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A CHARACTERIZATION OF UPPER-EMBEDDABLE GRAPHS

BY

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ABSTRACT. It is proved that a pseudograph G is upper-embeddable if and only if it has a spanning tree T such that $G - T$ has at most one component with an odd number of edges. This result is then used to show that all 4-edge connected graphs are upper-embeddable.

I. Introduction. In 1971, Nordhaus, Stewart and White [3] introduced the idea of the maximum genus $\gamma_M(G)$ of a connected graph G. They defined this parameter as the greatest genus among those orientable surfaces into which the graph has a 2-cell embedding. They also obtained the upper bound (3). Ringeisen, who has studied the maximum genus extensively, [4], [5], [6], gave the name upper-embeddable to those graphs for which equality holds in (3). In this paper, a characterization of upper-embeddable graphs is presented and together with a result of Kundu is used to show that all 4-edge connected graphs are upper-embeddable. It is worth remarking that the analogous question for nonorientable surfaces has been considered and solved by Ringel [7]. He has shown that every graph is upper-embeddable in the nonorientable sense.

II. Preliminaries. In this paper, the term graph will be used to include what are often referred to as pseudographs; that is, multiple adjacencies and loops are permitted. If G is a graph, $V(G)$ is the vertex set of G, $E(G)$ the edge set, and $D(G)$ the directed edge set. The elements of $D(G)$, which are called *arcs*, are pairs consisting of an edge together with one of the two possible directions which it can be given. Thus, there are two arcs arising from each edge, and each arc has an underlying edge, as well as an initial vertex and a terminal vertex. A *circuit* in G is a cyclic sequence of arcs such that the terminal vertex of a given arc in the sequence is the initial vertex of the following arc. If x_0 and x_1 are distinct arcs in a circuit C, let n_i be the number of arcs following x_i but preceding x_{i+1} in C, subscript addition being modulo 2. Then the *separation* between x_0 and x_1 in C is $\min(n_0, n_1)$.

The term *surface* is used to denote a compact oriented 2-manifold. A 2-cell

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embedding of a graph G in a surface S is a polyhedron on S, the 1-skeleton of which is isomorphic to G. In the followin, 2-cell embeddings are referred to simply as embeddings. If ε is an embedding, $F(\varepsilon)$ denotes the set of faces of ε. The face boundaries, together with the orientation induced on them by the orientation on the surface, are circuits in the embedded graph. The embedding is completely determined by the face boundary circuits.

If G is a graph, a *rotation* on G is a permutation σ of $D(G)$ such that if $x \in D(G)$, the orbit of x under σ is precisely the set of all arcs with the same initial vertex as x. Let τ be the permutation of $D(G)$ which sends each arc x to the arc obtained by assigning the opposite direction to the underlying edge of x. Then if σ is a rotation, define σ_0 to be $\sigma \circ \tau$. The orbits of σ_0 are circuits in G. The following result of Heffter and Edmonds [1] relates the set of embeddings of G to the set of rotations of G.

THEOREM 1 (HEFFTER-EDMONDS). *Let G be a graph. There is a* 1-1 *correspondence between the set of rotations on G and the set of embeddings of G. A rotation σ corresponds to an embedding ε such that the face boundary circuits of ε are the orbits of σ_0.*

If S is a surface, $\gamma(S)$ is the genus of S and $\chi(S)$ is the Euler characteristic of S, we have the following well-known fact:

$$\gamma(S) = 1 - \chi(S)/2. \tag{1}$$

If ε is an embedding of a graph G into S then

$$\chi(S) = |V(G)| - |E(G)| + |F(\varepsilon)|. \tag{2}$$

It is easily seen from (1) and (2) that for a given graph G, an embedding which minimizes $|F(\varepsilon)|$ will maximize $\gamma(S)$. Thus for the *maximum genus* $\gamma_M(G) = \max\{\gamma(S) | \exists$ an embedding of G in $S\}$ one obtains

$$\gamma_M(G) \leqslant [\beta(G)/2], \tag{3}$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ is the first Betti number of G, and $[\alpha]$ is the greatest integer not greater than α. The graph G is called *upper-embeddable* if equality holds in (3). The following lemma is easily obtained from the above:

LEMMA 1. *Let G be a graph such that $\beta(G)$ is even (odd). Then G is upper-embeddable if and only if G has an embedding with one (two) face(s).*

III. The main theorem. The following edge adding lemma is due to Ringeisen [6].

LEMMA 2. *Let G be a graph, u and v vertices of G and G' the graph obtained from G by adding an edge incident to u and v. Suppose ε is an embedding of G in which F_u is a face incident to u and F_v is a face incident to v. Then there is an* *embedding* ε' *of* G' *where*

$$|F(\varepsilon')| = |F(\varepsilon)| + 1 \text{ if } F_u = F_v,$$
$$|F(\varepsilon')| = |F(\varepsilon)| - 1 \text{ if } F_u \neq F_v.$$

COROLLARY. *Let* G *be a graph,* e_1 *and* e_2 *adjacent edges of* G *and* $G' = G - \{e_1, e_2\}$. *Suppose* G' *has an embedding with one face. Then* G *has an embedding with one face.*

PROOF. Since G' is embedded with one face, all vertices of G' are incident to that face, in particular those incident in G to e_1. Thus e_1 may be added to the embedding, dividing the single face in two. Let v be the vertex incident to both e_1 and e_2, and let u be the other vertex incident to e_2. Since v is incident to both faces in the embedding of $G' + e_1$ just obtained, it is incident to a face distinct from some face to which u is incident. Applying Lemma 2 produces a single face embedding of $G = G' + \{e_1, e_2\}$.

A type of converse to this corollary is the following:

LEMMA 3. *Let* G' *be a graph,* e_1 *and* e_2 *edges of* G', *and* $G = G' - \{e_1, e_2\}$. *Suppose there is an embedding* ε' *of* G' *with a face* F' *having boundary circuit of the form*

$$. x_1 \cdots x_2 \cdots \tau(x_1) \cdots \tau(x_2) \ldots$$

where x_i *is an arc arising from* e_i, $i = 1, 2$. *Then there is an embedding* ε *of* G *with* $|F(\varepsilon)| = |F(\varepsilon')|$.

PROOF. Let λ be the rotation on G' corresponding to ε' as given by Theorem 1. Note that $D(G) = D(G') - A$, where $A = \{x_1, x_2, \tau(x_1), \tau(x_2)\}$. Define a rotation σ on G by $\sigma(x) = \lambda^n(x)$ where n is the smallest positive integer such that $\lambda^n(x) \notin A$. Such an n exists since $\lambda^m(x) = x \notin A$ for some m. Let ε be the embedding of G corresponding to σ under Theorem 1. If y lies in an orbit of λ_0 not containing elements of A, then $\lambda \circ \tau(y) = \lambda_0(y) \notin A$. Thus $\sigma_0(y) = \sigma \circ \tau(y) = \lambda \circ \tau(y) = \lambda_0(y)$. Hence the orbits of λ_0 not containing elements of A are also orbits of σ_0, and it follows that the faces of ε' distinct from F are also faces of ε. Call these the fixed faces. By hypothesis, the orbit of λ_0 containing A is of the form $x_1P_1x_2P_2\,\tau(x_1)\,P_3\tau(x_2)P_4$, where the P_i are possibly empty sequences of arcs not in A. It is easily checked from the definition of σ that this implies the existence of an orbit of σ_0 having the form $P_4P_3P_2P_1$. Moreover, as $P_1, \ldots, P_4$ exhaust those elements of $D(G)$ not contained in orbits corresponding to fixed faces, it follows that the face corresponding to the orbit $P_4P_3P_2P_1$ is the only face in ε which is not a fixed face. (Note that if $P_4P_3P_2P_1$ is empty, F is the only face of ε, $E(G') = \{e_1, e_2\}$ and G is a single vertex. Thus $P_4P_3P_2P_1$ still represents a face; specifically the face of the embedding of the single vertex G in the sphere.)

Thus ε and ε' each have exactly one nonfixed face. One concludes $|F(\varepsilon)| = |F(\varepsilon')|$, as desired.

Call a graph G *edge even* (*odd*) if $|E(G)|$ is even (odd). Let T be a spanning tree of G. Then T *splits* G or is a splitting tree for G, if no more than one component of $G - T$ is edge odd. It is easily checked that if T splits G, all components of $G - T$ are edge even if and only if $\beta(G)$ is even. In order to relate this concept to the edge adding techniques, the following lemma is helpful:

LEMMA 4. *Let G be a graph each of the components of which is edge even. Then $E(G)$ may be partitioned into adjacent pairs.*

PROOF. Use induction on $|E(G)|$. The result is clear for $|E(G)| = 0$. Suppose it holds for all graphs with fewer edges than G. Let $e \in E(G)$. $G - e$ is edge odd and has at most two components which are not components of G. Thus exactly one component of $G - e$, say G', is edge odd. Let v be a vertex in G' which is incident to e. Let $e_1, \ldots, e_k$ be the edges in G' incident to v. We claim that one of these, without loss e_1, is such that each of the components of $G' - e_1$ is edge even. Suppose not. Then for each i, $G' - e_i$ is edge even but has an edge odd component. Since $G' - e_i$ has at most two components, it must in fact have two, both of which are edge odd. Let H_i be the component of $G' - e_i$ not containing v. For $i \neq j$, $H_i \cap H_j = \emptyset$. Otherwise H_i would be connected to v via H_j and e_j. Every vertex in G' is connected to v via some e_i, so $G' = \bigcup_i \{e_i, H_i\}$. Thus $|E(G')| = \Sigma_i |\{e_i\} \cup E(H_i)|$. But $|E(G')|$ is odd, whereas each $|\{e_i\} \cup E(H_i)|$ is even. This contradiction establishes the claim. Now the components of $G - \{e, e_1\}$ are the components of $G - e$ distinct from G', together with the components of $G' - e_1$. All of these are edge even. By induction, $E(G - \{e, e_1\})$ may be partitioned into adjacent pairs. Adjoining the adjacent pair $\{e, e_1\}$ to this partition produces the desired partition of $E(G)$.

It is now possible to prove the main result:

THEOREM 2. *Let G be a graph. Then G is upper-embeddable if and only if G has a splitting tree.*

PROOF. Suppose G has a splitting tree T. First assume $\beta(G)$ is even. Then each component of $G - T$ is edge even, and by Lemma 4, $E(G - T)$ may be partitioned into adjacent pairs. T is embeddable in the sphere with a single face. Repeated application of the corollary to Lemma 2 allows one to add the adjacent pairs in the partition while maintaining a single face embedding. Thus G has a single face embedding and by Lemma 1 is upper-embeddable. Now assume $\beta(G)$ is odd. Let H be the edge odd component of $G - T$. If H is such that removal of any edge disconnects it, H is a tree. In this case let e

be an edge incident to a vertex of degree one in H. Otherwise, let e be an edge the removal of which does not disconnect H. In either case each component of $G - e$ is edge even. Thus T splits $G - e$. Since $\beta(G - e)$ is even it follows from the above that $G - e$ has a single face embedding. Lemma 2 then implies that G has an embedding with two faces, showing by Lemma 1 that G is upper-embeddable.

Now suppose G is upper-embeddable. First assume $\beta(G)$ is even. Embeddability implies connectedness, so that if $|E(G)| = 0$, G is a single vertex and is a splitting tree for itself. Proceeding by induction assume all upper-embeddable graphs with even Betti number and fewer edges than G have splitting trees. By Lemma 1, G has a single face embedding ε. Let σ be the rotation corresponding to ε. Then σ_0 has a single orbit. Let e be an edge such that the separation S in this orbit between the two arcs x and $\tau(x)$ arising from e is minimal. If $S = 0$, then without loss, $\sigma \circ \tau(x) = \sigma_0(x) = \tau(x)$ so that σ fixes $\tau(x)$. Thus there is no other arc with the same initial vertex as $\tau(x)$. It follows that e is incident to a vertex v of degree one. One can easily delete e and v from ε to produce a single face embedding of $G - v$. As $\beta(G - v)$ is even, induction shows that $G - v$ has a splitting tree T. Then $T + e$ splits G. If $S > 0$, there is an arc y adjacent to x in that portion P of the boundary circuit between x and $\tau(x)$ which has length S. If $\tau(y)$ is also in P, the separation between y and $\tau(y)$ is less than S, contradicting the choice of x. Thus the boundary circuit is of the form $\ldots x\, y \ldots \tau(x) \ldots \tau(y) \ldots$. Let e_1 be the edge underlying y. Then by Lemma 3, $G - \{e, e_1\}$ has a single face embedding. $\beta(G - \{e, e_1\})$ is even. By induction, $G - \{e, e_1\}$ has a splitting tree T. Since e and e_1 are adjacent, the components of $G - T$ are of the forms H_1, $H_1 + \{e, e_1\}$, $H_1 + H_2 + \{e, e_1\}$, $H_1 + H_2 + H_3 + \{e_1, e_2\}$, or $\{e, e_1\}$, where the H_i are components of $G - \{e, e_1\} - T$. Since the H_i are edge even, the components of $G - T$ are edge even, and T splits G.

Now assume $\beta(G)$ is odd. By Lemma 1, G has an embedding with two faces. Let e be an edge incident to both faces. Deleting e produces a single face embedding of $G - e$. By the above there is a tree T which splits $G - e$. Each of the components of $G - T$ will also be a component of $G - e - T$, except for the component which contains e. Thus all but this one component are edge even, and T splits G.

IV. **Applications.** Theorem 2 permits one to demonstrate the existence of a single circuit by exhibiting a splitting tree. This may prove useful in the construction of current graphs, especially those with vertices of high degree, where use of the standard clockwise-counter clockwise convention can be awkward. Another application depends on the following result of Kundu [2].

THEOREM 3 [KUNDU]. *Let G be a graph with edge connectivity e, and let t be* *the maximum number of edge disjoint spanning trees in G. Then*

$$(e-1)/2 \leqslant t \leqslant e.$$

An immediate consequence is that if G is 4-edge connected, G has a spanning tree T which is edge disjoint from some other spanning tree, so that $G - T$ is connected. Thus T splits G. Applying Theorem 2 we have

THEOREM 4. *If G is a 4-edge connected graph, G is upper-embeddable.*

This theorem is best possible, in the sense that there are 3-edge connected graphs which are not upper embeddable. A perhaps minimal example is given in Figure 1 One may check, fairly readily, that this graph has no splitting tree

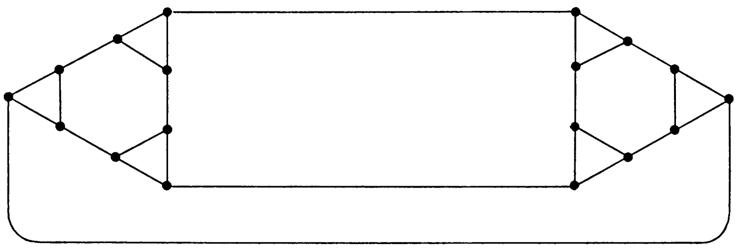

FIGURE 1

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