

# A NOTE ON THE OPERATOR $X \rightarrow AX - XB$ <sup>1</sup>

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**ABSTRACT.** If  $A$  and  $B$  are bounded linear operators on an infinite dimensional complex Hilbert space  $\mathcal{H}$ , let  $\tau(X) = AX - XB$  ( $X$  in  $\mathcal{L}(\mathcal{H})$ ). It is proved that  $\sigma(\tau) = \sigma(\tau|C_p)$  ( $1 < p < \infty$ ), where, for  $1 < p < \infty$ ,  $C_p$  is the Schatten  $p$ -ideal, and  $C_\infty$  is the ideal of all compact operators in  $\mathcal{L}(\mathcal{H})$ . Analogues of this result for the parts of the spectrum are obtained and sufficient conditions are given for  $\tau$  to be injective. It is also proved that if  $A$  and  $B$  are quasisimilar, then the right essential spectrum of  $A$  intersects the left essential spectrum of  $B$ .

**1. Introduction.** Let  $\mathcal{Q}$  denote a complex Banach algebra with identity and for fixed elements  $A$  and  $B$  in  $\mathcal{Q}$  let  $\tau \equiv \tau(A, B)$  denote the operator on  $\mathcal{Q}$  defined by  $\tau(X) = AX - XB$  ( $X$  in  $\mathcal{Q}$ ). In [19] M. Rosenblum proved that

$$\sigma(\tau) \subset \sigma(A) - \sigma(B) \equiv \{\alpha - \beta: \alpha \text{ in } \sigma(A), \beta \text{ in } \sigma(B)\},$$

with equality in case  $\mathcal{Q}$  is the algebra of all bounded linear operators on a Banach space. Several authors have subsequently studied conditions on  $A$  and  $B$  which insure that  $\tau$  is injective or surjective, and the purpose of this note is to give additional results concerning the parts of the spectrum of  $\tau$ .

Let  $\mathcal{H}$  denote a complex, separable, infinite dimensional Hilbert space and let  $\mathcal{L}(\mathcal{H})$  ( $= \mathcal{Q}$ ) denote the algebra of all bounded linear operators on  $\mathcal{H}$ . For  $A$  and  $B$  in  $\mathcal{L}(\mathcal{H})$ , Davis and Rosenthal [5] proved that  $\tau(A, B)$  is surjective if and only if  $\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$ , and that  $\tau$  is bounded below if and only if  $\sigma_\pi(A) \cap \sigma_\delta(B) = \emptyset$  (see below for notation). In §3 we give several reformulations of these results which show that the spectral properties of  $\tau$  are closely related to the spectral properties of the restrictions of  $\tau$  to certain norm ideals in  $\mathcal{L}(\mathcal{H})$ , such as the compact operators  $\mathcal{K}(\mathcal{H})$ , the trace class ( $TC$ ), and the intermediate Schatten  $p$ -ideals  $C_p$  ( $1 < p < \infty$ ). We prove that

$$\sigma(\tau) = \sigma(\tau| \mathcal{K}(\mathcal{H})) = \sigma(\tau| (TC)) = \sigma(\tau| C_p).$$

In particular, the following conditions are equivalent: (i)  $\tau$  is surjective; (ii)

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$\tau|_{\mathcal{K}(\mathcal{H})}$  is surjective; (iii)  $\tau|_{(TC)}$  is surjective; (iv) the range of  $\tau$  contains  $\mathcal{K}(\mathcal{H})$ ; additionally, we obtain analogous equivalent conditions for the case when  $\tau$  is bounded below.

We also study spectral properties of  $\tau(A, B)$  when  $A$  and  $B$  are quasisimilar operators on  $\mathcal{H}$ . Recall that  $A$  and  $B$  are quasisimilar if there exist operators  $X$  and  $Y$ , both of which are injective and have dense range, such that  $AX = XB$  and  $YA = BY$  [12]. Similar operators are quasisimilar and have equal spectra; thus if  $A$  and  $B$  are similar, then  $\tau(A, B)$  is not surjective since  $\partial\sigma(A) \subset \sigma_\delta(A) \cap \sigma_\pi(B)$ . It is known that quasisimilar operators may have different spectra [12], but in [8] and [24] it was proved that the essential spectra of quasisimilar operators intersect. In Theorem 2.1 we sharpen this result by proving that if  $A$  and  $B$  are quasisimilar, then the right essential spectrum of  $A$  intersects the left essential spectrum of  $B$ . Thus  $\tau(A, B)$  is not surjective, and these results give further evidence of the strength of the quasisimilarity relation.

An ordered pair of operators  $(A, B)$  is said to form a *disjoint pair* in case  $\tau(A, B)$  is injective [3], [6], [7]. Interest in disjoint pairs arises in part from the fact that if  $\tau(A, B)$  is injective, then  $\mathcal{H} \oplus \{0\}$  is a hyperinvariant subspace for each operator on  $\mathcal{H} \oplus \mathcal{H}$  whose operator matrix is of the form  $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ . If  $A$  and  $B$  have disjoint spectra then clearly  $(A, B)$  and  $(B, A)$  are disjoint pairs, but the converse is false. In [7] Douglas and Pearcy gave necessary and sufficient conditions for two normal operators to form a disjoint pair; they gave an example of a diagonalizable normal operator  $A$  and a normal operator without eigenvalues  $B$  such that  $(A, B)$  and  $(B, A)$  are disjoint pairs and  $\sigma(A) = \sigma(B)$ . In §4 we extend this example as follows:  $\tau(A, B)$  has no eigenvalues if  $A$  is a hyponormal operator without eigenvalues or a spectral operator whose scalar part has no eigenvalues, and if  $B$  is such that  $\mathcal{H}$  is spanned by the linear manifolds  $\{x: \|(B - \lambda)^n x\|^{1/n} \rightarrow 0\}$  ( $\lambda$  in  $\mathbb{C}$ ); we give several examples of such operators.

We conclude this section with some terminology and notation. For an operator  $T$  on a Banach space,  $\sigma(T)$ ,  $\sigma_\pi(T)$ , and  $\sigma_\delta(T)$  denote, respectively, the spectrum, approximate point spectrum, and approximate defect spectrum of  $T$ , i.e.

$$\sigma_\delta(T) = \{\lambda: T - \lambda \text{ is not surjective}\}.$$

If  $A$  is an operator on  $\mathcal{H}$ , then  $\tilde{A}$  denotes the image of  $A$  under the canonical mapping of  $\mathcal{L}(\mathcal{H})$  onto the Calkin algebra  $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ . Thus  $\tau(A, B)$  induces an operator  $\tilde{\tau}$  on the Calkin algebra by  $\tilde{\tau}(\tilde{X}) = \tau(\tilde{X})$ . We let  $\tau_K$  denote the restriction of  $\tau$  to  $\mathcal{K}(\mathcal{H})$ . For  $T$  in  $\mathcal{L}(\mathcal{H})$ ,  $\sigma_e(T)$  denotes the essential spectrum of  $T$ ;  $\sigma_{le}(T)$  and  $\sigma_{re}(T)$  denote, respectively, the left and right essential spectra of  $T$  [16]. An operator  $T$  is a *spectral operator* if it has a spectral measure;  $T$  is spectral if and only if  $T = S + Q$ , where  $S$ , called the *scalar part* of  $T$ , is similar to a normal operator, and  $Q$  is a quasinilpotent

operator that commutes with  $S$ ; this canonical decomposition is unique [4], [12]. For an operator  $T$  we let

$$\mathfrak{N}(T) = \{x \text{ in } \mathfrak{H}: \|T^n x\|^{1/n} \rightarrow 0\};$$

thus  $T$  is quasinilpotent if and only if  $\mathfrak{N}(T) = \mathfrak{H}$  [4, Lemma, p. 28].

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**2. On the essential spectra of quasisimilar operators.** In this section we give the following refinement of the results of [8] and [24].

**THEOREM 2.1.** *If  $A$  and  $B$  are quasisimilar, then  $\sigma_{le}(B) \cap \sigma_{re}(A) \neq \emptyset$  and (by symmetry)  $\sigma_{le}(A) \cap \sigma_{re}(B) \neq \emptyset$ .*

The outlines of the proof are modeled after the proof given in [8]. We will divide the proof into a sequence of lemmas dealing with special cases depending on the topological properties of the spectra of  $A$  and  $B$ . We note that the cases we consider are not necessarily mutually exclusive.

**LEMMA 2.2.** *If  $A$  and  $B$  are quasisimilar, then  $\sigma_{le}(B) \cap \sigma(A) \neq \emptyset$ .*

**PROOF.** If  $\sigma_{le}(B)$  is open in  $\sigma(B)$ , then since  $\sigma_{le}(B)$  is a nonempty closed subset of  $\sigma(B)$ , the result follows from [8, Theorem 2.5]. If  $\sigma_{le}(B)$  is not open in  $\sigma(B)$ , there exists  $t$  in  $\sigma_{le}(B)$  and  $\{t_n\}_{n=1}^\infty \subset \sigma(B) \setminus \sigma_{le}(B)$  such that  $t_n \rightarrow t$ . Since for each  $n$ ,  $\tilde{B} - t_n$  is left invertible but  $B - t_n$  is not invertible, either  $\ker(B - t_n)$  or  $\ker((B - t_n)^*) \neq \{0\}$ . Since  $A$  is quasisimilar to  $B$ , each  $t_n$  is in  $\sigma(A)$ , and thus  $t$  is in  $\sigma_{le}(B) \cap \sigma(A)$ .

**LEMMA 2.3.** *If  $A$  and  $B$  are quasisimilar and  $Z = \sigma_{le}(B) \cap \sigma(A)$ , then  $Z \not\subset \sigma_e(A) \setminus \sigma_{re}(A)$ .*

**PROOF.** Let  $U = \sigma_e(A) \setminus \sigma_{re}(A)$  and suppose  $Z \subset U$ ; since  $Z$  is closed and  $U$  is open there exists  $t$  in  $U \setminus Z$ . Since  $t \in U \subset \sigma(A)$ , then  $t \notin \sigma_{le}(B)$ , and thus  $B - t$  has finite dimensional null space. Now  $t \in U$  implies that  $A - t$  is semi-Fredholm with  $\text{index}(A - t) = +\infty$ . In particular,  $\dim \ker(B - t) = \dim \ker(A - t) = +\infty$ , and this contradiction completes the proof.

We will use results from [16] concerning essential spectra and semi-Fredholm operators; some of these results were summarized in [8, Lemma 2.7], to which we will refer. Let  $X$  denote a nonempty, bounded, open, connected subset of the plane; let  $\mathfrak{U}(X)$  denote the unbounded component of the complement of the closure of  $X$ , and let  $\beta(X) = \text{bdry}(\mathfrak{U}(X))$ ; note that

$\beta(X) \subset \text{bdry}(X)$  and that  $\beta(X)$  is connected [8]. In particular, if  $T$  is in  $\mathcal{E}(\mathcal{H})$  and  $\beta(X) \subset \sigma(T) \setminus \sigma_e(T)$ , then the connectedness of  $\beta(X)$  implies that  $\beta(X)$  is contained in some component  $H$  of  $C \setminus \sigma_e(T)$ ; since  $\beta(X)$  is uncountable, [8, Lemma 2.7] implies that  $H$  is a hole in  $\sigma_e(T)$ .

**LEMMA 2.4.** *If  $A$  and  $B$  are quasisimilar and if there is a hole  $H_0$  in  $\sigma_e(A)$  such that  $H_0 \subset \sigma(A)$ , then  $\sigma_{le}(B) \cap \sigma_{re}(A) \neq \emptyset$ .*

**PROOF.** We will give a proof by contradiction. Assuming  $\sigma_{le}(B)$  and  $\sigma_{re}(A)$  are disjoint we will obtain two sequences  $\{H_i\}_{i \geq 0}$  and  $\{K_i\}_{i \geq 1}$  such that the following properties are satisfied:

- (i)  $H_i$  is a hole in  $\sigma_e(A)$ ;  $\beta(H_i) \subset \sigma_{re}(A)$  ( $i \geq 0$ );
- (ii)  $K_i$  is a hole in  $\sigma_e(B)$ ;  $\beta(K_i) \subset \sigma_{le}(B)$  ( $i \geq 1$ );
- (iii)  $\beta(H_i) \subset K_{i+1}$ ,  $\beta(K_{i+1}) \subset H_{i+1}$  ( $i \geq 0$ );
- (iv)  $\mathcal{U}(H_i)^- \subset \mathcal{U}(K_i)$ ,  $\mathcal{U}(K_i)^- \subset \mathcal{U}(H_{i+1})$  ( $i \geq 1$ );
- (v)  $K_i \cap K_j = \emptyset$ ,  $H_i \cap H_j = \emptyset$  for all  $i \neq j$ .

Let us first show how to use the above sequences to obtain a contradiction. The idea of the proof is that the sequence  $H_1, K_2, H_3, K_4, \dots$  is a sequence of "annular" domains contained in a common bounded set, and in which adjacent terms "overlap". This will allow us to obtain a common limit point for  $\{\beta(H_i)\}$  and  $\{\beta(K_i)\}$ , which will thus be a point in  $\sigma_{le}(B) \cap \sigma_{re}(A)$ . To be precise, conditions (iii) and (iv) imply that  $\beta(H_i) \cap \beta(H_j) = \emptyset$  for  $i \neq j$ . Let  $\{h_i\}$  ( $i \geq 0$ ) be a sequence (of necessarily distinct points) with  $h_i \in \beta(H_i)$ . Let  $h_{i_k} \rightarrow h$  be a convergent subsequence; (i) implies that  $h$  is in  $\sigma_{re}(A)$ . Since  $i_k > i_{k-1}$ , (iv) implies that

$$\mathcal{U}(H_{i_k}) \subset \mathcal{U}(K_{i_k}) \subset \mathcal{U}(H_{i_{k-1}}) \subset \dots \subset \mathcal{U}(H_{i_{k-1}});$$

if  $L$  denotes the line segment from  $h_{i_k}$  to  $h_{i_{k-1}}$ , it follows that  $L$  contains a point  $g_k$  from  $\beta(K_{i_k})$ . Clearly  $g_k \rightarrow h$ , and (ii) now implies that  $h$  is in  $\sigma_{le}(B) \cap \sigma_{re}(A)$ , which is a contradiction.

It remains to prove the existence of the  $H_i$ 's and  $K_i$ 's satisfying (i)–(v). We repeat our assumption that  $\sigma_{le}(B)$  and  $\sigma_{re}(A)$  are disjoint. Since  $H_0 \subset \sigma(A)$ , then  $H_0 \subset \sigma(B)$  and so

$$\begin{aligned} \beta(H_0) &\subset \text{bdry}(H_0) \cap \sigma(B) \subset \text{bdry}(\sigma_e(A)) \cap \sigma(B) \\ &\subset \sigma_{re}(A) \cap \sigma(B) \subset \sigma(B) \setminus \sigma_{le}(B) = (\sigma(B) \setminus \sigma_e(B)) \cup (\sigma_e(B) \setminus \sigma_{le}(B)). \end{aligned}$$

We claim that  $\beta(H_0) \cap (\sigma_e(B) \setminus \sigma_{le}(B)) = \emptyset$ ; for otherwise, since  $\sigma_e(B) \setminus \sigma_{le}(B)$  is open, there is a point  $t$  in  $(\sigma_e(B) \setminus \sigma_{le}(B)) \cap \beta(H_0)$ . Thus  $B - t$  is semi-Fredholm with index  $(B - t) = -\infty$ , while  $A - t$  is Fredholm, and this contradicts the assumption that  $A$  and  $B$  are quasisimilar.

Now  $\beta(H_0) \subset \sigma(B) \setminus \sigma_e(B)$ , and the remarks following Lemma 2.3 imply that there exists a hole  $K_1$  in  $\sigma_e(B)$  such that  $\beta(H_0) \subset K_1$ ; a connectedness argument implies that  $\mathcal{U}(K_1)^- \subset \mathcal{U}(H_0)$ .

Since  $\beta(H_0)$  is an uncountable subset of  $\sigma(B)$ , the remarks after Lemma 2.3 imply that  $K_1 \subset \sigma(B)$  (whence  $K_1 \subset \sigma(B) \cap \sigma(A)$ ) and that

$$\begin{aligned}\beta(K_1) &\subset \text{bdry}(\sigma_e(B)) \cap \sigma(A) \subset \sigma_{le}(B) \cap \sigma(A) \\ &\subset \sigma(A) \setminus \sigma_{re}(A) = (\sigma(A) \setminus \sigma_e(A)) \cup (\sigma_e(A) \setminus \sigma_{re}(A)).\end{aligned}$$

Using an argument analogous to that above, it follows that  $\beta(K_1)$  and  $\sigma_e(A) \setminus \sigma_{re}(A)$  are disjoint, and thus  $\beta(K_1) \subset \sigma(A) \setminus \sigma_e(A)$ . Now  $\beta(K_1)$  is an uncountable connected subset of  $\sigma_e(B)$ ; thus, as above, there exists a hole  $H_1$  in  $\sigma_e(A)$  such that  $\beta(K_1) \subset H_1, \mathcal{U}(H_1)^- \subset \mathcal{U}(K_1)$ , and (using a connectedness argument as in the proof of [8, Lemma 2.9]),  $H_1$  and  $H_0$  are disjoint. The above procedure may now be used inductively to define the sequences satisfying (i)–(v).

**LEMMA 2.5.** *If  $A$  and  $B$  are quasisimilar and if there exists an infinite sequence  $\{z_n\}$  of distinct isolated points of  $\sigma(A)$  such that  $\ker(A - z_n)$  or  $\ker((A - z_n)^*) \neq \{0\}$  for each  $n$ , then  $\sigma_{re}(A) \cap \sigma_{le}(B) \neq \emptyset$ .*

**PROOF.** Since  $A$  and  $B$  are quasisimilar,  $\{z_n\} \subset \sigma(B)$ , and we can assume that  $z_n \rightarrow z$ . Since  $z$  is a limit point of isolated points of  $\text{bdry}(\sigma(A))$ , [16, Corollary 1.26] implies that  $z$  is in  $\sigma_{re}(A)$ . Clearly  $z$  is in  $\sigma(B)$  and we claim that  $z$  is in  $\sigma_{le}(B)$ . If  $\tilde{B} - z$  is left invertible but not invertible, then  $B - w$  is semi-Fredholm with  $\text{index}(B - w) = -\infty$  for all  $w$  in some disk  $D$  centered at  $z$ ; now  $D \subset \sigma(A)$ , which contradicts the assumption that the  $z_n$ 's are isolated in  $\sigma(A)$ . Thus, if  $\tilde{B} - z$  is left invertible, then  $z$  is in  $\sigma(B) \setminus \sigma_e(B)$ . Since  $z$  is a limit point of  $\sigma(B)$ , [8, Lemma 2.7] implies that there is an open disk  $D_1$  centered at  $z$  such that  $B - w$  or  $(B - w)^*$  is noninjective for each  $w$  in  $D_1$ . Thus  $D_1 \subset \sigma(A)$ , which again contradicts the fact that the  $z_n$ 's are isolated in  $\sigma(A)$ ; now the proof is complete.

The following result is a slightly stronger version of [8, Lemma 2.11]; we will omit the proof, since it is very similar to that of the latter result.

**LEMMA 2.6.** *Let  $A, B$ , and  $X$  be in  $\mathcal{L}(\mathcal{H})$ , with  $X$  injective and  $AX = XB$ . Let  $H$  be a component of  $\mathbb{C} \setminus \sigma_e(A)$  and let  $K$  be a nonempty closed-and-open subset of  $\sigma(B)$ . If  $K \subset H$  and  $K \cap \sigma_e(B) \neq \emptyset$ , then  $H \subset \sigma(A)$ .*

**PROOF OF THEOREM 2.1.** By Lemma 2.4 we may assume that if there is a hole  $H$  in  $\sigma_e(A)$ , then  $H \not\subset \sigma(A)$ , for otherwise the proof is complete. We may also assume from [8, Lemma 2.7] and Lemma 2.5 that  $H \cap \sigma(A)$  is at most finite, and that if  $K$  is the unbounded component of  $\mathbb{C} \setminus \sigma_e(A)$ , then  $K \cap \sigma(A)$  is at most finite. Let  $X = \sigma_{le}(B) \cap \sigma(A)$ ; Lemma 2.2 implies that  $X$  is nonempty, and we must prove that  $X \cap \sigma_{re}(A)$  ( $= \sigma_{le}(B) \cap \sigma_{re}(A)$ ) is nonempty. If  $X \cap \sigma_{re}(A) = \emptyset$ , then by Lemma 2.3 there is a component  $H$  of  $\mathbb{C} \setminus \sigma_e(A)$  such that  $X \cap H \neq \emptyset$ ; the preceding remarks imply that  $H \cap \sigma(A)$  is a finite set.

Since

$$(\sigma_{le}(B) \cap H)^- \cap \text{bdry}(H) \subset \sigma_{le}(B) \cap \sigma_{re}(A) = \emptyset,$$

there is an open set  $U$  such that  $\sigma_{le}(B) \cap H \subset U \subset U^- \subset H$ ; in particular,  $Y = \sigma_{le}(B) \cap H$  is a closed subset of  $\sigma(B)$ . We claim that  $Y$  is also an open subset of  $\sigma(B)$ ; for otherwise there is an infinite sequence of distinct points  $\{z_n\} \subset \sigma(B) \setminus Y$  such that  $z_n \rightarrow z$ , where  $z$  is some point in  $Y$ . By excluding at most a finite number of points, we have  $\{z_n\} \subset U \subset H$ , and thus each  $z_n$  is in  $\sigma(B) \setminus \sigma_{le}(B) \subset \sigma(A)$ . Thus  $\{z_n\} \subset \sigma(A) \cap H$ , which contradicts the fact that  $H \cap \sigma(A)$  is finite. Now  $Y$  is a nonempty closed-and-open subset of  $\sigma(B)$ , and since  $Y \subset \sigma_{le}(B) \subset \sigma_e(B)$ , Lemma 2.6 implies that  $H \subset \sigma(A)$ . This again contradicts the fact that  $H \cap \sigma(A)$  is finite; thus  $X \cap \sigma_{re}(A) \neq \emptyset$ , and the proof is complete.

**REMARK.** In [8] it was proven that if  $A$  and  $B$  are quasisimilar, then the spectrum of each invariant part of  $A$  intersects the spectrum of  $B$ ; in particular, each nonempty closed-and-open subset of  $\sigma(A)$  intersects  $\sigma(B)$ . In [25] we gave an example to show that the essential spectrum of a part of  $A$  may be disjoint from  $\sigma_e(B)$ . However, we are unable to resolve the question, also raised by Williams [24]: If  $A$  and  $B$  are quasisimilar, does each nonempty closed-and-open subset of  $\sigma_e(A)$  intersect  $\sigma_e(B)$ ? In view of Theorem 2.1, one may also ask whether each nonempty closed-and-open subset of  $\sigma_{re}(A)$  intersects  $\sigma_{le}(B)$ . One limiting factor in the preceding spectral intersection theorems is the known result that the boundaries of the spectra (and the boundaries of the essential spectra) of quasisimilar operators may be disjoint [12].

**3. Parts of the spectrum.** In this section we study relationships between the parts of the spectra of  $\tau$ ,  $\tilde{\tau}$ , and  $\tau|C_p$  ( $1 < p < \infty$ ). We state for reference several results of [5], [15] and [19].

**THEOREM 3.1** (ROSENBLUM [19]). (i)  $\sigma(\tau) \subset \sigma(A) - \sigma(B)$ ; (ii) if  $\tau - z$  is invertible, then

$$(\tau - z)^{-1}(X) = \frac{1}{2\pi i} \int_{\gamma} (A - z - w)^{-1} X (w - B)^{-1} dw$$

for each  $X$  in  $\mathcal{Q}$  (where  $\gamma$  is a suitable contour independent of  $X$ ).

**THEOREM 3.2** (KLEINECKE [15], [19]). If  $\mathcal{Q} = \mathcal{L}(\mathcal{X})$ , where  $\mathcal{X}$  is a Banach space, then  $\sigma(\tau) = \sigma(A) - \sigma(B)$ .

In the sequel, unless otherwise noted,  $\mathcal{H}$  is a separable infinite dimensional Hilbert space and  $\mathcal{Q} = \mathcal{L}(\mathcal{H})$ .

**THEOREM 3.3** (DAVIS AND ROSENTHAL [5]). (i)  $\sigma_{\pi}(\tau) = \sigma_{\pi}(A) - \sigma_{\delta}(B)$ ; (ii)  $\sigma_{\delta}(\tau) = \sigma_{\delta}(A) - \sigma_{\pi}(B)$ .

Before proceeding to characterize the parts of the spectrum of  $\tilde{\tau}$ , we note that Theorem 3.2 does not apply when  $\mathcal{Q}$  is the Calkin algebra, since the Calkin algebra is not algebraically isomorphic to  $\mathcal{L}(\mathcal{X})$  for any Banach space  $\mathcal{X}$ . To see this, we rely on the following lemma.

**LEMMA 3.4.** *If  $S$  is in  $\mathcal{L}(\mathcal{H})$  and  $f: \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{C}$  is a function such that  $STS - f(T)S$  is compact for each  $T$  in  $\mathcal{L}(\mathcal{H})$ , then  $S$  is compact.*

**PROOF.** The hypothesis implies that  $S^*S(TS^*)S - f(TS^*)S^*S$  is compact for each operator  $T$ . Let  $P = S^*S$ , let  $g(T) = f(TS^*)$ , and let  $\gamma = g(1_{\mathcal{H}})$ . Since  $PTP - g(T)P$  is compact, then  $\tilde{P}^2 - \gamma\tilde{P} = 0$ , and the spectral mapping theorem for elements of the Calkin algebra implies that  $\sigma_e(P) = \sigma(\tilde{P}) \subset \{0, \gamma\}$ . If  $\sigma_e(P) = \{0, \gamma\}$  and  $\gamma \neq 0$ , then there exists an orthogonal decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , with  $\mathcal{H}_1$  and  $\mathcal{H}_2$  infinite dimensional, such that if  $Q = O_{\mathcal{H}_1} \oplus \gamma \cdot 1_{\mathcal{H}_2}$ , then  $Q - P$  is compact. Thus  $QTQ - g(T)Q$  is compact for each operator  $T$ , and by considering  $T$  of the form  $O_{\mathcal{H}_1} \oplus R$  ( $R$  in  $\mathcal{L}(\mathcal{H}_2)$ ), it follows that  $\gamma^2R - g(T)\gamma$  is compact for every  $R$ , which is impossible. A similar argument also shows that  $\sigma_e(P) \neq \{\gamma\}$ ; thus  $\sigma_e(P) = \{0\}$  and so  $\|\tilde{P}\| = r(\tilde{P}) = 0$ . Since  $P$  is compact, so is  $S$ , and the proof is complete.

Suppose now that  $\mathcal{X}$  is a Banach space and that  $\varphi: \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{Q}$  ( $= \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ ) is an algebraic isomorphism. Let  $x$  be a nonzero vector in  $\mathcal{X}$ , let  $f$  in  $\mathcal{X}^*$  be such that  $f(x) \neq 0$ , and define  $R$  in  $\mathcal{L}(\mathcal{X})$  by  $R(y) = f(y)x$ . A calculation shows that  $RTR = f(T(x))R$  for each  $T$  in  $\mathcal{L}(\mathcal{X})$ , and thus

$$\varphi(R)\varphi(T)\varphi(R) = f(T(x))\varphi(R).$$

If  $\tilde{S} = \varphi(R)$ , then  $S$  is noncompact, and since  $\varphi$  is surjective, it follows that  $STS - f(\varphi^{-1}(\tilde{T})(x))S$  is compact for each operator  $T$ , which contradicts Lemma 3.4. (We are grateful to P. Rosenthal for suggesting that we use the identity  $RTR = f(T(x))R$ .)

Despite the inapplicability of Theorem 3.2 to the setting of the Calkin algebra, the direct analogues of Theorems 3.2 and 3.3 for  $\tilde{\tau}$  are still valid, as we will show presently. Thus, one might conjecture that these results hold more generally in the setting of  $C^*$ -algebras, but we now show that this is not the case.

**EXAMPLE 3.5.** Let  $A = 1_{\mathcal{H}} \oplus 0$  and let  $B = 0 \oplus 1_{\mathcal{H}}$  acting on  $\mathcal{H} \oplus \mathcal{H}$ . Let  $\mathcal{Q} \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$  denote the (commutative)  $C^*$ -algebra with identity generated by  $A$  and  $B$ , i.e.  $\mathcal{Q} = \{\alpha 1_{\mathcal{H}} \oplus \beta 1_{\mathcal{H}}: \alpha, \beta \in \mathbb{C}\}$ . It is clear that

$$\sigma(\tau(A, B)) = \sigma(A) - \sigma(B) = \{-1, 0, 1\}.$$

On the other hand it is easily verified that  $\sigma(\tau|\mathcal{Q}) = \{-1, 1\}$  and  $\sigma_{\mathcal{Q}}(A) = \sigma_{\mathcal{Q}}(B) = \{0, 1\}$ , so that  $\sigma(\tau|\mathcal{Q}) \neq \sigma_{\mathcal{Q}}(A) - \sigma_{\mathcal{Q}}(B)$ . Thus Theorem 3.2 (and similarly Theorem 3.3) fails for  $C^*$ -algebras and (since  $\mathcal{Q}$  is weakly closed)

even for commutative von Neumann algebras.

Recall that if  $T$  is in  $\mathcal{L}(\mathcal{H})$ , then  $\alpha$  is in  $\sigma_{le}(T)$  if and only if there exists an infinite rank projection  $T_{l,\alpha}$  such that  $(T - \alpha)T_{l,\alpha}$  is compact; thus  $\beta$  is in  $\sigma_{re}(T)$  if and only if there exists an infinite rank projection  $T_{r,\beta}$  such that  $T_{r,\beta}(T - \beta)$  is compact. It is clear that if  $P$  and  $Q$  are infinite rank projections, then there is a partial isometry  $V$  such that  $PVQ$  is noncompact.

**THEOREM 3.6.**  $\sigma(\tilde{\tau}) = \sigma_e(A) - \sigma_e(B)$ .

**PROOF.** Theorem 3.1 applied to the Calkin algebra implies that  $\sigma(\tilde{\tau}) \subset \sigma_e(A) - \sigma_e(B)$ . For the reverse inclusion let  $\alpha \in \sigma_e(A)$  and  $\beta \in \sigma_e(B)$ . Motivated by the proof of Theorem 3.2, we consider several cases for the locations of  $\alpha$  and  $\beta$ .

(i)  $\alpha \in \sigma_{le}(A)$ ,  $\beta \in \sigma_{re}(B)$ . Let  $V$  be a partial isometry such that  $W = A_{l,\alpha}VB_{r,\beta}$  is noncompact. Since  $(A - \alpha)W - W(B - \beta)$  is compact,  $\tilde{\tau} - (\alpha - \beta)$  is noninjective.

(ii)  $\alpha \in \sigma_{le}(A)$ ,  $\beta \in \sigma_{le}(B)$ . Let  $V$  be a partial isometry such that  $W = A_{l,\alpha}VB_{l,\beta}$  is noncompact. If  $\tilde{\tau} - (\alpha - \beta)$  is invertible, let  $X$  be such that  $(\tilde{\tau} - (\alpha - \beta))(X) = \tilde{A}_{l,\alpha}\tilde{V}$ . Now

$$(\tilde{\tau} - (\alpha - \beta))((\tilde{A} - \alpha)\tilde{X}) = (\tilde{A} - \alpha)\tilde{A}_{l,\alpha}\tilde{V} = 0,$$

and thus  $(\tilde{A} - \alpha)\tilde{X} = 0$ . Since

$$\tilde{W} = ((\tilde{A} - \alpha)\tilde{X} - \tilde{X}(\tilde{B} - \beta))\tilde{B}_{l,\beta} = (\tilde{A} - \alpha)\tilde{X}\tilde{B}_{l,\beta} = 0,$$

we have a contradiction.

(iii)  $\alpha \in \sigma_{re}(A)$ ,  $\beta \in \sigma_{le}(B)$ . Let  $V$  be a partial isometry such that  $W = A_{r,\alpha}VB_{l,\beta}$  is noncompact. If  $\tilde{\tau} - (\alpha - \beta)$  is surjective, let  $X$  be such that  $(\tilde{\tau} - (\alpha - \beta))(X) = \tilde{V}$ ; now

$$\tilde{W} = \tilde{A}_{r,\alpha}((\tilde{A} - \alpha)\tilde{X} - \tilde{X}(\tilde{B} - \beta))\tilde{B}_{l,\beta} = 0,$$

which is a contradiction.

(iv)  $\alpha \in \sigma_{re}(A)$ ,  $\beta \in \sigma_{re}(B)$ . Let  $V$  be a partial isometry such that  $W = A_{r,\alpha}VB_{r,\beta}$  is noncompact. If  $\tilde{\tau} - (\alpha - \beta)$  is invertible, choose  $X$  such that  $(\tilde{\tau} - (\alpha - \beta))(X) = \tilde{V}\tilde{B}_{r,\beta}$ ; note that

$$\begin{aligned} (\tilde{\tau} - (\alpha - \beta))(\tilde{X}(\tilde{B} - \beta)) &= ((\tilde{\tau} - (\alpha - \beta))(X))(\tilde{B} - \beta) \\ &= \tilde{V}\tilde{B}_{r,\beta}(\tilde{B} - \beta) = 0. \end{aligned}$$

Thus  $\tilde{X}(\tilde{B} - \beta) = 0$  and so

$$\tilde{W} = \tilde{A}_{r,\alpha}((\tilde{A} - \alpha)\tilde{X} - \tilde{X}(\tilde{B} - \beta)) = 0,$$

which is a contradiction.

Let  $\mathcal{Q}$  and  $\mathcal{B}$  denote the operators on the Calkin algebra defined by  $\mathcal{Q}(\tilde{X}) = \tilde{A}\tilde{X}$  and  $\mathcal{B}(\tilde{X}) = \tilde{X}\tilde{B}$ .



LEMMA 3.7. (i)  $\sigma_\pi(\mathcal{Q}) = \sigma_{le}(A)$ ; (ii)  $\sigma_\pi(\mathcal{B}) = \sigma_{re}(B)$ ; (iii)  $\sigma_\delta(\mathcal{Q}) = \sigma_{re}(A)$ ; (iv)  $\sigma_\delta(\mathcal{B}) = \sigma_{le}(B)$ .

PROOF. We prove only parts (i) and (iii) since the other parts follow in an analogous fashion.

(i) If  $\alpha \in \sigma_{le}(A)$ , then  $(\tilde{A} - \alpha)\tilde{A}_{l,\alpha} = 0$ , so  $\alpha \in \sigma_p(\mathcal{Q})$  and thus  $\sigma_{le}(A) \subset \sigma_\pi(\mathcal{Q})$ ; the reverse inclusion is clear.

(iii) If  $\alpha \in \sigma_{re}(A)$ , then  $\tilde{A}_{r,\alpha}(\tilde{A} - \alpha) = 0$ . If  $X$  is an operator such that  $(\tilde{A} - \alpha)\tilde{X} = \tilde{A}_{r,\alpha}$ , then  $0 = \tilde{A}_{r,\alpha}(\tilde{A} - \alpha)X = \tilde{A}_{r,\alpha}^2 = \tilde{A}_{r,\alpha}$ , which is a contradiction. Thus  $\sigma_{re}(A) \subset \sigma_\delta(\mathcal{Q})$ , and the reverse inclusion follows readily.

THEOREM 3.8. (i)  $\sigma_\pi(\tilde{\tau}) = \sigma_p(\tilde{\tau}) = \sigma_{le}(A) - \sigma_{re}(B)$ ; (ii)  $\sigma_\delta(\tilde{\tau}) = \sigma_{re}(A) - \sigma_{le}(B)$ .

PROOF. (i) Theorem 2 of [5] and Lemma 3.7 imply that

$$\sigma_\pi(\tilde{\tau}) \subset \sigma_\pi(\mathcal{Q}) - \sigma_\pi(\mathcal{B}) = \sigma_{le}(A) - \sigma_{re}(B).$$

The proof of Theorem 3.6(i) shows that

$$\sigma_{le}(A) - \sigma_{re}(B) \subset \sigma_p(\tilde{\tau}) \subset \sigma_\pi(\tilde{\tau}),$$

and (i) follows. (ii) Corollary 1 of [5] and Lemma 3.7 imply that

$$\sigma_\delta(\tilde{\tau}) \subset \sigma_\delta(\mathcal{Q}) - \sigma_\delta(\mathcal{B}) = \sigma_{re}(A) - \sigma_{le}(B);$$

the reverse inclusion is proved in Theorem 3.6(iii).

COROLLARY 3.9. If  $A$  and  $B$  are quasisimilar, then there exist noncompact operators  $X$  and  $Y$  such that  $AX - XB$  and  $BY - YA$  are compact.

PROOF. Theorem 2.1 implies that  $\sigma_{le}(A) \cap \sigma_{re}(B)$  and  $\sigma_{re}(A) \cap \sigma_{le}(B)$  are nonempty, so the result follows from Theorem 3.8(i).

REMARK. The following question was suggested to the author by P. Rosenthal; this question motivated our interest in Theorem 2.1 and the preceding corollary.

Question 3.10. If  $A$  and  $B$  are quasisimilar, do there exist noncompact quasiaffinities  $X$  and  $Y$  such that  $AX = XB$  and  $BY = YA$ ?

An affirmative answer to this question (together with Theorem 3.8) would imply Theorem 2.1 as an immediate corollary. From Corollary 3.9 there exist compact operators  $L$  and  $M$  such that (i)  $AX - XB = L$  and (ii)  $BY - YA = M$ ; it is natural to attempt to "lift" these equations to obtain a positive solution to Question 3.10. If  $\tau_K(A, B)$  is surjective, then there exists a noncompact operator  $X_1$  such that  $AX_1 - X_1B = 0$ , and a similar reduction in (ii) can be made if  $\tau_K(B, A)$  is surjective. However, as we remark below, the assumption that  $A$  and  $B$  are quasisimilar implies that neither  $\tau_K(A, B)$  nor  $\tau_K(B, A)$  is surjective. The next example shows that the one-sided version of Question 3.10 has a negative answer.

EXAMPLE 3.11. If  $U$  is a unilateral shift of multiplicity one, then  $U$  is a quasiaffine transform of a quasinilpotent operator  $Q$  [2]. Since  $\sigma_e(U) \cap \sigma_e(Q) = \emptyset$ , Theorem 3.8(i) implies that there exists no noncompact operator  $X$  such that  $UX - XQ$  or  $QX - XU$  is compact.

We next give several conditions that are each equivalent to the surjectivity of  $\tau$ . We begin with an example in which  $\tau$  is surjective (although not necessarily invertible) and in which we may explicitly solve the equation  $AX - XB = Y$ .

EXAMPLE 3.12. Let  $L, R$  and  $T$  be in  $\mathcal{L}(\mathcal{H})$  with  $LR = 1$  and  $r(R)r(T) < 1$ . For  $Y$  in  $\mathcal{L}(\mathcal{H})$ , the series  $X = \sum_{n=0}^{\infty} R^{n+1}YT^n$  converges in norm. Indeed, since  $\delta = r(R)r(T) < 1$ , there is an integer  $N > 0$  such that for  $n \geq N$ ,

$$\|R^n\|^{1/n} \|T^n\|^{1/n} < \varepsilon = (1 + \delta)/2,$$

and thus

$$\sum_{n=N}^m \|R^{n+1}YT^n\| \leq \|R\| \|Y\| \sum_{n=N}^m \varepsilon^n \text{ for } m > N.$$

A calculation now shows that  $\tau(L, T)(X) = Y$  so that  $\tau = \tau(L, T)$  is surjective.

It is noteworthy that the restrictions of  $\tau$  to certain  $\tau$ -invariant norm ideals in  $\mathcal{L}(\mathcal{H})$  are also surjective. Thus if  $Y$  is compact, then clearly  $X$  is compact and so  $\tau_K$  is surjective. Let  $Y$  be trace class and let  $\|\cdot\|_1$  denote the trace norm of a trace class operator; thus  $\|Y\| \leq \|Y\|_1$  and for  $A$  in  $\mathcal{L}(\mathcal{H})$ ,

$$\|AY\|_1 \leq \|A\| \|Y\|_1 \quad \text{and} \quad \|YA\|_1 \leq \|A\| \|Y\|_1$$

[21, Lemma 8, p. 39]. Now

$$\begin{aligned} \left\| \sum_{n=N}^m R^{n+1}YT^n \right\|_1 &\leq \sum_{n=N}^m \|R^{n+1}YT^n\|_1 \\ &\leq \sum_{n=N}^m \|R^{n+1}\| \|Y\|_1 \|T^n\| \leq \|R\| \|Y\|_1 \sum_{n=N}^m \varepsilon^n, \end{aligned}$$

and the completeness of the trace norm [21, p. 42] implies that  $X$  is trace class. In particular, if  $(TC)$  denotes the ideal of all trace class operators on  $\mathcal{H}$  equipped with the trace norm, and  $\tau_{(TC)}$  denotes the restriction of  $\tau$  to  $(TC)$ , then  $\tau_{(TC)}$  is surjective.

In contrast to the above cases, the restriction of  $\tau(L, T)$  to the ideal of all finite rank operators on  $\mathcal{H}$  need not be surjective, even if  $\tau$  is invertible. To see this, let  $\{e_n\}_{n=-\infty}^{\infty}$  denote an orthonormal basis for  $\mathcal{H}$  and define  $L$  and  $T$  as follows:

(i)  $Le_n = e_{n+1}$  ( $-\infty < n < \infty$ );

(ii)  $Te_n = (1/n)e_{n-1}$ ,  $n \geq 1$ ;  $Te_n = (1/(2-n))e_{n-1}$ ,  $n \leq 0$ .

$L$  is unitary; since  $T$  is a compact injective weighted shift,  $T$  is quasinilpotent,

and thus  $\tau(L, T)$  is invertible. Let  $Y$  denote the rank one projection onto the subspace spanned by  $e_0$ . A calculation shows that  $X = \sum_{n=0}^{\infty} L^{*n+1} Y T^n$  is given by the following relations:  $Xe_n = 0$  ( $n < 0$ ),  $Xe_0 = e_{-1}$ , and  $Xe_n = (1/n!)e_{-n-1}$  ( $n > 0$ ). Thus  $X$  is not of finite rank, and since  $\tau$  is injective and  $\tau(X) = Y$ , it follows that the restriction of  $\tau$  to the finite rank operators is not surjective.

The preceding example motivates the following basic result.

**THEOREM 3.13.** *The following are equivalent for  $\tau = \tau(A, B)$ : (i)  $\tau$  is surjective; (ii)  $\tau_K$  is surjective; (iii)  $\tau_{(TC)}$  is surjective; (iv) the range of  $\tau$  contains  $\mathcal{K}(\mathcal{H})$ .*

We defer the proof briefly for a preliminary calculation. We recall from [21] that  $\mathcal{K}(\mathcal{H})^* = (TC)$ , where a trace class operator  $L$  is identified with the functional  $f_L(K) = \text{trace}(KL)$  ( $K$  in  $\mathcal{K}(\mathcal{H})$ ). Further,  $(TC)^* = \mathcal{L}(\mathcal{H})$ ; an operator  $T$  on  $\mathcal{H}$  is identified with the functional  $g_T(f_L) = \text{trace}(TL)$  ( $L$  in  $(TC)$ ). Let  $A$  and  $B$  be in  $\mathcal{L}(\mathcal{H})$ ; under the last identification,  $\tau(A, B)$  corresponds to the operator  $\tau(g_T) = g_{AT-TB}$ . If  $\tau_{(TC)}(A, B)$  denotes the restriction of  $\tau$  to  $(TC)$ , then  $\tau_{(TC)}$  is bounded since  $\|AL - LB\|_1 \leq (\|A\| + \|B\|)\|L\|_1$  for  $L$  in  $(TC)$ . If  $L$  is trace class and  $K$  is compact, then

$$\begin{aligned} ((\tau_K(A, B))^*(f_L))(K) &= f_L(\tau_K(A, B)(K)) = \text{trace}((AK - KB)L) \\ &= \text{trace}((LA - BL)K) = (f_{LA-BL})(K), \end{aligned}$$

and thus

$$\tau_K(A, B)^* = -\tau_{(TC)}(B, A).$$

Now

$$\begin{aligned} ((\tau_K(A, B))^*(g_T))(f_L) &= g_T(\tau_K(A, B)^*(f_L)) = g_T(f_{LA-BL}) \\ &= \text{trace}(T(LA - BL)) = \text{trace}(L(AT - TB)) \\ &= g_{AT-TB}(f_L) = (\tau(A, B)(g_T))(f_L), \end{aligned}$$

and thus

$$\tau(A, B) = \tau_K(A, B)^{**}.$$

We recall that for an operator  $T$  on a Banach space  $\mathcal{X}$  we have the duality  $\sigma_\pi(T) = \sigma_\delta(T^*)$  and  $\sigma_\delta(T) = \sigma_\pi(T^*)$  [20, Theorems 4.12–4.15]; these results imply that  $\sigma_\delta(T) = \sigma_\delta(T^{**})$ .

**PROOF OF THEOREM 3.13.** Since  $\tau_K^{**} = \tau$ , an application of the identity  $\sigma_\delta(T) = \sigma_\delta(T^{**})$  when  $T = \tau_K$  yields the equivalence of (i) and (ii).

We next apply the duality theorem to  $T = -\tau_{(TC)}(A, B)$ ; thus,  $\tau_{(TC)}(A, B)$  is surjective if and only if  $\tau(B, A)$  is bounded below, or equivalently (from Theorem 3.3(i)), if  $\sigma_\pi(B) \cap \sigma_\delta(A) = \emptyset$ . Theorem 3.3(ii) implies that the last

condition is equivalent to  $\tau(A, B)$  being surjective, and thus (i) and (iii) are equivalent.

Let  $A_1$  and  $B_1$  be any operators in  $\mathcal{L}(\mathcal{H})$ . Since  $\tau_K(A_1, B_1)^{**} = \tau(A_1, B_1)$ , [13, Corollary 1.2] implies that if  $\mathcal{R}(\tau_K(A_1, B_1)) \subset \mathcal{R}(\tau)$ , then  $\mathcal{R}(\tau(A_1, B_1)) \subset \mathcal{R}(\tau)$ . If  $\mathcal{K}(\mathcal{H}) \subset \mathcal{R}(\tau)$ , then  $\mathcal{R}(\tau_K(A_1, B_1)) \subset \mathcal{K}(\mathcal{H}) \subset \mathcal{R}(\tau)$ , and thus  $\mathcal{R}(\tau(A_1, B_1)) \subset \mathcal{R}(\tau)$  for all pairs of operators  $A_1$  and  $B_1$ . If we set  $A_1 = B_1$ , it follows that  $\mathcal{R}(\tau)$  contains every commutator in  $\mathcal{L}(\mathcal{H})$ ; since each operator is the sum of two commutators [10, Corollary 2, p. 131], the proof is complete.

**ACKNOWLEDGMENT.** The author is grateful to J. P. Williams for directing the author's attention to [13] and for suggesting our use of [13, Corollary 1.2] in the present context.

**REMARK.** Returning to the case when  $A$  and  $B$  are quasisimilar (cf. Question 3.10), we may now show that  $\tau_K(A, B)$  is never surjective. Indeed, since  $\tau(B, A)$  is not injective then  $\sigma_\delta(A) \cap \sigma_\pi(B) \neq \emptyset$ ; thus  $\tau(A, B)$  is not surjective and the result follows from Theorem 3.13.

Theorem 3.13 allows us to extend a result of [14]. Let  $T$  be an operator on an  $n$ -fold copy of  $\mathcal{H}$  with an upper triangular operator matrix  $(T_{ij})_{1 \leq i, j \leq n}$ , i.e.  $T_{ij} = 0$  for  $1 \leq j < i \leq n$ . It is proved in [14, Theorem 4] that if  $T_{11}$  and  $T_{nn}$  have disjoint spectra, then there exists a compact operator  $K$  such that  $TK - KT$  is a rank one operator. The proof relies only on the fact that if  $\tau(T_{11}, T_{nn})$  is an invertible operator, then it maps  $\mathcal{K}(\mathcal{H})$  onto itself. It follows that the result of [14] is valid more generally whenever  $\tau_K$  is surjective, i.e.,  $\sigma_\delta(T_{11}) \cap \sigma_\pi(T_{nn}) = \emptyset$ ; moreover, in this case, the operator  $K$  may be taken to be trace class.

Using the duality theorem  $\sigma_\pi(T) = \sigma_\pi(T^{**})$  and the fact that  $\tau(A, B)$  is bounded below if and only if  $\tau(B, A)$  is surjective, we have the following analogue of Theorem 3.13 for the approximate point spectrum.

**PROPOSITION 3.14.** *The following are equivalent: (i)  $\tau$  is bounded below; (ii)  $\tau_K$  is bounded below; (iii)  $\tau_{(TC)}$  is bounded below (with respect to the trace norm).*

**COROLLARY 3.15.**  $\sigma(\tau) = \sigma(\tau_K) = \sigma(\tau_{(TC)})$ .

**PROOF.** Apply Theorem 3.13 and Proposition 3.14.

We next seek analogues of the preceding results for the Schatten  $p$ -ideals  $C_p$  equipped with the  $p$ -norm  $\| \cdot \|_p$  ( $1 \leq p < \infty$ ) [21]. Thus  $C_1$  is the trace class and  $C_\infty$  is  $\mathcal{K}(H)$ ; since these two cases have just been discussed we restrict attention to  $1 < p < \infty$ . We recall several well-known results from [21] and [26].  $C_p$  is complete in the  $p$ -norm and the ideal  $\mathcal{F}$ , consisting of all finite rank operators in  $\mathcal{L}(\mathcal{H})$ , is  $p$ -norm dense in  $C_p$ . If  $A$  and  $B$  are in  $\mathcal{L}(\mathcal{H})$  and  $K$  is in  $C_p$ , then  $\|AKB\|_p \leq \|A\| \|K\|_p \|B\|$ ; if  $K$  is a rank one operator, then  $\|K\| = \|K\|_p$ . For  $1/p + 1/q = 1$ , we have  $C_p^* = C_q$ . Clearly  $\mathcal{F}$  and  $C_p$  are invariant for  $\tau$  and we let  $\tau_{\mathcal{F}}$  and  $\tau_p$  denote the respective restrictions of  $\tau$  to

these ideals. We begin with another condition that is equivalent to  $\tau$  being bounded below.

LEMMA 3.16.  $\tau = \tau(A, B)$  is bounded below if and only if there exists a constant  $M > 0$  such that  $\|AX - XB\| \geq M\|X\|$  for each rank one operator  $X$ .

PROOF. If  $\tau$  is not bounded below, then from Theorem 3.3 there exists  $\lambda$  in  $\sigma_\pi(A) \cap \sigma_\delta(B)$ , and, in particular,  $(B - \lambda)^*$  is not bounded below. For  $\varepsilon > 0$ , there exist vectors  $e$  and  $f$  such that  $\|e\| = \|f\| = 1$ ,  $\|(A - \lambda)e\| < \varepsilon$ , and  $\|(B - \lambda)^*f\| < \varepsilon$ . Let  $e \otimes f$  denote the rank one operator defined by  $(e \otimes f)(x) = (x, f)e$ . Then

$$\tau(e \otimes f)(x) = (x, f)(A - \lambda)e - (x, (B - \lambda)^*f)e.$$

Thus for  $\|x\| = 1$ ,

$$\|\tau(e \otimes f)(x)\| \leq \|(A - \lambda)e\| + \|(B - \lambda)^*f\| < 2\varepsilon,$$

and since  $\|e \otimes f\| = 1$ , it follows that the restriction of  $\tau$  to the set of all rank one operators is not bounded below. Since the converse is obvious, the proof is complete.

LEMMA 3.17. If  $\tau_p$  is bounded below (with respect to  $\|\cdot\|_p$ ), then  $\tau$  is bounded below.

PROOF. If  $X$  is in  $C_p$ , then  $\|X\|_p = \| |X| \|_p$ , where  $|X| = (X^*X)^{1/2}$  [21]. If  $X$  is a rank one operator, then  $\tau(X)$  has rank  $\leq 2$ , so  $|\tau(X)|$  is a positive diagonalizable operator whose initial space has dimension  $\leq 2$ . It now follows that

$$\|\tau(X)\|_p = \| |\tau(X)| \|_p \leq 2^{1/p} \| |\tau(X)| \| = 2^{1/p} \|\tau(X)\|.$$

If  $M > 0$  is such that  $\|\tau(X)\|_p \geq M\|X\|_p$  for  $X$  in  $C_p$ , then for each rank one operator we have

$$\|\tau(X)\| \geq 2^{-1/p} \|\tau(X)\|_p \geq 2^{-1/p} M \|X\|_p = 2^{-1/p} M \|X\|.$$

Thus  $\tau$  is bounded below on the rank one operators, and the result follows from Lemma 3.16.

LEMMA 3.18. If  $\tau_p$  is surjective, then  $\tau$  is surjective.

PROOF. Since  $\tau_p$  is surjective,  $\tau_p^*$ , acting on  $C_q$ , is bounded below; under the correspondence of  $C_p^*$  with  $C_q$ , [26, Theorem 12.3, p. 132] implies  $\tau_p^* = -\tau_q(B, A)$ . Lemma 3.17 now implies that  $\tau(B, A)$  is bounded below and the proof is completed by an application of Theorem 3.3(i), (ii).

COROLLARY. 3.19.  $\sigma(\tau) \subset \sigma(\tau_p)$ .

PROOF. If  $\tau_p$  is bounded below and onto, the preceding two lemmas imply that  $\tau$  is invertible.

THEOREM 3.20.  $\sigma(\tau) = \sigma(\tau_p)$ .

PROOF. It remains to show that  $\sigma(\tau_p) \subset \sigma(\tau)$ . If we assume that  $\tau - \lambda$  is invertible, then clearly  $\tau_p - \lambda$  is injective, and we will use Rosenblum's resolvent formula (Theorem 3.1(ii)) to show that  $\tau_p - \lambda$  is surjective. We may assume  $\lambda = 0$ ; if  $X$  is in  $C_p$  we must show that  $\tau^{-1}(X)$  is also in  $C_p$ , where

$$\tau^{-1}(X) = \frac{1}{2\pi i} \int_{\gamma} (A - z)^{-1} X (z - B)^{-1} dz.$$

In this integral, the contour  $\gamma$  separates  $\sigma(A)$  from  $\sigma(B)$  and is disjoint from both; in particular, the resolvents  $(A - z)^{-1}$  and  $(z - B)^{-1}$  are uniformly continuous on  $\gamma$ . Moreover,  $\gamma$  is composed of a finite number of closed rectifiable Jordan curves, no two of which intersect [19]. For simplicity of notation (and with no loss of generality in the following argument) we assume there is only one curve, of finite length  $L$ , and parameterized continuously by  $\{z(t): 0 \leq t \leq 1\}$ .

For  $n > 0$ , let  $P_n$  denote a partition of  $[0, 1]$  into  $2^n$  equal subintervals, and let  $R_n$  denote the corresponding Riemann sum of  $\tau^{-1}(X)$  using left-hand endpoints, i.e.

$$R_n = \sum_{k=0}^{2^n-1} (A - \tilde{z}_k)^{-1} X (\tilde{z}_k - B)^{-1} (\tilde{z}_{k+1} - \tilde{z}_k),$$

where  $\tilde{z}_k = z(k/(2^n))$  ( $0 \leq k \leq 2^n - 1$ ). Now  $\|R_n - \tau^{-1}(X)\| \rightarrow 0$ , and since  $C_p$  is complete and  $\|K\| \leq \|K\|_p$  for each  $K$  in  $C_p$ , it suffices to verify that  $\{R_n\}$  is Cauchy in  $C_p$ .

For  $m > n$ ,  $P_m$  refines  $P_n$ ; for each  $k$ ,  $0 \leq k \leq 2^n - 1$ , let the points in  $P_m$  between  $(k/2^n)$  and  $((k+1)/2^n)$  be labelled

$$t_1 = k/2^n < t_2 < \cdots < t_{s-1} < t_s = (k+1)/2^n.$$

Let  $z_i = z(t_i)$ ,  $1 \leq i \leq s$ . Then in the sum for  $R_n$ , the term

$$R_{n,k} = (A - \tilde{z}_k)^{-1} X (\tilde{z}_k - B)^{-1} (\tilde{z}_{k+1} - \tilde{z}_k)$$

may be rewritten as

$$(A - \tilde{z}_k)^{-1} X (\tilde{z}_k - B)^{-1} ((z_s - z_{s-1}) + \cdots + (z_2 - z_1)),$$

and this term may be compared to

$$S_{m,k} = \sum_{i=1}^{s-1} (A - z_i)^{-1} X (z_i - B)^{-1} (z_{i+1} - z_i)$$

in the sum for  $R_m$ . Thus

$$\|S_{m,k} - R_{n,k}\|_p \leq \sum_{i=1}^{s-1} \|(A - z_i)^{-1}X(z_i - B)^{-1} - (A - \tilde{z}_k)^{-1}X(\tilde{z}_k - B)^{-1}\|_p |z_{i+1} - z_i|.$$

Now

$$\begin{aligned} & \|(A - z_i)^{-1}X(z_i - B)^{-1} - (A - \tilde{z}_k)^{-1}X(\tilde{z}_k - B)^{-1}\|_p \\ & \leq \|(A - z_i)^{-1}X(z_i - B)^{-1} - (A - z_i)^{-1}X(\tilde{z}_k - B)^{-1}\|_p \\ & \quad + \|(A - z_i)^{-1}X(\tilde{z}_k - B)^{-1} - (A - \tilde{z}_k)^{-1}X(\tilde{z}_k - B)^{-1}\|_p \\ & \leq \|(A - z_i)^{-1}\| \|X\|_p \|(z_i - B)^{-1} - (\tilde{z}_k - B)^{-1}\| \\ & \quad + \|(A - z_i)^{-1} - (A - \tilde{z}_k)^{-1}\| \|X\|_p \|(\tilde{z}_k - B)^{-1}\|. \end{aligned}$$

Due to the uniform continuity of the resolvents  $(A - z(t))^{-1}$  and  $(z(t) - B)^{-1}$  on  $[0, 1]$ , given  $\varepsilon > 0$ , there exists  $N > 0$ , such that for  $m > n > N$ ,  $\tilde{z}_k$  and each  $z_i$  are so close that the right-hand side of the last inequality is less than  $\varepsilon/L$ . Thus

$$\|S_{m,k} - R_{n,k}\|_p \leq \sum_{i=1}^{s-1} \frac{\varepsilon}{L} |z_{i+1} - z_i|$$

and it follows that

$$\|R_m - R_n\|_p = \left\| \sum_{k=0}^{2^n-1} (S_{m,k} - R_{n,k}) \right\|_p \leq \sum_{k=0}^{2^n-1} \frac{\varepsilon}{L} L_k,$$

where  $L_k$  is the length of the arc determined by  $\tilde{z}_k$  and  $\tilde{z}_{k+1}$ . Since the last sum is less than  $\varepsilon$ ,  $\{R_n\}$  is Cauchy in  $C_p$  and the proof is complete.

REMARK. The preceding results suggest some interesting questions that we hope to pursue elsewhere:

(i) Are the converses of Lemmas 3.17 and 3.18 true? In the case of Example 3.12 it is readily seen that  $\tau_p$  is surjective.

(ii) If the range of  $\tau$  contains  $C_p$ , is  $\tau$  surjective?

(iii) If  $\tau_{\mathfrak{F}}$  is surjective, is  $\tau$  surjective? If the range of  $\tau$  contains  $\mathfrak{F}$ , is  $\tau$  surjective? (Cf. the last part of Example 3.12.)

Recall that a Banach space operator  $T$  has dense range if and only if  $T^*$  is injective; moreover, if  $T^*$  has dense range, then  $T$  is injective [20, pp. 94–96]. These considerations give the following result.

COROLLARY 3.21. (i)  $\tau_{(TC)}(B, A)$  has dense range if and only if  $\tau(A, B)$  is injective; (ii)  $\tau_{(TC)}(B, A)$  is injective if and only if  $\tau_K(A, B)$  has dense range; (iii)  $\tau_{(TC)}(B, A)$  is injective if  $\tau(A, B)$  has dense range; (iv)  $\tau_K(A, B)$  is injective if  $\tau_{(TC)}(B, A)$  has dense range.

**4. Unique solutions to  $AX - XB = Y$ .** In this section we give several diverse sufficient conditions for  $\tau$  to be injective or to have no eigenvalues. We begin with an analogue of Example 3.12.

**PROPOSITION 4.1.** *If  $LR = 1$  and  $r(L)r(T) < 1$ , then  $\tau \equiv \tau(R, T)$  is bounded below; moreover, if  $\tau(X)$  is in a norm ideal  $\mathcal{G}$ , then so is  $X$ .*

**PROOF.** If  $X$  and  $Y$  are operators such that  $RX = XT + Y$ , then  $X = LXT + LY$ , and repeated substitution gives the identity

$$X = L^n X T^n + \sum_{i=0}^{n-1} L^{i+1} Y T^i.$$

Since  $r(L)r(T) < 1$ , it follows, as in the proof of Example 3.12, that  $X = \sum_{i=0}^{\infty} L^{i+1} Y T^i$  (uniform convergence). Thus  $\tau$  is injective, and clearly  $X$  is in  $\mathcal{G}$  if  $Y$  is. That  $\tau$  is actually bounded below follows from the inequality

$$\|X\| < \left( \sum_{i=0}^{\infty} \|L^{i+1}\| \|T^i\| \right) \|\tau(X)\|$$

(where the series again converges because  $r(L)r(T) < 1$ ).

**COROLLARY 4.2.** *If  $R$  is bounded below and  $Q$  is quasinilpotent, then  $\tau \equiv \tau(R, Q)$  is bounded below and  $\tau^{-1}(\mathcal{G}) \subset \mathcal{G}$ .*

**LEMMA 4.3.** *Let  $T$  and  $S$  be in  $\mathcal{L}(\mathcal{H})$ . Let  $X$  and  $T_1$  be operators such that  $T_1 X = XT$ ,  $X$  is injective, and  $\mathfrak{N}(T_1 - \alpha) = \{0\}$ ; let  $Y$  and  $S_1$  be operators such that  $SY = YS_1$ ,  $Y$  has dense range, and  $\mathfrak{N}(S_1 - \beta)^{\perp} = \mathcal{H}$ . Then  $\tau(T, S) - (\alpha - \beta)$  is injective.*

**PROOF.** Let  $Z$  be an operator such that  $(T - \alpha)Z = Z(S - \beta)$ ; then  $(T_1 - \alpha)XZY = XZY(S_1 - \beta)$ . For a vector  $t$  in  $\mathfrak{N}(S_1 - \beta)$ ,

$$\|(T_1 - \alpha)^n XZYt\|^{1/n} = \|XZY(S_1 - \beta)^n t\|^{1/n} \rightarrow 0;$$

since  $\mathfrak{N}(T_1 - \alpha) = \{0\}$ ,  $XZYt = 0$ , and thus  $ZYt = 0$ . Since  $\mathfrak{N}(S - \beta)$  and the range of  $Y$  are dense, it follows that  $Z = 0$ , and the proof is complete.

**LEMMA 4.4.** *If  $T^n x \rightarrow 0$  for each  $x \neq 0$  and  $\{x: S^n x \rightarrow 0\}$  is dense in  $\mathcal{H}$ , then  $\tau(T, S)$  is injective.*

**PROOF.** The proof is similar to that of Lemma 4.3.

**REMARK.** The preceding lemma gives further evidence of the essential arbitrariness of the spectra of operators comprising a disjoint pair. An example of Sz.-Nagy and Foiaş shows that there exists a contraction  $T$  satisfying  $T^n x \rightarrow 0$  and  $T^{*n} x \rightarrow 0$  for each  $x \neq 0$ , whose spectrum is the closed unit disk [23, p. 262]. Moreover, an operator  $S$  satisfies  $\mathfrak{N}(S)^{\perp} = \mathcal{H}$  if and only if  $S^*$  is a quasiaffine transform of some compact quasinilpotent



operator [2], and examples are given in [8] to show that the spectrum of such an operator  $S$  can be any compact, connected subset of the plane that contains the origin.

The proof of the following result is very similar to that of Lemma 4.3 and will be omitted. In the sequel we will denote by  $\mathfrak{U}(T)$  the closed span  $\bigvee_{\lambda \in \mathbb{C}} \mathfrak{N}(T - \lambda)$ . Since  $\mathfrak{N}(T - \lambda) = \{0\}$  for each  $\lambda \notin \sigma(T)$ , it suffices to take the span over  $\lambda \in \sigma(T)$ .

**LEMMA 4.5.** *If  $\mathfrak{U}(T) = \{0\}$  and  $\mathfrak{U}(S) = \mathcal{H}$ , then  $\tau(T, S)$  has no eigenvalues.*

**LEMMA 4.6.** *If  $T$  is a hyponormal operator in  $\mathcal{L}(\mathcal{H})$ , then  $\mathfrak{N}(T - \lambda) = \ker(T - \lambda)$  ( $\lambda \in \mathbb{C}$ ).*

**PROOF.** For each  $\lambda$ ,  $T - \lambda$  is hyponormal and [22] implies that  $\mathfrak{N}(T - \lambda)$  is a closed invariant subspace of  $T - \lambda$ . Since  $\|(T - \lambda)^n x\|^{1/n} \rightarrow 0$  for each  $x$  in  $\mathfrak{N}(T - \lambda)$ , [4, p. 28] implies that  $(T - \lambda)|_{\mathfrak{N}(T - \lambda)}$  is quasinilpotent, and so  $\|(T - \lambda)|_{\mathfrak{N}(T - \lambda)}\| = r((T - \lambda)|_{\mathfrak{N}(T - \lambda)}) = 0$ . Thus  $\mathfrak{N}(T - \lambda) \subset \ker(T - \lambda)$  and since the reverse inclusion is clear the proof is complete.

**LEMMA 4.7.** *If  $T$  is a hyponormal operator in  $\mathcal{L}(\mathcal{H})$ , then  $\mathfrak{U}(T) = \mathcal{H}$  if and only if  $T$  is a diagonalizable normal operator.*

**PROOF.** The result follows from the preceding lemma and the fact that if  $T$  is hyponormal, then  $T|(\bigvee_{\lambda \in \sigma(T)} \ker(T - \lambda))$  is normal [9, p. 12].

**LEMMA 4.8.** *Let  $S = J^{-1}NJ + Q$  denote the canonical decomposition of the spectral operator  $S$ . Then  $\mathfrak{N}(S - \lambda) = \ker(J^{-1}NJ - \lambda)$ .*

**PROOF.** Since  $J^{-1}NJ$  commutes with  $Q$ ,  $S$  is quasinilpotent equivalent to  $J^{-1}NJ$ , and thus  $\mathfrak{X}_S(\mathcal{F}) = \mathfrak{X}_{J^{-1}NJ}(\mathcal{F})$  for each closed set  $\mathcal{F}$  [4, Corollary 3.5, p. 52]. Thus

$$\mathfrak{N}(S - \lambda) = \mathfrak{N}(J^{-1}NJ - \lambda) = J^{-1}\mathfrak{N}(N - \lambda),$$

and Lemma 4.6 implies that

$$J^{-1}\mathfrak{N}(N - \lambda) = J^{-1}\ker(N - \lambda) = \ker(J^{-1}NJ - \lambda).$$

**LEMMA 4.9.** *A spectral operator  $S$  satisfies  $\mathfrak{U}(S) = \mathcal{H}$  if and only if its scalar part is similar to a diagonalizable normal operator.*

**PROOF.** Apply Lemmas 4.7 and 4.8.

**LEMMA 4.10.** *If  $S$  has finite spectrum, then  $\mathfrak{U}(S) = \mathcal{H}$ .*

**PROOF.** Let  $\{\lambda_1, \dots, \lambda_n\}$  denote the distinct elements of  $\sigma(S)$ . The result is clear if  $n = 1$  since in this case  $S - \lambda_1$  is quasinilpotent. We give a proof by induction on  $n$ ; for  $n > 1$  let  $E$  denote the spectral idempotent for  $S$

associated with  $\{\lambda_1\}$ . Let  $J$  be an invertible operator such that  $F = J^{-1}EJ$  is an orthogonal projection; since  $S$  commutes with  $E$ ,  $T = J^{-1}SJ$  commutes with  $F$ . Thus, with respect to the decomposition  $\mathcal{H} = (F\mathcal{H}) \oplus (1 - F)\mathcal{H}$ ,  $T$  is of the form  $T = A \oplus B$ ; in particular,  $A$  is similar to  $S|_{E\mathcal{H}}$  and  $B$  is similar to  $S|(1 - E)\mathcal{H}$ . From the Riesz decomposition theorem,  $\sigma(A) = \{\lambda_1\}$  and  $\sigma(B) = \{\lambda_2, \dots, \lambda_n\}$ . By induction  $\mathfrak{N}(B) = (1 - F)\mathcal{H}$  and, clearly,  $\mathfrak{N}(A - \lambda_1) = F\mathcal{H}$ . Thus

$$\mathcal{H} = J\mathcal{H} = J\mathfrak{N}(T) \subset \bigvee J\mathfrak{N}(J^{-1}SJ - \lambda) = \mathfrak{N}(S),$$

and the proof is complete.

REMARK. We are unable to characterize the class of operators  $S$  for which  $\mathfrak{N}(S) = \mathcal{H}$ , but we wish to make several observations in this connection. Using [1, Proposition 3.3] it follows immediately that if  $\mathfrak{N}(S) = \mathcal{H}$ , then  $S$  is quasitriangular. On the other hand, and in contrast to the preceding two lemmas, there exists a compact operator  $T$  with countably infinite spectrum such that  $\mathfrak{N}(T) \neq \mathcal{H}$ . Let  $\{e_n\}_{n=0}^\infty$  denote an orthonormal basis for  $\mathcal{H}$  and let  $R$  be defined by

$$Re_0 = 0, \quad Re_n = (1/n)e_0 + (1/n^2)e_n \quad (n \geq 1).$$

Let  $T = R^*$ ;  $T$  is compact,  $\sigma(T) = \{0\} \cup \{1/n^2\}_{n=1}^\infty$ , and a straightforward calculation shows that  $\mathfrak{N}(T) = \{0\}$  and  $\mathfrak{N}(T - 1/n^2) = \bigvee \{e_n\}$ . Thus  $\mathfrak{N}(T) = \bigvee_{n=1}^\infty \{e_n\} \neq \mathcal{H}$ . We note also that since  $\ker(R) = \bigvee \{e_0\}$  and  $\ker(R - 1/n^2) = \bigvee \{e_0 + (1/n)e_n\}$  ( $n \geq 1$ ), then  $\mathfrak{N}(R) = \mathcal{H}$ .

If there exists a finite set  $\{\lambda_1, \dots, \lambda_n\}$  such that  $\mathcal{H} = \bigvee_{i=1}^n \mathfrak{N}(T - \lambda_i)$ , there is little that can be said in general about the spectrum of  $T$ ; indeed, as remarked previously, there exist operators  $T$ , whose spectra are connected and contain 0 but are otherwise arbitrary, such that  $\mathfrak{N}(T)^- = \mathcal{H}$ . We next show that there is a large class of operators for which the converse of Lemma 4.10 is valid. For a closed subset of the plane  $\mathcal{F}$  and an operator  $T$  in  $\mathcal{L}(\mathcal{H})$  satisfying the single valued extension property, let  $\mathcal{K}_T(\mathcal{F}) = \{x \in \mathcal{H} : \text{there exists an analytic function } f: \mathbb{C} \setminus \mathcal{F} \rightarrow \mathcal{H} \text{ such that } (T - \alpha)f(\alpha) \equiv x\}$  [4, p. 1]. An operator  $T$  is said to satisfy condition C if  $\mathcal{K}_T(\mathcal{F})$  is closed for every closed set  $\mathcal{F}$  [22]. All hyponormal operators [22] and all decomposable operators (including normal operators, spectral operators, and operators with countable spectrum) [4] satisfy condition C. It is known that  $\mathcal{K}_T(\{0\}) = \mathfrak{N}(T)$  and it follows readily that  $\mathcal{K}_T(\{\lambda\}) = \mathfrak{N}(T - \lambda)$ .

PROPOSITION 4.11. *Let  $T$  be an operator satisfying property C; let  $\{\lambda_1, \dots, \lambda_n\}$  be a finite sequence of distinct scalars such that  $\mathcal{H} = \bigvee_{i=1}^n \mathfrak{N}(T - \lambda_i)$  and such that  $\mathfrak{N}_i \equiv \mathfrak{N}(T - \lambda_i) \neq \{0\}$  for each  $i$ . Then  $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$  and  $\mathfrak{N}(T - \lambda_i)$  is equal to the Riesz subspace for  $T$  corresponding to  $\{\lambda_i\}$ .*

PROOF. Since  $T$  satisfies property C, each  $\mathfrak{M}_i = \mathfrak{X}_T(\{\lambda_i\})$  is a closed  $T$ -invariant subspace. Since  $\|(T - \lambda_i)^n x\|^{1/n} \rightarrow 0$  for each  $x$  in  $\mathfrak{M}_i$ , [4, p. 23] implies that  $\sigma(T|_{\mathfrak{M}_i}) = \{\lambda_i\}$ . Since there are only finitely many  $\lambda_i$ 's and  $\mathfrak{H} = \bigvee_{i=1}^n \mathfrak{M}_i$ , a theorem of D. A. Herrero [11, Theorem 2] implies that  $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$ . Thus each  $\lambda_i$  is an isolated point of  $\sigma(T)$ , and [18, p. 424] implies that the Riesz subspace for  $T$  corresponding to the isolated point  $\lambda_i$  is  $\mathfrak{M}(T - \lambda_i)$ .

We now return to the question of the injectivity of  $\tau$  and we give our extension of the example of Douglas and Pearcy mentioned in the Introduction. Lemmas 4.5–4.10 yield the following result.

**THEOREM 4.12.** *Let  $T$  be a hyponormal operator with no eigenvalues or a spectral operator whose scalar part has no eigenvalues. Then  $\tau(T, S)$  has no eigenvalues if  $S$  satisfies any of the following conditions: (i)  $\sigma(S)$  is finite; (ii)  $S$  is a diagonalizable normal operator; (iii)  $S$  is a spectral operator whose scalar part is similar to a diagonalizable normal operator; (iv)  $\mathfrak{U}(S) = \mathfrak{H}$ .*

**THEOREM 4.13.** *Let  $T$  be a hyponormal operator in  $\mathcal{L}(\mathfrak{H})$ . Let  $S$  be an operator and  $\lambda$  a scalar such that  $\mathfrak{M}(S - \lambda) \neq \{0\}$ . Let  $A$  be in  $\mathcal{L}(\mathfrak{H})$  and let  $R$  be the operator on  $\mathfrak{H} \oplus \mathfrak{H}$  whose operator matrix is  $\begin{pmatrix} S & A \\ 0 & T \end{pmatrix}$ . Then  $R$  has a proper hyperinvariant subspace or  $R$  is a scalar multiple of the identity.*

PROOF. Since for each scalar  $\alpha$ ,  $\dim \ker(R^* - \alpha) \geq \dim \ker(T^* - \alpha)$ , we may assume that  $T^*$  has no eigenvalues; the hyponormality of  $T$  now implies that  $T$  has no eigenvalues. Now  $\mathfrak{M}(R - \lambda)^-$  is a hyperinvariant subspace for  $R$ , and we will show that this subspace is proper. Since  $\mathfrak{M}(S - \lambda) \neq \{0\}$  and  $\mathfrak{M}(S - \lambda) \oplus \{0\} \subset \mathfrak{M}(R - \lambda)$ , we have  $\mathfrak{M}(R - \lambda) \neq \{0\}$ . If a vector  $t \oplus s$  is in  $\mathfrak{M}(R - \lambda)$ , then a matrix calculation shows that  $s$  is in  $\mathfrak{M}(T - \lambda)$ . Since, from Lemma 4.6,  $\mathfrak{M}(T - \lambda) = \ker(T - \lambda) = \{0\}$ , we have  $s = 0$ , and thus

$$\{0\} \subsetneq \mathfrak{M}(R - \lambda)^- = \mathfrak{M}(S - \lambda)^- \oplus \{0\} \neq \mathfrak{H} \oplus \mathfrak{H},$$

which complete the proof.

**REMARK.** Lemmas 4.7, 4.9, and 4.10 furnish examples of operators  $S$  for which  $\mathfrak{U}(S) = \mathfrak{H}$ . It follows from [18, p. 424] that if  $\sigma(S)$  contains an isolated point  $\lambda$ , then  $\mathfrak{M}(S - \lambda) \neq \{0\}$ . Thus Theorem 4.13 applies to any operator  $S$  whose spectrum contains an isolated point, and, in particular, to any operator  $S$  whose spectrum is countable.

We conclude this section by giving a sufficient condition for the injectivity of  $\tau(T, S)$  in the case when  $T$  or  $S$  is a spectral operator. The following result was stated (without proof) in [8, Proposition 2.3].

**THEOREM 4.14.** *Let  $T, X$ , and  $S$  be in  $\mathcal{L}(\mathfrak{H})$  with  $S$  spectral and  $TX = XS$  or  $XT = SX$ . Let  $E(\cdot)$  denote the spectral measure of  $S$ . If  $E(\sigma(T)) = 0$ , then*

$X = 0$ , i.e.  $\tau(T, S)$  and  $\tau(S, T)$  are injective.

PROOF. Suppose that  $TX = XS$ . Let  $\{\mathcal{G}_n\}_{n=0}^\infty$  denote a collection of mutually disjoint Borel subsets of  $\sigma(S)$  such that  $\bigcup_{n=0}^\infty \mathcal{G}_n = \sigma(S) \setminus \sigma(T)$  and  $\mathcal{G}_n^- \cap \sigma(T) = \emptyset$  for each  $n$ ; e.g., let

$$\mathcal{G}_0 = \{x \in \sigma(S) : \text{dist}(x, \sigma(T)) > 1\},$$

and

$$\mathcal{G}_n = \{x \in \sigma(S) : 1/(n+1) < \text{dist}(x, \sigma(T)) \leq 1/n\} \quad \text{for } n \geq 1.$$

If  $E_n = E(\mathcal{G}_n)$ , then  $TXE_n = XSE_n = XE_nS$  [12, §2], and thus  $T(XE_n|_{E_n\mathcal{H}}) = (XE_n|_{E_n\mathcal{H}})(S|_{E_n\mathcal{H}})$ . Since  $\sigma(S|_{E_n\mathcal{H}}) \subset \mathcal{G}_n^-$  and  $\mathcal{G}_n^- \cap \sigma(T) = \emptyset$ , an application of a version of Rosenblum's Theorem [17, Theorem, §1] implies that  $XE_n|_{E_n\mathcal{H}} = 0$ , and thus  $XE_n = 0$ . Since  $\sigma(S) = \bigcup_{n=0}^\infty \mathcal{G}_n \cup (\sigma(S) \cap \sigma(T))$ , and  $E(\sigma(T)) = 0$ , the countable additivity of  $E(\cdot)$  in the strong operator topology [4, p. xvi] implies that  $y = \sum_{n=0}^\infty XE_n y$  for each  $y$  in  $\mathcal{H}$ . Now  $Xy = \sum_{n=0}^\infty XE_n y = 0$ , and so  $X = 0$ .

If  $XT = SX$ , then  $T^*X^* = X^*S^*$ ; if  $F(\cdot)$  denotes the spectral measure of  $S^*$ , then  $F(\sigma(T^*)) = E(\sigma(T))^* = 0$ , so the result follows from the previous case.

**COROLLARY 4.15.** *If  $\sigma(T)$  is countable and  $N$  is a normal operator with no eigenvalues in  $\sigma(T)$ , then  $\tau(N, T)$  and  $\tau(T, N)$  are injective.*

PROOF. If  $\sigma(T) = \{\lambda_i\}_{i=1}^M$ , then  $E_N(\{\lambda_i\}) = 0$  for each  $i$  since  $N$  has no eigenvalues in  $\sigma(T)$ ; thus  $E_N(\sigma(T)) = 0$ , and the result follows from Theorem 4.14.

**REMARK.** Since  $\sigma_\pi(\tau(N, T)) = \sigma_\pi(N) - \sigma_\delta(T)$  and since  $\sigma(N) = \sigma_\pi(N)$  for each normal operator  $N$ , we may use Theorem 4.14 to give examples of the case when  $\tau(N, T)$  is injective but not bounded below. For example, let  $T$  be an operator whose spectrum is the closed unit disk and let  $N$  be a diagonalizable normal operator  $\text{diag}(\{\lambda_n\})$  such that  $|\lambda_n| > 1$  for each  $n$  and such that  $\sigma(N)$  contains the unit circle (or any nonempty subset of the circle). Thus

$$\sigma(N) \cap \{\lambda : |\lambda| = 1\} \subset \sigma(N) \cap \text{bdry}(\sigma(T)) \subset \sigma_\pi(N) \cap \sigma_\delta(T)$$

and so  $\tau(N, T)$  is not bounded below. Moreover, since  $\mathcal{H}$  is spanned by eigenvectors for  $N$  corresponding to eigenvalues of modulus greater than one, the spectral subspace for  $N$  corresponding to the unit circle is the zero subspace, so Theorem 4.14 implies that  $\tau(N, T)$  is injective. An essentially different example of the case when  $\tau(T, S)$  is injective but not bounded below can be obtained using Lemma 4.4. For example  $\tau(T, S)$  will have this property if  $T$  and  $S$  satisfy the hypotheses of Lemma 4.4 and  $\sigma(T) = \sigma(S) = \{\lambda : |\lambda| \leq 1\}$  (cf. the remark after Lemma 4.4).

**5. Added in proof.** Since this paper was completed we have learned of several recent, related papers and results, listed below.

(i) A. Brown and C. Pearcy [27] have independently proved analogues of Lemmas 3.17–3.18 and Theorem 3.20 for *arbitrary* norm ideals; thus  $\sigma_\pi(\tau) \subset \sigma_\pi(\tau_{\mathcal{J}})$ ,  $\sigma_\delta(\tau) \subset \sigma_\delta(\tau_{\mathcal{J}})$ , and  $\sigma(\tau) = \sigma(\tau_{\mathcal{J}})$  for each norm ideal  $\mathcal{J}$ .

(ii) In a forthcoming paper [28] we answer affirmatively questions (i)–(iii) (following Theorem 3.20) as follows:  $\sigma_\pi(\tau) = \sigma_\pi(\tau_{\mathcal{J}})$  and  $\sigma_\delta(\tau) = \sigma_\delta(\tau_{\mathcal{J}})$  for each norm ideal  $\mathcal{J}$ . Additionally,  $\tau$  is surjective if and only if the range of  $\tau$  contains each rank one operator.

(iii) J. G. Stampfli has independently proved a result concerning the injectivity of  $\tau$  when  $A$  or  $B$  is decomposable [29, Appendix]. Our Theorem 4.14 for spectral operators follows from Stampfli's result. [29] also contains a separate proof for the case when  $A$  or  $B$  is normal [29, Lemma 5] using a method different from ours.

(iv) The operators  $R$  and  $T$  in the example following Lemma 4.10 were shown to the author by D. A. Herrero (private communication) in connection with a different topic.

(v) Let  $A$ ,  $B$ , and  $C$  be in  $\mathcal{L}(\mathcal{H})$  and let  $T$  be the operator on  $\mathcal{H} \oplus \mathcal{H}$  with operator matrix  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ . Two consequences of Rosenblum's Theorem are that if  $\sigma(A) \cap \sigma(B) = \emptyset$ , then  $T$  is similar to  $A \oplus B$  and the commutant of  $A \oplus B$  splits [30, Corollaries 0.14–0.15, pp. 8–9]. P. Rosenthal (private communication) has pointed out that Theorem 3.8 yields the following analogues of these results: If  $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ , then  $T$  is similar to a compact perturbation of  $A \oplus B$ ; if, additionally,  $\sigma_{le}(A) \cap \sigma_{re}(B) = \emptyset$ , then each operator commuting with  $A \oplus B$  is a compact perturbation of an operator of the form  $X \oplus Y$ . The interested reader may formulate analogues of these results for  $n \times n$  operator matrices. P. Rosenthal also independently showed that the one-sided version of Question 3.10 has a negative answer (cf. Example 3.11).

(vi) Various types of splitting theorems for  $A \oplus B$  may be found in the recent papers [31] and [32].

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