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4-MANIFOLDS, 3-FOLD COVERING SPACES AND RIBBONS

BY

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ABSTRACT. It is proved that a PL, orientable 4-manifold with a handle presentation composed by 0-, 1-, and 2-handles is an irregular 3-fold covering space of the 4-ball, branched over a 2-manifold of ribbon type. A representation of closed, orientable 4-manifolds, in terms of these 2-manifolds, is given. The structure of 2-fold cyclic, and 3-fold irregular covering spaces branched over ribbon discs is studied and new exotic involutions on S^4 are obtained. Closed, orientable 4-manifolds with the 2-handles attached along a strongly invertible link are shown to be 2-fold cyclic branched covering spaces of S^4. The conjecture that each closed, orientable 4-manifold is a 4-fold irregular covering space of S^4 branched over a 2-manifold is reduced to studying $\gamma \# S^1 \times S^2$ as a nonstandard 4-fold irregular branched covering of S^3.

1. Introduction. We first remark that the foundational paper [8] might be useful as an excellent account of definitions, results and historical notes.

Let F be a closed 2-manifold (not necessarily connected nor orientable) locally flat embedded in S^4. To each transitive representation ω: $\pi_1(S^4 - F) \to \mathbb{S}_n$ into the symmetric group of n letters there is associated a closed, orientable, PL 4-manifold $W^4(F, \omega)$ which is a n-fold covering space of S^4 branched over F. This paper deals with the problem of representing each closed, orientable PL 4-manifold W^4 as a n-fold covering space of S^4 branched over a closed 2-manifold.

I. Berstein and A. L. Edmonds proved [9] that in some cases (for instance $S^1 \times S^1 \times S^1 \times S^1$) n has to be at least 4. They also pointed out to the author that, for $\mathbf{C}P^2$, F must be nonorientable (using the Euler characteristic number). More generally, S. Cappell and J. Shaneson pointed out that, if F is orientable, the signature of W^4 must be zero.

We conjecture that each such W^4 is *an irregular simple 4-fold covering space of S^4 branched over a closed surface F* (*simple* means that the representation ω, where $W^4 \cong W^4(F, \omega)$, sends meridians into transpositions).

The manifold W^4 admits a handle representation $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$

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$\cup \gamma H^3 \cup H^4$. Thus, by duality, W^4 is obtained by pasting together two manifolds $V^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, $U^4 = H^0 \cup \gamma H^1$ along their common boundary, which is $\gamma \# S^1 \times S^2$.

Our idea is to represent V^4 and U^4 as coverings of D^4 branched over a 2-manifold with boundary in S^3, and then match the two coverings.

In this paper we prove that a manifold with presentation $H^0 \cup \lambda H^1 \cup \mu H^2$ is, in fact, a *dihedral* 3-*fold* covering of D^4 branched over a 2-manifold of a special type (which we call a ribbon manifold because it is a natural generalization of ribbon discs).

In the case that μH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2 = \partial(H^0 \cup \lambda H^1)$, then we show that the closed 4-manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is actually a 2-*fold cyclic* branched covering of S^4. For the case of 4-fold irregular branched coverings of S^4, we show our conjecture reduces to studying $\gamma \# S^1 \times S^2$ as a nonstandard 4-fold irregular branched covering of S^3.

It is shown in [7] that each manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is uniquely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$. From this point of view, our presentation of $H^0 \cup \lambda H^1 \cup \mu H^2$ as an irregular 3-fold covering space also provides a representation for closed, orientable 4-manifolds.

We study also the structure of the 2- and dihedral 3-fold covering spaces of ribbon discs. We obtain in this way some contractible 4-manifolds of Mazur, and this allows us to find many 2-knots in S^4 with the same 2-fold cyclic covering space, even S^4 itself, thus obtaining new examples of exotic involutions on S^4.

Lastly, we note some possible applications of these results to the study of 3-manifolds and classical knots.

I am indebted to Robert Edwards, Charles Giffen and Cameron Gordon for helpful conversations.

2. A simple case. We begin with the simple case of a manifold which is presented by one 0-handle, one 1-handle, and one 2-handle of a special type, both in order to obtain special results for this case and to illustrate the method.

Consider $S^1 \times B^3$ with presentation $H^0 \cup H^1$, i.e. one 0-handle plus one 1-handle. Its boundary, $S^1 \times S^2$, is illustrated in Figure 1.

We consider a knot K, contained in $S^1 \times S^2$, such as the one shown in Figure 1, which is *strongly invertible*, i.e. reflection u in the E-axis induces on K an involution with two fixed points. Now we add a 2-handle to $S^1 \times B^3$ so that K is the attaching sphere. More precisely, we have an embedding h: $\dot{B}^2 \times B^2 \to \partial(S^1 \times B^3)$ so that $h(\dot{B}^2 \times 0)$ equals K, and let $W^4 = S^1 \times B^3 \cup_h B^2 \times B^2$.

Let U: $S^1 \times B^3 \to S^1 \times B^3$ be the involution, with two discs D_1, D_2 as

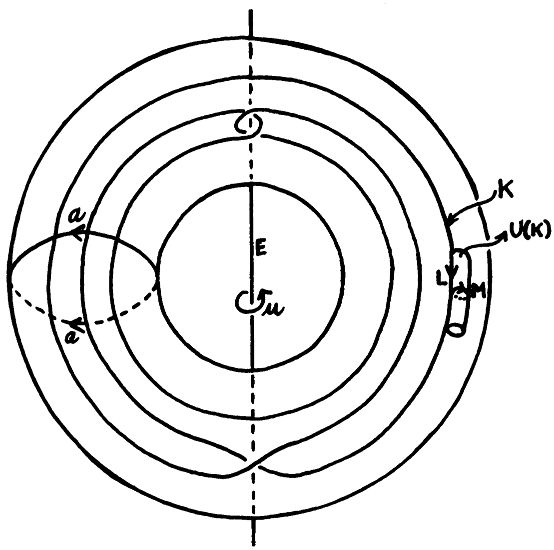


FIGURE 1

fixed-point set, which canonically extends the reflection u above.

Let V: $B^2 \times B^2 \to B^2 \times B^2$ be the reflection in $D = B^1 \times B^1$. We can represent $S^3 = \partial(B^2 \times B^2)$ by stereographic projection onto $R^3 + \infty$ in such a way that V induces on S^3 the reflection v in the y-axis. In this representation $\dot{B}^2 \times B^2$ is a regular neighborhood, X, of the unit circle C in the (x, y)-plane; the belt-sphere is the z-axis, and the belt-tube is $Y = S^3 - \operatorname{int} X$. Finally, let (m, l) be a meridian-longitude system on ∂X (see Figure 2).

Up to isotopy, we may suppose that $uh = hv$, so that (U, V) is an involution on $W^4 = S^1 \times B^3 \cup_h B^2 \times B^2$. The fixed-point set is the disc $(D_1 \cup D_2) \cup_h D$, where h pastes ∂D to $\partial(D_1 \cup D_2)$ along $\alpha \cup \beta$ of Figure 2.

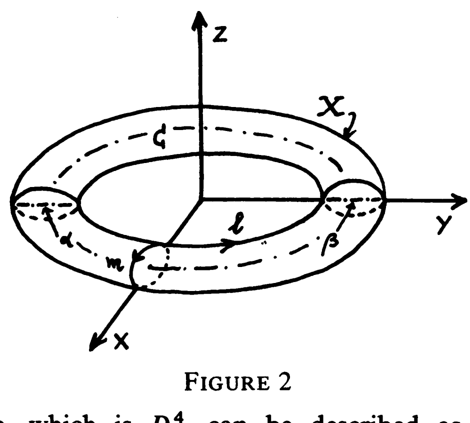


FIGURE 2

The orbit-space, which is D^4, can be described as follows. First, p: $S^1 \times B^3 \to S^1 \times B^3/U$ is the 2-fold branched covering space of D^4

branched over two disjoint discs D_1', D_2' (see Figure 3(a)) such that $p|S^1 \times S^2$: $S^1 \times S^2 \to S^3$ is the standard covering over $\partial(D_1' \cup D_2')$. In this covering $p(K)$ is an arc with its endpoints in $\partial D_1'$ and $\partial D_2'$. Second, q: $B^2 \times B^2 \to B^2 \times B^2/V$ is the 2-fold branched covering space of D^4 branched over a disc D' such that $p|\partial(B^2 \times B^2)$: $\partial(B^2 \times B^2) \to S^3$ is the standard covering over the trivial knot $\partial D'$. In this covering, $q(C)$ is the arc shown in Figure 3(c). We must paste along regular neighborhoods $q(X)$ and $p(U(K))$ of $q(C)$ and $p(K)$, respectively, by the map phq^{-1}, obtaining $D^4 = S^1 \times B^3/U \cup_{phq^{-1}} B^2 \times B^2/V$.

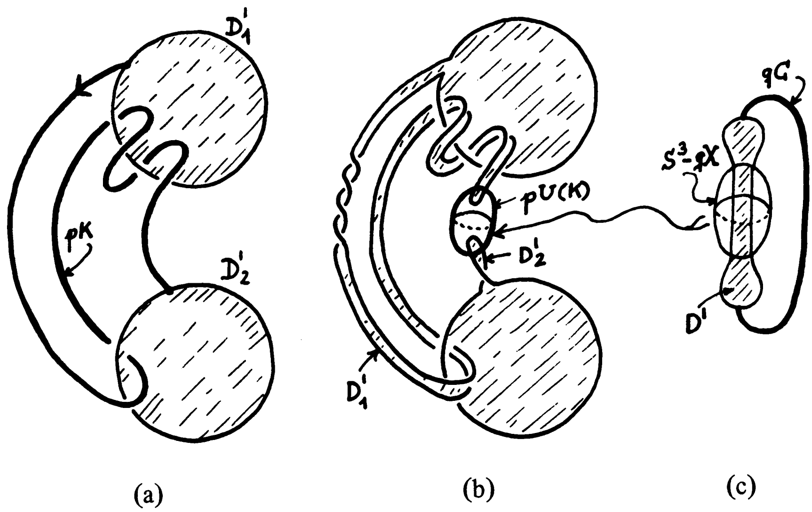


FIGURE 3

The branching set is $(D_1' \cup D_2') \cup_{phq^{-1}} D'$, which can be visualized as follows. Deform D_1' and D_2' by isotopy as illustrated in Figure 3(b), thus obtaining the "ribbon" $D_1' \cup D_2'$ (and, in fact, if we pull $D_1' \cup D_2'$ back into S^3 in the way suggested by the shaded part of Figure 3(b), then we obtain a ribbon immersion of $D_1' \cup D_2'$). Pasting $B^2 \times B^2/V$ to $D^4 = S^1 \times B^3/U$ along the balls $q(X)$ and $p(U(K))$ and then "absorbing" the bulge $B^2 \times B^2/V$ on D^4 back into D^4, we obtain the branch set in the aspect of Figure 3(b) $\cup$ 3(c) joined by the arrow. Of course, the number of twists in the boundary of the ribbon depends on the number of times that $h(l)$ goes around $\partial(U(K))$. The ribbon of Figure 3(b) corresponds to the choice $h(l) \sim L + 5M$ (on $\partial U(K)$). Note that the number of components of the branch-set is one if and only if K connects the two components of $D_1 \cup D_2$, in which case the branch set is a ribbon disc.

We collect together these results in the following theorem.

THEOREM 1. *The manifold* $W^4 = H^0 \cup H^1 \cup H^2$, *where* H^2 *is attached* *along a strongly invertible knot of $S^1 \times S^2$, is a 2-fold covering space of D^4, branched over a ribbon disc or over the union of a disc and either an annulus or a Möbius band.*

3. Ribbons, Mazur manifolds and exotic involutions. We have immediately an amusing result. Note that the manifold W^4 corresponding to the knot K, in Figure 1, is the one discovered by B. Mazur [6], which has the property that $W^4 \times I \approx B^5$. So, the double $2W^4 \cong S^4$ and W^4 is contractible. Then, the *ribbon 2-knot* R corresponding to the ribbon of Figure 3(b) (i.e., the 2-knot obtained by pushing a copy of the ribbon disc into each of the two sides of S^3 in S^4) has $2W^4 = S^4$ as 2-fold covering space. This is another example of exotic involution in S^4, first discovered by C. Gordon [1]. (It is easily checked that $\pi_1(S^4 - R) \neq Z$, showing that R is not the trivial knot in S^4.)

In order to state these results with more generality, let us quote now the description of ribbon knots given by T. Yajima [10]. Let $C_0, C_1, \ldots, C_\lambda$ be unlinked trivial circles in R^3. Take disjoint small arcs $\alpha_1, \ldots, \alpha_\lambda$ on C_0, and a small arc γ_i on C_i $(i = 1, \ldots, \lambda)$. For every i, connect α_i with γ_i by a nontwisting narrow band B_i which may run through C_j $(j = 0, \ldots, \lambda)$ or may get tangled with itself or with other bands. Then, each ribbon knot is of this type for some λ. We shall say that a presentation of this form has *type* $(C_0, C_1, \ldots, C_\lambda)$.

Consider a ribbon R of type (C_0, C_1) and let Δ_0, Δ_1 be disjoint discs with boundary C_0, C_1 respectively. We may assume that the center line path β of the band B_1 from C_0 to C_1 cuts Int $\Delta_0 \cup \Delta_1$ transversally, thus partitioning β into a composition of (nontrivial) paths which we write as $\beta = \beta_1 * \cdots * \beta_k$ (some k). Let $\# R = \Sigma(-1)^j$ where j runs over those indices from 1 to k such that the subpath β_j connects C_0 and C_1. Refer to Figure 3(a), where $\# R = 1$. Note that $\# R$ is always odd. We see that the 2-fold covering space branched over the disc R is a manifold $W^4 = H^0 \cup H^1 \cup H^2$, where H^2 is added along a strongly-invertible knot, which is homologous to $\# R$ times a generator of $H_1(S^1 \times B^3)$.

We now use the trick of Mazur [6] to describe $2W^4$. The manifold $W^4 \times I$ is obtained by adding a 2-handle to $S^1 \times B^4$ along a curve w in $S^1 \times S^3$ which is homologous to $\# R$ times a generator of $H_1(S^1 \times S^3)$. This defines the handle addition uniquely up to PL-homeomorphism since 1-knot theory of $S^1 \times S^3$ is essentially trivial.[1] Thus, $2W^4 = \partial(W^4 \times I)$ is obtained by a spherical modification of $S^1 \times S^3$ along a curve which runs $\# R$ times the generator of $H_1(S^1 \times S^3)$. Hence, we have

[1] A "Dehn-twist" along a belt-sphere of $S^1 \times B^1$ changes the framing of w by a map $w \to SO(3)$ which represents the nontrivial element of $\pi_1(SO(3))$ if and only if $\# R$ is odd. Thus, we cannot worry about framings here.

THEOREM 2. *All the ribbon 2-knots, corresponding to ribbon knots of type* (C_0, C_1) *with the same number* $|\# R|$, *have the same 2-fold covering space.*

In particular, if $\# R = \pm 1$, then $2W^4 = S^4$, and so we have

COROLLARY 1. *All the ribbon 2-knots of type* (C_0, C_1), *with* $\# R = \pm 1$, *have* S^4 *as 2-fold covering space.*

We see that, in contrast with the 3-dimensional analogue, the family of 2-knots in S^4 with the same 2-fold covering space is very large indeed.

More generally, if $\# R = 2m + 1$ we have $\pi_1(2W^4) \cong Z_{2m+1}$ and the universal covering space of $2W^4$ is $2m \# S^2 \times S^2$. This composite $2m \# S^2 \times S \to 2W^4 \to S^4$ is a regular dihedral branched cover over the ribbon 2-knot.

4. 2-fold coverings. We generalize these results to 4-manifolds with several 1-handles, being somewhat less explicit than before in the description of geometrical constructions.

The manifold $\lambda \# S^1 \times B^3$ has the presentation $H^0 \cup \lambda H^1$ (if $\lambda = 0$, $\lambda \# S^1 \times B^3 = B^4$). Its boundary $\lambda \# S^1 \times S^2$ is represented by the handlebody of Figure 4, with points on the boundary identified by reflection in the (x, y)-plane.

The reflection u in the x-axis is the restriction of an involution U in $\lambda \# S^1 \times B^3$ which has $\lambda + 1$ disjoint discs as fixed-point set. The orbit space of U is D^4, and we show the branch set, $C = C_0 \cup \cdots \cup C_\lambda$, in ∂D^4 in Figure 4.

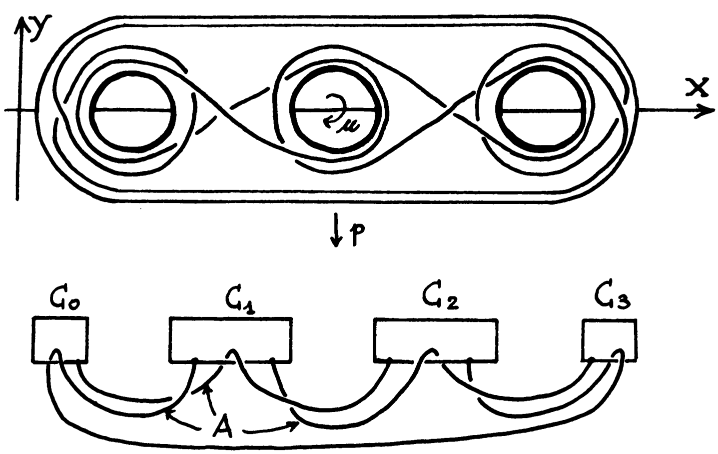


FIGURE 4

Let p be the projection, and consider now a system $A = A_1 \cup \cdots \cup A_\mu$

of disjoint, simple arcs in S^3, meeting C only in their endpoints. It is clear that $p^{-1}A$ is a *strongly invertible link* in $\lambda \# S^1 \times S^2$ (which means each $p^{-1}A_i$ is strongly-invertible with respect to u). We have the following theorem.

THEOREM 3. *The manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, where μH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$, is a 2-fold cyclic covering space of D^4 branched over a 2-manifold.*

REMARK. The branching set is easily obtained in a way similar to that in §2.

Utilizing the ribbon presentation explained in §3 we have immediately the following theorem:

THEOREM 4. *If R is a ribbon knot of type $(C_0, C_1, \dots, C_\lambda)$, the 2-fold covering space branched over the corresponding ribbon disc in D^4 is the manifold $W^4 = H^0 \cup \lambda H^1 \cup \lambda H^2$, where λH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$.*

REMARK. The manifold W^4 in the statement of the theorem has a spine composed by a bouquet of λ 1-cells $\{a_1, \dots, a_\lambda\} = A$ and λ 2-cells $\{w_1, \dots, w_\lambda\}$ so that the boundary of w_i, attached to A, is the word $T_i a_i T_i^t$, where T_i is a word in the alphabet $A \cup A^{-1}$ and T_i^t is the same word read backwards. This follows from the Yajima representation of a ribbon. Of course, from this property of the spine we see immediately that $H_*(W^4; Z/2) = 0$.

REMARK. If we want to know the structure of the 2-fold covering space of a 2-ribbon knot we have to look to the manifold $2W^4 = \partial(W^4 \times I)$. The triviality of $\pi_1 W^4$ implies that $W^4 = H^0 \cup \lambda H^1 \cup \lambda H^2$ is contractible, but in order to assure that the homotopy 4-sphere $2W^4$ is S^4 it is necessary and sufficient that $W^4 \times I$ be B^5 or, alternatively, that the Heegaard diagram provided by Lemma 1 in [7] goes to $(S^3; \varnothing)$ by Heegaard moves.

Another consequence of Theorem 3 is the following result.

THEOREM 5. *The closed 4-manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$, where μH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$, is a 2-fold cyclic covering space of S^4 branched over a closed 2-manifold.*

PROOF. Theorem 3 says that there is a 2-fold cyclic covering $p\colon H^0 \cup \lambda H^1 \cup \mu H^2 \to D^4$ branched over a 2-manifold F with boundary $\partial F \subset S^3 = \partial D^4$. But the cover $p|\partial(H^0 \cup \lambda H^1 \cup \mu H^2) = \gamma \# S^1 \times S^2 \to S^3$, branched over ∂F, must be standard (see [4]). Thus, ∂F is a system of $\gamma + 1$ unknotted and unlinked curves in S^3. Put D^4 in S^4 and fill up the curves ∂F with discs in $S^4 - D^4$. We get a closed 2-manifold $F' \subset S^4$ and the corresponding 2-fold cyclic branched covering space is $H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma \# S^1 \times B^3$. But this manifold is W^4 because of the results in [7]. $\square$

EXAMPLES. (a) Take $\mathbf{C}P^2 = H^0 \cup H^2 \cup H^4$. Here H^2 is attached along a trivial knot in $\partial H^0 = S^3$ with framing $+1$. Then $H^0 \cup H^2$ is a 2-fold cyclic covering of D^4 branched over a Möbius band F with its boundary in S^3. The surface F' of the theorem is a projective plane.

(b) Take $S^2 \times S^2 = H^0 \cup 2H^2 \cup H^4$. Here $2H^2$ is attached along a link of two components, simply linked, each with framing 0. Then $H^0 \cup 2H^2$ is a 2-fold cyclic covering of D^4 branched over a (torus-open disc). The surface F' is now a torus in S^3(!).

(c) Take $\mathbf{C}P^2 \# -\mathbf{C}P^2 = S^2 \times S^2 = H^0 \cup 2H^2 \cup H^4$. Here $2H^2$ is attached along a link of two components, simply linked, with framings 0 and 1, respectively. Then $H^0 \cup 2H^2$ is a 2-fold cyclic covering of D^4 branched over a (Klein-bottle-open disc). The surface F' is now a Klein bottle.

These examples explain beautifully why $\mathbf{C}P^2 \# S^2 \times S^2 = \mathbf{C}P^2 \# S^2 \times S^2$, because forming connected sum of the real projective plane with torus or Klein bottle produces the same result.

6. 3-fold coverings. There now remains the case of attaching μH^2 to $H^0 \cup \lambda H^1$ along a general system of curves. In this case W^4 need no longer be a 2-fold branched covering space, but we show in the following that it is an irregular 3-fold covering of D^4.

Firstly, we have to define $\lambda \# S^1 \times B^3 = H^0 \cup \lambda H^1$ as a *standard* irregular 3-fold covering space of D^4 branched along $\lambda + 2$ unlinked and unknotted copies of D^2. In general, we represent $\lambda \# S^1 \times B^n$ in the following way. Let s: $\mathbf{R}^n \circlearrowleft$ defined by $s(x_1, \ldots, x_n) = (x_1, \ldots, -x_n)$, and consider $A = \{(x_1, \ldots, x_n) \in [-1, 1]^n | x_1 = 1 \text{ and } x_n \in \pm[(2i-1)/3\lambda, 2i/3\lambda], \text{ for some } 0 < i \leqslant \lambda\}$. Then $\lambda \# S^1 \times B^n$ is $[-1, 2] \times [-1, 1]^n$ with the following identifications $\mathfrak{R}$ among elements $(y, x) \in [-1, 2] \times [-1, 1]^n$: $(y, x)\, \mathfrak{R}\, (y, sx)$ if and only if $y = -1, 2$ or $x \in A$ and $y \in [-\frac{1}{2}, \frac{1}{2}] \cup [\frac{3}{2}, 2]$. We suggest that the reader draw a picture to illustrate and understand the case $n = 2$.

We can represent D^{n+1} by $[0, 1] \times [-1, 1]^n$ with the following identifications: $(y, x)\, \mathfrak{R}'\, (y, sx)$ if and only if $y = 0, 1$ or $x \in A$ and $y \in [0, \frac{1}{2}]$.

We define the following map $\hat{p}_1$: $[-1, 2] \to [0, 1]$ by the following rule:

$$\hat{p}_1(t) = \begin{cases} -t & \text{if } -1 \leqslant t \leqslant 0, \\ t & \text{if } 0 \leqslant t \leqslant 1, \\ -t + 2 & \text{if } 1 \leqslant t \leqslant 2, \end{cases}$$

and the folding map $\hat{p}_{n+1}$: $[-1, 2] \times [-1, 1]^n \to [0, 1] \times [-1, 1]^n$ by $\hat{p}_{n+1} = \hat{p}_1 \times s$. Because $\hat{p}_{n+1}$ is compatible with $\mathfrak{R}$ and $\mathfrak{R}'$, it defines p_{n+1}: $\lambda \# S^1 \times B^n \to D^{n+1}$. This is an irregular 3-fold covering space branched over $\{0, 1\} \times [-1, 1]^{n-1} \cup \{0\} \times A/\mathfrak{R}'$. Clearly, p_{n+1} corresponds to the (simple) representation ω: $\pi_1(D^{n+1}\text{-branching set}) \to \mathfrak{S}_3$ such that $\omega(x) = (01)$

if x is a meridian around $\{0\} \times ([-1, 1]^{n-1} \cup A)/\mathfrak{R}'$ or $\omega(x) = (02)$ if x is a meridian around $\{1\} \times [-1, 1]^{n-1}/\mathfrak{R}'$.

In case $n = 3$, $p_4|\lambda \# S^1 \times S^2\colon \lambda \# S^1 \times S^2 \to S^3$ can be visualized by means of Figure 5, where the boundary of the handlebody X and the ball D^3 are identified by reflection in the (x, y)-plane, and p_4 identifies points by reflection through the axes R_{02} and P_{01}.

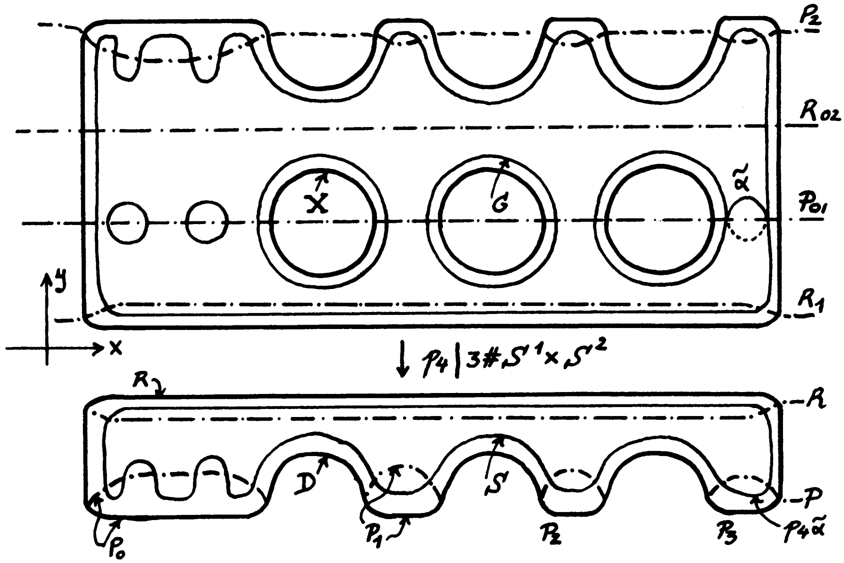


FIGURE 5

Here the boundary of the branching set is the union of $\partial(\{0\} \times ([-1, 1]^2 \cup A)/\mathfrak{R}') = P$ and $\partial(\{1\} \times [-1, 1]^2/\mathfrak{R}') = R$.

LEMMA 1. *Let L be a link of n components in $\lambda \# S^1 \times S^2$. Then, after an isotopy of L there exists a system $A = A_1 \cup \cdots \cup A_n$ of disjoint simple arcs in S^3 with the following properties*:

(a) *Each arc A_i does not meet R and meets P only in its endpoints.*

(b) *$p_4^{-1}A$ consists of L and a system A' of simple arcs.*

(c) *$p_4|L$ is a 2-fold branched covering over A.*

(d) *$p_4|A'\colon A' \to A$ is a homeomorphism.*

PROOF. It is easily checked that if A satisfies (a) and also if $p_4^{-1}A$ contains n closed components, then A satisfies (c) and (d). In order to find such a system A we divide the proof into several steps (we refer to Figure 5).

Step 1. *Putting L in the interior of X.*

Isotope L so that $\partial X \cap L$ is a system of points symmetric with respect to reflection in the (x, y)-plane (see Figure 5). Connect each pair of symmetric points of this system with a selfsymmetric arc lying in ∂X, and use regular neighborhoods of these arcs to isotope L into the interior of X.

Step 2. *Putting L onto a symmetric surface.*

Put L in normal projection with respect to the (x, y)-plane (by small isotopy), and consider the "checkerboard surface" F spanned by the link L. That is, we color one set of regions into which the (x, y)-plane is divided black, and the complementary set white, in such a way that any two regions with a common boundary are colored differently. Suppose that the region which contains ∞ is colored white, then join each two black regions, with a common double point, by a ribbon with a half-twist in the natural way. By a (gross) isotopy of L if necessary, we may assume that the normal projection of L is connected and separates the "holes" of X from one another and from ∞. In addition, we put a small "kink" in each component of L, which impedes into a black region of F, and reconstruct F from L in this new form. If a black region intersects $R^3 - \operatorname{int} X$ delete the interior of a small regular neighborhood of this intersection (which is a disc) so that the new surface is now contained in the interior of X. Call the deleted surface F again.

We construct an orientable surface G, containing L as follows. Consider a (relative) regular neighborhood V of F rel ∂F in X. Then $G = \partial V$ is an orientable surface containing ∂F, and hence $L \subset \partial F \subset G$. Because of the kinks, no component of L separates G. Because of the hole separating condition on L, G is parallel to ∂X, except for a number of extra-holes. By isotopy, position G such that these extra-holes are over the P_{01} axis and so that the new surface, still called G, is equal to $p_4^{-1}S$, where S is a 2-sphere contained in the interior of the ball D (as the one shown in Figure 5). The surface G contains each component of L as a nonseparating curve.

Step 3. *Putting each component of L onto a symmetric surface.*

Let us call $L_1, L_2, \ldots, L_n$ the components of L and consider n 2-spheres S_i, parallel to S. Call $G_i = p_4^{-1}S_i$. We can isotope each L_i onto G_i. Hence, in each surface G_i we have a nonseparating knot L_i.

Step 4. *Symmetrizing L_i.*

There is an orientation preserving homeomorphism of G_i sending the nonseparating curve $\tilde{\alpha}_i$ corresponding to $\tilde{\alpha}$ in Figure 5 onto L_i (the proof can be done using W. B. R. Lickorish's methods [5]). This homeomorphism is isotopic in G_i to $\tilde{f}_i$, which is a lifting of a homeomorphism f_i: $(S_i, P \cap S_i, R \cap S_i)\circlearrowleft$ (H. M. Hilden [2]). So, after an isotopy in G_i we can suppose that $\tilde{f}_i\tilde{\alpha}_i = L_i$. This isotopy can now be extended to X, using a regular neighborhood of G_i which does not meet the other surfaces. Then $f_i(p_4\tilde{\alpha}_i)$ is a simple arc which meets P exactly in its endpoints, does not meet R and $p_4^{-1}f_i(p_4\tilde{\alpha}_i)$ contains $\tilde{f}_i\tilde{\alpha}_i = L_i$.

The conditions of the lemma are fulfilled by $A = \bigcup_i f_i(p_4\tilde{\alpha}_i)$. $\square$

Let $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, where $\mu H^2 = H_1^2 \cup \cdots \cup H_\mu^2$, and let h_i:

$\dot{B}^2 \times B^2 \to \partial(H^0 \cup \lambda H^1) = \lambda \# S^1 \times S^2$ be the attaching map of the 2-handle H_i^2. Put the link $L = \bigcup_i h_i(\dot{B}^2 \times (0, 0))$ in $\lambda \# S^1 \times S^2$ as in Lemma 1, so that $P_4(\bigcup_i h_i(\dot{B}^2 \times B^2))$ is a regular neighborhood $U(A)$ of $A = p_4 L$. As in §2, suppose that the involution $h_i V h_i^{-1}$ preserves fibers of p_4: $\lambda \# S^1 \times S^2 \to S^3$. Now, add H_i^2 to $H^0 \cup \lambda H^1$ using h_i. Calling g_i the natural projection $H_i^2 \to H_i^2 | V$, add $H_i^2 | V$ to D^4 using the composition $p_4 h_i g_i^{-1}$, and add $H_i^2 | V$ to $p_4^{-1}(U(A_i)) - U(L_i)$ using $p_4 h_i g_i^{-1}$ followed by the inverse of the homeomorphism $p_4 | p_4^{-1} U(A_i) - U(L_i) \to U(A_i)$. Thus we obtain W^4 as $H^0 \cup \lambda H^1 \cup \mu H^2 \cup (\bigcup_i H_i^2 / V)$, and $p = p_4 \cup \bigcup_i g_i \cup \bigcup_i(\text{id}: H_i^2 / V \to H_i^2 / V)$ is a 3-fold covering over $D^4 \cup \bigcup_i H_i^2 / V \approx B^4$.

The branching set of p, lying over D^4, is a system of disjoint discs $\hat{P}_0 \cup \cdots \cup \hat{P}_\lambda \cup \hat{R}$ which intersect S^3 in a system $P_0 \cup \cdots \cup P_\lambda \cup R$ of unlinked and unknotted curves (see Figures 5 and 6). The branching set of p lying over H_i^2 / V is a disc which can be visualized as a band B_i, attached to $\hat{P}_0 \cup \cdots \cup \hat{P}_\lambda$ along two different arcs in the boundary. Pushing B_i into S^3, this band, with center line A_i, links $P_0 \cup \cdots \cup P_\lambda \cup R$ as A_i does, producing ribbon singularities. This shows that the branching set is an obvious generalization of a ribbon disc if we allow, in Yajima's description of ribbon knots (see §3), an arbitrary number of bands. We call such surfaces *ribbon manifolds*. So we have

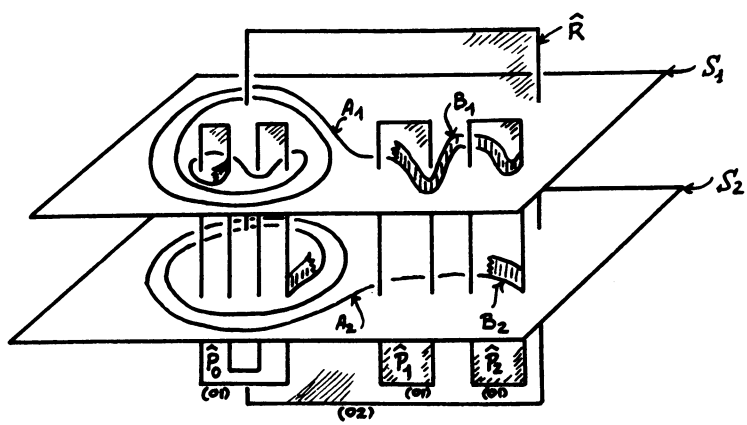


FIGURE 6

THEOREM 6. *Each $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$ is a 3-fold irregular covering space of D^4, the branching set being a ribbon manifold.*

As an immediate consequence of Theorem 6 we have

COROLLARY 2. *The double $2V^4$ of an orientable 4-manifold V^4 with 2-spine is a 3-fold irregular covering space of S^4 branched over a closed 2-manifold.* □

REMARKS. (1) The branching set which results from the proof of this theorem is a ribbon manifold of a special type as shown in Figure 6, because the arc A_i links $\hat{R}$ in a special way as a result of the application of Hilden's Theorem in Step 4 of Lemma 2. The branched cover corresponds to a representation π_1 (D^4-ribbon manifold) $\to \mathfrak{S}_3$ which sends Wirtinger generators linking $\hat{P}_i$ to (01) and the ones linking $\hat{R}$ to (02) (see Figure 6).

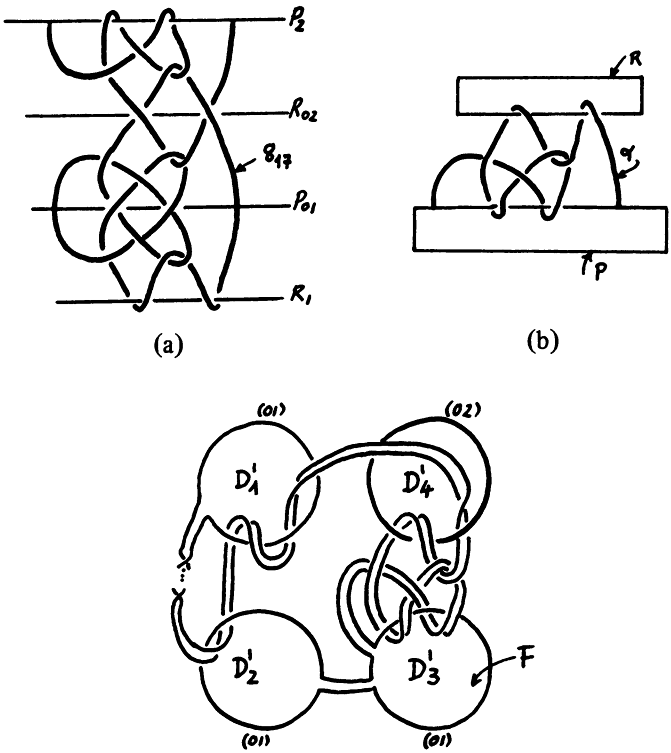
We call such a representation of a ribbon manifold a *colored ribbon manifold*. Note that if a colored ribbon disc is given it is very easy to exhibit a handle presentation for the corresponding 3-fold cover.

(2) A pseudo-handlebody structure on W^4 is a representation $W^4 = H^0 \cup \lambda H^1 \cup \mu H_s^2$, where H_s^2 means that $H_s^2 \cap (H^0 \cup \lambda H^1)$ is a *knotted* solid torus in $\partial H_s^2 = S^3$. Because W^4 has a 2-spine, W^4 has a handlebody representation with 0-, 1- and 2-"true" handles, and it is a 3-fold covering space of D^4 branched over a 2-manifold.

But we can obtain this 3-fold covering directly from the pseudo-handlebody structure of W^4, and also an *explicit* true handlebody structure for W^4. The reason is that we can define in H_s^2 a 3-fold covering space p over D^4 (instead of a 2-fold one), and use the lemma to modify the attaching knot in ∂H_s^2 to be symmetric with respect to p and with two fixed points. Now in an equivariant way we can match this projection with the one defined in $H^0 \cup \lambda H^1$ to get the result.

EXAMPLE. Let W^4 be the manifold $H^0 \cup H^1 \cup H_s^2$ where the attaching sphere of H_s^2 is the curve in Figure 1 and $(H^0 \cup H^1) \cap H_s^2$ is a regular neighborhood of the knot 8_{17}, in ∂H_s^2. (We choose this knot because its invertibility is not known.) We symmetrize 8_{17} in Figure 7(a) and have only to paste the ball $p(U(K))$ in Figure 3(b) to a regular neighborhood of the arc α in Figure 7(b). The resulting ribbon manifold F is shown in Figure 7(c). Of course, the covering over the discs D_1^1, $D_2^1 \# D_3^1$ and D_4^1 gives $S^1 \times B^3$ (compare Remark (1)) and we can lift the core of the ribbon band of F to get a handlebody representation $H^0 \cup H^1 \cup H^2$ for W^4.

(3) (Application to 3-dimensional topology.) Each closed, orientable 3-manifold M^3 can be obtained by special Dehn surgery on a link $L = L_1 \cup \cdots \cup L_\mu$ in S^3; we mean by "special" that the new meridian of the surgery (in L_i, for instance) goes *one* time around a longitude on $\partial U(L_i)$. Consider $W^4 = H^0 \cup \mu H^2$ where μH^2 is attached to $\partial H^0 = S^3$ along L using the framing corresponding to the Dehn-surgery in L. Then there exists a 3-fold dihedral covering $p\colon W^4 \to D^4$ branched over a ribbon manifold R, and $p|(\partial W^4 = M^3)\colon M^3 \to S^3$ is a 3-fold dihedral covering space branched over ∂R. Here, $\pi_1 W^4 = 1$. The ribbon manifold R consists of a disc D_1 and a disc with bands D_2. The representation of $\pi_1(D^4 - R) \to \mathfrak{S}_3$, corresponding to the cover, sends meridians of D_1 (resp. D_2) on the transposition (01) (resp. (02)). (Compare this representation of 3-manifolds with the one in [3].)



(c)

FIGURE 7

(4) Lemma 1 seems interesting in its own right because it gives a procedure for "symmetrizing" knots so that they can be represented by an arc (see Figure 7) with its endpoints in P and linking R a number of times. The minimum of this number is a measure of the strong noninvertibility of the knot.

7. Final remarks. (1) In [7] it is shown that each 4-manifold, represented by $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is completely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$. This shows that the result of Theorem 6 is not as special as it might appear, inasmuch as it provides a surjection from a subset of colored ribbon manifolds (see Remark (1) in §6) to the set of all closed, orientable, PL 4-manifolds.

We call a colored ribbon manifold *allowable* if the boundary of the corresponding 3-fold covering space is $\gamma \# S^1 \times S^2$ for some γ. The enumeration of colored ribbon manifolds which are representatives of closed 4-manifolds corresponds to the following problem:

Problem 1. *When is a colored ribbon manifold allowable?*

Thus allowable colored ribbon manifolds provide a representation of *PL*, closed, orientable 4-manifolds. In order that this representation be more useful it would be convenient to translate, in terms of colored ribbon manifolds, the concept of homeomorphism between 4-manifolds. Hence we state

Problem 2. *Given two colored ribbon manifolds which represent the same closed* 4-*manifold, find a combinatorial way of passing from one to the other.*

(2) Let $H^0 \cup \lambda H^4 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ be a handle presentation for a closed, orientable 4-manifold W^4. By duality W^4 is obtained by pasting together two manifolds $V^4 = H^0 \cup \lambda H^1 \cup \mu H^2$ and $U^4 = \gamma \# S^1 \times B^3$. It is important to remark that the manifold $W^4 = V^4 \cup \gamma \# S^1 \times B^3$ is *independent* of the way of pasting the boundaries together [7].

We have 3-fold irregular branched coverings q_1: $\gamma \# S^1 \times B^3 \to D^4$ and q_2: $V^4 \to D^4$, provided by Theorem 6, which are special in the sense of Remark (1) in §6. But in some cases, as Berstein and Edmonds pointed out, these 3-fold covering spaces cannot be pasted together (see §1).

Now V^4 and $\gamma \# S^1 \times B^3$ "a fortiori" have irregular 4-fold covering presentations p_1: $\gamma \# S^1 \times B^3 \to D^4$ and p_2: $V^4 \to D^4$, which can be obtained by adding to the branching set of q_1 (resp. q_2) a new properly embedded trivial disc, unlinked with the branching set, and by sending its meridian into the transposition $(03) \in \mathfrak{S}_4$.

The conjecture that each W^4 is a 4-fold irregular covering space of S^4 branched over a closed 2-manifold, follows from the next conjecture, where p_1', p_2' stand for the restriction to the boundary of p_1, p_2, respectively.

CONJECTURE. *The coverings p_1' and p_2' are cobordant, i.e. there is a* 4-*fold irregular covering* P: $(\lambda \# S^1 \times S^2) \times I \to S^3 \times I$, *which is equal to p_i' in* $(\lambda \# S^1 \times S^2) \times \{i\}$, $i = 1, 2$, *and branched over a* 2-*manifold with boundary equal to the union of the branching sets of* p_1', p_2'.

In solving this conjecture the following criterion may be useful:

LEMMA 2. *Let* p: $\lambda \# S^1 \times S^2 \to S^3$ *be a special covering* (*in the sense of Remark* (1) *of* §6) *such that $\hat{R}$ bounds a disc which does not cut any other component of the branching set*; *then p is standard.*

PROOF. It is clear that, by the conditions of the lemma, $\lambda \# S^1 \times S^2$ is a 2-fold covering space branched over (branching set of $p - \hat{R}$). Because this cover is standard (see [4]), it consists of a system of $\lambda + 1$ unknotted and unlinked components. □

Note that the solution of the above conjecture is closely related to the solution of Problem 1.

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