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DESARGUESIAN KLINGENBERG PLANES

BY

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ABSTRACT. Klingenberg planes are generalizations of Hjelmslev planes. If R is a local ring, one can construct a projective Klingenberg plane $\mathbf{V}(R)$ and a derived affine Klingenberg plane $\mathbf{A}(R)$ from R. If V is a projective Klingenberg plane, if R_1, R_2 and R_3 are local rings, if s_1, s_2 and s_3 are the sides of a nondegenerate triangle in V, and if each of the derived affine Klingenberg planes $\mathcal{Q}(V, s_i)$ is isomorphic to $\mathbf{A}(R_i)$, $i = 1, 2, 3$, then the rings R_1, R_2 and R_3 are isomorphic, and V is isomorphic to $\mathbf{V}(R_1)$; also, if g is a line of V, then the derived affine Klingenberg plane $\mathcal{Q}(V, g)$ is isomorphic to $\mathbf{A}(R_1)$. Examples are given of projective Klingenberg planes V, each of which has the following two properties: (1) V is not isomorphic to $\mathbf{V}(R)$ for any local ring R; and (2) there is a flag (B, b) of V, and a local ring S such that each derived affine Klingenberg plane $\mathcal{Q}(V, m)$ is isomorphic to $\mathbf{A}(S)$ whenever $m = b$, or m is a line through B which is not neighbor to b.

1. Desarguesian projective Klingenberg planes.

1.1 DEFINITION. Let $S_V = (\mathfrak{P}, \mathfrak{g}, \mathrm{I})$ be an incidence structure, and let $\sim$ be an equivalence relation on the points and lines of S_V such that no point is related to any line. We say that $V = (S_V, \sim)$ is a *projective Klingenberg plane* (abbreviated *PK-plane*) and that $\sim$ is the *neighbor* relation of V whenever there is a map φ: $V \to V'$ satisfying the following condition.

(PK) φ is a surjective incidence structure homomorphism from V to a projective plane $V' = (S_{V'}, \sim')$ where $\sim'$ is the equality relation on the points and lines of $S_{V'}$ such that the following four conditions are satisfied for all $P, Q \in \mathfrak{P}$; $g, h \in \mathfrak{g}$.

(i) If $\varphi P \neq \varphi Q$, then there is a unique line of V joining P and Q. We denote this line by PQ or $P \vee Q$.

(ii) If $\varphi g \neq \varphi h$, then g and h meet in a unique point of V. We denote this point by $g \cap h$.

(iii) $P \sim Q \Leftrightarrow \varphi P = \varphi Q$.

(iv) $g \sim h \Leftrightarrow \varphi g = \varphi h$.

REMARK. It is easily seen that any projective plane with neighbor elements [12, D0] together with its neighbor relation is a projective Klingenberg plane. Projective planes with neighbor elements are frequently called projective

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Hjelmslev planes. Lenz and Drake [14] give a slightly different definition of (projective) Klingenberg plane.

1.2 DEFINITION. Let R be a local ring with maximal ideal N. If x_1, x_2, $x_3 \in R$, let $r(x_1, x_2, x_3)$ denote the set $\{(a_1, a_2, a_3) | \exists t \in R \setminus N \ni a_j = tx_j$ for $j = 1, 2, 3\}$. If $y_1, y_2, y_3 \in R$, let $(y_1, y_2, y_3)s$ denote the set $\{(b_1, b_2, b_3) | \exists t \in R \setminus N \ni b_j = y_j t$ for $j = 1, 2, 3\}$. Let $\mathfrak{P} = \{r(x_1, x_2, x_3) | \exists j \ni x_j \notin N\}$ and let $\mathfrak{g} = \{(y_1, y_2, y_3)s | \exists j \ni y_j \notin N\}$. Define incidence by $r(x_1, x_2, x_3)\ \mathrm{I}\ (y_1, y_2, y_3)s$ if and only if $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$. Incidence can be seen to be well defined. We frequently write $r(x_1, x_2, x_3)$ as rx_i, and $(y_1, y_2, y_3)s$ as $y_i s$. Let $\mathbf{S}_V(R) = (\mathfrak{P}, \mathfrak{g}, \mathrm{I})$. If θ: $R \to R'$ is a local ring homomorphism, let $\mathbf{S}_V(\theta)$: $\mathbf{S}_V(R) \to \mathbf{S}_V(R')$ be defined by $\mathbf{S}_V(\theta)(rx_i) = r(\theta x_1, \theta x_2, \theta x_3)$, and similarly for lines. Let ν: $R \to R/N$ be the quotient map. Define a relation $\sim$ by letting $rx_i \sim ra_i$ whenever $\mathbf{S}_V(\nu)(rx_i) = \mathbf{S}_V(\nu)(ra_i)$ and $y_i s \sim b_i s$ whenever $\mathbf{S}_V(\nu)(y_i s) = \mathbf{S}_V(\nu)(b_i s)$. Let $\mathbf{V}(R) = (\mathbf{S}_V(R), \sim)$. If θ: $R \to R'$ is a local ring homomorphism, let $\mathbf{V}(\theta)$: $\mathbf{V}(R) \to \mathbf{V}(R')$ have the same action on points and lines of $\mathbf{V}(R)$ that $\mathbf{S}_V(\theta)$ does.

1.3 PROPOSITION [5, PROPOSITION 5.2.3]. *Let R be a local ring with maximal ideal N, and let ν: $R \to R/N$ be the quotient map; then* $\mathbf{V}(R)$ *is a PK-plane and* $\mathbf{V}(\nu)$: $\mathbf{V}(R) \to \mathbf{V}(R/N)$ *satisfies condition* (PK).

1.4 DEFINITION. Let V, V' be PK-planes. We define a *PK-plane homomorphism* ω: $V \to V'$ to be an incidence structure homomorphism which preserves the neighbor relation; that is, $P\ \mathrm{I}\ g$ in V implies $\omega P\ \mathrm{I}\ \omega g$ in V'; $P \sim Q$ in V implies $\omega P \sim \omega Q$ in V', and $g \sim h$ in V implies $\omega g \sim \omega h$ in V'.

1.5 DEFINITION. Let B be an invertible 3×3 matrix over a local ring R. We define a map σ_B: $\mathbf{V}(R) \to \mathbf{V}(R)$ by $\sigma_B(rx_i) = r((x_1, x_2, x_3)B)$ and $\sigma_B(y_i s) = ((B^{-1}((y_1, y_2, y_3)^\dagger))^\dagger)s$ where † is the transpose operator. Observe that σ_B is well defined. We call σ_B a *projective map* of $\mathbf{V}(R)$.

1.6 PROPOSITION [5, PROPOSITION 5.2.5]. *Let R be a local ring with maximal ideal N. A projective map σ_B of* $\mathbf{V}(R)$ *is a PK-plane automorphism.*

1.7 DEFINITION. Let V be a PK-plane, and let g be a line of V. We say a point P is *near* the line g, and write $P \simeq g$, if there is a point Q on g such that Q is neighbor to P. We denote the negations of $\simeq$ and $\sim$ by $\not\sim$.

The following proposition is easily shown.

1.8 PROPOSITION. *Let V be a PK-plane, and let φ: $V \to V'$ be a map satisfying condition* (PK) *of the definition of PK-plane. Then $P \simeq g$ in V if and only if $\varphi P\ \mathrm{I}\ \varphi g$ in V'.*

1.9 DEFINITION. Let V be a PK-plane. We say that a sequence (P_1, P_2, P_3) of three pairwise nonneighbor points of V is a *nondegenerate triangle* in V, and that P_1P_2, P_2P_3 and P_3P_1 are the *sides* of (P_1, P_2, P_3) if there is no line g of V such that all three points are near g.

1.10 DEFINITION. Let V be a PK-plane. We say a sequence of four distinct points of V is a nondegenerate quadrangle if any subsequence of three distinct points is a nondegenerate triangle.

1.11 DEFINITION. Let R be a local ring. In $\mathrm{V}(R)$ we denote the point $r(1, 0, 0)$ by O'; $r(0, 1, 0)$ by O''; $r(0, 0, 1)$ by O''', and $r(1, 1, 1)$ by E.

1.12 LEMMA [5, LEMMA 5.2.9]. *Let R be a local ring, let N be the maximal ideal of R, and let $D = R/N$. Then a matrix A in $R_{n\times n}$ is nonsingular (that is, invertible) if and only if its image in $D_{n\times n}$ under the quotient map ν: $R_{n\times n} \to D_{n\times n}$ is nonsingular.*

1.13 PROPOSITION. *Let R be a local ring. The projective maps of* $\mathrm{V}(R)$ *are transitive on the nondegenerate quadrangles of* $\mathrm{V}(R)$.

PROOF. Observe that (O', O'', O''', E) is a nondegenerate quadrangle of $V(R)$. Since the projective maps form a group under composition, it will suffice to show that for any given nondegenerate quadrangle (P_1, P_2, P_3, P_4) there is a projective map which takes (P_1, P_2, P_3, P_4) to (O', O'', O''', E). Form a matrix Z by letting the ith row of Z be an element of P_i for $i = 1, 2, 3$. By Lemma 1.12, a 3×3 matrix over a local ring R is invertible if and only if its image in R/N is invertible. Hence, Z is invertible. Let (x_1, x_2, x_3) be an element of $\sigma_{Z^{-1}}(P_4)$; then, since $\sigma_{Z^{-1}}$ is an automorphism, $\sigma_{Z^{-1}}(P_4)$ is not near $O'O''$, $O'O'''$ and $O''O'''$; hence $x_j \in R \setminus N$ for $i = 1, 2, 3$. Let B be the diagonal matrix with $B_{jj} = x_j^{-1}$ for $j = 1, 2, 3$. Then $\sigma_{Z^{-1}B}$ is the desired projective map.

1.14 DEFINITION. Let V be a PK-plane. We say V is *desarguesian* if there is a local ring R and a PK-plane isomorphism μ: $V \to \mathrm{V}(R)$. We call such an isomorphism a (*desarguesian*) *representation* of V.

2. Desarguesian derived affine Klingenberg planes.

2.1 DEFINITION. Let V be a PK-plane, and let g be a line of V. We obtain an incidence structure $S_{\mathcal{A}(V,g)}$ by removing all the points near g and lines neighbor to g from V and restricting the incidence relation appropriately. We define an equivalence relation $\|$ on the lines of $S_{\mathcal{A}(V,g)}$ by letting $h \| k$ whenever h, k and g have a point in common in V. We restrict the neighbor relation of V to the points and lines of $S_{\mathcal{A}(V,g)}$, and we denote this new relation by $\sim$. We let $\mathcal{A}(V, g) = (S_{\mathcal{A}(V,g)}, \|, \sim)$, and we call $\mathcal{A}(V, g)$ a *derived affine Klingenberg plane* (abbreviated *derived AK-plane*). We call $\|$ the *parallel* relation, and $\sim$ the *neighbor* relation of $\mathcal{A}(V, g)$.

2.2 DEFINITION. Let A, A' be derived AK-planes. A map ω: $A \to A'$ is said to be a *derived AK-plane homomorphism* if ω is an incidence structure homomorphism which preserves both the neighbor and parallel relations.

2.3 DEFINITION. Let $A = \mathcal{Q}(V, g)$ be a derived AK-plane. We say A is *desarguesian* if there is a local ring R and a derived AK-plane isomorphism α: $A \to \mathcal{Q}(\mathbf{V}(R), (1, 0, 0)s)$, and we call such an isomorphism a (*desarguesian*) *representation* of A. For any local ring R, let $\mathbf{A}(R) = \mathcal{Q}(\mathbf{V}(R), (1, 0, 0)s)$.

2.4 DEFINITION. Let R be a local ring. Observe that those points and lines of $\mathbf{V}(R)$ which are in $\mathbf{A}(R)$ are $r(1, x, y)$ for $x, y \in R$; $(d, m, -1)s$ for $d, m \in R$, and $(v, -1, u)s$ for $v, u \in R$. For convenience, we denote $r(1, x, y)$ by (x, y); $(d, m, -1)s$ by $[m, d]$, and $(v, -1, u)s$ by $[u, v]'$ for all $x, y, m, d, u, v \in R$. We denote the set of lines of $\mathbf{A}(R)$ which meet at the point $r(0, 1, m)$ on $(1, 0, 0)s$ by (m) and the set of lines of $\mathbf{A}(R)$ which meet at the point $r(0, u, 1)$ on $(1, 0, 0)s$ by $(u)'$. We denote the line $[0, 0]$ by g_x, the line $[0, 0]'$ by g_y and the point $(1, 1)$ by E.

3. The Triangle Theorem.

3.1 LEMMA. *Let V be a PK-plane. Let (P', P'', P''', F) be a nondegenerate quadrangle in V; let $k = P'' \vee P'''$, and let α: $\mathcal{Q}(V, k) \to \mathbf{A}(R)$ be a representation of $A(V, k)$. Then there is a representation α': $\mathcal{Q}(V, k) \to \mathbf{A}(R)$ such that $\alpha'(P') = (0, 0)$; $\alpha'(F) = (1, 1)$; $\alpha'(P' \vee P'') = [0, 0]$, and $\alpha'(P' \vee P''') = [0, 0]'$.*

PROOF. In $\mathbf{V}(R)$,

$$Z = (\alpha P', \alpha(P' \vee P'') \cap (1, 0, 0)s, \alpha(P' \vee P''') \cap (1, 0, 0)s, \alpha F)$$

is a nondegenerate quadrangle; hence by Proposition 1.13, there is a projective map σ_B of $\mathbf{V}(R)$ which takes Z to (O', O'', O''', E). Since σ_B is an automorphism of $\mathbf{V}(R)$ which fixes the line $(1, 0, 0)s$, σ_B induces an automorphism α_B of $\mathbf{A}(R)$. Let $\alpha' = \alpha_B\alpha$. Then α' has the desired properties.

3.2 DEFINITION. Assume we have desarguesian representations of AK-planes which are distinguished by superscripts; for example, assume we have α': $A' \to \mathbf{A}(S')$ and α'': $A'' \to \mathbf{A}(S'')$. Then we denote $(\alpha')^{-1}(x, y)$ by $'(x, y)$ and use other similar notation. For example, $''g_x = (\alpha'')^{-1}g_x$ where $g_x = (0, 0, 1)s$. If $A' = \mathcal{Q}(V, k')$, we denote $(\alpha')^{-1}[m, 0] \cap k'$ by $'(m)$ and so on.

3.3 LEMMA. *Let V be a PK-plane, let k' and k'' be nonneighbor lines of V, and let $A' = \mathcal{Q}(V, k')$ and $A'' = \mathcal{Q}(V, k'')$. Assume that there are local rings H' and H'' such that A' is isomorphic to $\mathbf{A}(H')$, and A'' is isomorphic to $\mathbf{A}(H'')$. Then the following conditions hold.*

(i) *H'' is isomorphic to H'; hence A'' is isomorphic to $\mathbf{A}(H')$ and A'.*

(ii) *If β': $A' \to \mathbf{A}(S)$ is a desarguesian representation of A', and if $'g_y = k''$, then there is a desarguesian representation β'': $A'' \to \mathbf{A}(S)$ of A'' such that $''g_x = k'$; $''g_y = 'g_x$ and $'(1, y) = ''(y, 1)$ for every $y \in S$.*

(iii) *If α': $A' \to \mathbf{A}(R)$ and α'': $A'' \to \mathbf{A}(R)$ are desarguesian representations of*

A' and A'' such that $'g_y = k''$; $''g_x = k'$; $''g_y = 'g_x$ and $'(1, y) = ''(y, 1)$ for every $y \in R$, then $'[m, d] = ''[d, m]'$ for all d, $m \in R$, and if $x \in R \setminus N$ where N is the maximal ideal of R, then $'(x, y) = ''(x^{-1}y, x^{-1})$.

PROOF. (i) Let φ: $V \to V'$ be a map satisfying condition (PK) of the definition of PK-plane. Since φ is surjective, there is, by Proposition 1.8, a point K'' on k'' which is not near k', and a point K' on k' which is not near k''. Let $k = K' \vee K''$, and let $O = k' \cap k''$. Let Z be a point not near k', k'', k; such a point exists as can be seen by looking at the images of k', k'', k in V'. Then (K'', K', O, Z) is a nondegenerate quadrangle in V as can be seen by looking at its image in V'. Let ξ: $A' \to \mathrm{A}(H')$ be a derived AK-plane isomorphism. By Lemma 3.1, there is a representation ξ': $A' \to \mathrm{A}(H')$ such that $\xi'(k'') = \xi'(K'' \vee O) = g_y$.

(ii) and a continuation of (i). Assume A' has a desarguesian representation β': $A' \to \mathrm{A}(M')$ such that $'g_y = k''$. The construction of the preceding paragraph satisfies this requirement. Let $G'' = 'g_x \cap k'$, and $G''' = 'g_y \cap k'$. Then $(G'', G''', 'O', 'E)$ is a nondegenerate quadrangle in V. Let β: $A'' \to \mathrm{A}(M'')$ be a representation of A''. By Lemma 3.1, there is a representation β'': $A'' \to \mathrm{A}(M'')$ such that $''O' = G''$; $''E = 'E$; $''g_x = k'$, and $''g_y = 'g_x$. The line $'[0, 1]'$ is equal to the line $''[0, 1]$. Observe that $'(1, n) \sim '(1, 0)$ if and only if n is in the maximal ideal of M', and that $''(m, 1) \sim ''(0, 1)$ if and only if m is in the maximal ideal of M''. Thus we can change the symbols of M'' to match those of M' in such a way that the maximal ideals are equal and $'(1, y) = ''(y, 1)$ for all $y \in M'$.

(iii) and a continuation of (i) and (ii). Assume we have representations α': $A' \to \mathrm{A}(R')$ and α'': $A'' \to \mathrm{A}(R'')$ such that $'g_y = k''$; $''g_x = k'$; $''g_y = 'g_x$; there is a set R which is the symbol set of both R' and R''; the maximal ideals of R' and R'' are equal, and $'(1, y) = ''(y, 1)$ for every $y \in R$. The construction of the preceding paragraph satisfies these requirements. Let $R' = (R, +, \#)$ and $R'' = (R, *, \cdot)$, and write $r \# s$ as rs for all $r, s \in R$. Let $E = 'E = ''E$. Let N be the maximal ideal of R'. For each $x \in R \setminus N$, there is a multiplicative inverse x' in R' and a multiplicative inverse x'' in R''. We wish to show that $x' = x''$.

Let $'(m) = '[m, 0] \cap k'$; let $'(u)' = '[u, 0]' \cap k'$; let $''(m) = ''[m, 0] \cap k''$, and let $''(u)' = ''[u, 0]' \cap k''$ for all $m, u \in R$. The line $'[m, 0]$ joins $'(0, 0)$ and $'(1, m)$. Also,

$$'(m) = '[m, 0] \cap k' = (''(m, 1) \vee ''(0)') \cap ''[0, 0] = ''(m, 0).$$

Once we have shown a relationship between the coordinates of A' and A'', we have, by the symmetry of the situation, that a similar relationship holds with the roles of A' and A''; R' and R''; k' and k'', and the second and third positions of the triples interchanged. Since $'[m, 0] \cap k' = '(m) = ''(m, 0)$, we have that $'(0, m) = ''(m)'$ for all $m \in R$. Also,

$$'[m, b] = '(m) \vee '(0, b) = ''(m, 0) \vee ''(b)' = ''[b, m]'$$

for all $m, b \in R$. Since $'[m, b] = ''[b, m]'$, and since $'(1, y) = ''(y, 1)$ for all $m, b, y \in R$, and since OE where $O = k' \cap k''$ meets $'[m, b]$ at $'(1, m + b)$ and $''[b, m]'$ at $''(m * b, 1)$, we have $m + b = m * b$ for all $m, b \in R$.

Let $x \in R \setminus N$. Then since x' is the inverse of x in R'; x'', in R'', we have

$$'(x, 1) = '[x', 0] \cap '[0, 1] = ''[0, x']' \cap ''[1, 0]' = ''(x', x').$$

Using this, we have

$$'[0, x]' = '(x, 1) \vee O = ''(x', x') \vee O = ''[0, x'];$$

hence

$$'(x, 0) = '[0, x]' \cap '[0, 0] = ''[0, x'] \cap ''[0, 0]' = ''(0, x').$$

Using the interchange symmetry, we have that $''(0, y) = '(y'', 0)$ for all $y \in R \setminus N$. Hence, $'(x, 0) = ''(0, x') = '((x')'', 0)$. Thus, $x' = x''$ for all $x \in R \setminus N$. Let $x^{-1} = x' = x''$ for all $x \in R \setminus N$.

Let $x \in R \setminus N$ and $y \in R$. Then,

$$'(x, y) = '[0, x]' \cap '[0, y] = ''[0, x^{-1}] \cap ''[y, 0]' = ''(x^{-1} \cdot y, x^{-1}).$$

By the interchange symmetry, $''(x, y) = '(y^{-1}, y^{-1}x)$.

We wish to show that $a \cdot b = ab$ for all $a, b \in R$. Let $a \in R \setminus N$; $b \in R$. From our relations above we have with $x = a^{-1}$; $y = b$, that $'(a^{-1}, b) = ''(a \cdot b, a)$, and with $x = a \cdot b, y = a$, that $''(a \cdot b, a) = '(a^{-1}, a^{-1}(a \cdot b))$. Hence $b = a^{-1}(a \cdot b)$, and $ab = a \cdot b$ when $a \in R \setminus N$; $b \in R$. Let $a \in N$; $b \in R$. Then since $1 \notin N$, and since $(N, +)$ is a subgroup of $(R, +)$, we have $a - 1 \notin N$; hence

$$a \cdot b = ((a - 1) + 1) \cdot b = (a - 1) \cdot b + b = (a - 1)b + b = ab.$$

Hence, $a \cdot b = ab$ for all $a, b \in R$. Thus, $R' = (R, +, \#) = (R, *, \cdot) = R''$, and we are done.

3.4 THE TRIANGLE THEOREM. *Let V be a projective Klingenberg plane which has a nondegenerate triangle with sides k', k'' and k''' such that each of the derived affine Klingenberg planes $A' = \mathfrak{A}(V, k')$, $A'' = \mathfrak{A}(V, k'')$ and $A''' = \mathfrak{A}(V, k''')$ is desarguesian. Then V is desarguesian, and if R', R'' and R''' are local rings such that A' is isomorphic to* $\mathbf{A}(R')$; *A'' to* $\mathbf{A}(R'')$, *and A''' to* $\mathbf{A}(R''')$, *then R', R'' and R''' are isomorphic, and V is isomorphic to* $\mathbf{V}(R')$.

PROOF. If R', R'' and R''' are local rings such that A' is isomorphic to $\mathbf{A}(R')$; A'' to $\mathbf{A}(R'')$, and A''' to $\mathbf{A}(R''')$, then by Lemma 3.3, R', R'' and R''' are isomorphic.

Let α: $A' \to \mathbf{A}(R')$ be a desarguesian representation of A', and let $R = R'$.

Let $\varphi: V \to V'$ be a map satisfying condition (PK). Let F be a point such that φF is not on $\varphi k'$, $\varphi k''$, $\varphi k'''$. It is easily seen that $(k' \cap k''', k''' \cap k'', k'' \cap k', F)$ is a nondegenerate quadrangle of V. By Lemma 3.1, there is a representation $\alpha': A' \to \mathrm{A}(R)$ of A' with $'g_y = k''$; $'g_x = k'''$, and $'E = F$.

Let N be the maximal ideal of R.

By Lemma 3.3, there is a representation $\alpha'': A'' \to \mathrm{A}(R)$ of A'' such that $''g_y = 'g_x = k'''$; $''g_x = k'$; $''E = 'E = F$, and $'(1,y) = ''(y,1)$ for all $y \in R$. Thus, if $P \not\sim k', k''$, and if $P = '(x,y)$, then $x \notin N$, and by Lemma 3.3, $'(x,y) = ''(x^{-1}y, x^{-1})$. Similarly, by Lemma 3.3, there is a representation $\alpha''': A''' \to \mathrm{A}(R)$ of A''' such that $'''g_y = ''g_x = k'$; $'''g_x = k'' = 'g_y$; $'''E = F$, and if $Q \not\sim k'', k'''$, and $Q = ''(x,y)$, then $x \notin N$, and $''(x,y) = '''(x^{-1}y, x^{-1})$.

We wish to show that if $P \not\sim k''', k'$ and $P = '''(x,y)$, then $x \notin N$, and $'''(x,y) = '(x^{-1}y, x^{-1})$. Let Q be a point of V which is not near k', k'', k'''. Then $Q = '''(x,y)$ where $x, y \notin N$. Using the relations above, $Q = ''(y^{-1}, y^{-1}x) = '(x^{-1}y, x^{-1})$. If $x, y \notin N$, then $'''(x,y) \not\sim k', k'', k'''$. Let $n \in N$, $d \notin N$. The line $'''[0,d]$ contain the points $'O'$ and $'''(d,d) = '(1, d^{-1})$; hence $'''[0,d] = '[d,0]'$, and $'''(0,d) = '(d)'$. Since $n \in N$ and $d \notin N$, we have $n + d \notin N$; so that $'''(1, n+d) = '(n+d, 1)$, and since $'''(0,d) = '(d)'$, we have $'''[n,d] = '[d,n]'$; so that $'''(n) = '(n,0)$, and hence $'''[n,0] = '[0,n]'$. Thus,

$$'''(1,n) = '''[n,0] \cap '''[0,1]' = '[0,n]' \cap '[0,1] = '(n,1).$$

If $G \not\sim k''', k'$, and if $G = '''(x,y)$, then $x \notin N$, and by Lemma 3.3, $'''(x,y) = '(x^{-1}y, x^{-1})$. Thus the relationships between the three representations α', α'' and α''' can now be treated in a cyclically symmetric fashion.

For each i, define $\lambda^i: (S^i, \sim^i) \to \mathrm{V}(R)$ where S^i is the incidence structure of A^i; $\sim^i$, the neighbor relation of A^i, by $\lambda'('(x,y)) = r(1,x,y)$; $\lambda''(''(x,y)) = r(y,1,x)$ or $\lambda'''('''(x,y)) = r(x,y,1)$, and correspondingly for lines. Observe that since each of the α^i is an automorphism, each of the λ^i is injective and preserves and reflects both the incidence and neighbor relations.

Let $\lambda: V \to \mathrm{V}(R)$ be defined by $\lambda P = \lambda^i(P)$ and $\lambda g = \lambda^i(g)$ whenever P, g are in S^i. We wish to show that λ is well defined. Since k', k'', k''' are the sides of a nondegenerate triangle, the map λ is defined for all points and lines of V. Let P be a point in A' and in A''. Assume $P = '(x,y)$. Then $P = ''(x^{-1}y, x^{-1})$ and $\lambda'(P) = \lambda''(P)$. By symmetry, λ is well defined on points. Let g be a line in A' and in A''. Since $g = '[m,d]$ if and only if $g = ''[d,m]'$, there are two cases. Assume first that $g = '[m,d]$. Then $\lambda'(g) = (d, m, -1)s = \lambda''(g)$. If g cannot be written as $'[m,d]$ for any $m, d \in R$, then $g = '[u,v]'$ for some $u \in N$, $v \in R$; similarly, if g cannot be written as $''[d,m]'$ for any $d, m \in R$, then $g = ''[w,z]$ for some $w \in N$, $z \in R$. Assume now that $g = '[u,v]'$, $u \in N$, and $g = ''[w,z]$, $w \in N$. Let $P = g \cap 'g_x = g \cap ''g_y$.

Then $P = '(v, 0) = ''(0, z)$; hence $z = v^{-1}$. Let $Q = g \cap '[1, 0] = g \cap ''[0, 1]'$. Then

$$Q = '\left(v(1-u)^{-1}, v(1-u)^{-1}\right) = ''(1, w + v^{-1}).$$

Thus, $Q = ''(1, (1-u)v^{-1})$; hence $w = -uv^{-1}$. Observe that

$$\lambda'(g) = (v, -1, u)s = (-1, v^{-1}, -uv^{-1})s = \lambda''(g).$$

By symmetry, λ is well defined on lines. Thus, λ is well defined.

Let $h' = (1, 0, 0)s$; $h'' = (0, 1, 0)s$ and $h''' = (0, 0, 1)s$. It is easily seen that λ takes the points and lines of A^i onto the points and lines of $\mathcal{Q}(\mathbf{V}(R), h^i)$. Assume P I g in V. Since P is not near one of k', k'', k''', say k^j, both P and g are in A^j. Since λ^j preserves incidence, λP I λg in $\mathbf{V}(R)$; hence λ preserves incidence. Assume λQ I λf in $\mathbf{V}(R)$. Since λQ is not near one of the h^i, say h^j, both λQ and λf are in $\mathcal{Q}(\mathbf{V}(R), h^j)$. Since λ^j reflects incidence, Q I f in V. Hence λ reflects incidence. Similar arguments show that λ preserves and reflects the neighbor relation.

Observe that any point of $\mathbf{V}(R)$ can be written as rx_i with one of the x_i equal to 1, and that any line of $\mathbf{V}(R)$ can be written as $y_i s$ with one of the y_i equal to -1; hence λ is surjective. Assume that $\lambda P = \lambda Q$; $P \neq Q$. Then P and Q are not in the same A^i. Assume that P is in A' and that Q is in A''. Then $P = '(x, y)$ where $x \in N$. Thus, $\lambda P = rx_i$, then $x_2 \in N$, a contradiction to Q being in A''. The other cases are similar by symmetry; hence λ is injective on points. For each i, each line of S^i has at least two pairwise nonneighbor points on it by Proposition 1.8; hence λ is injective on lines. Hence λ: $V \to \mathbf{V}(R)$ is an isomorphism and V is desarguesian.

3.5 Proposition. *Let V be a desarguesian PK-plane with a representation μ: $V \to \mathbf{V}(R)$, and let g be a line of V. Then $\mathcal{Q}(V, g)$ is desarguesian, and there is a representation α: $\mathcal{Q}(V, g) \to \mathbf{A}(R)$.*

Proof. One can construct a nondegenerate quadrangle $Z = (Q', Q'', Q''', F)$ of V such that $g = Q'' \vee Q'''$. By Proposition 1.13, there is a projective map σ_B of $\mathbf{V}(R)$ which takes Z to (O', O'', O''', E). Then $\sigma_B\mu$ induces a derived AK-plane isomorphism α: $\mathcal{Q}(V, g) \to \mathbf{A}(R)$.

3.6. Proposition. *Let R and S be local rings. Then, $\mathbf{A}(R)$ is isomorphic to $\mathbf{A}(S)$ if and only if R is isomorphic to S.*

Proof. If R is isomorphic to S, then $\mathbf{A}(R)$ is isomorphic to $\mathbf{A}(S)$.

Assume ξ: $\mathbf{A}(R) \to \mathbf{A}(S)$ is an isomorphism. Observe that $Z = (\xi O', \xi E, \xi g_x \cap (1, 0, 0)s, \xi g_y \cap (1, 0, 0)s)$ is a nondegenerate quadrangle in $\mathbf{V}(S)$. By Proposition 1.13, there is a projective map σ_B which takes Z to (O', E, O'', O'''). The map σ_B induces an automorphism α_B of $\mathbf{A}(S)$. Then $\eta = \alpha_B \xi$ takes $(0, 0)$, $(1, 1)$, g_x and g_y of $\mathbf{A}(R)$ to $(0, 0)$, $(1, 1)$, g_x and g_y of

$\mathbf{A}(S)$. Also, η takes the points of $[1, 0]$ to the points of $[1, 0]$. Define $\varphi: R \to S$ by letting $\varphi(a) = b$ whenever $\varphi(a, a) = (b, b)$. Then using the lines through O'' and O''', one can see that $\eta(x, y) = (\varphi x, \varphi y)$ for all $x, y \in R$. The line $[m, 0]$ joins O' and $(1, m)$ in $\mathbf{A}(R)$, and $[\varphi m, 0]$ joins O' and $(1, \varphi m)$ in $A(S)$; then $\eta[m, 0] = [\varphi m, 0]$. The line $[m, d]$ goes through $(0, d)$ and is parallel to $[m, 0]$ in $\mathbf{A}(R)$, and $[\varphi m, \varphi d]$ goes through $(0, \varphi d)$ and is parallel to $[\varphi m, 0]$ in $A(S)$; thus $\eta[m, d] = [\varphi m, \varphi d]$. Thus, $(\varphi x, \varphi(xm + d)) = (\varphi x, \varphi x(\varphi m) + \varphi d)$ for all $x, m, d \in R$; hence φ is a ring homomorphism. Since φ is a bijection, R is isomorphic to S.

3.7 PROPOSITION. *Let R and S be local rings. Then $\mathbf{V}(R)$ is isomorphic to $\mathbf{V}(S)$ if and only if R is isomorphic to S.*

PROOF. If R is isomorphic to S, then $\mathbf{V}(R)$ is isomorphic to $\mathbf{V}(S)$.

Assume that $\mu: \mathbf{V}(R) \to \mathbf{V}(S)$ is an isomorphism. Then $Z = (\mu O', \mu O'', \mu O''', \mu E)$ is a nondegenerate quadrangle in $\mathbf{V}(S)$. By Proposition 1.13, there is a projective map σ_B which takes Z to (O', O'', O''', E). Then $\sigma_B \mu$ induces an isomorphism from $\mathbf{A}(R)$ to $\mathbf{A}(S)$. By Proposition 3.6, R is isomorphic to S.

4. **Desarguesian Hjelmslev planes.** The term *affine Hjelmslev plane* (abbreviated *AH-plane*) is defined in [17, Definition 2.3]; also see [17, Satz 2.6] for an equivalent definition.

The term *projective plane with neighbor elements* is defined in [12, D0]. We will usually call a projective plane with neighbor elements a *projective Hjelmslev plane* (abbreviated *PH-plane*).

4.1 DEFINITION. A local ring S with maximal ideal N is said to be an *AH-ring* if every element of N is both a right and left zero divisor, and whenever $x, y \in N$, there is an $m \in S$ such that $y = xm$, or there is a $u \in S$ such that $x = yu$. An AH-ring S' with maximal ideal N' is said to be an *H-ring* if whenever $m, n \in N'$, there is an $x \in S$ such that $n = xm$, or there is a $y \in S$ such that $m = yn$.

4.2 PROPOSITION. *Let R be a local ring. Let $R' = (R, +, \cdot)$ be the ring anti-isomorphic to R; that is, $a \cdot b = ba$ for all a, $b \in R$. The incidence structure of $\mathbf{V}(R')$ is the dual of that of $\mathbf{V}(R)$ and the neighbor relation of $\mathbf{V}(R')$ is the same as that of $\mathbf{V}(R)$.*

4.3 PROPOSITION. *Let R be a local ring with maximal ideal N. Any two neighbor points of $\mathbf{V}(R)$ are joined by at least one line if and only if whenever x, $y \in N$, there is an $m \in R$ such that $y = xm$, or there is a $u \in R$ such that $x = yu$.*

PROOF. By the transitivity on nondegenerate quadrangles, it suffices to consider the pairs of points $r(1, 0, 0)$, $r(1, x, y)$ such that $x, y \in N$.

4.4 PROPOSITION. *Let R be a local ring with maximal ideal N. No two neighbor lines of* $\mathrm{V}(R)$ *meet in exactly one point if and only if whenever $n \in N$, there is a $k \in R \setminus \{0\}$ such that $kn = 0$.*

PROOF. By the transitivity on nondegenerate quadrangles, it suffices to consider the incidences $r(1, 0, 0), r(1, k, 0)$ I $(0, 0, -1)s, (0, n, -1)s$ such that $n \in N$.

By Proposition 4.2, the following two propositions follow immediately.

4.5 PROPOSITION. *Let R be a local ring with maximal ideal N. No two neighbor points of* $\mathrm{V}(R)$ *are joined by exactly one line if and only if whenever $n \in N$, there is a $k \in R \setminus \{0\}$ such that $nk = 0$.*

4.6 PROPOSITION. *Let R be a local ring with maximal ideal N. Any two neighbor lines of* $\mathrm{V}(R)$ *meet in at least one point if and only if whenever m, $n \in N$, there is an $x \in R$ such that $n = xm$, or there is a $y \in R$ such that $m = yn$.*

By transitivity on nondegenerate quadrangles, propositions corresponding to Propositions 4.3, 4.4 and 4.5 hold in $\mathrm{A}(R)$; we call these new propositions Propositions 4.3a, 4.4a and 4.5a. Baker [7] shows Propositions 4.3a and 4.4a, and a result closely related to Proposition 4.5a.

REMARK. Lorimer [15] proves that if R is an AH-ring, then $\mathrm{A}(R)$ is an AH-plane. Another proof of this result is given in Bacon [4]; also see Lorimer and Lane [16].

The following three propositions now follow easily from Propositions 4.3–4.6 and 4.3a–4.5a.

Baker did not state the following proposition in [7]; however it is an easy consequence of results she obtains there.

4.7 PROPOSITION. *Let R be a local ring. If* $\mathrm{A}(R)$ *is an AH-plane, then R is an AH-ring.*

4.8 PROPOSITION. *Let R be a local ring. If* $\mathrm{V}(R)$ *is a PH-plane, then R is an H-ring.*

4.9 PROPOSITION. *If R is an H-ring, then* $\mathrm{V}(R)$ *is a PH-plane.*

REMARK. Klingenberg [12, S 28] claims to have shown the above result. There is an error in his argument for S 28 where he says that the lines $(u_1, 1, u_3)s$ and $(u_1, 1 + \bar{c}, u_3)s$ are distinct. This fails as follows: let K be a field, and let $\bar{c}, u_1, u_3$ be in the maximal ideal $\{kX \mid k \in K\}$ of the ring $K[X]/(X^2)$; then, by multiplying the triple $(u_1, 1 + \bar{c}, u_3)s$ on the right by $(1 - \bar{c})$, we see that the two lines are equal.

REMARK. There are AH-rings which are not H-rings; see [4, Construction 1.1].

4.10 Proposition. *Let V be a projective Hjelmslev plane, and let s_1, s_2, s_3 be the sides of a nondegenerate triangle in V. If there are local rings R_1, R_2, R_3 such that $\mathcal{Q}(V, s_i)$ is isomorphic to $\mathbf{A}(R_i)$ for $i = 1, 2, 3$, then V is isomorphic to $\mathbf{V}(R_1)$, and R_2 and R_3 are isomorphic to R_1. Thus, R_1 is an H-ring.*

Proof. This is immediate by the Triangle Theorem and Proposition 4.8.

Remark. The preceding proposition is quoted and used in Dugas [10].

4.11 Proposition. *Let V be a PH-plane, and let k' and k'' be nonneighbor lines of V. If $A' = \mathcal{Q}(V, k')$ and $A'' = \mathcal{Q}(V, k'')$ are desarguesian, and if α: $A' \to \mathbf{A}(S)$ is a representation of A', then S is an H-ring.*

Proof. Obviously V is a PK-plane. One can construct a nondegenerate quadrangle (Q', Q'', Q''', F) such that $k' = Q'' \vee Q'''$ and $k'' = Q' \vee Q'''$. By Lemmas 3.1 and 3.3, if α: $A' \to \mathbf{A}(S)$ is a representation, then there are representations α': $A' \to \mathbf{A}(S)$ and α'': $A'' \to \mathbf{A}(S)$ such that $'g_y = k''$; $''g_x = k'$; $''E = 'E = F$, and $'[m, d] = ''[d, m]'$ for all $m, d \in S$.

It is well known (see Lüneburg [17, p. 260]) that A' and A'' are AH-planes whose neighbor relations are induced from V. Since A' is isomorphic to $\mathbf{A}(S)$, we have, by Proposition 4.7, that S is an AH-ring. Let $n, m \in S$. The lines $'[0, 0]$ and $'[m, -n]$ must meet in A' or in A''. Let P be a point on both lines. If P is in A', then $P = '(x, 0)$ for some $x \in S$; hence $xm = n$. If P is in A'', then $P = ''(0, y)$ for some $y \in S$; hence $yn = m$. Thus S is an H-ring.

Remark. Part of the preceding argument is adapted from Klingenberg's argument for S 26 in [12].

5. Examples.

5.1. Examples. Let r be an integer such that there is a field F with r elements and an affine plane A of order r which is not isomorphic to $\mathbf{A}(F)$. Let k be an integer greater than 1, and let $R = F[X]/(X^{k+1})$. Then R is a commutative H-ring and $\mathbf{V}(R)$ is a PH-plane. Let H be the set of points which are joined to O' by r^k or more lines. Then the lines of $\mathbf{V}(R)$ induce an affine plane A' of order r isomorphic to $\mathbf{A}(F)$ on the points of H. Let $h'' = (0, 1, 0)s$, and observe that O' is on h''. The incidences of $\mathbf{V}(R)$ associated with the points of H can be changed in such a way that the resulting incidence structure together with the original neighbor relation is a PH-plane W, that the lines of W induce an affine plane on the points of H which has the same parallel classes as A', and which is isomorphic to A (and hence is not isomorphic to $\mathbf{A}(F)$), and that H is the set of points which are joined to O' by r^k or more lines. Moreover, this change can be made in such a way that if g, k are lines of $\mathbf{V}(R)$, then $P \mathrel{\mathrm{I}} h'', g, k$ in W if and only if $P \mathrel{\mathrm{I}} h'', g, k$ in $\mathbf{V}(R)$. Thus, $\mathcal{Q}(W, h'') = \mathcal{Q}(\mathbf{V}(R), h'')$ is isomorphic to $\mathbf{A}(R)$. If m is a line such that O' is near m, and m does not meet H, then

$\mathcal{A}(W, m) = \mathcal{A}(\mathbf{V}(R), m)$ is isomorphic to $\mathbf{A}(R)$. It is routine to show that W is not desarguesian. Let $B = r(1, 0, X)$. Then $B \mathrm{I} h''$, and there are at least $r^{k+1} + r^k - r^2 + 1$ lines d through B such that $\mathcal{A}(W, d)$ is isomorphic to $\mathbf{A}(R)$, and the less than r^2 remaining lines are pairwise neighbor.

The above examples were inspired by results in Artmann [1], [2]; also see Bacon [3], [4].

5.2 Examples. There are constructions in Drake [8, Theorem 5.2] and Artmann [2, Satz 2] (see Bacon [4, Proposition 1.3 and Theorem 2.1]) which can be used to construct nondesarguesian PH-planes V each of which has a line g such that $\mathcal{A}(V, g)$ is isomorphic to $\mathbf{A}(R)$ for some commutative H-ring R.

6. **Counterexamples and applications.** Examples 5.1 and 5.2 are counterexamples to the theorem in Klingenberg [12, p. 110] by S 29 of [12]. Examples 5.1 and 5.2 are counterexamples to Hauptsatz 2 of Klingenberg [13] by the last four lines of Hauptsatz 2 [13, p. 191, lines 1–4] which can be routinely checked in a manner similar to the proof of S 29 of [12]. Examples 5.1 and 5.2 are counterexamples to S 5.17 of Klingenberg [11] by Lemma 6.1 below.

In Bacon [5], various characterizations are given of desarguesian and pappian desarguesian AK- and PK-planes. For example, it is shown that an AK-plane V is desarguesian if and only if the automorphisms of V are (P, g_∞)-, (Γ, g_∞)- and (Σ, g_∞)-transitive for some point P and some nonneighbor directions Γ and Σ. The characterizations of desarguesian and pappian desarguesian PK-planes are applications of the Triangle Theorem. In [5], various characterizations are also given of translation AK-planes, affinely moufang AK-planes, and moufang PK-planes. Some of these characterizations involve biternary rings. Also, PK-planes are classified by elation type in [5].

6.1. Lemma. *Let R be a commutative H-ring. Then* $\mathbf{A}(R)$ *satisfies axioms δ and Π as defined in* [11].

Proof. One can use automorphisms of the type $\sigma(x, y) = (x + a, y + b)$ to show axiom δ, and one can use a routine calculation with $g = [0, 0]$; $g' = [0, 0]'$; $P_1' = (0, 1)$; $P_2 = (-a, 0)$; $P_3 = (-b, 0)$; $p_{21} = [a, -a]'$; $p_{23} = [c, ac]$, and so on, together with Proposition 1.13 above to show axiom Π.

Hauptsatz 1 and the last four lines of Hauptsatz 2 of Klingenberg [13] can be used with the Triangle Theorem to obtain the following theorem.

6.2 Theorem. *Let V be a projective Klingenberg plane, and let s_1, s_2, s_3 be the sides of a nondegenerate triangle of V. Then V is isomorphic to* $\mathbf{V}(R)$ *for some local ring R if and only if each $\mathcal{A}(V, s_i)$ for $i = 1, 2, 3$ satisfies axioms* (d) *and* (D).

In Bacon [5, Theorem 11.2.4] it is shown that if W is a group of automorphisms of a PK-plane V and if W is (P, p)-, (Q, q)- and (R, r)-transitive where $P \mathrm{I} p$, $Q \mathrm{I} q$, $R \mathrm{I} r$ and (P, Q, R) is a nondegenerate triangle of V, then W has a star spine center; if, in addition, (p, q, r) is a nondegenerate triangle in the dual of V, then W is moufang, and it is shown, using the Triangle Theorem, that if W is moufang, and (G, h)-transitive for some G, h such that $G \not\sim h$, then V is desarguesian, and W is projectively desarguesian [5, Proposition 11.4.4], [6, Correction A.11.1].

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