TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 243, September 1978

VECTOR VALUED EIGENFUNCTIONS OF ERGODIC TRANSFORMATIONS

BY

E. FLYTZANIS

ABSTRACT. We study the solutions X, T, of the eigenoperator equation

$$X(h(\cdot)) = TX(\cdot) \text{ a.e.},$$

where h is a measurable transformation in a σ-finite measure space (S,Σ,m), T is a bounded linear operator in a separable Hilbert space H and X: $S \to H$ is Borel measurable. We solve the equation for some classes of measure preserving transformations. For the general case we obtain necessary conditions concerning the eigenoperators, in terms of operators induced by h in the scalar function spaces over the measure space. Finally we investigate integrability properties of the eigenfunctions.

Introduction. A measure space (S,Σ,m) will denote a separable, σ-finite measure space. It is called a probability space if $m(S) = 1$. A transformation h: $S \to S$ denotes a measurable invertible nonsingular transformation. It is a measure preserving transformation (*m.p.t.*) if $m(h(A)) = m(h^{-1}(A)) = m(A)$ for all A in Σ. H denotes a separable complex Hilbert space, an operator T: $H \to H$ denotes an invertible bounded linear operator, and X: $S \to H$ a Borel measurable function. We say that X,T is a solution of the *eigenoperator* equation for h if

$$X(h(\cdot)) = TX(\cdot) \quad \text{a. e.} \tag{1}$$

Considered as a generalization of the notion of eigenvalues, eigenoperators present interest for their use in the study of H-valued sequences of random variables and also as possible invariants of transformations, in particular measure preserving transformations.

The eigenoperator equation for *m.p.t.* has been solved completely for the cases where H is finite dimensional or T is unitary, [1], [2]. In a sense these cases are not very interesting because the eigenoperator equation reduces to eigenvalue equations.

In §1 of this work we find necessary conditions for the eigenoperators $\{T\}$ of a transformation h in terms of a class of Hilbert space operators $\{V\}$

Received by the editors July 7, 1976.

AMS (MOS) subject classifications (1970). Primary 28A65; Secondary 47A10.

Key words and phrases. Eigenvalue, measure preserving transformation, eigenoperator, vector valued eigenfunction, measures in Hilbert space, weighted shift operators.

induced by h in $L_2(m)$ where $Vf(\cdot) = f(h(\cdot))$. These conditions are shown to be also sufficient for some classes of *m.p.t.* and solutions X,T for which X is square integrable. In §2 we study integrability properties of eigenfunctions for ergodic transformations.

In this work we make use of some basic notions from the theory of *m.p.t.* and the unitary operators induced by them [3], [4], also from the theory of Hilbert-Schmidt operators [5]. Concerning conventions we note the following: Many statements referring to the measure space hold modulo sets of measure zero, but usually we will not mention this explicitly. If X,T is a solution of the eigenoperator equation we denote by $H(X)$ the subspace of H spanned by the essential range of X and then $H(X)$ is invariant under both T and T^{-1}. Also $X(\cdot) \in H(X)$ a.e. and clearly we can study only the restriction of T to $H(X)$.

1. **Eigenoperators.** h, (S,Σ,m) are as in the introduction. We say that h is *m-bounded* if $m(h^{-1}(A)) < cm(A)$ and $m(h(A)) < cm(A)$ for some $c > 1$ and all A in Σ. In this case we say also that m is *h-bounded.* We note that given h and m we can always find a probability measure m' equivalent to m ($m' \sim m$) and h-bounded. Thus, if m is already a probability measure we can define m' by

$$m'(A) = \sum m\big(h^i(A)\big)/3 \cdot 2^{|i|}$$

for every A in Σ, otherwise we find first an equivalent probability measure, which we can do because the space is σ-finite, and then proceed as above.

LEMMA 1. *If m', m'' are two equivalent σ-finite measures and m'' is h-bounded, then m' is also h-bounded iff the positive (a.e.) functions*

$$w(h^{-1}(\cdot))/w(\cdot) \quad \textit{and} \quad w(h(\cdot))/w(\cdot)$$

are (ess.) bounded, where $w(\cdot) = dm'/dm''$ is the Lebesgue derivative of m' with respect to m''.

PROOF. Omitted.

LEMMA 2. *h is a transformation in a σ-finite measure space (S,Σ,m) and X,T is a solution of the eigenoperator equation. Then we can find a probability measure $m' \sim m$ such that:*

a. *m' is h-bounded.*

b. *X is square integrable with respect to m', i.e. $\int \|X\|^2 dm' < \infty$.*

PROOF. First we find as indicated above a probability measure $m'' \sim m$ that is also h-bounded. Then we define a probability measure $m' \sim m''$ by its Lebesgue derivative $cw(\cdot) = dm'/dm''$ where:

$$w(s) = 1 \quad \text{if } s \in S' = \{s\colon \|X(s)\| < 1\}$$

and

$$w(s) = 1/\|X(s)\|^2 \quad \text{if } s \notin S'.$$

We have $w(\cdot) \leq 1$ and c is chosen so that $m'(S) = 1$. Concerning the ratio $w(h^{-1}(\cdot))/w(\cdot)$ we have: if $s \in S'$ then $w(s) = 1$ and

$$w(h^{-1}(s))/w(s) = w(h^{-1}(s)) \leq 1;$$

if $s \notin S'$ and $h^{-1}(s) \notin S'$, then

$$w(h^{-1}(s))/w(s) = \|X(s)\|^2/\|X(h^{-1}(s))\|^2 \leq \|T\|^2$$

by the eigenoperator equation; finally if $s \notin S'$ and $h^{-1}(s) \in S'$ then $\|X(h^{-1}(s))\| \leq 1$ and

$$w(h^{-1}(s))/w(s) = 1/w(s) = \|X(s)\|^2 = \left\|X\left(h\left(h^{-1}(s)\right)\right)\right\|^2$$
$$\leq \|T\|^2 \|X(h^{-1}(s))\|^2 \leq \|T\|^2$$

by the eigenoperator equation. Therefore

$$w\left(h^{-1}(\cdot)\right)/w(\cdot) \leq \max\left\{1, \|T\|^2\right\}.$$

In the same way we show

$$w(h(\cdot))/w(\cdot) \leq \max\left\{1, \|T^{-1}\|^2\right\}$$

and by Lemma 1 it follows that m' is h-bounded. Concerning b we have:

$$\int \|X\|^2 dm' = \int_{S'} \|X\|^2 dm' + \int_{\bar{S}'} \|X\|^2 dm' = m'(S') + m''(\bar{S}') < \infty. \quad \text{Q.E.D.}$$

Given a transformation h in a σ-finite measure space (S,Σ,m) we define a class of Hilbert space operators as follows: if m is h-bounded it induces in $L_2(m)$ an invertible bounded linear operator V_h, where $V_h f(\cdot) = f(h(\cdot))$; in general we denote by L_h the class $\{V_{h,m'}\}$ of Hilbert space invertible bounded linear operators, where $m' \sim m$ is a h-bounded probability measure and $V_{h,m'} f(\cdot) = f(h(\cdot))$.

PROPOSITION 1. *h is a transformation in a σ-finite measure space (S,Σ,m). X,T is a solution to the eigenoperator equation for h with $H(X) = H$. Then*

(i) *There exists an operator $V \in L_h$ and a Hilbert-Schmidt operator K whose range is dense in H such that*

$$KV^* = TK \tag{2}$$

where V is the adjoint of V.

(ii) *If m is h-bounded and X is square integrable with respect to m then in* (2) *we can replace V by V_h, i.e.*

$$KV_h^* = TK. \tag{3}$$

PROOF. (i) First we find a probability measure m' as in Lemma 1. Next we define K: $L_2(m') \to H$ by the strong integral

$$Kf = \int f(\cdot)X(\cdot)dm', \qquad f \in L_2(m').$$

A similar operator appears also in the study of probability measure in Hilbert spaces [5]. K is well defined as a bounded linear operator, in fact

$$\|K\|^2 < \int \|X(\cdot)\|^2 dm' < \infty$$

by Lemma 2. To show that K is Hilbert-Schmidt we consider an orthonormal basis $\{f_i\}$ in $L_2(m')$. Let also $\{v_j\}$ be an orthonormal basis in H. Then

$$\sum_i \|Kf_i\|^2 = \sum_i \sum_j |\langle Kf_i, v_j\rangle|^2 = \sum_i \sum_j \left|\int f_i(\cdot) < X(\cdot), v_j > dm'\right|^2$$

$$= \sum_j \int |\langle X(\cdot), v_j\rangle|^2 dm' = \int \|X(\cdot)\|^2 dm' < \infty.$$

It follows that K is Hilbert-Schmidt. To show that the range of K is dense in H we note that if v is a vector in H then $\langle X(\cdot), v\rangle \in L_2(m')$ and not the zero a.e. function because of $H(X) = H$. Therefore for some $f_i \in \{f_i\}$ we have

$$\langle Kf_i, v\rangle = \int f(\cdot)\langle X(\cdot), v\rangle dm' \neq 0.$$

Finally we note that

$$TKf = \int f(\cdot)TX(\cdot)dm' = \int f(\cdot)X(h(\cdot))dm'$$

$$= \int f(h^{-1}(\cdot))X(\cdot)d(m' \circ h^{-1}) = KV^*f,$$

because $V^*f = f(h^{-1}(\cdot)) \cdot (d(m' \circ h^{-1})/dm')$.

(ii) We can replace m' by m in the proof for (i). Q.E.D.

Unfortunately not much is known about the operators in L_h, while V_h is well understood especially if m is h-invariant in which case V_h is unitary. For the rest of this section we consider *m.p.t.* on σ-finite measure spaces. Also we consider only solution to the eigenoperator equation having square integrable eigenfunctions. From Proposition 1 (ii) we can derive a complete characterization of eigenoperators in this case. An operator Λ: $H \to H$ is called an S-operator [5] if it is selfadjoint, positive semidefinite and its eigenvalues have finite sum.

THEOREM 1. *T is an eigenoperator of a m.p.t. in a σ-finite measure space having square integrable eigenfunction with $H(X) = H$ iff there exists an S-operator Λ: $H \to H$ injective and with dense range such that $T\Lambda T^* = \Lambda$.*

PROOF. ($\Leftarrow$) In H we define the Gaussian probability measure having covariance operator Λ and zero mean. Then T preserves this measure [5] and the identity function on H is the required eigenfunction. ($\Rightarrow$) From equation $KV_h^* = TK$ in Proposition 1 (ii) it follows that the orthogonal complement of the kernel of K is invariant under both V_h and $V_h^{-1} = V_h^*$. Restricting the equation to this subspace we have that K is injective and has dense range. Taking adjoints we find $V_h K^* = K^* T$ where K^* is also injective and has dense range. Multiplying the two equations we obtain $T\Lambda T^* = \Lambda$ where $\Lambda = KK^*$ has all the required properties. Q.E.D.

REMARK. It follows from the proof above that we can obtain all eigenoperators of *m.p.t.* with square integrable eigenfunctions, by considering only probability spaces.

The next result, obtained in [2] by a different approach follows now directly from Theorem 1.

COROLLARY 1. *If H is finite dimensional or if T is unitary then T is an eigenoperator of a m.p.t., having an eigenfunction as in Theorem* 1, *iff H is spanned by eigenvectors of T having eigenvalues of norm* 1. *Then it follows from Proposition* 1 *that they are also eigenvalues of h.*

Next we solve the eigenoperator equation for some classes of *m.p.t.* For convenience we consider only ergodic *m.p.t.* $L_2'(m)$ denotes the subspace of $L_2(m)$ consisting of functions with zero mean, i.e. $\int f(\cdot)dm = 0$. By the spectrum of h we mean the spectrum of the restriction of V_h to $L_2'(m)$. We note that Proposition 1 (ii) implies that the images under K of the orbits of V_h^* are orbits of T.

THEOREM 2. *h is an ergodic m.p.t. having complete point spectrum. Then T is an eigenoperator of h having square integrable eigenfunction with $H(X) = H$, iff H is spanned by eigenvectors of T having distinct eigenvalues that are also eigenvalues of h.*

PROOF. ($\Rightarrow$) It follows from Proposition 1 (ii). ($\Leftarrow$)$\{v_i: i = 1,2,\dots\}$, $\{c_i: i = 1,2,\dots\}$ are the eigenvectors and their corresponding eigenvalues, as assumed in the theorem. If $\{f_i: i = 1,2,\dots\}$ are the corresponding eigenfunctions of h then

$$X(\cdot) = \sum_i f_i(\cdot)v_i/2^{|i|}$$

has all the required properties. Q.E.D.

Next we consider ergodic transformations with continuous spectrum. In this case we assume for convenience that the eigenfunction has zero mean, i.e. $\int X(\cdot)dm = 0$, and then we can take the restriction of the equation $KV_h^* = TK$ to the subspace $L_2'(m)$. We assume also $H(X) = H$. We can derive various orbit structure properties of T by using known properties of $V_h^* = V_h^{-1}$, [4]. The collection $\{v\} \subset H$ consisting of the nonzero elements of $KL_2'(m)$ is dense in H. Thus for every v in a dense subset $\{v\}$ of H we have:

a. The orbits $\{T^i v: i = 1,2,\dots\}$, $\{T^i v: i = -1,-2,\dots\}$ are infinite dimensional, precompact and have the zero vector as an accumulation point.

b. They have the ergodic property,

$$\left(\sum_1^n T^i v/n\right) \to \text{zero vector strongly,} \qquad \text{as } n \to \pm\infty.$$

c. If h has absolutely continuous spectrum then $T^i v \to$ zero vector strongly, as $i \to \pm\infty$.

In deriving the conditions above we only use the fact that K is compact. Perhaps the work in [6] can be used to derive some stronger conditions using also the properties of K as a Hilbert-Schmidt operator.

In the special case where h has σ-Lebesgue spectrum, which arises often in applications, we can in fact derive necessary and sufficient conditions.

THEOREM 3. *h is an ergodic m.p.t. having σ-Lebesgue spectrum. T is an eigenoperator of h having a square integrable eigenfunction with zero mean and $H(X) = H$, iff there exists a countable collection $\{v_i: i = 1,2,\dots\} \subset H$ such that*:

a. $\{T^i v_j: i = 1,2,\dots, j = 0,\pm 1,\pm 2,\dots\}$ *spans H, and*

b. $\Sigma_i \Sigma_j \|T^i v_j\|^2 < \infty$.

PROOF. If h has σ-Lebesgue spectrum we can find a countably infinite collection $\{f_k: k = 1,2,\dots\} \subset L_2'(m)$ such that

$$\{f_k(h^j(\cdot)): k = 1,2,\dots, j = 0,\pm 1,\dots\}$$

is an orthonormal basis for $L_2'(m)$. Then: ($\Rightarrow$) follows from Proposition 1 (ii) using the defining property of Hilbert-Schmidt operators. ($\Leftarrow$) The function

$$X(\cdot) = \sum_i \sum_j f_i(h^{-j}(\cdot))T^j v_i$$

satisfies all the requirements. Q.E.D.

If in the operators of Theorem 3 we impose also the condition that $\{T^i v_j\}$ is an orthogonal collection then we obtain basically weighted shift operators which are eigenoperators of every *m.p.t.* although it is not clear what $H(X)$ would be in each case.

2. **Eigenfunctions.** As it appears that the integrability properties of eigenfunctions are important we examine this question. We assume now that h is an ergodic transformation in a probability space (S,Σ,m). If X,T is a solution and $\|T\| \leqslant 1$ or $\|T^{-1}\| \leqslant 1$ then X must have constant norm (a.e.) and in

particular it is p-integrable for every $p > 0$ [1], as it is also if T or T^{-1} are totally bounded, i.e.

$$\|T^n\|, \quad n = 0,1,2,\ldots, \quad \text{or} \quad \|T^{-n}\|, \quad n = 0,1,2,\ldots,$$

are uniformly bounded. In order to state a result for the general case we introduce first some notation. The spectral radius of an operator $T: H \to H$ is defined by $r(T) = \limsup \|T^n\|^{1/n}$. Also we define the spectral radius $r(h)$ for an ergodic transformation in a probability space as follows. For every A with $m(A) > 0$ we define the sets

$$A_n = \bigcup_{-n}^{n} h^i(A), \quad n = 0,1,2,\ldots, \qquad \triangle A_n = A_n - A_{n-1}, \quad n = 1,2,\ldots.$$

The sets $\triangle A_n$ are disjoint and $m(U\triangle A_n) = 1$. The radius of convergence of the power series $\Sigma c^n m(\triangle A_n)$ is given by

$$r(A,h) = 1/\limsup(m(\triangle A_n))^{1/n}.$$

We define $r(h) = \inf\{r(A,h): A \in \Sigma, m(A) > 0\}$. Clearly $r(h) \geqslant 1$.

PROPOSITION 2. *h is an ergodic transformation in a probability space and $r(h)$ its spectral radius.*

(i) *If X,T is a solution of the eigenoperator equation and*

$$r(T), r(T^{-1}) < (r(h))^{1/p}$$

then X is p-integrable. Also if $r(A,h) > r(h)$ for all A then the statement holds also with equality.

(ii) *For every $\varepsilon > 0$ and $p > 0$ there exists a solution X,T such that*

$$\|T\|, \|T^{-1}\| \leqslant (r(h))^{1/p} + \varepsilon$$

and X is not p-integrable, provided H is infinite dimensional. Also if $r(A,h) = r(h) = 1$ for some A then we can find the same solution for all p.

PROOF. (i) We set $f(\cdot) = \|X(\cdot)\|^p$ and $c_n = \max\{\|T^n\|^p, \|T^{-n}\|^p\}$. From the eigenoperator equation we have $f(h^n(\cdot)), f(h^{-n}(\cdot)) \leqslant c_n f(\cdot)$. Also for some $\alpha > 0$ we have $m(A = \{s: f(s) \leqslant a\}) > 0$. We define the function $g(\cdot)$ by $g(s) = ac_n$ for $s \in \triangle A_n$, $n = 0,1,\ldots$. Then $\int g(\cdot)dm = a\Sigma c_n m(\triangle A_n) < \infty$ because

$$\limsup(c_n m(\triangle A_n))^{1/n} \leqslant \limsup(c_n)^{1/n} \limsup(m(\triangle A_n))^{1/n}$$
$$\leqslant (r(T))^p / r(h) < 1.$$

(If $r(A,h) > r(h)$ we can replace the last $\leqslant$ by $<$.) Since $f(\cdot) \leqslant g(\cdot)$ by the definition of $g(\cdot)$ it follows that $f(\cdot)$ is integrable and hence $X(\cdot)$ is p-integrable.

(ii) If $r(h) = \infty$ there is nothing to prove. Otherwise we choose $c = (r(h))^{1/p} + \varepsilon/2$ and a set $A \in \Sigma$ so that $r(A,h) < r(h) + p\varepsilon/2$. We define

$g(\cdot)$ by $g(s) = c_n$ for $s \in \triangle A_n$. Then

$$\int g^p(\cdot)dm = \sum (c^p)^n m(\triangle A_n) = \infty \text{ because } c^p > r(A,h).$$

(If $r(A,h) = r(h) = 1$ we would take this A.) Next we choose

$$a = (r(h))^{1/p} + \varepsilon = c + \varepsilon/2$$

and if $\{v_n: n = 0, \pm 1, \dots\}$ is an orthonormal set in H we define $T: H \to H$ by $Tv_n = v_{n+1}/a$ for $n \geq 0$ and $Tv_n = av_{n+1}$ for $n < 0$. The H-valued function

$$X(\cdot) = \sum g(h^{-n}(\cdot))v_n/a^{|n|}$$

is well defined because

$$\|X(\cdot)\| \leq \sum g(h^{-n}(\cdot))a^{-|n|} \leq g(\cdot)\sum c^{|n|}/a^{|n|} < \infty \text{ a.e.}$$

X is not p-integrable because $\|X(\cdot)\|^p \geq g^p(\cdot)$. Finally we note that X,T is a solution and

$$\|T\| = \|T^{-1}\| = a = (r(h))^{1/p} + \varepsilon. \quad \text{Q.E.D.}$$

It follows that h has all its eigenfunctions integrable with respect to m iff $r(h) = \infty$. However no such transformation or even transformations with $r(h) \neq 1$ are known. One can also compute $r(h)$ using intersections of sets and then if A is such that

$$\{h^i(A): i = 0, \pm 1, \dots\}$$

are independent, where h is *m.p.t.*, then

$$r(A,h) = 1/m(\bar{A})^2,$$

and if there exists such sets with measure arbitrarily close to zero we get $r(h) = 1$. Thus if h is the two-sided shift on the infinite product space of the unit interval then $r(h) = 1$, and we can apply Proposition 2 (ii).

We add that Proposition 2 holds also for ergodic transformations on infinite σ-finite measure spaces if in this case we define $r(h) = 1$.

References

1. A. Beck, *Eigenoperators of ergodic transformations*, Trans. Amer. Math. Soc. **94** (1960), 118–129. MR **27** #2606.

2. E. Flytzanis, *Linear dynamical systems*, Proc. Amer. Math. Soc. **55** (1976), 367–370.

3. P. R. Halmos, *Lectures on ergodic theory*, Chelsea, New York, 1960. MR **20** #3958; **22** #2677.

4. K. Jacobs, *Lecture notes on ergodic theory*, 1962/63, Part I, Mat. Inst. Aarhus Univ., Aarhus, 1963. MR **28** #3138; **28**, p. 1247.

5. H. H. Kuo, *Gaussian measures in Banach spaces*, Springer-Verlag, Berlin and New York, 1975.

6. U. Krengel, *Weakly wandering vectors and weakly independent partitions*, Trans. Amer. Math. Soc. **164** (1972), 199–226. MR **44** #7353.

School of Technology, Aristotelian University of Thessaloniki, Thessaloniki, Greece