^1 \ldots V^j \in \mathcal{V}$ with $Y_n \subseteq V_n^1 \cup \ldots \cup V_n^j$ and

(ii) $K_m = K_m(s)$ for all $m \neq n$.

Then (3) holds, since (b)(i) implies that $(K_n)^{(\beta_n)} = 0$, whence rank$(K_n) < \beta_n$. To check (1), fix $p \in K(s)$. If p fails to be in any of the $K(s\mu)$ of type (b) then, since Y_n is finite, for each n there must be a $q_n \in Y_n$ with $\forall V \in \mathcal{V}$ $(q_n \in V_n \to p_n \in V_n)$. Let $q = \langle q_0 q_1 \ldots \rangle \in \Pi_n Y_n$. Then $\forall V \in \mathcal{V}\,(q \in V \to p \in V)$, so, taking V containing q, p is in cl$(V) \cap K(s)$, which is one of the $K(s\mu)$ of the type (a).

Thus, the $K(s)$ may indeed be defined to satisfy (1)–(3), which completes the proof of 5.1.

5.1 can be improved by using a modified Cantor-Bendixon analysis. Let $w(Z)$ denote the weight of the space Z. For any X, let $X' = X - \bigcup\{N: N$ is open and $w(N) < c\}$; let $X^{[0]} = X$, $X^{[\alpha+1]} = (X^{[\alpha]})'$, and take intersections at limits as before. Call X almost dispersed iff $X^{[\alpha]}$ is empty for some α (equivalently, iff every subspace contains a neighborhood of weight $\leq c$). The class of almost dispersed spaces contains all dispersed spaces and all spaces of weight $< c$, and is closed under countable box products.

5.2. THEOREM. *If each X_n is compact and almost dispersed, then $\Pi_n X_n$ is c-Lindelöf.*

The proof is as for 5.1, with the appropriate modification in the notion of rank. In the justification for the existence of $\mathcal{V}$, one uses that $\Pi_n Y_n$ has weight $< c$, rather than cardinality $< c$. In the argument establishing (1), use the fact that the Y_n^i are compact, rather than finite.

Then, by 3.1, we have

5.3. THEOREM (CH). *If each X_n is compact and almost dispersed, then $\Pi_n X_n$ is paracompact.*

6. Products of locally compact spaces. Compactness can be replaced by local compactness plus paracompactness in the positive results of §4 and §5 by

6.1. THEOREM. *Assume that each X_n is paracompact and has an open cover $\mathfrak{U}_n$ such that whenever $U_n \in \mathfrak{U}_n$ $(n \in \omega)$, $\Pi_n \mathrm{cl}(U_n)$ is paracompact. Then $\Pi_n X_n$ is paracompact.*

Thus, e.g., by 4.1, 4.4 and 5.3,

6.2. COROLLARY. *Assume each X_n is a paracompact and locally compact.*

(a) (CH) *If each X_n is locally of weight $\leq \omega_1$, then $\Pi_n X_n$ is paracompact.*

(b) (MA) *If each X_n is first countable, then $\Pi_n X_n$ is paracompact.*

(c) (CH) *If each X_n is dispersed, then $\Pi_n X_n$ is paracompact.*

The assumption of paracompactness of the X_n cannot be dropped from these results, since each X_n is homeomorphic to a closed subspace of $\Pi_n X_n$. Also, under CH, $\omega_1 \times (\omega + 1)^\omega$ is not even normal (see **[Ku]**). A special case of 6.2(b) is that under MA, the box product of locally compact metric spaces is paracompact. Under CH, this is essentially due to Rudin **[R 1]**, as was pointed out by van Douwen **[vD 1]**. **[vD 1]** also shows that the box product of non-locally-compact separable metric spaces need not be normal.

To prove 6.1, note first that we can, by criterion (c) of Michael's Theorem 1.1, assume that each $\mathfrak{U}_n$ is countable; if not, we pass to σ-discrete refinements of the $\mathfrak{U}_n$ and note that a discrete union of paracompact spaces is paracompact. Again using paracompactness of the X_n, we may assume that each $\mathcal{U}_n$ is locally finite.

Say $\mathcal{U}_n = \{U_n^k: k < \omega\}$. By normality, we can find closed $F_n^k \subseteq U_n^k$ such that $\bigcup_k F_n^k = X_n$. Then, there are open $V_n^{k,i}$ $(i < \omega)$ such that

$$U_n^k \supseteq \operatorname{cl} V_n^{k,0} \supseteq V_n^{k,0} \subseteq \operatorname{cl} V_n^{k,1} \supseteq \cdots \supseteq F_n^k.$$

Whenever $f \in \omega^\omega$, let

$$A_f = \bigcap_i E\Big(\prod_n V_n^{f(n),i}\Big) = \bigcap_i E\Big(\prod_n \operatorname{cl} V_n^{f(n),i}\Big).$$

Then the A_f cover $\Pi_n X_n$. Each A_f is closed, as it is the intersection of closed sets, and it is open, as it is the intersection of ω open sets and $\Pi_n X_n/\sim$ is ω_1-open.

Furthermore, each A_f is paracompact in its relative topology. To see this, note that A_f is covered by the interiors of the $\Pi_n \operatorname{cl}(U_n^{g(n)})$, where g ranges over the countably many elements of ω^ω which are eventually equal to f. Also, each $\Pi_n \operatorname{cl}(U_n^{g(n)})$ is paracompact by assumption. Paracompactness of A_f now follows immediately from the following easy general fact:

6.3. Lemma. *If $Y = \bigcup_j \operatorname{int}(K_j)$, where each K_j is closed and paracompact, then Y is paracompact.*

Proof. If $\mathcal{U}$ is an open cover of Y, for each j let $\mathcal{V}_j$ be a family of open (in K_j) subsets of K_j which refines $\mathcal{U}$, covers K_j, and is locally finite in K_j. Let $\mathcal{W}_j = \{V \cap \operatorname{int}(K_j): V \in \mathcal{V}_j\}$. $\mathcal{W}_j$ is a locally finite and open (in Y) refinement of $\mathcal{U}$ which covers $\operatorname{int}(K_j)$ so the $\mathcal{W}_j$ for $j < \omega$ form a σ-locally finite open refinement of $\mathcal{U}$ which cover Y. Hence by Michael's Theorem 1.1, Y is paracompact.

Finally, the family of A_f for $f \in \omega^\omega$ is closure-preserving; equivalently, since the A_f are closed, each p has a neighborhood W which is disjoint from all the A_f not containing p. To see this, let $W = \Pi_n W_n$, where

$$W_n = X_n - \bigcup \{\operatorname{cl} V_n^{k,i}: i, k \in \omega \text{ and } p(n) \notin \operatorname{cl} V_n^{k,i}\}.$$

W_n is open since $\mathcal{U}_n$ is locally finite. If $p \notin A_f$, then for some i, $p \notin E(\Pi_n \operatorname{cl} V_n^{f(n),i})$. Thus, $W_n \cap \operatorname{cl} V_n^{f(n),i} = 0$ for infinitely many n, so $W \cap A_f = 0$.

Thus, if we let f_α $(\alpha < c)$ enumerate ω^ω and let

$$B_\alpha = A_{f_\alpha} - \bigcup \{A_{f_\beta}: \beta < \alpha\},$$

then $\Pi_n X_n$ is expressed as $\bigcup_\alpha B_\alpha$, a disjoint union of clopen paracompact subspaces, and is hence paracompact.

References

[A] A. V. Arhangel'skiĭ, *The power of bicompacta with first axiom of countability*, Dokl. Akad. Nauk SSSR **187** (1969), 967–968 = Soviet Math. Dokl. **10** (1969), 951–955.

[vD 1] E. K. van Douwen, *The box product of countably many metrizable spaces need not be normal*, Fund. Math. **88** (1975), 127–132.

[vD 2] ______, *Another non-normal box product*, General Topology and Appl. **7** (1977), 71–76.

[vD 3] ______, *Separation and covering properties of box products and products* (to appear).

[J] T. Jech, *Lectures in set theory*, Lecture Notes in Math., vol. 217, Springer-Verlag, Berlin and New York, 1971.

[Kn] C. J. Knight, *The box topology*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 41–54.

[Ku] K. Kunen, *Some comments on box products*, Colloq. Math. Soc. János Bolyai, Infinite and finite sets, Vol. 10, Keszthely, 1973, pp. 1011–1016.

[M 1] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.

[M 2] ______, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. **8** (1957), 822–828.

[P] R. Pol, *Short proofs of two theorems on cardinality of topological spaces*, Bull. Acad. Polon Sci. **22** (1974), 1245–1249.

[R 1] M. E. Rudin, *The box product of countably many compact metric spaces*, General Topology and Appl. **2** (1972), 293–298.

[R 2] ______, *Countable box products of ordinals*, Trans. Amer. Math. Soc. **192** (1974), 121–128.

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706