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CURVES WITH LARGE TANGENT SPACE

BY

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ABSTRACT. THEOREM. *Let V be a complex analytic variety irreducible at a point $p \in V$. Given any integer l, there exists an analytic curve C_l on V passing through p and irreducible at p such that the germs of C_l and V at p are isomorphic up to order l.*

In [6] Hironaka and Rossi have proven the following result: Let V be a complex analytic variety of pure dimension r and p an isolated singular point of V. Then there exists an integer ν_0 so that if p' is an isolated singular point of a complex analytic variety V' of pure $\dim r$ and if $\mathcal{O}_p(V)/m_p^\nu \simeq \mathcal{O}_{p'}(V')/m_{p'}^\nu$, for some $\nu \geqslant \nu_0$, where the isomorphism is as $\mathbf{C}$ algebras and m_p is the maximal ideal of the local ring $\mathcal{O}_p(V)$ of V at p, then $\mathcal{O}_p(V) \simeq \mathcal{O}_{p'}(V')$. We show here that it is necessary for V and V' to be of the same dimension.

THEOREM. *Let V be a complex analytic variety irreducible at a point $p \in V$. Given any integer l, there exists an analytic curve C_l on V passing through p and irreducible at p such that $\mathcal{O}_p(V)/m_p(V)^l \simeq \mathcal{O}_p(C_l)/m_p(C_l)^l$ as $\mathbf{C}$ algebras. In particular, there exists an irreducible analytic curve C_2 on V having the same tangent space at p as V does, where the tangent space to V at $p = m_p(V)/m_p(V)^2$.*

In §§2 and 3, we generalize this result to complete domains and finally to arbitrary analytically irreducible local Noetherian rings to read as follows: Let R be a local Noetherian ring, whose completion $\hat{R}$ with respect to the maximal ideal M is a domain. Then for every integer $l > 0$, there exists a prime ideal P_l in R, with $P_l \subset M^l$ and $\dim R/P_l = 1$. The results of §§1 and 2 can be deduced from those of §3; we include both proofs to illustrate the various techniques. The argument presented in Theorem 3 is due to M. Hochster, and the authors are grateful for his permission to reproduce it here.

In §4, we give an application of these results to local differentiable

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embeddings of a real or complex analytic variety. We show how to recover several other known results, due to Risler [12] and Merrien [8]. Also these curves have application to the solution of power series equations [5].

1.

PROOF OF THEOREM. We may assume that a neighborhood of p in V is embedded in $\mathbf{C}^n$, n = embedding dim of V at p, and p is the origin in $\mathbf{C}^n$. We consider the blowing up of the maximal ideal, that is the quadratic transform B of V with center at p; B is the closure in $V \times \mathbf{P}^{n-1}$ of the set B' in $V \times \mathbf{P}^{n-1}$ where $B' = \{(y_1, \ldots, y_n, z_1, \ldots, z_n) \in V - \{0\} \times \mathbf{P}^{n-1}: z_i y_j = z_j y_i$ for all $1 \leq i, j \leq n\}$. It is known that the natural projection $\phi: B \to V$ has the following properties: $F = \phi^{-1}(p) \subset \mathbf{P}^{n-1}$ is the tangent cone to V at p so $\dim F = r - 1$. ϕ: $B - \phi^{-1}(p) \to V - p$ is a biholomorphism, so $B - \phi^{-1}(p)$ is topologically connected and $B = \overline{B - \phi^{-1}(p)}$ is of pure dim r.

We also consider the normalization X of B. There exists a holomorphic map $\pi: X \to B$ such that: π is a proper map with finite fibers. If S is the singular locus of B, then $\pi^{-1}(B - S)$ is dense in X, $\pi: \pi^{-1}(B - S) \to B - S$ is biholomorphic, and X is locally irreducible. The local ring of X at any point x, $\mathcal{O}_x(X)$ is the integral closure of $\mathcal{O}_{\pi(x)}(B)$ in its full quotient ring. If $S(X)$ denotes the singular locus of X, then $\dim_x S(X) < \dim_x X - 2$. $\dim \pi^{-1}(F) = r - 1$ because π preserves dim.

Hence there exist $x \in \pi^{-1}(F)$ such that x is a simple point of X. Now $\mathcal{O}_x(X) \simeq \mathbf{C}\{x_1, \ldots, x_r\}$, the convergent power series in r variables so there exist convergent power series $f_1(x_1, \ldots, x_r), \ldots, f_n(x_1, \ldots, x_r)$ all vanishing at the origin such that $\mathcal{O}_p(V) = \mathbf{C}\{y_1, \ldots, y_n\} = \mathbf{C}\{f_1, \ldots, f_n\}$. We now show that $\mathcal{O}_p(V)$ is a subring of $\mathcal{O}_x(X)$, that is the canonical homomorphism $\mathcal{O}_p(V) \to \mathcal{O}_x(X)$ is an injection. Let $B_{\pi(x)} = B_1 \cup B_2 \cup \cdots \cup B_n$ be a decomposition into germs of irreducible components (which all have the same dimension since the connected manifold $\phi^{-1}(\operatorname{Reg} V)$ is dense in B) such that the germ of X at x is the normalization of B_1. $\phi | X - \phi^{-1}(F)$ is an open map and $\pi | B_1 - F$ is an open map so $\pi \circ \phi(X)$ contains an open set of V; hence any analytic function vanishing on $\pi(B_1)$ must vanish identically on the irreducible variety V, so it is an injection.

It is clear that $m_p(V) = m_x(X) \cap \mathcal{O}_p(V)$. The Krull topology of $\mathcal{O}_x(X)$ defined by powers of the maximal ideal induces a topology T_2 on $\mathcal{O}_p(V)$ which is clearly stronger than the natural topology T_1 on $\mathcal{O}_p(V)$.

LEMMA 1. *$T_1 = T_2$, that is there is an increasing function h: $\mathbf{Z}^+ \to \mathbf{Z}^+$ such that $m_x(X)^{h(i)} \cap \mathcal{O}_p(V) \subset m_p(V)^i$.*

LEMMA 2. *Given integers r and $N > 0$, there exist integers $n_1, n_2, \ldots, n_r$ all > 0 so that for any formal power series $h(x_1, \ldots, x_r)$, $h(t^{n_1}, \ldots, t^{n_r}) \equiv 0$ implies* $\operatorname{ord} h(x_1, \ldots, x_r) \geq N$.

We will finish deducing the theorem before giving the proofs of these lemmas. We will pick suitable convergent power series $x(t) = (x_1(t), \dots, x_r(t))$ in one variable t without constant terms and let C_l be the image of $f(x(t))$ in V; then the domain

$$T = \mathbf{C}\{f_1(x_1(t), \dots, x_r(t)), \dots, f_n(x_1(t), \dots, x_r(t))\}$$

will be the local ring $\mathcal{O}_p(C_l)$ in the theorem. Let $R = \mathcal{O}_p(V)$, $\psi: R \to T$ be the canonical surjection given by substitution ($\phi \to \phi(f(x(t)))$), and I_l denote the kernel of ψ. For $g \in R$, by ord g we mean the order of g considered as an element of $\mathcal{O}_x(X)$.

Let $N = h(l) + 1$. Now choose the monomials $t^{n_1}, \dots, t^{n_r}$ which satisfy the property in Lemma 2. If $g \in I_l$, then $g(f(t^{n_1}, \dots, t^{n_r})) = 0$ so by Lemma 2, ord $g \geqslant N > h(l)$, so $g \in m_p(V)^l$. Hence $I_l \subset m_p(V)^l$. Now $T = R/I_l \simeq \mathcal{O}(C_l)$ and the maximal ideal $m(T)$ of T is $m(R)/I_l$. Hence

$$\begin{aligned}\mathcal{O}_p(C_l)/m_p(C_l)^l &= T/m(T)^l = (R/I_l)/\left(m(R)/I_l\right)^l \\ &\simeq R/\left(m(R)^l + I_l\right) = R/m(R)^l\end{aligned}$$

as $I_l \subset m(R)^l$. Q.E.D.

Proof of Lemma 2. Suppose $n_1, \dots, n_r$ are any positive integers and $h(x_1, \dots, x_r)$ is a nonzero formal power series such that $h(t^{n_1}, \dots, t^{n_r}) = 0$. Then there exist two monomials $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $x_1^{\beta_1} \cdots x_1^{\beta_r}$ appearing in h such that $\alpha_1 n_1 + \cdots + \alpha_r n_r = \beta_1 n_1 + \cdots + \beta_r n_r$ and each $\alpha_i < N$. Letting $\gamma_i = \alpha_i - \beta_i$, we have each $\gamma_i < N$ too. So it is enough to choose n_i so that $\Sigma \gamma_i n_i = 0$ and $\gamma_i < N$ implies each $\gamma_i = 0$.

Let p be a prime $> N$, l_i, $1 \leqslant i \leqslant r$, be primes chosen inductively, $l_1 > p$, so that $rp^2 l_i < l_{i+1}$, and $n_i = p^{r-i} l_i$. Suppose $\Sigma_{i=1}^r \gamma_i n_i = 0$ and each $\gamma_i < p$, let k be the largest integer for which $\gamma_k \neq 0$. By our choice p^{r-k+1} divides $n_1, \dots, n_{k-1}$ so p^{r-k+1} divides $n_k \;\; = p^{r-k} l_k \gamma_k$ so p divides $l_k \gamma_k$; since p and l_k are relatively prime, p divides γ_k. We know $\gamma_k < p$ so a contradiction will follow when we show $\gamma_k > 0$.

Now $n_i = p^{r-i} l_i$, $n_{i+1} = p^{r-i-1} l_{i+1}$, and $l_{i+1} > rp^2 l_i \Rightarrow rpn_i < n_{i+1} \Rightarrow n_i \leqslant n_k$ for $i \leqslant k-1$. Since $\gamma_i < p$ for all i, we have

$$\begin{aligned}\left|\sum_{i=1}^{k-1} \gamma_i n_i\right| &\leqslant \sum_{i=1}^{k-1} |\gamma_i n_i| \leqslant p(k-1) n_{k-1} \\ &\leqslant p(k-1) n_k / rp < n_k.\end{aligned}$$

Hence $\Sigma_{i=1}^k \gamma_i n_i = 0$ implies that $\gamma_k n_k > 0$ and so $\gamma_k > 0$. Q.E.D.

Proof of Lemma 1. Let $q = \pi(x)$, $S' = \mathcal{O}_{\pi(x)}(B)$, and $S_1' = \mathcal{O}_{\pi(x)}(B_1)$. Since $\mathcal{O}_x(X)$ is the integral closure of $\mathcal{O}_{\pi(x)}(B_1)$ in its field of quotients, $\mathcal{O}_x(X)$ is a finite module over $\mathcal{O}_{\pi(x)}(B_1)$. Hence the Krull topology of S_1' is induced by the Krull topology of $\mathcal{O}_x(X)$. Therefore it suffices to show that T_1 is induced by the Krull topology of S_1'. We now recall:

(A) ZARISKI SUBSPACE THEOREM [2]. *Let R be an analytically irreducible domain* (*completion is a domain*) *and S a local ring birational with R* (*means same quotient field*) *and dominating R, and S a spot over R* (*means S is the localization of a finite algebra over R*), *then R is a subspace of S.*

(B) *The ring of convergent power series at an irreducible point of a complex analytic variety is analytically irreducible.* [7, *p.* 89].

(C) [3, Lemma 1.11]. *Let R be an analytic local ring and S a spot over R. Then there is a smallest analytic ring S' containing S and $\hat{S} = \hat{S}'$, where hat denotes completion in the maximal ideal topology.*

Now let $(p; a_1, a_2, \ldots, a_n)$ denote the coordinates of the point $\pi(x)$ on B. By linear change of coordinates, we may assume $a_1 = 1$. Let

$$S = \mathcal{O}_p(V)[y_2/y_1, \ldots, y_n/y_1](z_1 - a_1, \ldots, z_n - a_n),$$

where $z_i = y_i/y_1$. We first show that $\mathcal{O}_{\pi(x)}(B)$, the local analytic ring of B at the point $\pi(x)$, is the smallest analytic ring S' containing S. We have a commutative diagram:

$$\begin{array}{ccc} S & \subset & \mathcal{O}_{\pi(x)}(B) \\ \cap & \nearrow_{\phi} & \\ S' & & \end{array}$$

Since $y_1, \ldots, y_n, z_1 - a_1, \ldots, z_n - a_n$ generate the maximal ideal of both S and $\mathcal{O}_{\pi(x)}(B)$ we have that $\phi(M_{S'}) = M_{\mathcal{O}_{\pi(x)}(B)}$. Since S' and $\mathcal{O}_{\pi(x)}(B)$ are both analytic rings, it follows [9] that $\phi(S') = \mathcal{O}_{\pi(x)}(B)$.

Now S is a spot over $\mathcal{O}_p(V)$ so by (B) and (A) above, $\mathcal{O}_p(V)$ is a subspace of S. Since S is obviously a subspace of $\hat{S}$, we have $\mathcal{O}_p(V)$ is a subspace of $\hat{S}$. Next by part (C), we have $\hat{S} = \hat{S}'$. Hence $\mathcal{O}_p(V)$ is a subspace of $\hat{S}'$. But $\mathcal{O}_p(V) \subset S'$ so it follows that $\mathcal{O}_p(V)$ is a subspace of S'. By similar techniques we could show that $\mathcal{O}_p(V)$ is a subspace of S_1'.

REMARK. Alternately we could go directly to the normalization and show $\mathcal{O}_p(V)$ is a subspace of $\mathcal{O}_x(X)$ by applying (A), (B), and (C). This proof would proceed by using the fact that $\mathcal{O}_x(X)$ is a finite $\mathcal{O}_{\pi(x)}(B)$ module and constructing a spot over $\mathcal{O}_p(V)$ for which $\mathcal{O}_x(X)$ is the smallest analytic ring containing the spot. Q.E.D.

2. By techniques similar to those employed in §1, one can prove a formal version of the theorem.

THEOREM 2. *If k is a field of characteristic zero and $R = k[[y_1, \ldots, y_n]]/I$ is a complete domain, and m is the maximal ideal of R, then for every integer*

$l > 0$, *there is a prime ideal* P_l *of depth one with* $P_l \subset m^l$.

Before starting on the proof of this we need to establish some preliminaries.

For an arbitrary Noetherian ring S and field $k \subset S$, let $D^1(S/k)$ be the module of Kahler differentials of S over k. If S is local with max ideal m, let

$$\hat{D}(S/k) = D^1(S/k) / \bigcap_{i=1}^{\infty} m^i D^1(S/k).$$

It is well known that for a local ring S containing its residue field of characteristic zero, with $\hat{D}(S/k)$ a finite S module, we have $\hat{D}(S/k)$ is a free S module if and only if S is regular. If S is finitely generated over k (respectively complete) then $D(S/k)$ (respectively $\hat{D}(S(k))$) is a finite module over S. If M is a finitely generated S module, generated by say $e_1, \ldots, e_m$, let (a_{ij}) be a matrix with entries in S such that for each i, $\sum_{j=1}^{n} a_{ij} e_i = 0$, and such that any row vector $(a_1, a_2, \ldots, a_n)$ for each $\Sigma a_j e_j = 0$ is a linear combination with coefficients in S of the rows of (a_{ij}); in other words the rows of (a_{ij}) generate the module of relations of $e_1, \ldots, e_n$. For an integer $p \geqslant 0$, the $(n-p) \times (n-p)$ minors of (a_{ij}) generate an ideal $I_p(M)$; by convention $I_p(M) = S$ for $p \geqslant n$. The ideal does not depend on either the choice of basis $\{e_1, \ldots, e_n\}$ or the relation matrix (a_{ij}); $I_p(M)$ is the pth fitting of M. We have $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = S$. By the jacobian ideal J of S we will mean the first nonzero fitting ideal of $D(S/k)$. One knows that S is regular if and only if J is the unit ideal. We will denote $\hat{D}(S/k)$ by $\Omega(S/k)$.

It follows immediately from the Serre criteria for normality that if S is a normal Noetherian ring and $k \subset S$, then every minimal prime of $\mathrm{Jac}(\Omega(S/k))$ has height $\geqslant 2$. (We will denote this by saying ht $J \geqslant 2$.)

LEMMA 3. *Let* S *be a Noetherian ring,* $k \subset S$, $J = \mathrm{Jac}(\Omega(S/k))$, M *a maximal ideal of* S *with* $J \not\subset M$, $S/M = F$, $F \subset S$, *and* F *a finite field extension of* k, *then* S_M *is regular.*

PROOF. $\mathrm{Jac}(\Omega(S_M/k)) = \mathrm{Jac}(\Omega(S/k)) \cdot S_M = JS_M = S_M$ as $J \not\subset M$. In char 0 all extensions are separable and for a finite separable field extension, $\Omega(F/k) = 0$. So by the exact sequence $0 \to S \otimes_F \Omega(F/k) \to \Omega(S/k) \to \Omega(S/F) \to 0$ we see that $\Omega(S/k) = \Omega(S/F)$ so $\mathrm{Jac}(\Omega(S_M/F)) = S_M$. Hence S_M is regular.

LEMMA 4. *Let* $k \subset S \subset F[[t]]$, *where* $k \subset F$ *are fields,* $[F : k] < \infty$, $S \not\subset F$, *and* S *is a Noetherian complete ring. Then* Krull dim $S \leqslant 1$.

PROOF. We will show that $F[[t]]$ is a finite extension over S and hence integral over S so by the going up and down theorem for integral extensions, Krull dim S = Krull dim $F[[t]]$.

Because $[F : k] < \infty$, it is easy to see that the map

$$v: S \to F[[t]] \overset{\text{evaluation}}{\to} F$$

is not injective: Let $g \in S$, $g \notin F$, and $v(g) \neq 0$, then $v(g)$ satisfies some minimal polynomial over k, $v(g)^r + k_1 v(g)^{r-1} + \cdots + k_r = 0$ with $k_i \in k$ and $k_n \neq 0$. So $f = g^r + k_1 g^{r-1} + \cdots + k_{r-1} g + k_r \in S$, $v(f) = 0$, and $f \neq 0$. To see the last statement, let $g = \Sigma a_i t^i$, $a_i \in F$, and a_m be the first nonzero term with $m > 0$; then the mth coefficient of f is $a_m(k_r + 2k_{r-1}a_0 + 3k_{r-2}a_0^2 + \cdots + rk_0 a_0^{r-1})$ which is nonzero since the polynomial $\Sigma_{i=1}^r i k_{r-i} a_0^{i-1}$ has degree less than the minimal polynomial for a_0.

Since $1 \in S$, $m(F[[t]]) \cap S \subset m(S)$. Let g be an element of the kernel of v, $g = t^n u(t)$, $n > 0$, u a unit in $F[[t]]$. We can subject $F[[t]]$ to an F isomorphism $x = \phi(t)$ so that $\phi(g) = x^n$; hence we may assume $k \subset S \subset F[[x]]$ and $x^n \in S$. Now $x^n \in m(F[[x]])$ and $x^n \in S$ so $x^n \in m(S)$; since S is complete, $k[[x^n]] \subset S$. Let $f_1, \ldots, f_r$ be a basis of F over k. Then $\{f_j x^i | 1 \leqslant j \leqslant r, 1 \leqslant i \leqslant n\}$ is a basis of $F[[x]]$ over $k[[x^n]]$. Since $k[[x^n]] \subset S$, $F[[x]]$ is also a finite S module.

REMARK. The hypothesis of complete is critical here as the following example shows. There exist in $k[[t]]$ an infinite set $\{f_i\}_{i=1}^{\infty}$ of algebraically independent elements [1]; like $e^x - 1$, $e^{x^2} - 1, \ldots$ for example. Then $k[f_1, f_2, f_3, \ldots]$ is not Noetherian and has infinite Krull dim. And $S' = k[f_1, \ldots, f_n]_{(f_1, \ldots, f_n)}$ is local, Noetherian, not complete, and has Krull dim n. Note that $k[[f_1, \ldots, f_n]]$ has Krull dim one. (Any two power series in one variable are analytically dependent.) The problem here is that the completion of $k[f_1, \ldots, f_n]_{(f_1, \ldots, f_n)}$ does not inject into $k[[f_1, \ldots, f_n]]$.

PROOF OF THEOREM 2. Let B be the blowing up of the maximal ideal in R, and N be the integral closure of B in its quotient field. Clearly B and N are contained in the quotient field of R, and B is a finite algebra over N. That N is a finite B module follows from the below fact.

Fact D [9, 36.1, 36.5]. A ring A is pseudogeometric if A is Noetherian and if for every prime ideal p of A and ring E, $A/p \subset E$, with the quotient field of E a finite extension of the quotient field of A/p, and E integral over A/p, we have that E is a finite A/p module. In characteristic zero, complete rings are pseudogeometric. If A is pseudogeometric, then every localization of a finite algebra over A is pseudogeometric.

Let $M(R) = (y_1, \ldots, y_n)R$, $B = R[y_2/y_1, \ldots, y_n/y_1]$, and $I = (y_1, \ldots, y_n)B = y_1 B$ and IN be the extension of I to N. N is a finite B module so N/IN is a finite B/I module. However B/I does not inject into N/IN.

EXAMPLE. $R = k[[x, y, z]]/(x^4 + y^4 - z^2)$, let $z = xv$, $y = xu$, $B = k[[x]][u, v]/(x^2(1 + u^4) - v^2)$. Then v/x is in the quotient field of B, $(v/x)^2 - (1 + u^4) = 0$ so $v/x \in N$. However $v/x = \sqrt{1 + u^4}$ is in $\hat{B}$ but not in B.

Hence x divides v in N, but x does not divide v in B. So v is an element of the kernel $B/I \to N/IN$.

Now I is principal and a proper ideal. Furthermore it can be shown that $\dim N/IN = \dim N - 1$. (It should be pointed out that B and N have problems with their Krull dimension. In opposition to the case of local rings or affine rings, B does not have the property that every maximal chain of primes has the same length.)

EXAMPLE. Let $R = k[[x,y]]$ and $B = k[[x,y]][y/x]$. Then $(0) \subset (x) \subset (x, y/x)$ is a chain of primes of length 2 in B. But $(0) \subset I = (1 - y(y/x))B$ is a maximal chain of primes of length one in B: I is principal so of height one; it suffices to show I is a maximal ideal. Now $x - y^2 \in I$ so $B/I = k[[y^2, y]][1/y] = k[[y]][1/y] = k((y))$ which is a field.

Let J be the jacobian ideal of N. Now $\operatorname{ht} J \geqslant 2$, and $\operatorname{ht} IN = 1$ so $J \not\subset IN$; hence $J' = J/IN$ is not zero in $N' = N/IN$. Clearly N' is an affine ring, so there exists a maximal ideal M' of N/IN with $J' \not\subset M'$. (This is easily seen by tensoring N' with the algebraic closure $\bar{k}$ of k, finding a maximal ideal M_0 of $N' \otimes_k \bar{k}$ with $J' \otimes_k \bar{k} \not\subset M_0$ via the Hilbert Nullstellensatz, and letting $M' = M_0 \cap N'$.) Now let M be the contraction of M' to N and $S = \hat{N}_M$. Clearly the contraction of M to R is the maximal ideal of R. Since N_M is a spot over R, the residue field F of S is finite over the residue field k of R. By Lemma 3, S is regular. By Cohen's structure theorem for complete local rings, $S = F[[t_1, \ldots, t_r]]$, where $r = \dim R$. Let $\eta: S \to F[[t]]$ be a homomorphism as in Lemma 2 with $\ker \eta \subset M(S)^l$. Let i be the inclusion $R \to B \to N \to S$, and $\psi = \eta(i)$. By Lemma 4, $\dim \psi(R) = 1$; hence $\dim R/\ker \psi = 1$. The theorem now follows from the Chevalley subspace theorem:

Fact E [**13**, p. 255]. Let $R \subset S$ be complete local rings with $m(S) \cap R = m(R)$. Then R is a subspace of S. Q.E.D.

3. We would now like to improve the results of §§1 and 2 to yield the following:

PROPOSITION 1. *If* $\mathbf{R}$ *is the field of real numbers and* $A = \mathbf{R}\{x_1, \ldots, x_n\}/I$ *is the local ring of germs of convergent power series over* $\mathbf{R}$ *at a point* p *on a real analytic variety* V, *and* $l > 0$, *then there is a curve* C_l *lying on* V *and passing through* p *such that* V *and* C_l *are isomorphic up to order* l *at* p. *This curve is given by a* $\mathbf{R}$ *algebra homomorphism* $A \to \mathbf{R}\{t\}$.

Of course the ideal I must be the ideal of a real analytic variety, that is the real Hilbert Nullstellensatz must hold for I. That is every $f \in A$ which vanishes on the locus of I belongs to I. It is known that this is equivalent to the following condition on I: for every $f_1, \ldots, f_p \in A$ with $f_1^2 + \cdots + f_p^2 \in I$, we have each $f_i \in A$.

Unfortunately this does not seem to follow immediately because when blowing up and normalizing the residue field is likely to change from **R** to **C**, yielding a complex analytic curve on the complexification, $V \otimes_{\mathbf{R}} C$ of V, rather than a real analytic curve on V. For this reason, we give a different proof of Theorem 2 which works for a more general class of rings.

THEOREM 3. *Let S be an analytically irreducible domain with residue field of characteristic zero. Then for every $l > 0$, there is a prime P_l with $P_l \subset M^l$ and* $\dim S/P_l = 1$.

REMARK. Actually this theorem holds in mixed characteristic as well, but we are not interested in that here.

We first recall [**13**, pp. 86–92, Vol. I] the properties of the norm map. Let K be a finite algebraic extension of a field k of degree n and $y \in K$. Fixing a basis $w_1, w_2, \ldots, w_n$ of K over k, we write $yw_i = \sum_{j=1}^n a_{ij}w_j$, $a_{ij} \in k$, $1 \leqslant i \leqslant n$, or in matrix notation $y\Omega = A\Omega$, where $A = (a_{ij})$ is an $n \times n$ matrix and Ω is the column matrix with entries w_i. The characteristic polynomial $\det(A - XI) = X^n + a_1X^{n-1} + \cdots + a_n$ has y as root. It is not necessarily the minimal polynomial of y over k. It is not hard to see that the polynomial does not depend on the choice of basis of K over k. Note that $a_n = (-1)^n \det A$. We set the norm of y, $N_{K/k}(y) = (-1)^n a_n$. The following properties are well known:

(a) $N_{K/k}(xy) = N_{K/k}(x)N_{K/k}(y)$.

(b) If $y \in k$, then $N_{K/k}(y) = y^n$.

(c) If $R \subset S$ are domains with quotient fields $k \subset K$ respectively, with S finite over R, R normal, and $y \in S$, then $N_{K/k}(y) \in R$ (because the norm is just the product of all the conjugates of x in some normal extension of K). Also if p is an ideal of S, then $N_{K/k}(p) \subset R \cap p$.

Also recall [**11**] Pfister and Poperin have generalized Artin approximation to show the following: Let (R, m) be a complete local ring. Let $f_1, \ldots, f_m \in R[x] = R[x_1, \ldots, x_n]$ be m polynomials in n variables over R. Then $\forall c \in N$, $\exists N_c \in \mathbf{N}$ (depending on c and $f_i, \ldots, f_m$) such that if $y = (y_1, \ldots, y_n) \in R^n$ and each $f_i(y) = 0 \bmod m^{N_c}$, then $\exists(\lambda_1, \ldots, \lambda_n) \in R^n$ such that $\lambda_i = y_i \bmod m^c$ for each i, and each $f_i(\lambda) = 0$.

LEMMA 5. *Let (R, m) be analytically irreducible local domain (i.e. $\hat{R}$ is a domain) and let $a_1, \ldots, a_n \in \mathbf{N}$ be given. Then there exist $N \in \mathbf{N}$ such that*

$$(R - m^{a_1}) \cdots (R - m^{a_n}) \subset R - m^N.$$

PROOF. Pick $c = \max\{a_1, \ldots, a_n\}$ and apply the above to the equation $X_1 \cdot X_2 \cdots X_n = 0$ over $\hat{R}$, i.e. let $h = 1$ and $f_1(X) = X_1 \cdots X_n$. Choose $N = N_c$ as guaranteed by the theorem. Now suppose $y_i \in R - m^{a_i}$. It then follows that $y_1 \cdot y_2 \cdots y_m \in m^N$. For if not $f_1(y) = 0 \bmod m^N\hat{R}$ and we could choose $\lambda_1, \ldots, \lambda_n \in \hat{R}$ such that $\lambda_i = y_i \bmod m^c\hat{R}$ and $\lambda_1 \cdots \lambda_n = 0$.

Now $\lambda_i = y_i \bmod m^c\hat{R}$, $y_i \notin m^{a_i}$, and $c \geqslant a_i$ implies $\lambda_i \notin m^{a_i}\hat{R}$ which implies $\lambda_i \neq 0$. But $\hat{R}$ is a domain. This is a contradiction. Q.E.D.

LEMMA 6. *Let (S, n) be a complete local domain, (R, m) be a normal local domain, with $R \subset S$, and S a finite R module. Let the quotient fields of R and S be F and G respectively. Let $\mathfrak{N} = \mathrm{Norm}_{G/F}$. Then $\mathfrak{N}(S) \subset R$ and for all $c \in \mathbf{N}$, there exist $N_c \in \mathbf{N}$ such that $\mathfrak{N}^{-1}(m^{N_c}) \subset n^c$.*

PROOF. We first assume that G is a normal field extension of F, that is the fixed field of the Galois group of G over F is precisely F. Let $\{\phi_1, \ldots, \phi_\nu\} = \mathrm{Gal}(G/F)$, say ϕ_1 = identity. Each ϕ_i induces an R automorphism of S. For all $s \in S$, $\mathfrak{N}(s) = \prod_{i=1}^{\nu}\phi_i(s)$. Hence if $s = \phi_1(s) \notin n^c$, we have $\phi_i(s) \notin n^c$ for each i and by Lemma 5, there exist N_c such that $(S - n^c)^n \subset S - n^{N_c}$. But then $\mathfrak{N}(s) \notin n^{N_c}$. Hence $\mathfrak{N}^{-1}(m^{N_c}) \subset n^c$.

In the general case, let H be a finite normal field extension of F containing G and let (T, q) = integral closure of R in H. Since T is finite over S, by the Artin Reese lemma, the natural Krull topology on S is the same as the topology induced from T. Hence for every c there exists d such that $q^d \cap S \subset n^c$. By the special case, there exists N so $\mathfrak{N}_{T/R}^{-1}(m^N) \subset q^d$. This N also works for S: If $s \in S$ and $\mathfrak{N}_{S/R}(s) \in m^N$, then

$$\begin{aligned}\mathfrak{N}_{T/R}(s) &= \mathfrak{N}_{S/R}(\mathfrak{N}_{T/S}(s)) = \mathfrak{N}_{S/R}(s^{[H:G]})\\ &= (\mathfrak{N}_{S/R}(s))^{[H:G]} \in m^{N[H:G]} \subset m^N \Rightarrow s \in q^d \cap S \subset n^c.\end{aligned}$$

PROOF OF THEOREM 3. Let the Krull dimension of $S = d$. We will prove that for all i, $0 \leqslant i < d$, and all c, there exists a prime p in S with $\mathrm{ht}\, p = i$ and $p \subset m^c$. By a trivial induction on i, we may assume $i = 1$. Let $X_1, \ldots, X_d$ be a system of parameters for S, then $\hat{S}$ is a finite extension of $R = k[[X_1, \ldots, X_d]]$. Let $\mathfrak{N} = \mathrm{Norm}_{\hat{S}/R}$. Pick N so large that $\mathfrak{N}^{-1}(m^N) \subset \hat{n}^c$. Let P_0 be the prime in R generated by $X_1^{p_1} - X_2^{p_2}$, where p_1, p_2 are large prime integers; $P_0 \subset m^N$. Let $\tilde{p}$ be a prime of $\hat{S}$ lying over P_0. Then $\tilde{p} \subset \hat{n}^c$, for $s \in \tilde{p} \Rightarrow \mathfrak{N}(s) \in \tilde{p} \cap R \subset P_0 \subset m^N \Rightarrow s = \mathfrak{N}^{-1}\mathfrak{N}(s) \subset \hat{n}^c$. Now let $p = \tilde{p} \cap S$. Clearly $p \subset n^c$, and $p \cap S \neq (0)$ because $X_1^{p_1} - X_2^{p_2} \in p \cap S$.

4. It is clear that by the techniques of the last section, we can prove Proposition 1. (The projection of the local parametrization of a real analytic variety is semianalytic so one just picks a high order curve in this semianalytic set not lying in the discriminant locii and lifts it to a real analytic curve on V. Details are omitted and left to the reader. See for instance [**8**].)

We now show how Proposition 1 can be used to study the ring of C^∞ and C^k functions on an irreducible real analytic set. Let X be a real analytic set in $\mathbf{R}^n$, irreducible at x, and $C_x^\infty(X)$ be the ring of germs at x of infinitely differentiable functions on X. Let T: $C_x^\infty(X) \to \mathbf{R}[[x_1, \ldots, x_n]]/I(X)$ be the Taylor map, taking a C^∞ function to its Taylor series. Then T is clearly an $\mathbf{R}$

algebra homomorphism. It is well known that T is surjective. Let m_1 be the maximal ideal of $\mathfrak{F} = \mathbf{R}[[x_1, \ldots, x_n]]/I(X)$ and m_2 be the maximal ideal of $C_x^\infty(X)$. Let $p_i \subset m_1^i$ be as in Proposition 1. Since there is a one to one correspondence between primes of $C_x^\infty(X)$ containing ker T and primes of $\mathfrak{F}$, $q_i = T^{-1}(p_i)$ is a prime of $C_x^\infty(X)$ of depth one with $q_i \subset m_2^i$.

COROLLARY 1. *Let X be germ of a real analytic subvariety in $\mathbf{R}^n$, irreducible at x, and $C_x(X)$ the ring of germs at x of real valued infinitely differentiable functions on X. Then there is an irreducible real analytic curve C in X passing through x so that their C^∞ tangent spaces are the same at x, that is $T(V, C_x^\infty) = T(C, C_x^\infty)$, where $T(V, C_x^\infty) = \{r \in \mathbf{R}^n | \Sigma_{i=1}^n r_i \partial_i f(x) = 0 \ \ \forall f \in I(V, C_x^\infty)\}$.*

REMARK. We have not made use of the fact that $T(V, \mathcal{Q}_x) = T(V, C_x^\infty)$ in the above corollary, and in fact our theorem can be used to prove results of this nature.

COROLLARY 2. *Let X be a germ of a real analytic subvariety in $\mathbf{R}^n$, irreducible at x, $C_x^k(X)$ the ring of germs at x of real valued k times continuously differentiable functions on X, and $\mathcal{Q}_x(X)$ the ring of germs of real analytic functions on X. Then there exist $k > 0$ so that $T(X, C_x^k) = T(X, \mathcal{Q}_x)$, where*

$$T(X, C_x^k) = \left\{ r \in \mathbf{R}^n | \sum_{i=1}^n r_i \partial_i f(x) = 0 \, \forall f \in I(X, C_x^k) \right\}$$

and

$$T(X, \mathcal{Q}_x^k) = \left\{ r \in \mathbf{R}^n | \sum_{i=1}^n r_i \partial_i f(x) = 0 \, \forall f \in I(X, \mathcal{Q}_x) \right\}.$$

PROOF. Pick a curve C in X as in Corollary 1. Then clearly:

$$\begin{array}{ccc} T(C, C_x^k) & \subset & T(X, C_x^k) \\ \cap & & \cap \\ T(C, \mathcal{Q}_x) & = & T(X, \mathcal{Q}_x) \end{array}$$

so it suffices to see that these exist $k > 0$ so that $T(C, C_x^k) = T(C, \mathcal{Q}_x)$. We sketch the proof of this fact below. Assume the point x is the origin. Let ϕ: $\mathbf{R} \to C$ be the desingularization of the irreducible curve C (obtained by complexifying C, normalizing the new complex curve and then restricting to the real part). Without loss of generality, one may assume that C is embedded real analytically in minimal dimension. One may make a real analytic change of coordinates so that ϕ has the form $\phi(t) = (t^{q_1}u_1(t), \ldots, t^{q_n}u(t))$, where the u_i are units $q_1 < q_2 < \cdots < q_n$, and there is no polynomial in $\phi_1(t), \ldots, \phi_{k-1}(t)$ whose order is precisely q_k. (If this condition is not satisfied, make inductively a sequence of transformations until it is.) Pick $k > q_n/q_1 + 1$. If $f \in I(X, C_x^k)$ and T_k is the kth order Taylor serves about 0, then $f - T_k = o(|x|^k)$ on C. Comparing with $\phi(t)$, we get $\Sigma_{|\alpha| \leqslant k} a_\alpha \phi(t)^\alpha = o(t^{q_1 k})$ where $T_k = \Sigma_{|\alpha| \leqslant k} a_\alpha x^\alpha$. Then T_k can have no linear terms because each $q_i < kq_1$ and no polynomial in the $\phi_j(t), j \neq i$, can have order q_i.

For additional details of the argument above, see [4].

Straightforward generalization of these arguments yields the following result claimed by the author in [4] and proven by Risler in [12]: Let X be the germ of a real analytic set in $\mathbf{R}^n$ and $I(X)$ the ideal of X in $\mathbf{R}\{x_1, \ldots, x_n\}$. Then there is a function λ: $\mathbf{N} \to \mathbf{N}$, with $\lim_{r\to\infty} \lambda(r) = r$, so that if F is a function of class C^r on R^n vanishing on X, then $T_{\lambda(r)}F \in I(X) + m^{\lambda(r)}$.

Corollary 3. *In* [8] *Merrien has proven that if I is an ideal of $\mathcal{O}_n = \mathbf{R}\{x_1, \ldots, x_n\}$ with $I =$* ideal(locus(I)), *then $I = \bigcap_{\gamma\in\Gamma} \ker \gamma$, where Γ is the set of all* $\mathbf{R}$ *algebra homomorphisms $\mathcal{O}_n \to \mathbf{R}\{t\}$ with $I \subset \ker \gamma$. This follows immediately from Proposition* 1 *because the condition on the ideal is precisely that I is the ideal of a real analytic variety; hence for every $i > 0$, there exists γ_i: $\mathcal{O}_n \to \mathbf{R}[[t]]$ with $I \subset \ker \gamma_i \subset I + m^i$. Hence*

$$I \subset \bigcap_{i=1}^{\infty} \ker \gamma_i \subset \bigcap_{i=1}^{\infty} (I + m^i) = I$$

as $\mathcal{O}_n$ is a local Noetherian ring.

Now suppose ϕ: $X \to Y$ is a map of irreducible analytic varieties and ϕ: $\mathcal{O}(Y, q) \to \mathcal{O}(X, p)$ the induced map on the associated local analytic rings. It is not necessarily true that the image of ϕ is closed in $\mathcal{O}(X, p)$ in the Krull topology. (The map $(x, y) \to (x, xy, xe^y)$ provides an easy counterexample.)

Definition. Let ϕ: $R \to S$ be a local algebra homomorphism of analytic domains. By a curve in S, we mean a local algebra hom γ: $S \to \mathcal{O}_1$. Clearly a curve in S induces a curve in R. We say that ϕ is nice if for every $f \in S - \phi R$, there is a nonzero curve γ in S such that $\gamma(f) \in \mathcal{O}_1 - \gamma\phi R$. That is a function factors through ϕ if and only if it factors through every curve.

Corollary 4. *ϕ is nice if and only if $\phi(R)$ is closed in S in the Krull topology.*

Proof. Suppose $\phi(R)$ is closed in S. Since the closure of $\phi(R)$ in S is $\bigcap_{k=1}^{\infty}(\phi(R) + M(S)^k)$, for every $f \in S - \phi(R)$, there exists a $k > 0$ so $f \notin \phi(R) + M(S)^k$. Now let γ be a curve in S so that $\ker \gamma \subset M(S)^k$. We have a commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ R/\ker \gamma\phi & \xrightarrow{\bar{\phi}} & S/\ker \gamma \end{array}$$

It is clear that $\pi_2(f) \notin \bar{\phi}(R/\ker \gamma\phi) = \pi_1\phi(R)$.

Conversely assume ϕ is nice. Let $f \in S - R$ and γ curve in S with $\gamma(f) \notin \gamma\phi R$. Now mappings of one dim analytic rings are always finite of zero, so by the Artin Reese lemma $\gamma\phi R$ is closed in γS. Since γ is continuous, $\gamma^{-1}\gamma\phi R$ is closed in S. Also $f \notin \gamma^{-1}\gamma\phi R$ and $\phi R \subset \gamma^{-1}\gamma\phi R$. Hence $\phi R = \bigcap_\gamma \gamma^{-1}\gamma\phi R$ is closed in S.

For additional applications of these ideas, see [5].

ADDED IN PROOF. After this paper was accepted for publication, the authors discovered that Theorem 3 had been proven by R. Berger (*Zur Ideal theorie analytisch normaler Stellenringe*, J. Reine Angew. Math. **201** (1958), 172–177) by different techniques.

REFERENCES

1. S. Abhyankar, *Two notes on formal power series*, Proc. Amer. Math. Soc. 7 (1956), 903–905.

2. ———, *Resolution of singularities of embedded algebraic surfaces*, Academic Press, New York.

3. S. Abhyankar and M. Van der Put, *Homomorphisms of analytic local rings*, Creile's J. **242** (1970), 26–60.

4. J. Becker, *Holomorphic and differentiable tangent spaces to a complex analytic variety*, J. Differential Geometry (to appear).

5. ———, *Applications of functional analysis to solving power series equations*. II, (in preparation).

6. H. Hironaka and H. Rossi, *On the equivalence of embeddings of exceptional complex spaces*, Math. Ann. **156** (1964), 313–333.

7. B. Malgrange, *Ideals of differentiable functions*, Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1967.

8. J. Merrien, *Idéaux de l'anneau des séries formelles à coefficients réels et variétés associées*, J. Math. Pures Appl. **50** (1971), 169–187.

9. M. Nagata, *Local rings*, Interscience, New York, 1962.

10. R. Narasimhan, *Introduction to the theory of analytic spaces*, Lecture Notes in Math. No. 25, Springer-Verlag, Berlin and New York, 1966.

11. G. Pfister and D. Popescu, *Die strenge Approximationseigenschaft lokaler Ringe*, Invent. Math. **30** (1975), 145–174.

12. J. J. Risler, *Sur la divisibilité des fonctions de classe C^r par les fonctions analytiques Réelles*, (preprint).

13. O. Zariski and P. Samuel, *Commutative algebra*, Vols. I and II, Van Nostrand, New York, 1960.

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