

EVEN TRIANGULATIONS OF S^3 AND THE COLORING OF GRAPHS

BY

JACOB ELI GOODMAN AND HIRONORI ONISHI

ABSTRACT. A simple necessary and sufficient condition is given for the vertices of a graph, planar or not, to be properly four-colorable. This criterion involves the notion of an "even" triangulation of S^3 and generalizes, in a natural way, a corresponding criterion for the three-colorability of planar graphs.

0. Introduction. With the Appel-Haken solution to the Four Color Problem [2], the question remains open of characterizing those graphs, planar or not, that are 4-colorable. This paper represents a step toward a solution by offering a new criterion for the 4-colorability of a graph embedded in 3-space, which was suggested by an analogous criterion for the 3-colorability of a graph embedded in the plane. The main result is that a graph in the 3-sphere S^3 is (vertex) 4-colorable if and only if it is a subcomplex of the 1-skeleton of an "even" triangulation of S^3 —one in which every edge has an even number of faces incident to it.

The corresponding result one dimension lower is well known [4, Theorem 7.4.3]. In §1, we present a summary of this theory with some auxiliary results, and in §2 we present the parallel theory in 3 dimensions.

Since the original submission of this paper, Robert D. Edwards has announced an (independent) proof of the main result, following an idea of P. Deligne, R. MacPherson, and J. Morgan (see Notices Amer. Math. Soc. **24** (1977), A-257).

The beautiful sequence of papers by Steve Fisk entitled *Geometric coloring theory*, which has begun appearing still more recently in *Advances in Math.* (**24** (1977), 298–340, et seq.), also contains ideas which overlap ours to some extent.

We express our gratitude to the referee for his helpful suggestions about tightening the exposition of the paper.

Received by the editors January 3, 1977 and, in revised form, October 25, 1977 and April 24, 1978.

AMS (MOS) subject classifications (1970). Primary 05C15; Secondary 05C10, 55A15.

© American Mathematical Society 1979

1. 3-coloring. (1) Let T be a triangulation of a region in the sphere S^2 . Suppose T is 3-colored, i.e., the vertices of T are colored by three or fewer colors so that no two adjacent vertices have the same color. Then it is clear that every interior vertex of T is *even*, i.e., has even degree: look at the alternating colors of its neighbors. The converse is also true, in the following sense:

THEOREM 1.1. *Let T be a triangulation of a disk or of S^2 . If T is even, i.e., every interior vertex of T is even, then T can be 3-colored and a 3-coloring is unique up to a permutation of the colors (cf. [4, Theorem 7.4.3]).*

PROOF. If the region is S^2 , remove one face from T ; there remains an even triangulation of a disk. Thus we may assume the region is a disk. We may also assume T has at least two faces. It is clear that T has a face whose removal leaves an even triangulation of a disk. Thus by induction on the number of faces we arrive easily at the conclusions.

(2) A topological property of the triangulated region needed in Theorem 1.1 is that it is simply-connected. For example, Figure 1.1 shows an even triangulation of an annulus which cannot be 3-colored. For the uniqueness of the 3-coloring, all we need is that the region be the closure of an open edge-connected set.

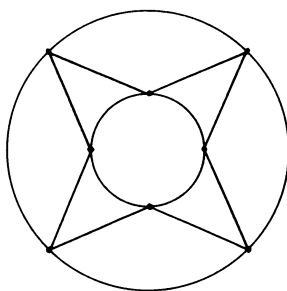


FIGURE 1.1

(3) Let T^1 be a triangulation of the circle S^1 . It is trivial to 3-color T^1 . Once T^1 is 3-colored, there are two types of vertices: one whose neighbors have different colors is of *type XYZ*, and one whose neighbors have the same color is of *type XYX*.¹ Clearly a 3-coloring of T^1 which induces a given typing is unique up to a permutation of the colors.

(4) Let T be a triangulation of the disk B^2 , and let T^1 be the induced boundary triangulation; we call T an *extension* of T^1 . It is clear that if T is 3-colored, then a vertex of T^1 is of type *XYZ* or *XYX* according as it is even or odd in T .

¹S. Fisk calls these “nonsingular” and “singular”, resp. (see Introduction).

(5) Let T and T^1 be as in (4). Suppose T^1 is 3-colored in such a way that a vertex of T^1 is of type XYZ if and only if it is even in T . If T is even, then there is a unique 3-coloring of T extending that of T^1 . In fact, take any 3-coloring of T and note that the new typing coincides with the original by (4). Then permute the colors.

(6) Let T^1 be a triangulation of S^1 and consider an assignment of the symbols E or O to the vertices of T^1 . Such an assignment satisfies the *mod 2 condition* if the number of O -vertices is even. Assume the mod 2 condition. Choose an E -vertex V_1 and assign $+1$ or -1 to each E -vertex V according as the number of O -vertices between V_1 and V is even or odd. Because of the mod 2 condition this is independent of the direction around T^1 . The E - O assignment satisfies the *mod 3 condition* if the sum of the numbers $+1$ and -1 over all E -vertices is a multiple of 3. The significance of these conditions is apparent in the following:

THEOREM 1.2. *Let T^1 be a triangulation of S^1 . Given an E - O assignment on (the vertices of) T^1 , there is a 3-coloring of T^1 which induces the assignment (i.e., a vertex is of type XYZ or XYX according as it is an E - or O -vertex) if and only if the E - O assignment satisfies the mod 2 and mod 3 conditions.*

PROOF. Suppose T^1 is 3-colored by A , B , and C . The cyclical orientation ABC induces an orientation of each edge. A vertex of T^1 is of type XYX if and only if the two edges incident to it are oppositely oriented. This gives the mod 2 condition on the induced E - O assignment. The mod 3 condition follows from the fact that the number of positively oriented edges and the number of negatively oriented edges are congruent mod 3, since the coloring can be thought of as a map from S^1 to the (oriented) triangle ABC .

Conversely, suppose that the E - O assignment satisfies the mod 2 and mod 3 conditions. Let $V_1, \dots, V_n$ be the vertices of T^1 in a cyclical order. Color V_1 A and V_2 B . Then color V_3 C or A according as V_2 is an E - or O -vertex. Continue this. The mod 2 and mod 3 conditions guarantee that this can be done consistently, i.e., the color of V_n is C or B according as V_1 is an E - or O -vertex. T^1 is now 3-colored and the coloring induces the given E - O assignment.

(7) Let T be an even triangulation of the disk B^2 . A 3-coloring of T induces an E - O assignment on the boundary T^1 as in Theorem 1.2. By (4), a vertex of T^1 is an E -vertex if and only if it is even in T . The converse is also true:

THEOREM 1.3. *Let T^1 be a triangulation of S^1 with an E - O assignment satisfying the mod 2 and mod 3 conditions. Then T^1 can be extended to an even triangulation T of the disk inside S^1 in such a way that a vertex of T^1 is even in T if and only if it is an E -vertex. Moreover, we can make sure that (i) and (ii) are satisfied or, alternatively, (iii):*

- (i) T has only one interior vertex if T^1 has no E -vertices. (This is trivial.)
- (ii) T has no interior vertices if T^1 has at least one E -vertex (cf. [3]).
- (iii) No interior edge of T joins vertices of T^1 .

(8) In view of Theorems 1.1 and 1.2, Theorem 1.3 is equivalent to the statement: Any 3-colored triangulation T^1 of S^1 can be extended to a 3-colored triangulation T of the disk inside S^1 (satisfying (i) and (ii), or (iii)).

(9) We prove (ii) of Theorem 1.3 in a slightly more general form: Let $k \geq 3$ and let T^1 be a k -colored triangulation of S^1 involving all k colors. Then T^1 can be extended to a k -colored triangulation T of the disk inside S^1 without any interior vertices.

PROOF. Let n be the number of vertices of T^1 . We may assume $n > 3$. Choose three consecutive vertices V_1, V_2, V_3 having three distinct colors A, B, C . If V_2 is the only vertex of color B , then join it to every other vertex of T^1 . If not, join V_1 and V_3 . This gives a triangle $V_1V_2V_3$ and a k -colored triangulation T_1^1 of a circle not containing V_2 . Since T_1^1 has $n - 1$ vertices and k colors, we can extend it by induction on n , and we obtain a desired triangulation T .

PROOF OF (iii). Let m be the number of E -vertices of T^1 . We may assume $m \geq 2$. Let $V_1, \dots, V_n$ be the vertices of T^1 in a cyclical order, and suppose that V_1 and V_k ($k \geq 2$) are E -vertices and that V_i for $1 < i < k$ are O -vertices. $V_1, \dots, V_k$ have only two colors, say A and B . Introduce an interior vertex W , color it C , and join W to all the V_i for $1 \leq i \leq k$. Consider the triangulation T_1^1 : $V_1WV_kV_{k+1} \cdots V_n$. Since V_k and V_1 are E -vertices, the color of V_{k+1} and V_n is C . Thus V_k and V_1 are O -vertices of T_1^1 . W is an E - or O -vertex according as k is even or odd. Thus the number of E -vertices of T_1^1 is $m - 1$ or $m - 2$ according as k is even or odd. In either case, by induction on m we obtain a desired 3-colored triangulation T .

(10) In the proof of (iii), if $n \not\equiv 0 \pmod{3}$, then we can choose V_1 such that k is odd, so that m decreases by 2. Thus by induction on m we obtain a desired T with at most $\lfloor 2m/3 \rfloor + 1$ interior vertices.

THEOREM 1.4. Any 3-colored graph G in S^2 can be extended to an even triangulation T of S^2 so that G is an induced subgraph of T .

PROOF. Embed G in a triangulation of S^2 (without disturbing the edges of G). A new edge may join two vertices of the same color. Introduce a new vertex on each such edge and color it by one of the two remaining colors. Extend each 3-colored "triangular" polygon to a 3-colored triangulation of its inside without any diagonals by (iii) of Theorem 1.3. This gives a desired T .

2. 4-coloring. (1) Let T be a triangulation of a region in the 3-sphere S^3 . Suppose T is (vertex) 4-colored. Then it is clear that every interior edge L of

T is even, i.e., the number of faces incident with L , called the *degree* of L , is even: look at the alternating colors of the vertices adjacent to L , i.e., adjacent to both ends of L .

THEOREM 2.1. *Let T be a triangulation of a simply-connected region R in S^3 (R being the closure of an open connected set). If T is even, i.e., every interior edge of T is even, then T can be 4-colored, and a 4-coloring of T is unique up to a permutation of the colors.*

PROOF. The condition that every interior edge of T is even implies that to every simple loop $J_1, \dots, J_d$ of 3-simplices around an interior edge, any 4-coloring of any J_i induces a unique 4-coloring of the "wheel" $J_1 \cup \dots \cup J_d$. Since R is simply-connected, every loop of 3-simplices is a sum of simple loops. Thus we arrive at the conclusion.

(2) Unlike the 2-dimensional case, this time the region can have holes. For example a spherical shell is simply-connected. But a solid torus is not, and it is easy to construct an even triangulation of a solid torus which cannot be 4-colored.

(3) Let T^2 be a triangulation of the sphere S^2 . It is not trivial to 4-color T^2 ; the Four Color Theorem [2] says it can be done. Once T^2 is 4-colored, there are two types of edges: one whose neighboring vertices have different colors we will call of *type XYZ*, and one whose neighboring vertices have the same color we will call of *type XYX*. A 4-coloring of T^2 which induces a given typing is unique up to a permutation of the colors.

(4) Let T be a triangulation of the ball B^3 and let T^2 be the induced boundary triangulation; we will call T an *extension* of T^2 . If T is 4-colored, then an edge of T^2 is of type XYZ or XYX according as it is even or odd in T .

(5) Let T and T^2 be as in (4). Suppose T^2 is 4-colored in such a way that an edge of T^2 is of type XYZ if and only if it is even in T . If T is even, then there is a unique 4-coloring of T extending that of T^2 . The reason is exactly the same as in (1.5).

(6) Let T^2 be a triangulation of S^2 and consider an assignment of E or O to the edges of T^2 . We say that such an assignment satisfies the *mod 2 condition* or the *mod 3 condition* if, for every vertex V of T^2 , the E - O assignment on the spokes of the wheel about V satisfies the mod 2 or mod 3 condition in the sense of (1.6).

THEOREM 2.2. *Let T^2 be a triangulation of S^2 . Given an E - O assignment on (the edges of) T^2 , there is a 4-coloring of T^2 which induces the assignment (i.e., an edge is of type XYZ or XYX according as it is an E - or O -edge) if and only if the assignment satisfies the mod 2 and mod 3 conditions. Moreover the 4-coloring is unique (up to a permutation of the colors).*

PROOF. Suppose that the E - O assignment is induced by a 4-coloring of T^2 . For each vertex V of T^2 , the rim R of the wheel about V is 3-colored and the induced E - O assignment on the vertices of R coincides with the E - O assignment on the corresponding spokes, and hence the mod 2 and mod 3 conditions are satisfied.

Conversely, suppose that the E - O assignment satisfies the mod 2 and mod 3 conditions. Take a vertex V and 3-color the rim about V so that it induces the E - O assignment on R corresponding to the E - O assignment of the spokes about V . Color V by the 4th color. Since S^2 is simply-connected, it follows that this 4-coloring of the wheel about V spreads to a unique 4-coloring of T^2 , and that the latter induces the given E - O assignment.

(7) Let T be an even triangulation of the ball B^3 . A 4-coloring of T induces an E - O assignment on the boundary T^2 as in Theorem 2.2. By (4), an edge of T^2 is an E -edge if and only if it is even in T .

THEOREM 2.3. *Let T^2 be a triangulation of S^2 with an E - O assignment satisfying the mod 2 and mod 3 conditions. Then T^2 can be extended to an even triangulation T of the ball inside S^2 in such a way that an edge of T^2 is even in T if and only if it is an E -edge. Moreover, we can make sure that (i) and (ii) are satisfied or, alternatively, (iii):*

- (i) T has only one interior vertex if T^2 has no E -edges. (This is trivial: T^2 is 3-colorable.)
- (ii) T has no interior vertices if T^2 has at least one E -edge.
- (iii) No interior edge of T joins vertices of T^2 .

(8) In view of Theorems 2.1 and 2.2, Theorem 2.3 is equivalent to the statement: Any 4-colored triangulation T^2 of S^2 can be extended to a 4-colored triangulation T of the ball inside S^2 (satisfying (i) and (ii), or (iii)).

(9) We prove (ii) of Theorem 2.3 in a slightly more general form: Let $k \geq 4$ and let T^2 be a k -colored triangulation of S^2 involving all k colors. Then T^2 can be extended to a k -colored triangulation T of the ball inside S^2 without any interior vertices.

PROOF. We use induction on n , the number of vertices of T^2 . If $n = 4$, there is nothing to do. Let $n > 4$, and consider three cases:

(a) Suppose T^2 contains a triangle $V_1V_2V_3$ colored (say) A, B, C which separates vertices of colors other than A, B, C . Then fill in face $V_1V_2V_3$ and use the induction hypothesis to get a colored triangulation of the ball bounded by the new face and each of the resulting hemispheres. The triangulations then patch together along face $V_1V_2V_3$.

(b) Suppose T^2 contains a triangle $V_1V_2V_3$ colored (say) A, B, C which separates vertices, but such that the only colors appearing on the hemisphere S_1 on one side of the triangle are A, B, C (see Figure 2.1(a)). Fill in face $V_1V_2V_3$ and use the induction hypothesis to get a colored triangulation T_2 of

the ball bounded by the new face and the opposite hemisphere S_2 . In T_2 , face $V_1V_2V_3$ is joined to a vertex V_0 , necessarily of a different color; let J be the tetrahedron $V_0V_1V_2V_3$ and let B_1 be the ball bounded by S_1 and face $V_1V_2V_3$. Now remove face $V_1V_2V_3$, and triangulate $B_1 \cup J$ by joining V_0 to every vertex in S_1 (see Figure 2.1(b)); call the resulting triangulation T_1 . Then $T_1 \cup (T_2 - J)$ is the desired triangulation T .

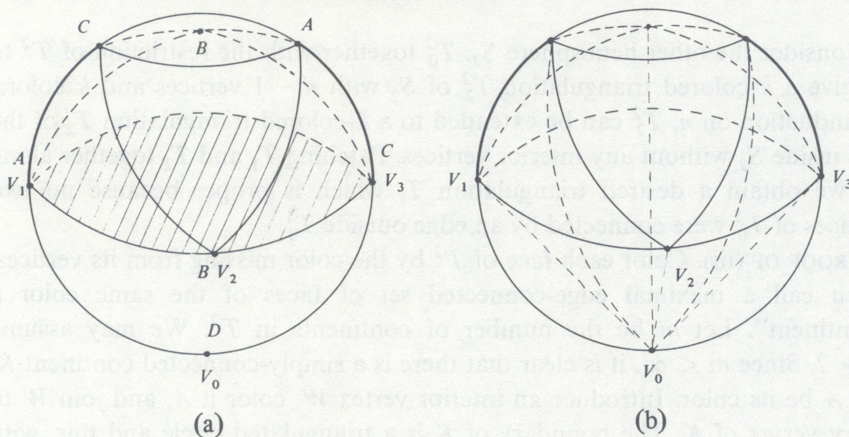


FIGURE 2.1

(c) Suppose, finally, that T^2 contains no triangle separating vertices. Then in the wheel around any vertex, no two vertices can be joined by an edge external to the wheel. Note that there is a vertex V_1 of T^2 such that the wheel about V_1 involves at least 4 colors. In fact, choose any two faces of T^2 which together involve at least 4 colors. By considering a chain of edge-connected faces between them, we see that some pair of adjacent faces together involve 4 colors, and we get a desired vertex V_1 .

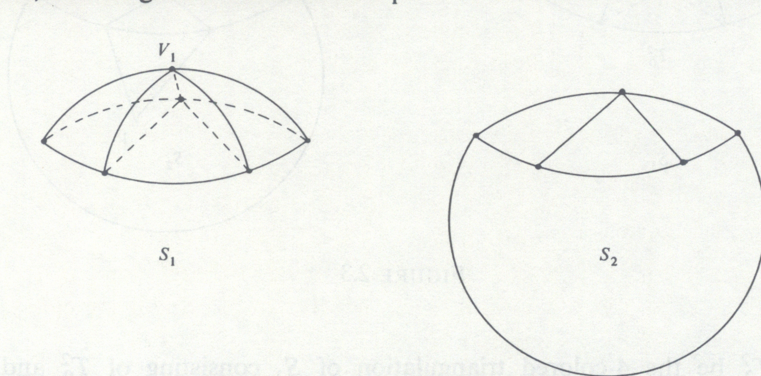


FIGURE 2.2

Let A be the color of V_1 . If V_1 is the only vertex of color A , then join V_1 to

every other vertex of T^2 . Suppose some other vertex is colored A . Consider the rim T_0^1 of the wheel about V_1 . T_0^1 has at least 3 colors. Thus by (1.9) T_0^1 can be extended to a colored triangulation T_0^2 of the disk B^2 inside T_0^1 , without any interior vertices (see Figure 2.2). T_0^2 together with the wheel about V_1 give a k -colored triangulation T_1 , without any interior vertices, of the ball inside the hemisphere S_1 bounded by the wheel and the "flat" disk B^2 .

Consider the other hemisphere S_2 . T_0^2 together with the restriction of T^2 to S_2 give a k -colored triangulation T_2^2 of S_2 with $n - 1$ vertices and k colors. By induction on n , T_2^2 can be extended to a k -colored triangulation T_2 of the ball inside S_2 without any interior vertices. Patching T_1 and T_2 together along T_0^2 we obtain a desired triangulation T , which is proper because no two vertices of T_0^1 were connected by an edge outside T_0^1 .

PROOF OF (iii). Color each face of T^2 by the color missing from its vertices. Then call a maximal edge-connected set of faces of the same color a "continent". Let m be the number of continents in T^2 . We may assume $m \geq 2$. Since $m < \infty$, it is clear that there is a simply-connected continent K . Let A be its color. Introduce an interior vertex W , color it A , and join W to every vertex of K . The boundary of K is a triangulated circle and this, with W , forms a wheel T_0^2 about W . The wheel T_0^2 cuts S^2 into two spheres, S_1 and S_2 ; say $S_1 = K \cup T_0^2$. Let T_1 be the triangulation of the ball inside S_1 induced by $K \cup T_0^2$ and the edges joining W to the vertices of K (see Figure 2.3).

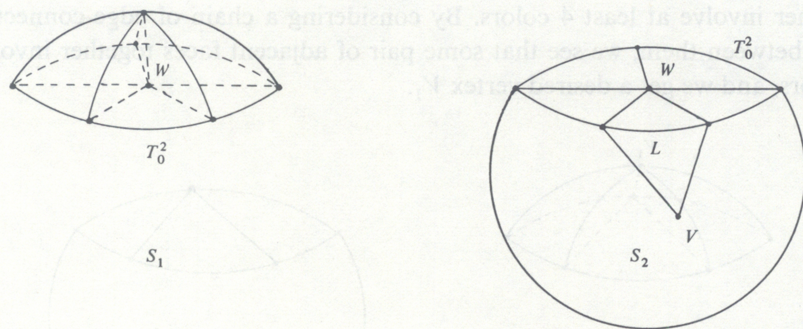


FIGURE 2.3

Let T_2^2 be the 4-colored triangulation of S_2 consisting of T_0^2 and the restriction of T^2 to S_2 . Let L be a boundary edge of T_0^2 . The vertex V of T_2^2 across L from W is colored A because L is on the continental boundary of K .

Thus L is not on a continental boundary in T_2^2 . It follows that T_2^2 has fewer than m continents. By induction on m , we obtain a 4-colored triangulation T_2 of the ball inside S_2 extending T_1^2 , and with no interior edge joining a pair of vertices of T_1^2 . Patching T_1 and T_2 together along T_0^2 , we obtain a desired T .

(10) It is clear from the proof of (iii) above that the triangulation T obtained has at most m interior vertices. It may have fewer than m , and in fact the number of continents may decrease by more than 1 in the induction step.

THEOREM 2.4. *Any 4-colored graph G in S^3 can be extended to an even triangulation T of S^3 so that G is an induced subgraph of T .*

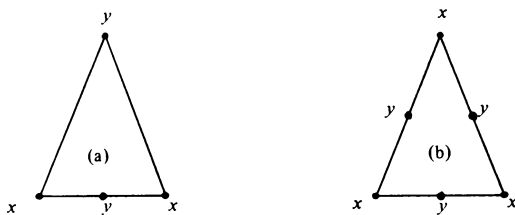


FIGURE 2.4

PROOF. By [1, Corollary VII.3], we can embed G in a triangulation T_1 of S^3 (without disturbing the edges of G). Assign colors to the new vertices in T , randomly using the 4 colors of G . On each xx -colored edge of T_1 (it cannot be an edge of G), introduce a new vertex and color it $y \neq x$. On each face of T_1 which (including the new vertices) has only 2 colors (necessarily as in Figure 2.4(a) or (b)), introduce a new vertex and color it $z \neq x, y$. In each 3-simplex of T which (including the new vertices) has only 3 colors, introduce a new vertex and give it the remaining color. Now subdivide the edges of T_1 at the new vertices (if any); subdivide the faces of T_1 according to the scheme in Figure 2.5; and subdivide the 3-simplices of T_1 (whose faces have now been triangulated and properly 4-colored) by Theorem 2.3(iii).² Since no two vertices of any 3-simplex are joined by an edge not on the boundary, we get a proper triangulation T of S^3 when we glue all the separate triangulations together along the common faces. And since no original edge has been tampered with, G is embedded in T . The evenness of T is a consequence of (2.1).

²Corresponding to the six possibilities for a face shown in Figure 2.5, there are 48 inequivalent cases of an (improperly) 4-colored 3-simplex whose edges and faces have been subdivided and the new vertices colored to provide a proper 4-coloring in accordance with the scheme given here.

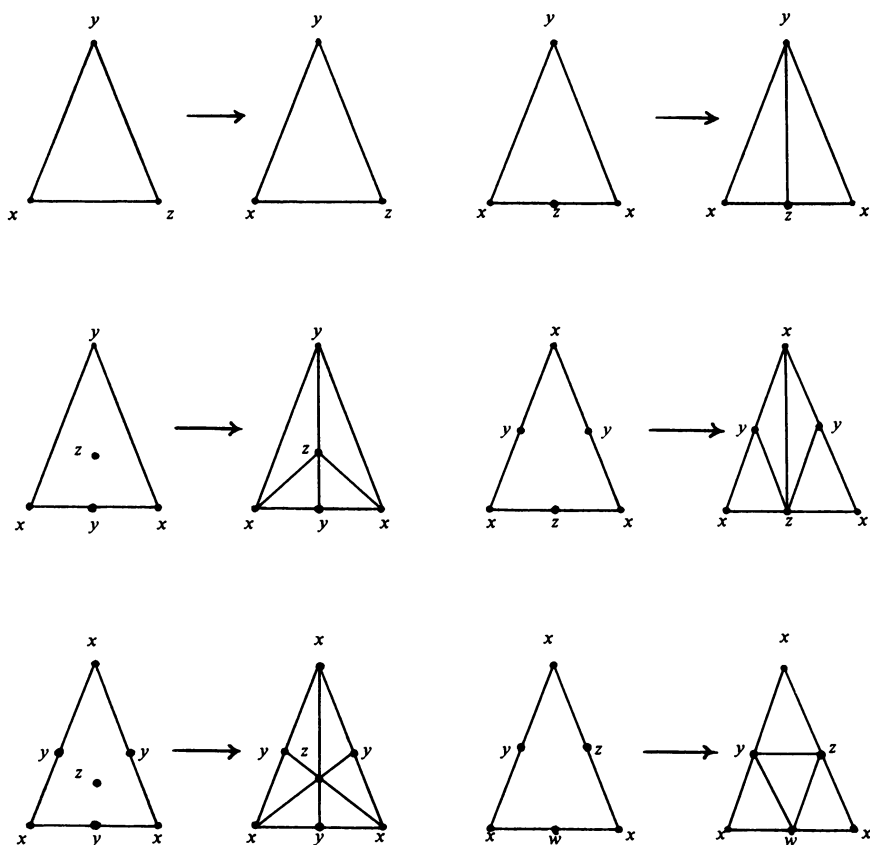


FIGURE 2.5

BIBLIOGRAPHY

1. E. Akin, *Manifold phenomena in the theory of polyhedra*, Trans. Amer. Math. Soc. **143** (1969), 413–473.
2. K. Appel and W. Haken, *Every planar map is four colorable*, Bull. Amer. Math. Soc. **82** (1976), 711–712; Illinois J. Math. **21** (1977), 429–567.
3. N. C. Dalkey, *Parity patterns on even triangulated polygons*, J. Combinatorial Theory **2** (1967), 100–103.
4. O. Ore, *The four color problem*, Academic Press, New York, 1967.

DEPARTMENT OF MATHEMATICS, CITY COLLEGE (CUNY), NEW YORK, NEW YORK 10031