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THE ASYMPTOTIC BEHAVIOUR OF CERTAIN INTEGRAL FUNCTIONS

BY

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ABSTRACT. Let $f(z)$ be an integral function satisfying

$$\int^{\infty}\{\log m(r,f)-\cos\pi\rho\log M(r,f)\}^{+}\frac{dr}{r^{\rho+1}}<\infty$$

and

$$0<\underline{\lim}_{r\to\infty}\frac{\log M(r,f)}{r^{\rho}}<\infty$$

for some ρ: $0<\rho<1$. It is shown that such functions have regular asymptotic behaviour outside a set of circles with centres ζ_i and radii t_i for which

$$\sum_{i=1}^{\infty}\frac{t_i}{|\zeta_i|}<\infty.$$

1. Introduction. For an integral function $f(z)$ let

$$M(r,f)=\max_{|z|=r}|f(z)|,\qquad m(r,f)=\min_{|z|=r}|f(z)|$$

and let $n(r,f)$ be the number of zeros of f in $|z|\leqslant r$. The *order* ρ of f is

$$\rho=\overline{\lim_{r\to\infty}}\frac{\log\log M(r,f)}{\log r}.$$

The following result appears in [6].

THEOREM A. *Let ρ be a positive number less than one and let $f(z)$ be an integral function of order ρ satisfying the following conditions*:

(i) *there is a finite constant K such that*

$$\overline{\lim_{\substack{r_2>r_1\\ r_1\to\infty}}}\int_{r_1}^{r_2}\{\log m(r,f)-\cos\pi\rho\log M(r,f)\}\frac{dr}{r^{\rho+1}}\leqslant K;$$

(ii) *there are numbers α and β, with $0<\alpha<\beta<\infty$, such that, for all large r,*

$$\alpha r^{\rho}\leqslant n(r,f)\leqslant\beta r^{\rho}.$$

Let

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(1.1) $$k = \left\{ \frac{2\beta}{\alpha} \right\}^{1/\rho}.$$

Then there is a curve C: $z = re^{i\phi(r)}$, *where* $\phi(r)$ *is a continuous function satisfying*

(1.2) $$|\phi(R_1) - \phi(R_2)| = o\left|\log \frac{R_2}{R_1}\right|^{1/2} \quad \textit{as } \min(R_1, R_2) \to \infty,$$

a function $\varepsilon(t)$ *satisfying* $\pi \geqslant \varepsilon(t) \geqslant 0$ *and* $\varepsilon(t) \to 0$ *as* $t \to \infty$, *and a function* $\nu(t)$ *satisfying* $1 \geqslant \nu(t) \geqslant 0$, $\nu(t) \to 0$ *as* $t \to \infty$ *and*

(1.3) $$\int^{\infty} \frac{\nu(t)}{t}\, dt < \infty,$$

for which the following is true. If ζ *is any point on C, then the set*

$$\{z\colon k^{-1}|\zeta| < |z| \leqslant k|\zeta| \textit{ and } |\arg z\zeta^{-1}| \geqslant \varepsilon(|\zeta|)\}$$

contains at most $\nu(|\zeta|)N(|\zeta|)$ *zeros of f, where* $N(|\zeta|)$ *is the number of zeros of f in*

$$\{z\colon k^{-1}|\zeta| < |z| \leqslant k|\zeta|\}.$$

The equation (1.2) is a consequence of the following: there is a constant $A = A(k)$ and a function $\Delta(t)$ satisfying $\pi \geqslant \Delta(t) \geqslant 0$, $\Delta(t) \to 0$ and

(1.4) $$\int^{\infty} \frac{\Delta(t)^2}{t}\, dt < \infty$$

for which

(1.5) $$|\phi'(t)| \leqslant A\, \frac{\Delta(t)}{t} \quad \text{for all large } t.$$

The reader is referred to [6] for details.

It will be shown here that this result leads to a precise description (outside a small exceptional set) of the asymptotic behavior of a certain class of integral functions. To be specific, let ρ be a positive number less than one and suppose that f is an integral function satisfying

(i)′ with the convention that $a^+ = \max(0, a)$ for any real number a,

$$\int^{\infty} \{\log m(r,f) - \cos \pi\rho \log M(r,f)\}^+ \frac{dr}{r^{\rho+1}} < \infty;$$

(ii)′ there is a finite nonzero constant β such that

$$0 < \beta = \varliminf_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} < \infty.$$

We shall prove here

THEOREM 1. *Let* ρ *be a positive number less than one and let* $f(z)$ *be an integral function satisfying conditions* (i)′ *and* (ii)′ *above. Then* $f(z)$ *satisfies the hypotheses of Theorem* A *and with* $\phi(r)$ *as in that theorem we have*

$$\left|r^{-\rho}\log\left|f(re^{i(\phi(r)+\theta-\pi)})\right| - \beta\cos\rho\theta\right| = o(1) \tag{1.6}$$

as r *tends to infinity outside a set of discs with centres* ζ_i, *radii* t_i, *for which*

$$\sum_{i=1}^{\infty}\frac{t_i}{|\zeta_i|} < \infty. \tag{1.7}$$

The exceptional set of Theorem 1 may be described briefly, following Hayman [7], as an *E-set*. Theorem 1 has much in common with results of Essén [4] and Essén and Lewis [5] on subharmonic functions. In [4] Essén is concerned with functions subharmonic in the plane slit along the negative real axis while [5] generalizes the considerations of [4] to functions subharmonic in d-dimensional cones and also establishes an improved estimate of the exceptional set. When restricted to integral functions the result of [4] combined with the estimate of the exceptional set of [5] may be viewed as a special case of Theorem 1, when $|f(r)| = M(r,f)$, $|f(-r)| = m(r,f)$ and $\log|f(-r)| \leqslant \cos\pi\rho\log|f(r)|$.

The condition (i)′ cannot be replaced with

$$\overline{\lim_{r_1, r_2\to\infty}}\int_{r_1}^{r_2}\{\log m(r,f) - \cos\pi\rho\log M(r,f)\}\frac{dr}{r^{\rho+1}} \leqslant 0, \tag{1.8}$$

a condition arising in the work of Anderson [1]. For in [1], Anderson shows that

$$\int_0^{\infty}\{\log|f(-r)| - \cos\pi\rho\log|f(r)|\}\frac{dr}{r^{\rho+1}}$$

exists (so that (1.8) certainly holds) for an integral function $f(z)$ with real negative zeros if

$$\frac{\log f(r)}{r^{\rho}} \to A \qquad (0 < A < \infty) \tag{1.9}$$

for some ρ: $0 < \rho < 1$. It will be shown in §9, however, that there exists an integral function $f(z)$ with real negative zeros satisfying (1.9) and such that, for some $\varepsilon > 0$,

$$\log|f(-r)| < (A\cos\pi\rho - \varepsilon)r^{\rho}$$

for all r in a set of infinite logarithmic measure. Since an E-set intersects every ray through the origin in a set of finite logarithmic measure (1.6) cannot hold outside an E-set.

2. Preliminaries. From (i)′ it follows that

$$\overline{\lim_{r_1, r_2 \to \infty}} \int_{r_1}^{r_2} \{\log m(r,f) - \cos \pi\rho \log M(r,f)\} \frac{dr}{r^{\rho+1}} \leq 0$$

and from this together with (ii)′ and the theorem of Anderson already mentioned [1, p. 154] we deduce that

$$\log M(r,f) \sim \beta r^{\rho}, \tag{2.1}$$

$$\log f_1(r) \sim \beta r^{\rho}, \tag{2.2}$$

where

$$f_1(z) = \prod_{1}^{\infty} \left(1 + \frac{z}{|a_n|}\right), \tag{2.3}$$

the numbers a_n, $n = 1, 2, 3, \ldots,$ being the nonzero zeros of f arranged in order of increasing magnitude. A well-known consequence of (2.2) is that

$$n(r,f) = n(r,f_1) \sim \pi^{-1}\beta \sin \pi\rho \, r^{\rho}, \tag{2.4}$$

so that functions satisfying (i)′ and (ii)′ are of order ρ and satisfy (i) and (ii) of Theorem A.

In the course of the proof of Theorem 1 we shall find it convenient to refer to a result due to Kolomiiceva [9]. A complete discussion of Kolomiiceva's theorem would involve us in needless complications but a simple consequence of it is

LEMMA 1. *Let $g(z)$ be an integral function satisfying*

$$\lim_{r \to \infty} \frac{\log M(r, g)}{r^{\rho}} = \beta,$$

where $0 < \rho < 1$ and $0 < \beta < \infty$, which is such that, for each $\eta > 0$, the number of zeros of g in

$$\{|z| \leq r\} \cap \{|\arg z| \leq \pi - \eta\}$$

is $o(r^{\rho})$ as $r \to \infty$. Then a necessary and sufficient condition that

$$\log|g(re^{i\theta})| = (\beta \cos \rho\theta + o(1))r^{\rho}$$

outside a set E is the following: given $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $r(\varepsilon)$ such that for all z outside E satisfying $|z| > r(\varepsilon)$,

$$\int_0^{\delta r} \frac{n_z(t, g)}{t} \, dt < \varepsilon r^{\rho}, \tag{2.5}$$

where $n_z(t, g)$ is the number of zeros of g contained in the open disc with centre z and radius t.

3. **An auxiliary function.** We suppose without loss of generality that $f(0) = 1$ so that

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right). \tag{3.1}$$

As was mentioned before the results of Theorem A hold for functions satisfying the hypotheses of Theorem 1. Choose

$$k = (1\,2)^{1/\rho} \tag{3.2}$$

and let C and $\phi(t)$ be as in Theorem A. We relabel C as C_π and for every θ satisfying $-\pi < \theta < \pi$ we define C_θ by

$$C_\theta\colon z = re^{i(\phi(r) + \theta - \pi)}. \tag{3.3}$$

Let us rearrange the zeros of f in the following way: if a_n is a zero of f lying on the curve C_θ say, we transfer it to the point $a'_n = |a_n|e^{i\theta}$ and define

$$F(z) = \prod_1^\infty \left(1 - \frac{z}{a'_n}\right). \tag{3.4}$$

Our first concern is to show that $\log|F(|z|e^{i\theta})|$ and $\log|f(z)|$ do not greatly differ. Later we shall show that $\log|F(z)|$ and $\log|f_1(z)|$ have similar asymptotic behavior and then, after estimating $\log|f_1(z)|$, we shall appeal to the intermediate character of F to estimate $\log|f(z)|$.

4. **Comparison of f and F.** We shall prove

LEMMA 2. *Given any number $\varepsilon > 0$, there exists a number $R(\varepsilon)$ such that, if $f(z) \neq 0$,*

$$\left|\log|f(z)| - \log\left|F\left(|z|e^{i\theta}\right)\right|\right| < \varepsilon r^\rho \tag{4.1}$$

whenever $|z| > R(\varepsilon)$, where θ satisfies $-\pi < \theta \leqslant \pi$ and is such that z lies on C_θ.

Throughout the proof we suppose that $z = re^{i\psi}$ is not a zero of f. We have, from (3.1) and (3.4),

$$\log\left|\frac{f(z)}{F(re^{i\theta})}\right| = \sum_1^\infty \log\left|\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1}\right| \tag{4.2}$$

and we examine the sum of (4.2) in three parts. First, with $a_n = r_n e^{i\phi_n}$ consider, for $p > 1$,

$$
\begin{aligned}
S_1 &= \prod_{r_n > k^p r} \log\left|\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1}\right| \\
&\leqslant \sum_{r_n > k^p r} \log\left(1 + \frac{r}{r_n}\right)\left(1 - \frac{r}{r_n}\right)^{-1} \\
&\leqslant 2r(1 - k^{-1})^{-1} \sum_{r_n > k^p r} r_n^{-1} \\
&= 2r(1 - k^{-1})^{-1} \int_{k^p r}^{\infty} \frac{dn(t)}{t}.
\end{aligned}
$$

Integrating by parts we obtain

$$
S_1 = O(k^{p(\rho-1)} r^{\rho}). \tag{4.3}
$$

Next consider

$$
\begin{aligned}
S_2 &= \sum_{r_n < k^{-p} r} \log\left|\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1}\right| \\
&\leqslant \sum_{r_n < k^{-p} r} \log\left\{\left(1 + \frac{r_n}{r}\right)\left(1 - \frac{r_n}{r}\right)^{-1}\right\} \\
&\leqslant 2r^{-1}(1 - k^{-1})^{-1} \sum_{r_n < k^{-p} r} r_n \\
&= 2r^{-1}(1 - k^{-1})^{-1} \int_0^{k^{-p} r} t\, dn(t) \\
&= O(k^{-p(\rho+1)} r^{\rho}).
\end{aligned} \tag{4.4}
$$

Finally we consider the remaining part of the sum, that for which $k^{-p} r \leqslant r_n \leqslant k^p r$. Since $\theta = \pi + \psi - \phi(r)$ and $a'_n = r_n e^{i(\pi + \phi_n - \phi(r_n))}$,

$$
\begin{aligned}
J_n &= \left|\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1}\right|^2 \\
&= \frac{\left(1 - \frac{r}{r_n}\right)^2 + \frac{4r}{r_n} \sin^2\left(\frac{\psi - \phi_n}{2}\right)}{\left(1 - \frac{r}{r_n}\right)^2 + \frac{4r}{r_n} \sin^2\left(\frac{\psi - \phi_n - \phi(r) + \phi(r_n)}{2}\right)}
\end{aligned}
$$

Let us write $t_n = r/r_n$, $\psi - \phi_n = \psi_n$, $\phi(r) - \phi(r_n) = \nu_n$. Then

$$J_n = \frac{(1-t_n)^2 + 4t_n\sin^2(\psi_n/2)}{(1-t_n)^2 + 4t_n\sin^2((\psi_n - \nu_n)/2)}$$

$$= 1 + \frac{4t_n\sin\left(\psi_n - \frac{1}{2}\nu_n\right)\sin\frac{1}{2}\nu_n}{(1-t_n)^2 + 4t_n\sin^2((\psi_n - \nu_n)/2)}.$$

Hence

$$\log J_n \leqslant \frac{4t_n\left|\sin\left(\psi_n - \frac{1}{2}\nu_n\right)\sin\frac{1}{2}\nu_n\right|}{(1-t_n)^2 + 4t_n\sin^2\left(\dfrac{\psi_n - \nu_n}{2}\right)}$$

(4.5)

$$\leqslant \frac{8t_n\left|\sin\left(\dfrac{\psi_n - \nu_n}{2}\right)\sin\frac{1}{2}\nu_n\right| + 8t_n\left|\sin\frac{1}{4}\nu_n\sin\frac{1}{2}\nu_n\right|}{(1-t_n)^2 + 4t_n\sin^2\left(\dfrac{\psi_n - \nu_n}{2}\right)}$$

since, for any real numbers a and b,

$$\left|\sin\left(a - \frac{1}{2}b\right)\right| \leqslant 2\left|\sin\left(\frac{1}{2}(a-b) + \frac{1}{4}b\right)\right|$$

$$\leqslant 2\left|\sin\frac{1}{2}(a-b)\right| + 2\left|\sin\frac{1}{4}b\right|.$$

Further, from (1.5),

$$|\nu_n| = |\phi(r) - \phi(r_n)| \leqslant \left|\int_{r_n}^{r}|\phi'(t)|dt\right|$$

(4.6)
$$\leqslant \left|\int_{r_n}^{r} A\frac{\Delta(t)}{t}dt\right| \leqslant A\left|\log\frac{r}{r_n}\right|\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)\right\}$$

$$\leqslant Ak^p\left|1 - \frac{r}{r_n}\right|\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)\right\}.$$

Substituting (4.6) into (4.5) we obtain

$$\log J_n \leqslant \frac{4Ak^p t_n|1-t_n|\left|\sin\left(\dfrac{\psi_n - \nu_n}{2}\right)\right|}{(1-t_n)^2 + 4t_n\sin^2\left(\dfrac{\psi_n - \nu_n}{2}\right)}\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)\right\}$$

$$+ A^2k^{2p}t_n\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)^2\right\}$$

$$\leqslant Ak^p t_n^{1/2}\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)\right\} + A^2k^{2p}t_n\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)^2\right\}$$

$$\leqslant A_1k^{3p}\left\{\sup_{t \geqslant k^{-p}r}\Delta(t)\right\},$$

where $A_1 = A + \pi A^2$. Hence, since from (2.4) the number of zeros of f in $|z| \leqslant k^p r$ is at most $2\alpha k^{p\rho} r^\rho$ for large r, where $\alpha = \beta\pi^{-1}\sin \pi\rho$,

$$\begin{aligned} S_3 &= \sum_{k^{-p}r \leqslant r_n \leqslant k^p r} \log\left|\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{re^{i\theta}}{a_n'}\right)^{-1}\right| \\ &\leqslant 2\alpha A_1 k^{p(\rho+3)} r^\rho \left\{\sup_{t \geqslant k^{-p}r} \Delta(t)\right\}. \end{aligned} \tag{4.7}$$

Given $\varepsilon > 0$ we may choose p sufficiently large that $S_1 + S_2 < \varepsilon r^\rho$ for all large r and with this p we may choose $r_0(\varepsilon)$ so that $S_3 < \varepsilon r^\rho$ for $r > r_0(\varepsilon)$, since $\Delta(t) \to 0$ as $t \to \infty$, which proves one half of Lemma 1. The second half, that

$$\log\left|\frac{F(|z|e^{i\theta})}{f(z)}\right| < \varepsilon|z|^\rho,$$

is proved similarly.

5. **The zeros of** $F(z)$. We shall prove

LEMMA 3. *Let δ be a fixed positive number less than π and let $n_z(t, F, \delta)$ be the number of zeros of F contained in*

$$\left\{\zeta\colon |\arg \zeta| \leqslant \pi - \frac{1}{2}\delta\right\} \cap \{\zeta\colon |\zeta - z| < t\}. \tag{5.1}$$

Then given any positive number $\varepsilon < \frac{1}{2}$ there exists a number $R(\varepsilon, \delta)$ such that, with $|z| = r$,

$$\int_0^{\varepsilon r} n_z(t, F, \delta)\frac{dt}{t} < \varepsilon r^\rho \tag{5.2}$$

for all z outside a set H_1 (where H_1 depends only on δ) and such that $|z| > R(\varepsilon, \delta)$. Moreover H_1 is covered by a set of discs C_i, centres ζ_i, radii t_i, $i = 1, 2, 3, \ldots$, such that $\sum_1^\infty t_i/|\zeta_i| < \infty$.

Throughout the proof of Lemma 3 we write $n_z(t)$, $n_z(t, \delta)$ instead of $n_z(t, F)$, $n_z(t, F, \delta)$.

We shall make use of an argument of Azarin [2] in which the following lemma is used.

LEMMA 4 ([10, Lemma 3.2]). *If a set E in the complex plane is covered by discs of bounded radii such that each point of the set is the centre of a disc, then from this one may select a subsystem of discs which covers the set, each point of the plane being covered no more than ν times by the discs of this subsystem, where ν is an absolute constant.*

Let $R_1 = R_1(\delta)$ be such that, for $r \geqslant R_1$ we have $\varepsilon(r) < \frac{1}{2}\delta$, where $\varepsilon(r)$ is the function occurring in Theorem A. (We may note that, if $z \in S(\delta, R_1)$, where

(5.3) $$S(\delta, R_1) = \{z: |z| \geqslant R_1 \text{ and } |\arg z| \leqslant \pi - \delta\},$$

then $n_z(t, \delta) = n_z(t)$ certainly for $0 < t < \frac{1}{4}\delta|z|$.) Let H_1 be the set of points z in $|z| \geqslant R_1$ at which, for some $t = t(z)$ satisfying $0 < t < \frac{1}{2}|z|$, we have

(5.4) $$n_z(t, \delta) \geqslant t|z|^{\rho - 1}.$$

Let E_n be the subset of H_1 contained in the annulus

$$\{z: 4^{n+1} > |z| \geqslant 4^n\}, \qquad n = 0, 1, 2, \ldots.$$

We surround each point z of H_1 by a disc of radius $t(z)$ and from the set of such discs surrounding points of E_n we select a subsystem K_n which covers E_n, while covering each point of the plane at most ν times. This can be done, by Lemma 4. We note that the members of K_m do not intersect the members of K_n if $|n - m| \geqslant 2$, and therefore $K = \bigcup_{n=1}^{\infty} K_n$ is a set of discs the members of which cover each point of the plane at most 2ν times.

Now, K is a countable set the members of which may be ordered: C_i, $i = 1, 2, 3, \ldots,$ where C_i is a disc with centre ζ_i and radius t_i, where $0 < t_i < \frac{1}{2}|\zeta_i|$, $i = 1, 2, 3, \ldots$; moreover, from (5.4) we have

$$n_{\zeta_i}(t_i, \delta) \geqslant t_i|\zeta_i|^{\rho - 1}, \qquad i = 1, 2, 3, \ldots.$$

Hence

(5.5) $$\sum_1^{\infty} \frac{t_i}{|\zeta_i|} \leqslant \sum_1^{\infty} \frac{n_{\zeta_i}(t_i, \delta)}{|\zeta_i|^{\rho}}.$$

Now, if z_n is one of the zeros of F contained in $S(\frac{1}{2}\delta, R_1)$ and also in one of the discs, say C_i, then $|z_n| - |\zeta_i| \leqslant |z_n - \zeta_i| < t_i < \frac{1}{2}|\zeta_i|$ so $|z_n| < \frac{3}{2}|\zeta_i|$. Hence, from (5.5) and the fact that K covers any point in the plane at most 2ν times,

(5.6) $$\sum_1^{\infty} \frac{t_i}{|\zeta_i|} \leqslant 2\nu\left(\frac{3}{2}\right)^{\rho} \sum \frac{1}{|z_n|^{\rho}},$$

where the sum on the right-hand side is taken over those zeros of F which are contained in $S(\frac{1}{2}\delta, R_1)$. We proceed to show that this sum is finite.

Let n be a nonnegative integer, and let b_n be a positive number satisfying $k^n R_1 \leqslant b_n < k^{n+1} R_1$ at which

(5.7) $$\nu(b_n)\log k \leqslant \int_{k^n R_1}^{k^{n+1} R_1} \nu(t)\,\frac{dt}{t},$$

where $\nu(t)$ is the function occurring in Theorem A and k is given by (3.2). The number of zeros of F in

$$\{z: k^n R_1 \leqslant |z| < k^{n+1} R_1 \text{ and } |\arg z| \leqslant \pi - \tfrac{1}{2}\delta\}$$

is no more than $\nu(b_n)N(b_n)$, where $N(b_n)$ is the number of zeros of F in $\{z: k^{-1}b_n \leqslant |z| < kb_n\}$. Hence, making use of (5.7) we have, for some constant A,

$$\begin{aligned}\sum \frac{1}{|z_n|^\rho} &\leqslant \sum_0^\infty \nu(b_n)N(b_n)(k^n R_1)^{-\rho} \\ &\leqslant \sum_0^\infty \nu(b_n)Ab_n^\rho (k^n R_1)^{-\rho} \leqslant Ak^\rho \sum_0^\infty \nu(b_n) \\ &\leqslant Ak^\rho(\log k)^{-1}\int_{R_1}^\infty \nu(t)\frac{dt}{t} < \infty,\end{aligned}$$

from Theorem A. The sum on the left-hand side of (5.6) is thus finite.

Suppose that z is a point outside H_1 and satisfying $|z| \geqslant R_1$. Then, given any positive number $\varepsilon < \frac{1}{2}$, $\int_0^{\varepsilon|z|} n_z(t, \delta)dt/t < \varepsilon|z|^\rho$. This proves Lemma 3.

6. **The behaviour of** $f_1(z)$. Let $f_1(z)$ be the function (2.3). Since

$$\log m(r,f) - \cos \pi\rho \log M(r,f) \geqslant \log m(r,f_1) - \cos \pi\rho \log M(r,f_1)$$

it follows from (i)′ and Kjellberg's Lemma [8, p. 193, formula (21)] that

$$\int^\infty |\log m(r,f_1) - \cos \pi\rho \log M(r,f_1)| \frac{dr}{r^{\rho+1}} < \infty. \tag{6.1}$$

Given a positive number $\varepsilon > 0$, it follows from (6.1) and (2.2) that

$$\log m(r,f_1) > \left(\beta \cos \pi\rho - \frac{1}{2}\varepsilon\right)r^\rho$$

for r outside a set $E = E(\varepsilon)$ of finite logarithmic measure. Hence, for $\delta = \varepsilon/2\beta\rho$,

$$\log|f_1(re^{i\theta})| > \log m(r,f_1) > (\beta \cos \rho\theta - \varepsilon)r^\rho \tag{6.2}$$

for $\pi \geqslant |\theta| \geqslant \pi - \delta$ and for r outside E.

It is well known (see e.g. [12, p. 272]) that

$$|r^{-\rho}\log|f_1(re^{i\theta})| - \beta \cos \rho\theta| \to 0 \tag{6.3}$$

as $r \to \infty$, uniformly for $|\theta| \leqslant \pi - \delta$. In particular

$$|r^{-\rho}\log|f_1(re^{i(\pi-\delta)})| - \beta \cos \rho(\pi - \delta)| \to 0$$

as $r \to \infty$. Hence, for $\pi \geqslant |\theta| \geqslant \pi - \delta$ and for sufficiently large r

$$\begin{aligned} r^{-\rho}\log|f_1(re^{i\theta})| &\leqslant r^{-\rho}\log|f_1(re^{i(\pi-\delta)})| \\ &\leqslant \beta \cos \rho(\pi - \delta) + \tfrac{1}{2}\varepsilon \\ &< \beta \cos \rho\theta + 2\beta \sin \tfrac{1}{2}\rho(|\theta| - \pi + \delta) + \tfrac{1}{2}\varepsilon \\ &< \beta \cos \rho\theta + \varepsilon. \end{aligned} \tag{6.4}$$

Taking (6.2) and (6.4) together, and taking account of (6.3) we obtain

LEMMA 5. *Let $f_1(z)$ be the integral function* (2.3). *Given $\varepsilon > 0$*

$$|r^{-\rho}\log|f_1(re^{i\theta})| - \beta\cos\rho\theta| < \varepsilon \qquad (-\pi < \theta \leqslant \pi)$$

for all large r outside E, a set of finite logarithmic measure.

From this together with Lemma 1 we deduce

LEMMA 6. *Given any $\varepsilon > 0$ there exist positive numbers $\delta = \delta(\varepsilon)$ and $r(\varepsilon)$ such that*

$$\int_0^{\delta r} n_{-r}(t, f_1)\,\frac{dt}{t} < \varepsilon r^\rho \tag{6.5}$$

for all $r > r(\varepsilon)$ lying outside a set E_0 of finite logarithmic measure, where $n_{-r}(t, f_1)$ is the number of zeros of f_1 in $(-r - t, -r + t)$. Further, E_0 is independent of ε and is a union of disjoint intervals each of which contains more than one point.

We need verify only that E_0 may be taken to be a union of disjoint intervals each containing more than one point; Lemma 6 certainly holds for some set E_1 of finite logarithmic measure and some functions $\delta(\varepsilon)$, $r(\varepsilon)$, by Lemma 1.

To this end, given $\varepsilon > 0$, let $\eta = \delta(\frac{1}{3}\varepsilon)$, where δ is the function known to exist and suppose that r_1 and r_2 are two points outside E_1 with $2r_1 \geqslant r_2 > r_1 > r(\frac{1}{3}\varepsilon)$ and such that f_1 has no zeros in $[r_1, r_2]$. Then, for $r_1 < r < r_2$,

$$\begin{aligned} I &= \int_0^{\eta r} n_{-r}(t, f_1)\,\frac{dt}{t} \\ &= \sum_1^{n_1}\log\frac{\eta r}{r + x_n} + \sum_1^{n_2}\log\frac{-\eta r}{r + y_n}, \end{aligned} \tag{6.6}$$

where $x_1, \ldots, x_{n_1}$ are the zeros of f_1 in $(-r, -r + \eta r)$ and $y_1, \ldots, y_{n_2}$ are the zeros of f_1 in $(-r - \eta r, -r)$. Now

$$\frac{r}{r + x_n} = 1 - \frac{x_n}{r + x_n} < 1 - \frac{x_n}{r_1 + x_n} = \frac{r_1}{r_1 + x_n},$$

so, in view of Lemma 1 and Lemma 5

$$\begin{aligned} \sum_1^{n_1}\log\frac{\eta r}{r + x_n} &< \sum_1^{n_1}\log\frac{\eta r_1}{r_1 + x_n} \\ &\leqslant \int_0^{\eta r_1} n_{-r_1}(t, f_1)\,\frac{dt}{t} < \frac{1}{3}\varepsilon r_1^\rho. \end{aligned} \tag{6.7}$$

Similarly, $-r/(r + y_n) < -r_2/(r_2 + y_n)$, so

$$\sum_1^{n_2}\log\frac{-\eta r}{r + y_n} \leqslant \int_0^{\eta r_2} n_{-r_2}(t, f_1)\,\frac{dt}{t} < \frac{1}{3}\varepsilon r_2^\rho < \frac{2}{3}\varepsilon r_1^\rho. \tag{6.8}$$

From (6.6), (6.7) and (6.8),

$$\int_0^{\eta r} n_{-r}(t, f_1)\,\frac{dt}{t} < \varepsilon r_1^{\rho} < \varepsilon r^{\rho} \tag{6.9}$$

for r in (r_1, r_2).

Let E_0 be the set obtained by removing from E_1 all points which are limit points both from the left and from the right of the complement of E_1. Then E_0 is a union of disjoint intervals each containing more than one point and is contained in E_1. Moreover, if $r > r(\frac{1}{3}\varepsilon)$ and r lies outside E_0, then (6.9) holds, with $\eta = \eta(\varepsilon) = \delta(\frac{1}{3}\varepsilon)$. This completely proves Lemma 6.

7. **Further consideration of the zeros of** F. We first observe that the set of intervals the union of which is E_0 is countable since the logarithmic measure of E_0 is finite and the logarithmic measure of each interval is positive. We may therefore regard each interval as closed without affecting the value of the logarithmic measure of E_0. We write $E_0 = \bigcup_1^{\infty} J_i$, where

$$J_i = [c_i, d_i], \quad c_{i+1} > d_i > c_i, \quad i = 1, 2, 3, \ldots.$$

Let ε be any positive number. With Lemmas 3 and 6 in view, let $\tau = \tau(\varepsilon) = \min(\frac{1}{2}, \varepsilon, \delta(\varepsilon))$ (where $\delta(\varepsilon)$ is the function of Lemma 6) and let $r_0(\varepsilon)$ be a number at least as large as $\max(r(\varepsilon), R(\tau, \frac{1}{8}\tau))$ (where $r(\varepsilon)$, $R(x, y)$ are respectively the functions of Lemmas 6 and 3) such that $r_0(\varepsilon) \notin E_0$ (E_0 is the set of Lemma 6) and for which $d_i < (1 + \frac{1}{4}\tau)c_i$ whenever $c_i > r_0(\varepsilon)$. Since E_0 is of finite logarithmic measure this choice of $r_0(\varepsilon)$ is possible.

Define, for $J_i = [c_i, d_i]$ where $c_i > r_0(\varepsilon)$,

$$B_i = \{z\colon |z| \in J_i\} \cap \{z\colon \pi \geqslant |\arg z| > \pi - \tfrac{1}{8}\tau\},$$

$$D_i = \{z\colon |z| \in J_i\} \cap \{z\colon \pi \geqslant |\arg z| > \pi - \tfrac{1}{8}\tau\} \setminus H_1$$

where H_1 is the exceptional set of Lemma 3 corresponding to $\delta = \frac{1}{8}\tau$. $H_1 = H_1(\tau) = H_1(\varepsilon)$.

Let z be any point in D_i. Then, with $r = |z|$,

$$I(z) = \int_0^{\tau r} n_z(t, F)\,\frac{dt}{t}$$

$$= \log \frac{(\tau r)^{m+n+p+q}}{\Pi_1^m |z - a_j| \Pi_1^n |z - b_j| \Pi_1^p |z - u_j| \Pi_1^q |z - v_j|},$$

where the a_j, b_j, u_j, v_j are zeros of F in $|\zeta - z| < \tau r$ which are respectively in B_i, in $\{z\colon |z| \in J_i\} \setminus B_i$, in $|z| < c_i$ and in $|z| > d_i$. Then we have, recalling Lemma 3 and Lemma 6,

$$I(z) \leqslant \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \log \frac{(\tau r)^n}{\prod_{j=1}^n (r - |u_j|)} + \log \frac{(\tau r)^p}{\prod_{j=1}^p (|v_j| - r)}$$

$$+ \int_0^{\tau r} n_z\left(t, F, \tfrac{1}{8}\tau\right) \frac{dt}{t}$$

$$\leqslant \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \log \frac{(\tau c_i)^n}{\prod_{j=1}^n (c_i - |u_j|)} + \log \frac{(\tau d_i)^p}{\prod_{j=1}^p (|v_j| - d_i)}$$

$$+ \int_0^{\tau r} n_z\left(t, F, \tfrac{1}{8}\tau\right) \frac{dt}{t}$$

$$(7.1) \qquad \leqslant \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \int_0^{\tau c_i} n_{-c_i}(t, f_1) \frac{dt}{t} + \int_0^{\tau d_i} n_{-d_i}(t, f_1) \frac{dt}{t}$$

$$+ \int_0^{\tau r} n_z\left(t, F, \tfrac{1}{8}\tau\right) \frac{dt}{t}$$

$$< \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \int_0^{\delta(\varepsilon) c_i} n_{-c_i}(t, f_1) \frac{dt}{t} + \int_0^{\delta(\varepsilon) d_i} n_{-d_i}(t, f_1) \frac{dt}{t}$$

$$+ \int_0^{\tau r} n_z\left(t, F, \tfrac{1}{8}\tau\right) \frac{dt}{t}$$

$$\leqslant \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \varepsilon\left[c_i^\rho + d_i^\rho\right] + \tau r^\rho$$

$$< \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + 4\varepsilon r^\rho.$$

Now, B_i is contained in a rectangle the sides of which have length $\frac{1}{4}\tau d_i < \frac{3}{8}\tau c_i$ and $d_i - c_i \cos \frac{1}{8}\tau$. Hence for any point z in D_i the circle $|\zeta - z| < \tau|z|$ contains all of B_i and so all the zeros of F in B_i (i.e. all the a_j) appear in (7.1). We can thus apply Cartan's Lemma [3, p. 75] to estimate (7.1) and obtain

$$\prod_1^m |z - a_j| \geqslant \left\{\tau d_i \exp\left(-\frac{\varepsilon d_i^\rho}{m}\right)\right\}^m$$

outside a set of at most m discs C_j', $j = 1, 2, \ldots, m$, the sum of the radii of which is at most $A = 2e\tau d_i \exp(-\varepsilon d_i^\rho / m)$. Hence, for all z in D_i outside these discs we have, from (7.1),

$$(7.2) \qquad \int_0^{\tau r} n_z(t, F) \frac{dt}{t} < 4\varepsilon r^\rho + \log\left(\frac{r}{d_i}\right)^m + \varepsilon d_i^\rho < 6\varepsilon r^\rho.$$

We must have $A \leqslant 2e(d_i - c_i)$. For suppose that $A > 2e(d_i - c_i)$. Then

$$I(\varepsilon, d_i) = \int_0^{\delta(\varepsilon)d_i} n_{-d_i}(t, f_1)\,\frac{dt}{t} \geqslant \int_0^{\tau d_i} n_{-d_i}(t, f_1)\,\frac{dt}{t}$$

$$\geqslant \log \frac{(\tau d_i)^m}{\prod_{j=1}^m (d_i - |a_j|)}$$

$$\geqslant \log \frac{(\tau d_i)^m}{(d_i - c_i)^m}$$

$$> \log \frac{(2e\tau d_i)^m}{A^m} = \varepsilon d_i^\rho,$$

a contradiction, since $I(\varepsilon, d_i) \leqslant \varepsilon d_i^\rho$, d_i being a boundary point of E_0. Hence

(7.3) $$A \leqslant 2e(d_i - c_i).$$

Suppose that C_j' has radius t_j' and centre ζ_j', $j = 1, 2, \ldots, m$.

$$\sum_{j=1}^m \frac{t_j'}{|\zeta_j'|} \leqslant \frac{1}{c_i} \sum_{j=1}^m t_j' \leqslant 2e\left(\frac{d_i - c_i}{c_i}\right).$$

Also, since $d_i < (1 + \frac{1}{4}\tau)c_i < 2c_i$ and since, for $x \geqslant 1$, $\log x \geqslant (x-1)/x$,

$$\frac{d_i - c_i}{c_i} = \frac{d_i}{c_i} \cdot \frac{d_i - c_i}{d_i} < 2 \log \frac{d_i}{c_i},$$

so

(7.4) $$\sum_{j=1}^m \frac{t_i'}{|\zeta_j'|} < 4e \log \frac{d_i}{c_i}.$$

We are thus able to prove

LEMMA 7. *Let ε be any positive number, and let $\tau = \min(\frac{1}{2}, \varepsilon, \delta(\varepsilon))$, where $\delta(\varepsilon)$ is the function of Lemma 6. Let $r_0(\varepsilon)$ be a positive number greater than* $\max(r(\varepsilon), R(\tau, \frac{1}{8}\tau))$ *such that $r_0(\varepsilon) \notin E_0$ and for which $d_i < (1 + \frac{1}{4}\tau)c_i$ whenever $c_i > r_0(\varepsilon)$, where $r(\varepsilon)$, $R(\tau, \frac{1}{8}\tau)$ are respectively the functions of Lemmas 6, 3, and E_0 is the set of Lemma 6. Then for all z in*

$$T\left(\tfrac{1}{8}\tau, r_0(\varepsilon)\right) = \left\{z\colon |z| \geqslant r_0(\varepsilon) \text{ and } \pi \geqslant |\arg z| \geqslant \pi - \tfrac{1}{8}\tau\right\}$$

we have, with $|z| = r$,

(7.5) $$\int_0^{\tau r} n_z(t, F)\,\frac{dt}{t} < 6\varepsilon r^\rho$$

except when z belongs to an E-set, $H_2 = H_2(\varepsilon)$.

Suppose first that z, in $T(\frac{1}{8}\tau, r_0(\varepsilon))$, lies in $\bigcup\{z\colon |z| \in J_i\}$, where the union is over those $J_i = [c_i, d_i]$ for which $c_i > r_0(\varepsilon)$. Then for all z outside H_1, the E-set of Lemma 3, and outside a set of discs centres ζ, radii t for which

$$\sum \frac{t}{|\zeta|} < 4e \sum \log \frac{d_i}{c_i} < 4e \, \log \text{meas } E_0 < \infty,$$

(7.5) holds. This follows from (7.2) and (7.4).

Suppose next that z, in $T(\frac{1}{8}\tau, r_0(\varepsilon))$, lies outside $\bigcup\{z\colon |z| \in J_i\}$. Then, with $|z| = r$, we have from Lemma 6

$$\int_0^{\tau r} n_z(t, F)\,\frac{dt}{t} \leqslant \int_0^{\delta(\varepsilon) r} \eta_{-r}(t, f_1)\,\frac{dt}{t} < \varepsilon r^\rho.$$

(7.5) thus holds for z in $T(\frac{1}{8}\tau, r_0(\varepsilon))$ outside an E-set,and Lemma 7 is proved.

We prove

LEMMA 8. *Let ε be any positive number and let $\sigma = \sigma(\varepsilon) = \frac{1}{32}\tau(\varepsilon)$, where $\tau(\varepsilon)$ is the function of Lemma 7. There exists a number $r_1(\varepsilon)$ and an E-set, $H_3 = H_3(\varepsilon)$, such that*

$$\int_0^{\sigma r} n_z(t, F)\,\frac{dt}{t} < 6\varepsilon r^\rho, \tag{7.6}$$

whenever $|z| = r > r_1(\varepsilon)$ and z lies outside H_3.

For z in $T(\frac{1}{8}\tau, r_0(\varepsilon))$ and outside $H_2(\varepsilon)$, (7.6) certainly holds, by Lemma 7.

Consider z outside $T(\frac{1}{8}\tau, r_1(\varepsilon))$, where $r_1(\varepsilon) = \max(r_0(\varepsilon), R(\frac{1}{32}\tau, \frac{1}{8}\tau))$. Let $H_4(\varepsilon)$ be the E-set $H_1(\frac{1}{8}\tau)$ of Lemma 3. Then, with $\sigma = \frac{1}{32}\tau$ and $r = |z|$, and z outside $H_4(\varepsilon)$, we have from Lemma 3

$$\int_0^{\sigma r} n_z\left(t, F, \tfrac{1}{8}\tau\right)\frac{dt}{t} < \sigma r^\rho < \varepsilon r^\rho.$$

But for $0 < t < \frac{1}{32}\tau r$ and $r \geqslant R(\frac{1}{32}\tau, \frac{1}{8}\tau)$, $n_z(t, F) = n_z(t, F, \frac{1}{8}\tau)$ for z outside $T(\frac{1}{8}\tau, r_1(\varepsilon))$. Hence $\int_0^{\sigma r} n_z(t, F)dt/t < \varepsilon r^\rho$ for z outside $T(\frac{1}{8}\tau, r_1(\varepsilon))$ and outside $H_4(\varepsilon)$, with $|z| = r > R(\frac{1}{32}\tau, \frac{1}{8}\tau)$.

Lemma 8 then follows with $H_3(\varepsilon) = H_2(\varepsilon) \cup H_4(\varepsilon)$.

The following is an immediate consequence of Lemma 8.

LEMMA 9. *Let ε be any positive number. There exist positive numbers $\alpha(\varepsilon)$, $r_2(\varepsilon)$ and an E-set H_5, independent of ε, such that*

$$\int_0^{\alpha(\varepsilon) r} n_z(t, F)\,\frac{dt}{t} < \varepsilon r^\rho$$

when $r = |z| > r_2(\varepsilon)$ and z lies outside H_5.

For z such that $r = |z| > r_1(\frac{1}{6})$ and z lies outside $H_3(\frac{1}{6})$, where r_1 and H_3 are as in Lemma 8, we have, with $\sigma = \sigma(\frac{1}{6})$, $\int_0^{\sigma r} n_z(t, F)dt/t < r^\rho$. Given any integer $n \geqslant 1$, suppose that $H_3(1/6n)$ is covered by the discs $C_i(n)$, radii $t_i(n)$ and centres $\zeta_i(n)$, $i = 1, 2, 3, \ldots$. Let $i_0 = i_0(n)$ be the smallest integer such that

$$(7.7)\qquad \sum_{i=i_0}^{\infty} \frac{t_i(n)}{|\zeta_i(n)|} \leq 2^{-n} \sum_{i=1}^{\infty} \frac{t_i(1)}{|\zeta_i(1)|}, \qquad n = 2, 3, \ldots .$$

Let $r_2(1) = r_1(\frac{1}{6})$ and, supposing $r_2(m)$ defined, $m \geq 1$, let $r_2(m+1)$ be the smallest number which is no less than $\max\{r_2(1/m) + 1, r_1(1/6(m+1))\}$ and such that

$$C_i(m+1) \subset \{z: |z| \leq r_2(1/(m+1))\}, \qquad i = 1, 2, \ldots, i_0(m+1) - 1.$$

Let H_4 be given by

$$H_4 = \left\{ \bigcup_1^{\infty} C_i(1) \right\} \cup \left\{ \bigcup_{n=2}^{\infty} \bigcup_{i=i_0(n)}^{\infty} C_i(n) \right\}.$$

From (7.7), H_4 is an E-set.

Given any number ε, $0 < \varepsilon < 1$, let m be the integer such that

$$(7.8)\qquad \frac{1}{m+1} < \varepsilon \leq \frac{1}{m};$$

define $r_2(\varepsilon) = r_2(1/(m+1))$, $\alpha(\varepsilon) = \sigma(1/6(m+1))$. Let ε be any positive number, $0 < \varepsilon < 1$, and let z be outside H_4 and such that $r = |z| > r_2(\varepsilon)$. Then, if m is the integer satisfying (7.8), z lies outside $H_3(1/6(m+1))$ and $r = |z| > r_1(1/6(m+1))$ so by Lemma 8,

$$\int_0^{\alpha(\varepsilon)r} n_z(t, F)\frac{dt}{t} = \int_0^{\sigma\left(\frac{1}{6(m+1)}\right)r} n_z(t, F)\frac{dt}{t}$$
$$< \frac{1}{m+1} r^{\rho} < \varepsilon r^{\rho}.$$

Lemma 9 is thus proved.

8. Completion of the proof of Theorem 1. By Lemma 1 and Lemma 9,

$$(8.1)\qquad |r^{-\rho}\log|F(re^{i\theta})| - \beta\cos\rho\theta| \to 0$$

as $z = re^{i\theta}$ tends to infinity outside H_4. From (8.1) and Lemma 2, Theorem 1 follows.

9. A counterexample. Let $f(z)$ be an integral function with real negative zeros. In [11] Titchmarsh proves that if

$$(9.1)\qquad \overline{\lim_{r\to\infty}} \frac{\log|f(r)|}{r^{\rho}} = A \qquad (0 < A < \infty)$$

for some ρ such that $0 < \rho < 1$ then

(i) $\overline{\lim}_{r\to\infty} \log|f(-r)|/r^{\rho} = A\cos\pi\rho$; and

(ii) given $\varepsilon > 0$,

$$(9.2)\qquad \log|f(-r)| > (A\cos\pi\rho - \varepsilon)r^{\rho}$$

for all r outside a set of linear density zero.

We shall show that the exceptional set of (ii) cannot be replaced with a set of finite logarithmic measure by constructing an integral function satisfying (9.1) for which (9.2) fails for some $\varepsilon > 0$ on a set of infinite logarithmic measure. The construction depends on Lemma 1.

Let A be any fixed positive number. Let (R_m) be an increasing sequence of positive numbers, let $\eta_m = (\log m)^{-1}$ and let $\delta_m = m^{-1/2}$, $m = 2, 3 \dots$. Let $f(z)$ be an integral function with real negative zeros for which the counting function $n(r, f)$ satisfies

$$n(r, f) \sim Ar^\rho.$$

We introduce an integral function $g(z)$ obtained from $f(z)$ by placing $1 + [\eta_m R_m^\rho]$ additional zeros at $-R_m$. It is clear that the sequence (R_m) may be chosen sparsely enough that $n(r, g)$, the counting function of the zeros of $g(z)$, satisfies $n(r, g) \sim Ar^\rho$.

With Lemma 1 in view let us consider, for $R_m < r < (1 - 1/m)^{-1}R_m$,

$$\int_0^{\delta_m r} \frac{n_{-r}(t, g)}{t}\, dt. \tag{9.3}$$

Since $r - R_m < r - r(1 - m^{-1}) < \delta_m r$, each zero at $-R_m$ contributes $\log\{\delta_m r/r - R_m\}$ to the integral (9.3). Hence we have, for $R_m < r < (1 - 1/m)^{-1}R_m$,

$$\begin{aligned} \int_0^{\delta_m r} \frac{n_{-r}(t, g)}{t}\, dt &\geqslant \eta_m R_m^\rho \log\left\{ \frac{\delta_m r}{r - R_m} \right\} \\ &\geqslant \eta_m R_m^\rho \log(m\delta_m) = \tfrac{1}{2} R_m^\rho \\ &> \tfrac{1}{4} r^\rho \end{aligned} \tag{9.4}$$

for all large m, taking account of the definitions of η_m and δ_m. Further

$$E = \bigcup_{m=3}^{\infty} \left\{ r\colon R_m < r < \left(1 - \frac{1}{m}\right)^{-1} R_m \right\}$$

is a set of infinite logarithmic measure.

Now we appeal to Lemma 1 to conclude that there must be a number $\varepsilon > 0$ such that

$$\log|f(-r)| < (A \cos \pi\rho - \varepsilon) r^\rho \tag{9.5}$$

for all large r in E. For suppose that there were a sequence (r_n) tending to infinity through E such that

$$\log|f(-r_n)| \geqslant (A \cos \pi\rho - o(1)) r_n^\rho.$$

From (i) of Titchmarsh's result, then,

$$\log|f(-r_n)| = (A \cos \pi\rho + o(1)) r_n^\rho \tag{9.6}$$

and so, from Lemma 1, there must exist $\delta > 0$ such that

$$\int_0^{\delta r_n} \frac{n_{-r_n}(t, g)}{t}\, dt < \tfrac{1}{4} r_n^\rho$$

for all large n, which contradicts (9.4). (9.5) thus holds for all large r in E.

Theorem 1 is an improved version of a result which forms part of a thesis submitted for the degree of Ph.D at the University of London. It is a pleasure to express my gratitude to Professor W. K. Hayman of Imperial College, London, for his generous advice and encouragement.

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