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# DISTRIBUTION OF EIGENVALUES OF A TWO-PARAMETER SYSTEM OF DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. In this paper two simultaneous Sturm-Liouville systems are considered, the first defined for the interval $0 < x_1 < 1$, the second for the interval $0 < x_2 < 1$, and each containing the parameters $\lambda$ and $\mu$. Denoting the eigenvalues and eigenfunctions of the simultaneous systems by $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}(x_1, x_2)$, respectively, $j, k = 0, 1, \ldots,$ asymptotic methods are employed to derive asymptotic formulae for these expressions, as $j + k \to \infty$, when $(j, k)$ is restricted to lie in a certain sector of the $(x, y)$-plane. These results constitute a further stage in the development of the theory related to the behaviour of the eigenvalues and eigenfunctions of multiparameter Sturm-Liouville systems and answer an open question concerning the uniform boundedness of the $\psi_{j,k}(x_1, x_2)$.

**1. Introduction.** The importance of multiparameter Sturm-Liouville problems in mathematical physics has led in recent years to a revival of interest in this area of investigation after a period of relative neglect. However, most investigations to date have been concerned with problems related to oscillation and expansion theory (cf. [**1**], [**3**], [**5**], and [**9**]), and it has been pointed out by Atkinson [**2**, §4] that such fundamental problems as the determination of the behaviour of the eigenvalues and of the eigenfunctions of multiparameter Sturm-Liouville systems have not yet been resolved.

Stimulated by Atkinson's remark, the author was led in an earlier paper [**6**] to investigate the behaviour of the eigenvalues and eigenfunctions of the simultaneous two-parameter systems

$$y_1'' + (\lambda A_1(x_1) - \mu B_1(x_1) + q_1(x_1))y_1 = 0, \quad 0 \leqslant x_1 \leqslant 1, ' = d/dx_1, \quad (1.1)$$

$$y_1(0)\cos\alpha_1 - y_1'(0)\sin\alpha_1 = 0, \qquad 0 \leqslant \alpha_1 < \pi,$$
$$y_1(1)\cos\beta_1 - y_1'(1)\sin\beta_1 = 0, \qquad 0 < \beta_1 \leqslant \pi, \quad (1.2)$$

and

$$y_2'' + (-\lambda A_2(x_2) + \mu B_2(x_2) + q_2(x_2))y_2 = 0, \ 0 \leqslant x_2 \leqslant 1, ' = d/dx_2, \ (1.3)$$

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$$y_2(0)\cos\alpha_2 - y_2'(0)\sin\alpha_2 = 0, \qquad 0 \leqslant \alpha_2 < \pi,$$
$$y_2(1)\cos\beta_2 - y_2'(1)\sin\beta_2 = 0, \qquad 0 < \beta_2 \leqslant \pi. \tag{1.4}$$

To explain the work of that paper (and also our present problem) in greater detail, let us recall that by an eigenvalue of the system (1.1)–(1.4) we mean a pair of numbers, $(\lambda^*, \mu^*)$, such that when $\lambda = \lambda^*$ and $\mu = \mu^*$, $(1.2i - 1)$ has, for $i = 1, 2$, a nontrivial solution, say $y_i(x_i, \lambda^*, \mu^*)$, which satisfies $(1.2i)$. Furthermore, we also know that the eigenvalues and normalized eigenfunctions of the system (1.1)–(1.4) may be represented in the form $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}(x_1, x_2)$, respectively, $j, k = 0, 1, \ldots,$ where $(\lambda_{j,k}, \mu_{j,k})$ denotes that eigenvalue of (1.1)–(1.4) for which $y_1(x_1, \lambda_{j,k}, \mu_{j,k})$ has precisely $j$ zeros in $0 < x_1 < 1$ and $y_2(x_2, \lambda_{j,k}, \mu_{j,k})$ has precisely $k$ zeros in $0 < x_2 < 1$, while

$$\psi_{j,k}(x_1, x_2) = C_{j,k} \prod_{i=1}^{2} y_i(x_i, \lambda_{j,k}, \mu_{j,k})$$

and $C_{j,k}$ denotes a normalization constant. In [6] the positive quadrant of the $(x, y)$-plane was divided into three disjoint sectors, $\Omega$, $\Omega_1$, and $\Omega_2$ (each with vertex at the origin), and asymptotic formulae were derived for the $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}$, as $j + k \to \infty$, when $(j, k)$ was restricted to lie in $\Omega$ and in $\Omega_1$, respectively (since the results for $\Omega_2$ are similar to those for $\Omega_1$). The choice of the central sector, $\Omega$, was motivated by the fact that for $(j, k)$ in that sector, the above formulae could be derived by means of standard techniques, that is, without having to appeal to transition point theory. However, in order to apply the same techniques to the case $(j, k) \in \Omega_1$, we imposed the further condition that $A_2'B_2 - A_2B_2' = 0$ for $0 \leqslant x_2 \leqslant 1$, and thus the results of [6] for $(j, k) \in \Omega_1$ are valid only under this hypothesis. Hence in this paper we propose to derive asymptotic formulae for the above expressions for $(j, k) \in \Omega_1$ under the hypothesis that $A_2'B_2 - A_2B_2' \neq 0$ for $0 \leqslant x_2 \leqslant 1$. The importance of the case considered here lies in the fact that the results obtained will enable us to answer an open question concerning the uniform boundedness of the $\psi_{j,k}(x_1, x_2)$.

In what follows it will be supposed that $A_1$, $B_1$, and $q_1$ are real and continuous in $0 \leqslant x_1 \leqslant 1$, with both $A_1$ and $B_1$ having absolutely continuous first derivatives in this interval, that $q_2$ is real and continuous in $0 \leqslant x_2 \leqslant 1$, that $A_2$ and $B_2$ are real and of class $C^3$ in some interval containing the interval $0 \leqslant x_2 \leqslant 1$ in its interior, that $A_2'B_2 - A_2B_2' \neq 0$ for $0 \leqslant x_2 \leqslant 1$, and lastly, that $\Delta = A_1B_2 - A_2B_1 \neq 0$ in $I^2$ (the product of the intervals $0 \leqslant x_1 \leqslant 1$, $0 \leqslant x_2 \leqslant 1$). Furthermore, there is no loss of generality in assuming henceforth that the $A_j$, $B_j$, and $\Delta$ are positive for all values of $x_1$ and $x_2$ in $I^2$; for we know (see [6, Appendix A]) that this can always be achieved, if necessary, by introducing a nonsingular transformation in the parameters $\lambda$ and $\mu$.

After introducing notation and assumptions in §2, we concern ourselves in §3 with the asymptotic integration of (1.3). Here we encounter problems involving a simple transition point whose position varies with $\mu/\lambda$; and techniques from transition point theory are used to obtain estimates for the solutions of (1.3) for the various positions of the transition point. In §4 we fix our attention upon certain disjoint subsets $\Omega_i^*$, $i = 1, \ldots, 5$, of the sector $\Omega_1$ introduced above, and use the results of §3 to obtain estimates for $\eta_{j,k}$ $(= \mu_{j,k}/\lambda_{j,k})$ for $(j, k)$ in each of the $\Omega_i^*$ (see Theorem 4.1). These results enable us to determine the position of the transition point hitherto mentioned corresponding to $\eta_{j,k}$ and hence to deduce from the results of §3 estimates for the solutions of (1.3) when $\lambda = \lambda_{j,k}$ and $\mu = \mu_{j,k}$. We remark that the $\Omega_i^*$ have been introduced since there are essentially five different forms of solutions of (1.3) to consider, depending upon the position of the transition points, and hence upon the $\eta_{j,k}$. Thus by restricting $(j, k)$ to a particular $\Omega_i^*$, we are able to restrict the $\eta_{j,k}$, and hence the corresponding transition points, to intervals for which a common formula may be derived to describe the solutions of (1.3) when $\lambda = \lambda_{j,k}$, $\mu = \mu_{j,k}$, and $(j, k) \in \Omega_i^*$. In §5 we employ the results of §§3 and 4 together with estimates for the solutions of (1.1) (which are obtained by standard arguments) to derive asymptotic formulae for $\lambda_{j,k}$, $\mu_{j,k}$, and $\psi_{j,k}$ for $(j, k)$ in each of the $\Omega_i^*$ (see Theorems 5.1–5.5). These results are then utilized in §6 to deduce some facts concerning the uniform boundedness of the $\psi_{j,k}$ and the dependence of the $\lambda_{j,k}$ and $\mu_{j,k}$ on the $q_i$; extensions of the foregoing results are also discussed here.

A novel feature of our work is that we are able to demonstrate that smoothness of the coefficients of the system (1.1)–(1.4) does not ensure the uniform boundedness of the $\psi_{j,k}(x_1, x_2)$. Indeed, although we were able to show in [**6**] that the $\psi_{j,k}$ are uniformly bounded when $(j, k)$ lies in the sector $\Omega$ (see above), we shall see in the sequel that this is certainly not the case for $(j, k)$ in $\Omega_1$. In fact, for this latter sector, we shall show instead that the $j^{-1/3}\psi_{j,k}$ are uniformly bounded, and that this result is false if 1/3 is replaced by any smaller number.

**2. Notation and assumptions.** In the sequel we shall put

$$P_1(x_1, \lambda, \mu) = \lambda A_1(x_1) - \mu B_1(x_1), \quad P_2(x_2, \lambda, \mu) = -\lambda A_2(x_2) + \mu B_2(x_2),$$
$$r(x_2) = A_2(x_2)/B_2(x_2),$$

and denote by $b_1$ and $b_2$ the infimum and supremum, respectively, of $A_1(x_1)/B_1(x_1)$ in $0 \leqslant x_1 \leqslant 1$ and by $a_1$ and $a_2$ the infimum and supremum, respectively, of $r(x_2)$ in $0 \leqslant x_2 \leqslant 1$. Moreover, we shall extend the definition of $q_2(x_2)$ to all real values of $x_2$ by putting $q_2(x_2) = q_2(0)$ for $x_2 < 0$ and $q_2(x_2) = q_2(1)$ for $x_2 > 1$. Observe that there exists the interval $[c, d]$, where $-1 < c < 0$ and $1 < d < 2$, such that in this interval $B_2(x_2)$, $r(x_2)$ are positive and of class $C^3$, $q_2(x_2)$ is continuous, $r'(x_2) \neq 0$ ($' = d/dx_2$), and $99a_1/100 < r(x_2) < a_2 + (b_1 - a_2)/100$.

To simplify matters, *we shall assume for the remainder of this paper that* $r'(x_2) < 0$ for $c \leqslant x_2 \leqslant d$ (see §6) and put $\delta = (r(c) - a_2)/2$. Then for $r(d) \leqslant t \leqslant r(c)$, we denote by $x(t)$ the point of $[c, d]$ at which $r(x_2) = t$ (note that in this interval $x(t)$ is of class $C^3$ and $x'(t) < 0$, where $' = d/dt$), put $c^* = x(a_2 + \delta)$, and let $\delta_1 = \min\{(c^* - c), (d - 1)\}$. For $c \leqslant x_2 \leqslant d$, $r(d) \leqslant t \leqslant r(c)$, we shall also put

$$\Psi(x_2, t) = \frac{3}{2} \int_{u=0}^{1} u^{1/2}\left[ B_2((1 - u)x(t) + ux_2) \times \int_{s=0}^{1} -r'((1 - su)x(t) + sux_2)\, ds \right]^{1/2} du,$$

$$\Psi_1(x_2, t) = \int_0^1 -r'((1 - s)x(t) + sx_2)\, ds,$$

$$\Psi_2(x_2, t) = B_2(x_2)\Psi_1(x_2, t),$$

$$\Phi(x_2, t) = 2\Psi(x_2, t)/3\Psi_1^{3/2}(x_2, t),$$

$$\Phi_1(x_2, t) = \Psi_2^{1/2}(x_2, t)\Psi^{-1/3}(x_2, t),$$

$$\Phi_2(x_2, t) = (3\Phi(x_2, t)/2)^{2/3},$$

where all fractional powers have their positive values (observe that these functions are all positive and of class $C^2$ in the given rectangle), and denote by $N_1$ and $N_2$ the infimum and supremum, respectively, of $\Phi(x_2, t)$ in this rectangle.

Finally, we shall hereinafter consider the differential equation (1.3) as being defined in the interval $[c, d]$.

**3. Asymptotic integration of (1.3).**

3.0. *Introduction*. Throughout this section we suppose that $\lambda > 0$, put $t = \mu/\lambda$, and assume that $t$ is held fixed at a value satisfying $a_1 < t < a_2 + \delta$. Then in order to obtain asymptotic formulae for the eigenvalues and eigenfunctions of the system (1.1)–(1.4), it is essential that we derive estimates for the solutions of (1.3) in $[c, d]$ for large values of $\lambda$ under the given restriction on $t$. Accordingly, it is to this topic that this section is devoted and the method which we shall employ in achieving our ends will be to approximate (1.3) by a certain related equation which may be solved explicitly with the use of Bessel functions (see subsections 3.2–3.3). However, before passing to these approximations, we shall firstly concern ourselves in subsection 3.1 with introducing certain auxiliary variables which will play an important role in the analysis to follow. In subsection 3.4 we derive some results concerning the zeros of solutions of (1.3).

*Notation.* (1) Throughout this section we shall write $x^*$ for $x(t)$ (see §2). Observe that $c^* < x^* < 1$.

(2) For $a$ any point of $[c, d]$, let $u(x_2, a)$ and $v(x_2, a)$ denote the solutions of (1.3) in $[c, d]$ satisfying $u(a, a) = 1$, $u'(a, a) = 0$ and $v(a, a) = 0$, $v'(a, a) = 1$, where $' = d/dx_2$.

(3) For $a$ any complex number, let $U(z, a)$ and $V(z, a)$ denote the solutions of the equation

$$y'' + zy = 0, \qquad ' = d/dz, \tag{3.1}$$

satisfying $U(a, a) = 1$, $U'(a, a) = 0$ and $V(a, a) = 0$, $V'(a, a) = 1$.

(4) Let $T(z)$, $S(z)$ denote the solutions of (3.1) defined by

$$T(z) = [\kappa^* U(z, 0) + V(z, 0)], \quad S(z) = 3^{1/2}[\kappa^* U(z, 0) - V(z, 0)],$$

where $\kappa^* = \Gamma(1/3)/3^{1/3}\Gamma(2/3)$. Observe that $W(T, S)(z) = -2 \times 3^{1/2}\kappa^*$, where $W$ denotes the Wronskian.

(5) For $z$ real let

$$\rho(z) = \left(T^2(z) + S^2(z)\right)^{1/2}, \quad \rho^*(z) = \left((T'(z))^2 + (S'(z))^2\right)^{1/2},$$

where $' = d/dz$ and all fractional powers have their positive values.

(6) Let $\gamma(z)$ and $\gamma^*(z)$ be defined for all real values of $z$ by the conditions that

$$\cos\gamma(z) = T(z)/\rho(z), \quad \sin\gamma(z) = S(z)/\rho(z)$$

and

$$\cos\gamma^*(z) = T'(z)/\rho^*(z), \quad \sin\gamma^*(z) = S'(z)/\rho^*(z),$$

with the indeterminate multiple of $2\pi$ in these definitions being fixed by choosing $\gamma(0) = \pi/3$, $\gamma^*(0) = -\pi/3$ and requiring that both $\gamma(z)$ and $\gamma^*(z)$ be continuous functions of $z$.

(7) Let $\kappa = (1/2 \times 3^{1/2}\kappa^*)^{1/2}$.

Finally, for the remainder of this section we shall drop subscripts and write $x$ for $x_2$, $B$ for $B_2$, $q$ for $q_2$, and $P$ for $P_2$ in $c < x < d$.

3.1. *Auxiliary variables.* Let $r_1(x) = (r(x) - r(x^*))/(x - x^*)$ if $x \neq x^*$, $r_1(x^*) = r'(x^*)$, where $' = d/dx$ (see §2). It is a simple matter to verify that $r_1(x)$ is of class $C^2$ in $[c, d]$ and that in this interval $|r_1(x)|$, $|1/r_1(x)|$, $|r_1'(x)|$, and $|r_1''(x)|$ all remain less than some bound independent of $x$, $t$, and $\lambda$. Writing $P(x)$ for $P(x, \lambda, \mu)$ in $c < x < d$, it now follows that

$$P(x) = \lambda\chi^2(x)(x - x^*), \tag{3.2}$$

where $\chi^2(x) = B(x)|r_1(x)|$. Then in order to deal with fractional powers of $P$ and of other functions that will appear below, let us henceforth agree to the following convention: we shall take the argument of a positive quantity to be zero, take arg $e^{i\nu}$ ($\nu$ real) to be $\nu$, take the argument of a complex number which is a product of factors whose arguments have been specified to be the sum of the arguments of its factors, and lastly, interpret the expression $a^\nu$ ($a \neq 0$), where arg $a$ has been specified and $\nu$ is real, in accordance with the rule $a^\nu = |a|^\nu \exp\{i\nu \arg a\}$. This convention enables us to determine arg $\lambda$, arg $\chi^2(x)$, and $\arg(x - x^*)$ for $x > x^*$, and if for $x < x^*$ we agree to take $\arg(x - x^*)$ to be $\pi$, then it is clear that the expressions

$$P^{1/2}(x) = \lambda^{1/2}\chi(x)(x - x^*)^{1/2}$$

and

$$w(x) = \int_{x^*}^{x} P^{1/2}(s)\, ds \tag{3.3}$$

are unambiguously defined in $[c, d]$. Furthermore, it is not difficult to verify that $w(x)$ may be expressed in the form

$$w(x) = (2/3)\lambda^{1/2}\chi_1(x)(x - x^*)^{3/2}, \tag{3.4}$$

where $\chi_1(x)$ is real, positive, and of class $C^2$ in $[c, d]$, and that in this interval $\chi_1(x)$, $1/\chi_1(x)$, $|\chi_1'(x)|$, and $|\chi_1''(x)|$, $' = d/dx$, all remain less than some bound, say $M$, independent of $x$, $t$, and $\lambda$.

Let us next introduce the function $z$ defined by

$$z(x) = \lambda^{1/3}\chi_1^{2/3}(x)(x - x^*) \tag{3.5}$$

for $c \leqslant x \leqslant d$. Then it follows from (3.4) that $w(x) = (2/3)z^{3/2}(x)$, and hence we conclude from (3.3) that

$$z^{1/2}(x)z'(x) = P^{1/2}(x) \tag{3.6}$$

for $c \leqslant x \leqslant d$, where $' = d/dx$. From (3.2) and (3.5)–(3.6) we now have

$$z'(x) = \lambda^{1/3}B^{1/2}(x)|r_1(x)|^{1/2}\chi_1^{-1/3}(x),$$

and hence $z'(x) > 0$ in $[c, d]$. Moreover, it is clear that in this inteval $\lambda^{-1/3}z'(x)$, $1/\lambda^{-1/3}z'(x)$, $|z''(x)/z'(x)|$, and $|\{z, x\}|$ all remain less than some bound independent of $x$, $t$, and $\lambda$, where $\{z, x\}$ denotes the Schwarzian derivative of $z$ with respect to $x$.

Observe that as $x$ runs from $c$ to $d$, $w(x)$ traces out a curve in the $w$-plane such that $\arg w(x) = 3\pi/2$ in $c \leqslant x < x^*$, and as $x$ runs from $c$ to $x^*$, $|w(x)|$ strictly decreases from a value exceeding $2\lambda^{1/2}M^\#/3$ (where $M^\# = \delta_1^{3/2}/M < 1$ and $\delta_1$ is defined in §2) when $x = c$ to zero when $x = x^*$, while in the interval $x^* < x \leqslant d$, $\arg w(x) = 0$, and as $x$ runs from $x^*$ to $d$, $|w(x)|$ strictly increases from zero to a value exceeding $2\lambda^{1/2}M^\#/3$ when $x = d$. Analogous results also hold for the path traced out by $z(x)$ as $x$ runs from $c$ to $d$.

3.2. *Estimates*. We are now going to use the results of subsection 3.1 to derive estimates for the solutions of (1.3) in $[c, d]$. Accordingly, *let* $x_1$ *be any point of* $[c, d]$. Then recalling the definition of $v(x, x_1)$ (as well as the other expressions) given in subsection 3.0, we may argue in a manner similar to that in [**4**, pp. 4–5] to verify that $v(x, x_1)$ satisfies the integral equation

$$(z')^{1/2}y(x) = (z_1')^{-1/2}V(z, z_1) + \int_{x_1}^{x} V(z, \zeta)(\zeta')^{-1/2}Q(s)y(s)\,ds, \quad (3.7)$$

$c \leqslant x \leqslant d$, where here $z = z(x)$, $z_1 = z(x_1)$, $\zeta = z(s)$, $z' = z'(x)$, $z_1' = z'(x_1)$, $\zeta' = z'(s)$, $Q(s) = (-q(s) + \{z, s\}/2)$, and $\{z, s\}$ denotes the Schwarzian derivative of $z$ with respect to $s$. *Assuming henceforth that* $x_2$ *is any point of* $[c, d]$ *satisfying* $x_2 \geqslant x_1$, and that $\lambda > (2 \times 10^3/M^{\#})$ (hence it follows from subsection 3.1 that both $|w(c)|$ and $w(d)$ exceed $10^3$), it is our intention to utilize (3.7) to arrive at an estimate for $v(x_2, x_1)$. In order to achieve this end, it is essential that we obtain approximations for $V(z_2, z_1)$ ($z_2 = z(x_2)$) for the various positions of $z_1$, $z_2$, and accordingly, we shall now pass to such approximations.

Let $w_j = w(x_j)$, $\rho_j = \rho(z_j)$, and $\gamma_j = \gamma(z_j)$ for $j = 1, 2$, $\nu = 5/72$, and $\nu^\dagger = 7/72$. Then from the formulae

$$V(z_2, z_1) = U(z_1, 0)V(z_2, 0) - V(z_1, 0)U(z_2, 0),$$
$$V'(z_2, z_1) = U(z_1, 0)V'(z_2, 0) - V(z_1, 0)U'(z_2, 0),$$

from the formulae given in [**4**, p. 82] expressing $U(z, 0)$, $V(z, 0)$ and their derivatives in terms of Bessel functions, and from the results given in [**10**. pp. 199, 202], it is easy to see that:

(1) if $|w(x)| \geqslant 1$ in $x_1 \leqslant x \leqslant x_2$, then

$$\begin{aligned} V(z_2, z_1) = z_1^{-1/4}z_2^{-1/4}\Big[&\sin W_{12}\{1 + O(|w|^{-2})\} \\ &+ \cos W_{12}\{\nu w_1^{-1} - \nu w_2^{-1} + O(|w|^{-3})\}\Big], \\ V'(z_2, z_1) = z_1^{-1/4}z_2^{1/4}\Big[&\cos W_{12}\{1 + O(|w|^{-2})\} \\ &- \sin W_{12}\{\nu w_1^{-1} + \nu^\dagger w_2^{-1} + O(|w|^{-3})\}\Big], \end{aligned} \quad (3.8)$$

where $W_{12} = w_2 - w_1$ and $|w| = \min\{|w_1|, |w_2|\}$;

(2) if $w_2 \geqslant 1$, then

$$\begin{aligned} V(z_2, z_1) = \kappa\rho_1 z_2^{-1/4}\Big[&\sin W_{12}^*\{1 + O(w_2^{-2})\} \\ &- \cos W_{12}^*\{\nu w_2^{-1} + O(w_2^{-3})\}\Big], \\ V'(z_2, z_1) = \kappa\rho_1 z_2^{1/4}\Big[&\cos W_{12}^*\{1 + O(w_2^{-2})\} \\ &- \sin W_{12}^*\{\nu^\dagger w_2^{-1} + O(w_2^{-3})\}\Big], \end{aligned} \quad (3.9)$$

where $W_{12}^* = w_2 + \gamma_1 - \pi/4$;

(3) if $|w_1| \geqslant 1$ and $\arg w_1 = 3\pi/2$, then

$$V(z_2, z_1) = C(x_1)\Big[T(z_2)\big\{1 - i\nu w_1^{-1} + O\big(|w_1|^{-2}\big)\big\} + S(z_2)\big\{O\big(|w_1|^{-3}\big)\big\}\Big],$$

$$V'(z_2, z_1) = C(x_1)\Big[T'(z_2)\big\{1 - i\nu w_1^{-1} + O\big(|w_1|^{-2}\big)\big\} + S'(z_2)\big\{O\big(|w_1|^{-3}\big)\big\}\Big], \tag{3.10}$$

where $C(x_1) = \kappa z_1^{-1/4}\exp\{i(w_1 + \pi/4)\}$ (note that in these equations the coefficients of $S(z_2)$ and $S'(z_2)$ are also $O(\exp\{-2|w_1|\})$).

Moreover, if $w_2 \geqslant 1$, then in (3.10) we may put

$$\kappa T(z_2) = z_2^{-1/4}\Big[\sin W_2\big\{1 + \alpha w_2^{-2} + O\big(w_2^{-4}\big)\big\} - \cos W_2\big\{\nu w_2^{-1} + \beta w_2^{-3} + O\big(w_2^{-5}\big)\big\}\Big],$$

$$\kappa S(z_2) = z_2^{-1/4}\Big[\cos W_2\big\{1 + O\big(w_2^{-2}\big)\big\} + \sin W_2\big\{\nu w_2^{-1} + O\big(w_2^{-3}\big)\big\}\Big],$$

$$\kappa T'(z_2) = z_2^{1/4}\Big[\cos W_2\big\{1 + \alpha^\dagger w_2^{-2} + O\big(w_2^{-4}\big)\big\} - \sin W_2\big\{\nu^\dagger w_2^{-1} + \beta^\dagger w_2^{-3} + O\big(w_2^{-5}\big)\big\}\Big],$$

$$\kappa S'(z_2) = -z_2^{1/4}\Big[\sin W_2\big\{1 + O\big(w_2^{-2}\big)\big\} + \cos W_2\big\{\nu^\dagger w_2^{-1} + O\big(w_2^{-3}\big)\big\}\Big], \tag{3.11}$$

where

$$W_2 = w_2 + \pi/4, \quad \alpha = p_1(1/3), \quad \beta = -p_2(1/3),$$

$$\alpha^\dagger = p_1(2/3), \quad \beta^\dagger = p_2(2/3),$$

$$p_1(s) = -(4s^2 - 1)(4s^2 - 9)/128, \quad p_2(s) = (4s^2 - 25)p_1(s)/24,$$

while if $|w_2| \geqslant 1$ and $\arg w_2 = 3\pi/2$, then in (3.10) we may put

$$T(z_2) = 2^{-1}C_1(x_2)\big\{1 + O\big(|w_2|^{-1}\big)\big\},$$

$$S(z_2) = C_2(x_2)\big\{1 + O\big(|w_2|^{-1}\big)\big\},$$

$$T'(z_2) = 2^{-1}C_3(x_2)\big\{1 + O\big(|w_2|^{-1}\big)\big\},$$

$$S'(z_2) = -C_4(x_2)\big\{1 + O\big(|w_2|^{-1}\big)\big\}, \tag{3.12}$$

where $\kappa C_j(x_2)$ equals $z_2^{-1/4}\exp\{i((-1)^j w_2 + \pi/4)\}$ for $j = 1, 2$ and $z_2^{1/4}\exp\{i((-1)^j w_2 - \pi/4)\}$ for $j = 3, 4$. Finally, we remark that in all of the above formulae (as well as in the formulae to follow) the constant implied in any one of the $O$ symbols is *independent* of $x_1$, $x_2$, $t$, and $\lambda$.

Let $x_1'$ and $x_2'$ ($> x_1'$) denote the points of $[c, d]$ at which $|w(x)| = 1$ and put $P_j = P(x_j)$ for $j = 1, 2$. Then in light of the foregoing results we may now argue with (3.7) and the Gronwall lemma to show that

$$v(x_2, x_1) = P_1^{-1/4}P_2^{-1/4}\left[z_1^{1/4}z_2^{1/4}V(z_2, z_1) + O\left(\lambda^{-1/2}E_{12}\right)\right], \quad (3.13)$$

and similarly we can show that

$$v'(x_2, x_1) = P_1^{-1/4}P_2^{1/4}\left[z_1^{1/4}z_2^{-1/4}V'(z_2, z_1) + O\left(\lambda^{-1/3}E_{12}\right)\right], \quad (3.14)$$

where $E_{12} = \exp\{\mathrm{Im}(w_2 - w_1)\}$, and it is to be understood that in (3.13)–(3.14) we are to replace $P_j^{\pm 1/4}$ by $(z_j')^{\pm 1/2}$ ($z_j' = z'(x_j)$) and $z_j^{\pm 1/4}$ by 1 when $x_1' < x_j < x_2'$ for $j = 1, 2$. Analogous results for $u$ and $u'$ can also be obtained in a similar manner.

3.3. *Further developments*. Due to future requirements we are now going to develop further the formulae (3.13)–(3.14). Accordingly, let $K$ be a number not less than 1 chosen large enough so that the constant implied in any one of the $O$ symbols appearing in (3.8)–(3.12) and in the formulae for $U$ and $U'$ analogous to (3.8)–(3.10) does not exceed $K$. Choose $R > 10^2K$ and let $\alpha$ be an arbitrary, but fixed positive number. Assuming henceforth that $\lambda > \max\{(R/\alpha)^{16/3}, (10^4(\alpha + R)/M^{\#})^4\}$ (in which case it follows from a simple argument involving (3.3) that $w(d) - w(1)$, $|w(c) - w(c^*)|$, and hence $w(d)$ and $|w(c)|$ all exceed $10^3(\alpha\lambda^{3/16} + R)$), let $x_1^{\#}$, $x_2^{\#}$ ($> x_1^{\#}$) denote the points of $[c, d]$ at which $|w(x)| = R$ and $x_3^{\#}$, $x_4^{\#}$ ($> x_3^{\#}$) the points of $[c, d]$ at which $|w(x)| = \alpha\lambda^{3/16}$. Then standard arguments involving the use of (3.7)–(3.12) and techniques similar to those used in [4, pp. 6–11], [6, Theorem 3.4] show that:

(1) when $x_1 \geqslant x_4^{\#}$ or when $x_2 \leqslant x_3^{\#}$,

$$v(x_2, x_1) = P_1^{-1/4}P_2^{-1/4}\Big[\sin\theta_{12}\left\{1 + O(|w|^{-2}) + o(\lambda^{-1/2})\right\}$$
$$+ \cos\theta_{12}\left\{\nu w_1^{-1} - \nu w_2^{-1} - \theta_{12}^* + o(\lambda^{-1/2})\right\}\Big], \quad (3.15)$$

$$v'(x_2, x_1) = P_1^{-1/4}P_2^{1/4}\Big[\cos\theta_{12}\left\{1 + O(\lambda^{-3/8})\right\}$$
$$+ \sin\theta_{12}\left\{O(\lambda^{-3/16})\right\}\Big], \quad (3.16)$$

where $\nu$ and $|w|$ are as in (3.8),

$$\theta_{12} = \theta(x_2, x_1), \quad \theta_{12}^* = \theta^*(x_2, x_1), \quad \theta(s_2, s_1) = \int_{s_1}^{s_2} P^{1/2}(s)\, ds,$$

and

$$\theta^*(s_2, s_1) = 2^{-1}\int_{s_1}^{s_2} Q(s)P^{-1/2}(s)\, ds$$

(see (3.7));

(2) when $x_1 > x_3^{\#}$ and $x_2 \geqslant x_4^{\#}$,

$$v(x_2, x_1) = \kappa\rho_1(z_1')^{-1/2}P_2^{-1/4}\Big[\sin W_{12}^*\{1 + O(w_2^{-2}) + o(\lambda^{-1/2})\} - \cos W_{12}^*\{\nu w_2^{-1} + \theta^*(x_2, \xi) + o(\lambda^{-1/2})\}\Big], \tag{3.17}$$

where $W_{12}^*$ is defined above (see (3.9)) and $\xi = \max\{x_1, x^*\}$;

(3) when $x_3^{\#} < x_1 \leqslant x_1^{\#}$ and $x_2 \leqslant x^*$,

$$v(x_2, x_1) = \kappa^2 P_1^{-1/4}(z_2')^{-1/2}z_1^{1/4}S(z_1) \times\big[T(z_2)\{1 + o(\lambda^{1/2})\} - S(z_2)T(z_1)/S(z_1)\big]; \tag{3.18}$$

(4) when $x_1 \leqslant x_3^{\#}$ and $E_{12}^* = \kappa|P_1|^{-1/4}(z_2')^{-1/2}\exp\{|w_1|\}$,

$$\begin{aligned} v(x_2, x_1) &= E_{12}^*\Big[T(z_2)\{1 - i\nu w_1^{-1} + i\theta_{12}^* + O(|w_1|^{-2}) + o(\lambda^{-1/2})\} \\ &\qquad + S(z_2)\{O(\exp\{-2|w_1|\})\}\Big] \qquad \text{if } x_2 \leqslant x^*, \\ &= E_{12}^*\Big[T(z_2)\{1 - i\nu w_1^{-1} + i\theta^*(x^*, x_1) + O(|w_1|^{-2}) + o(\lambda^{-1/2})\} \\ &\qquad - S(z_2)\{\theta^*(x_2, x^*) + |w_1|^{-3} + h(z_2)\}\Big] \qquad \text{if } x_2 > x^*, \end{aligned} \tag{3.19}$$

where $h(z_2)$ is $O(\lambda^{-2/3})$ for $x_2 < x_2^{\#}$, $(x_2 - x^*)^{1/2}O(\lambda^{-1/2})$ for $x_2^{\#} \leqslant x_2 < x_4^{\#}$, and $O(\lambda^{-29/48}) + (x_2 - x^*)^{1/2}o(\lambda^{-1/2})$ for $x_2 \geqslant x_4^{\#}$;

(5) when $x_1 \leqslant x_3^{\#}$, $x_2 \geqslant x_4^{\#}$, and $E_{12}^{\dagger} = \kappa|P_1|^{-1/4}P_2^{1/4}\exp\{|w_1|\}$,

$$\begin{aligned} v'(x_2, x_1) = E_{12}^{\dagger}\Big[& z_2^{-1/4}T'(z_2)\{1 - i\nu w_1^{-1} + i\theta^*(x^*, x_1) \\ &+ O(|w_1|^{-2}) + o(\lambda^{-1/2})\} \\ &- z_2^{-1/4}S'(z_2)\{\theta^*(x_2, x^*) + o(\lambda^{-1/2})\} \\ &+ z_2^{1/4}T(z_2)\{o(\lambda^{-3/8})\} + z_2^{1/4}S(z_2)\{o(\lambda^{-1/2})\}\Big], \end{aligned} \tag{3.20}$$

as $\lambda \to \infty$, *with each of these formulae holding uniformly in* $x_1$, $x_2$, *and* $t$. Finally, we remark that further results for $v'$ as well as analogous results for $u$ and $u'$ can also be obtained in a similar fashion.

3.4. *Zeros*. Guided by future requirements we are now going to derive some facts concerning the zeros of $v(x_2, x_1)$. Accordingly, we first note from [7, pp. 98–99] that the zeros of $T(z)$ and $T'(z)$ all lie on the positive real axis, and if we denote these zeros by $\tau_n$ and $\tau_n'$, respectively, $n \geqslant 1$, arranged in increasing order of magnitude, then $0 < \tau_1' < \tau_1 < \tau_2' < \tau_2 < \dots$, and

$$2\tau_n^{3/2}/3 = (n - 1/4)\pi + O(n^{-1}), \quad 2(\tau_n')^{3/2}/3 = (n - 3/4)\pi + O(n^{-1})$$

as $n \to \infty$. It can also be verified that the real zeros of $S(z)$ and $S'(z)$ are all positive and if we denote these zeros by $s_n$ and $s_n'$, respectively, $n \geqslant 1$, arranged in increasing order of magnitude, then $0 < s_1 < s_1' < s_2 < s_2' < \dots, s_1 < \tau_1 < s_2 < \tau_2 < \dots, \tau_1' < s_1' < \tau_2' < s_2' < \dots$, and

$$2s_n^{3/2}/3 = (n - 3/4)\pi + O(n^{-1}), \quad 2(s_n')^{3/2}/3 = (n - 1/4)\pi + O(n^{-1})$$

as $n \to \infty$. By arguing with these results, the definition of $\gamma(z)$, and (3.12), we may now show that when $z$ runs from $-\infty$ to $\infty$, $\gamma(z)$ strictly decreases from $\pi/2$ to $-\infty$, taking on the value $-(n-1)\pi$ at $z = s_n$ and $-(2n-1)\pi/2$ at $z = \tau_n$ for $n \geqslant 1$. Moreover, $d\gamma(z)/dz = -(1/\kappa\rho(z))^2$. Similarly, we can show that $\gamma^*(z)$ strictly increases from $-\pi/2$ to $-\pi/3$ when $z$ runs from $-\infty$ to 0 and strictly decreases from $-\pi/3$ to $-\infty$ when $z$ runs from 0 to $\infty$, taking on the value $-n\pi$ at $z = s_n'$ and $-(2n-1)\pi/2$ at $z = \tau_n'$ for $n \geqslant 1$. We also have $d\gamma^*(z)/dz = -z/(\kappa\rho^*(z))^2$. Since $\gamma(z) - \gamma^*(z)$ is never equal to a multiple of $\pi$ and $\gamma(\tau_1) = -\pi/2$, $-3\pi/2 < \gamma^*(\tau_1) < -\pi/2$, it therefore follows that $0 < \gamma(z) - \gamma^*(z) < \pi$ for all real $z$.

Next let $m$ denote the smallest value of $n$ for which $\tau_n' \geqslant R^*$ $(= (3R/2)^{2/3}$, where $R$ is defined at the beginning of subsection 3.3). Then it follows from the foregoing results and (3.11) that $m > 15$ and

$$p(n - 1/2) < \tau_n < p(n), \quad p(n-1) < \tau_n' < p(n - 1/2) \quad \text{for } n \geqslant m,$$

where $p(s) = (3\pi s/2)^{2/3}$. Observe also from (3.11) that

$$z^{1/4}|T(z)| > 1/2\kappa \quad \text{in } p(n-1) \leqslant z \leqslant p(n - 1/2)$$

and

$$z^{-1/4}|T'(z)| > 1/2\kappa \quad \text{in } p(n - 1/2) \leqslant z \leqslant p(n) \text{ for } n \geqslant m.$$

Now put

$$z_0^\dagger = 0, \quad z_1^\dagger = (\tau_1' + \tau_1)/2, \quad z_{2n}^\dagger = (\tau_{n+1}' + \tau_n)/2,$$

$$z_{2n+1}^\dagger = (\tau_{n+1}' + \tau_{n+1})/2 \quad \text{for } n = 1, \dots, (m-2),$$

$$z_{2n}^\dagger = p(n) \quad \text{and} \quad z_{2n+1}^\dagger = p(n + 1/2) \quad \text{for } n \geqslant (m-1).$$

Then there exists the positive constant $\beta$ such that $|T(z)| \geqslant \beta$ (resp. $|T'(z)| \geqslant \beta$) in each of the intervals $z_{2n}^\dagger \leqslant z \leqslant z_{2n+1}^\dagger$ (resp. $z_{2n+1}^\dagger \leqslant z \leqslant z_{2n+2}^\dagger$) for $n = 0, \dots, (m-2)$. Note from (3.11) that if $\zeta(z) = 1$ for $0 \leqslant z < p(m-1)$ and $\zeta(z) = z$ for $z \geqslant p(m-1)$, then the absolute values of $\zeta^{1/4}(z)T(z)$, $\zeta^{1/4}(z)S(z)$, $\zeta^{-1/4}(z)T'(z)$, and $\zeta^{-1/4}(z)S'(z)$ in the interval $0 \leqslant z < \infty$ all remain less than some bound independent of $z$.

(a) *We shall now employ the above results to deduce some facts concerning the zeros of* $v(x_2, x_1)$ *when* $x_1$ *is held fixed at a value satisfying* $x_1 \leqslant x_3^{\#}$ (see subsection 3.3). Indeed, by arguing with (3.8) and (3.14) for the interval $x_1 \leqslant x_2 \leqslant x_1^{\#}$ and with (3.12), (3.14), and the formula

$$V'(z_2, z_1) = \kappa^2 \rho_1 \rho_2^* \sin(\gamma_1 - \gamma_2^*)$$

for the interval $x_1^{\#} < x_2 \leqslant x^*$ (where $\rho_j^* = \rho^*(z_j)$, $\gamma_j^* = \gamma^*(z_j)$ for $j = 1, 2$), it is easy to see that in $[x_1, x^*]$, $v(x_2, x_1)$ can only vanish when $x_2 = x_1$ if $\lambda$ is sufficiently large (that is, if $\lambda$ exceeds a certain fixed positive number which of course does not depend upon $x_1$ or $t$). Turning to the case where $x_2 > x^*$, let $n^\dagger$ denote the largest value of $n$ for which $n\pi \leqslant w(d)$ and $x_n^\dagger$, $n = 0, \ldots, 2n^\dagger$, the points of $[c, d]$ satisfying $z(x_n^\dagger) = z_n^\dagger$ (it is not difficult to verify that $n^\dagger - m > 10^4$ and $x_{2n^\dagger}^\dagger > 1$). Then assuming $\lambda$ sufficiently large, we may argue with (3.10) and (3.13)–(3.14) to verify that $v(x_2, x_1)$ vanishes, but $v'(x_2, x_1)$ does not vanish, in each of the intervals $(x_{2n-1}^\dagger, x_{2n}^\dagger)$, $n = 1, \ldots, n^\dagger$, while in each of the intervals $[x_{2n}^\dagger, x_{2n+1}^\dagger]$, $n = 0, \ldots, (n^\dagger - 1)$, $v(x_2, x_1) \neq 0$. *Hence if* $\lambda$ *is sufficiently large, if* $x^* < x' \leqslant x_{2n^\dagger}^\dagger$, *and if* $v(x_2, x_1)$ *has precisely* $k$ *zeros in* $(x_1, x')$ *and vanishes at* $x_2 = x'$, *then* $x_{2k+1}^\dagger < x' < x_{2k+2}^\dagger$, $z_{2k+1}^\dagger < z(x') < z_{2k+2}^\dagger$, *and when* $k \geqslant m - 1$, $(k + 1/2)\pi < w(x') < (k + 1)\pi$.

(b) *We are now going to establish some facts concerning the zeros of* $v(x_2, x_1)$ *when* $x_1$ *is held fixed at a value satisfying* $x_3^{\#} < x_1 < x_4^{\#}$; and to this end we shall firstly investigate the behaviour of $V(z, z_1)$ and $V'(z, z_1)$ in the interval $I = \{z | z_1 \leqslant z < \infty\}$. Accordingly, put $\tau_0 = -\infty$ and suppose that for some $j \geqslant 0$ either (i) $\tau_j < z_1 \leqslant s_{j+1}$, or (ii) $s_{j+1} < z_1 \leqslant \tau_{j+1}$. Note that for case (i), $-j\pi \leqslant \gamma_1 < -(j - 1/2)\pi$, while for case (ii), $-(j + 1/2)\pi \leqslant \gamma_1 < -j\pi$; and a simple calculation also shows that the absolute value of $\gamma_1 \lambda^{-3/16}$ in $x_3^{\#} < x_1 < x_4^{\#}$ remains less than some bound independent of $x_1$ and $\lambda$. If we now express $V$, $V'$ in the form

$$V(z, z_1) = \kappa^2 \rho_1 \rho(z) \sin(\gamma_1 - \gamma(z)),$$
$$V'(z, z_1) = \kappa^2 \rho_1 \rho^*(z) \sin(\gamma_1 - \gamma^*(z)), \tag{3.21}$$

and denote by $t_n$, $n \geqslant 0$ (resp. $t_n'$, $n \geqslant 1$), the points of $I$ at which $\gamma(z) = \gamma_1 - n\pi$ (resp. $\gamma^*(z) = \gamma_1 - n\pi$), then it follows that $V(z, z_1)$ (resp. $V'(z, z_1)$) has a simple zero at each of the points $t_n$ (resp. $t_n'$) and vanishes nowhere else in $I$, and that $z_1 = t_0 < t_1' < t_1 < t_2' < t_2 < \ldots$, $t_1' > 0$, $\tau_{j+n} < t_n \leqslant s_{j+n+1}$ for $n \geqslant 0$ if $\tau_j < z_1 \leqslant s_{j+1}$, and $s_{j+n+1} < t_n \leqslant \tau_{j+n+1}$ for $n \geqslant 0$ if $s_{j+1} < z_1 \leqslant \tau_{j+1}$. Since it is clear from the definition of $m$ above that

$$p(n-1) < s_n < p(n - 1/2) < \tau_n < p(n) \quad \text{for } n \geqslant m - 15,$$

it therefore follows that if $m^\dagger$ denotes the smallest value of $n$ for which

$t'_n \geqslant R^*$ and $m^* = (m^\dagger - \min\{m^\dagger, 10\})$, then

$$j + m^\dagger \geqslant m - 2, \qquad m^\dagger < (3 + R/\pi),$$
$$p(j + n - 1/2) < t_n < p(j + n + 1/2) \quad \text{if } \tau_j < z_1 \leqslant s_{j+1},$$
$$p(j + n) < t_n < p(j + n + 1) \quad \text{if } s_{j+1} < z_1 \leqslant \tau_{j+1},$$

and hence

$$(n - 3/4)\pi < \big((2/3)t_n^{3/2} + \gamma_1 - \pi/4\big) < (n + 3/4)\pi \quad \text{for } n \geqslant m^*.$$

Since $((m^* - 1)\pi - \gamma_1 + \pi/4) > (m - 14)\pi > R/2$, it follows from (3.9) and the interlacing of the $t_n$ and $t'_n$ that

$$p^*(n - 1/4) < t_n < p^*(n + 1/4) \quad \text{for } n \geqslant m^*,$$
$$p^*(n - 3/4) < t'_n < p^*(n - 1/4) \quad \text{for } n \geqslant m^* + 1,$$

where $p^*(s) = [3((s + 1/4)\pi - \gamma_1)/2]^{2/3}$. Observe also from (3.9) that

$$z^{1/4}|V(z, z_1)| > \kappa\rho_1/2 \quad \text{in } p^*(n - 3/4) \leqslant z \leqslant p^*(n - 1/4) \quad \text{for } n \geqslant m^* + 1$$

and

$$z^{-1/4}|V'(z, z_1)| > \kappa\rho_1/2 \quad \text{in } p^*(n - 1/4) \leqslant z \leqslant p^*(n + 1/4) \text{ for } n \geqslant m^*.$$

Lastly, let $n^*$ denote the largest value of $n$ for which $((n + 1/2)\pi - \gamma_1) \leqslant w(d)$ and $x(n^*)$ the point of $[c, d]$ satisfying $z(x(n^*)) = p^*(n^* + 1/4)$ (it is not difficult to verify that $n^* - m^\dagger > 10^4$ and $x(n^*) > 1$). Then by appealing to the above properties of $\gamma(z)$ and $\gamma^*(z)$, to (3.21), and proceeding in a manner similar to that in part (a) above, it is not difficult to verify that *if* $\lambda$ *is sufficiently large, if* $x_1 < x' \leqslant x(n^*)$, *if* $v(x_2, x_1)$ *has precisely* $k$ *zeros in* $(x_1, x')$ *and vanishes at* $x_2 = x'$, *and if* $k \geqslant m^*$, *then*

$$(k + 3/4)\pi < (w(x') + \gamma_1 - \pi/4) < (k + 5/4)\pi.$$

**4. Preliminary results.** We are now going to use the results of §3 to derive some estimates for the eigenvalues of the system (1.1)–(1.4). These estimates will enable us in §5 to establish asymptotic formulae for the eigenvalues and eigenfunctions of the above system. Accordingly, recall from [6, §2] that by an eigenvalue of (1.1)–(1.4) we mean a pair of numbers, $(\lambda^*, \mu^*)$, such that for $\lambda = \lambda^*$ and $\mu = \mu^*$, (1.1) and (1.3) have nontrivial solutions satisfying (1.2) and (1.4), respectively. If $y_1(x_1, \lambda^*, \mu^*)$ and $y_2(x_2, \lambda^*, \mu^*)$ denote these solutions, respectively, then the product, $\prod_{i=1}^2 y_i(x_i, \lambda^*, \mu^*)$, is called an eigenfunction of the system (1.1)–(1.4) corresponding to $(\lambda^*, \mu^*)$. Important results pertaining to the eigenvalues and eigenfunctions of (1.1)–(1.4) were recorded in the reference just cited, and in particular, it was noted that the eigenvalues were all real and could be represented in the form $(\lambda_{j,k}, \mu_{j,k})$, $j, k = 0, 1, 2, \ldots,$ where, with $\phi_i$ $(i = 1, 2)$ denoting the solution of $(1.2i - 1)$

satisfying $\phi_i(0, \lambda, \mu) = \sin \alpha_i$, $\phi_i'(0, \lambda, \mu) = \cos \alpha_i$, $\phi_1(x_1, \lambda_{j,k}, \mu_{j,k})$ has precisely $j$ zeros in $0 < x_1 < 1$ and $\phi_2(x_2, \lambda_{j,k}, \mu_{j,k})$ has precisely $k$ zeros in $0 < x_2 < 1$.

*Notation.* (1) Let

$$h_1(t) = \int_0^1 (A_1(x_1) - tB_1(x_1))^{1/2}\, dx_1 \quad \text{for } -\infty < t \leqslant b_1,$$

$$h_2(t) = \int_0^1 (-A_2(x_2) + tB_2(x_2))^{1/2}\, dx_2 \quad \text{for } a_2 \leqslant t < \infty$$

(see §2 for terminology used here and below; also throughout this section we assume that fractional powers of positive quantities have their positive values).

(2) Let $g(t) = h_2(t)/h_1(t)$ for $a_2 \leqslant t < b_1$, $g(b_1) = h_2(b_1)/h_1(b_1)$ if $h(b_1) \neq 0$, and $g(b_1) = \infty$ otherwise. We remark that besides the properties of $g$ listed in [**6**, Subsection 3.0], we also have, as a consequence of the conditions imposed in this paper upon $A_2$ and $B_2$, the further properties that $g'(t)$ exists and is continuous and positive in $a_2 \leqslant t < b_1$ (here and below $' = d/dt$).

(3) Let $t_1 = a_2 + \delta/2$ and $t_2$ be any number satisfying $t_1 < t_2 < b_1$.

(4) Let $\theta_1^* = \tan^{-1} g(a_2)$ and $\theta_i = \tan^{-1} g(t_i)$ for $i = 1, 2$, where the principal branch of the inverse tangent is taken (observe that $0 < \theta_1^* < \theta_1 < \theta_2 < \pi/2$).

(5) Let $\Omega$ and $\Omega_1$ denote the sectors in the $(x, y)$-plane defined by the inequalities $\theta_1 \leqslant \theta \leqslant \theta_2$ and $0 \leqslant \theta < \theta_1$, respectively, where $\theta$ denotes the angle which a ray emanating from the origin makes with the positive $x$-axis.

(6) Let

$$H(t) = \int_{x(t)}^1 (-A_2(x_2) + tB_2(x_2))^{1/2}\, dx_2,$$

$$G(t) = H(t)/h_1(t) \quad \text{for } a_1 \leqslant t \leqslant a_2 + \delta.$$

It is clear that in this interval $G(t)$ is strictly increasing, $G(a_1) = 0$, $G(a_2) = g(a_2)$, and since $H(t) = (2/3)(1 - x(t))^{3/2}\Psi(1, t)$, we also see that $G(t)$ is of class $C^1$ in $[a_1, a_2 + \delta]$ and of class $C^2$ in $(a_1, a_2 + \delta]$. Furthermore, it is readily verified that $G'(t) > 0$ in $(a_1, a_2 + \delta]$ and $G'(a_2) = g'(a_2)$.

(7) Let $G_1(t) = (3G(t)/2)^{2/3}$ for $a_1 \leqslant t \leqslant a_2 + \delta$. Observe that in this interval $G_1(t)$ is of class $C^2$, $G_1'(t) > 0$, and $G_1(a_1) = 0$.

(8) Put

$$a = g'(a_2) = G'(a_2), \quad a^* = 2(G_1'(a_1))^{3/2}/3,$$

$$b = \left[1 + a(g(t_1) - g(a_2))^{-1} + (a + a^*)/g(a_2)\right]^6,$$

and let $\Omega_i^*$, $1 \leqslant i \leqslant 5$, denote that subset of $\Omega_1$ which is composed of points

$(x, y)$ for which $x > b$ and:

$$\left(\tan \theta_1^* + ax^{-5/12}\right) \leqslant y/x < \tan \theta_1 \quad \text{if } i = 1,$$

$$|\tan \theta_1^* - y/x| < ax^{-5/12} \quad \text{if } i = 2,$$

$$\left(\tan \theta_1^* - ax^{-1/6}\right) < y/x \leqslant \left(\tan \theta_1^* - ax^{-5/12}\right) \quad \text{if } i = 3,$$

$$a^* x^{-5/8} \leqslant y/x \leqslant \left(\tan \theta_1^* - ax^{-1/6}\right) \quad \text{if } i = 4,$$

and lastly, $0 \leqslant y/x < a^* x^{-5/8}$ if $i = 5$.

The sectors $\Omega$ and $\Omega_1$ defined above are precisely the sectors introduced in §1; and as stated there, asymptotic formulae were derived in **[6]** for $\lambda_{j,k}$ and $\mu_{j,k}$ as $j \to \infty$, $(j, k) \in \Omega$. To derive corresponding formulae for $\Omega_1$ (under the conditions assumed in this paper), it is essential that we first obtain estimates for $\lambda_{j,k}$, $\mu_{j,k}$ for $(j, k)$ in this sector. Accordingly, we observe from **[6**, Equation (4.2)] that we already have some information at our disposal, for we know that there exist the positive numbers $M_1$, $M_2$ such that

$$M_1^2 < \lambda_{j,k}/j^2 < M_2^2 \tag{4.1}$$

when $(j, k) \in \Omega_1$ and $j$ is sufficiently large. Putting $\eta_{j,k} = \mu_{j,k}/\lambda_{j,k}$, we may argue precisely as we did in **[8**, Lemma 1] to show also that

LEMMA 4.1. *If $j$ is sufficiently large, then $a_1 < \eta_{j,0} < \eta_{j,1} < \cdots < \eta_{j,k^*} < a_2 + \delta$, where $k^* = k^*(j)$ denotes the greatest integer less than $j$* $\tan \theta_1$.

We now come to our main result, namely,

THEOREM 4.1. *If $j$ is sufficiently large, then*

$$\left(a_2 + 2^{-1}j^{-5/12}\right) < \eta_{j,k} < \left(a_2 + \delta\right) \quad \textit{if } (j, k) \in \Omega_1^*,$$

$$|\eta_{j,k} - a_2| < 2j^{-5/12} \quad \textit{if } (j, k) \in \Omega_2^*,$$

$$\left(a_2 - 2j^{-1/6}\right) < \eta_{j,k} < \left(a_2 - 2^{-1}j^{-5/12}\right) \quad \textit{if } (j, k) \in \Omega_3^*,$$

$$\left(a_1 + 2^{-1}j^{-5/12}\right) < \eta_{j,k} < \left(a_2 - 2^{-1}j^{-1/6}\right) \quad \textit{if } (j, k) \in \Omega_4^*,$$

*and*

$$a_1 < \eta_{j,k} < \left(a_1 + 2j^{-5/12}\right) \quad \textit{if } (j, k) \in \Omega_5^*.$$

PROOF. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Moreover, we shall henceforth suppose that $(j, k) \in \Omega_1$ and that $j > [b + 2\delta^{-1} + 4(a_2 - a_1)^{-1}]^6$ and is large enough to ensure the validity of (4.1) and the assertion of Lemma 4.1. Also, in this proof we shall frequently introduce the order symbol $O$ in relation to functions which depend upon some or all of the variables $x_2, j, k$; and it is to be understood that in each case the constant implied in the $O$ symbol is independent of any of these variables.

We are now going to employ the results of §3 to obtain estimates for $k/j$ for various values of $\eta_{j,k}$, and the assertion of the theorem will follow from these estimates. Accordingly, in that section let us now take $\lambda = \lambda_{j,k}$ and $t = \eta_{j,k}$. Observe from (3.2)–(3.3) that (here and below we use the notation of §§2 and 3)

$$P(x) = \lambda B(x)(t - r(x)) = \lambda(x - x^*)\Psi_2(x, t),$$

$$w(x) = \int_{x^*}^{x} P^{1/2}(s)\, ds = (2/3)\lambda^{1/2}(x - x^*)^{3/2}\Psi(x, t), \tag{4.2}$$

for $c \leqslant x \leqslant d$, where $x^* = x(t)$ and $P^{1/2}$, $(x - x^*)^{3/2}$ are to be interpreted according to the convention given in the statements preceding (3.3). Then with $\beta$ a constant satisfying $1/2 \leqslant \beta \leqslant 2$ and putting

$$e_i = M_i N_i \tag{4.3}$$

for $i = 1, 2$ (see §2), let us firstly suppose that $\eta_{j,k} \geqslant t^\dagger = a_2 + \beta j^{-5/12}$. For this case it is clear from (4.1)–(4.2) that

$$\begin{aligned} w(0) &\geqslant (2/3)\lambda^{1/2}\big(-x(t^\dagger)\big)^{3/2}\Psi(0, t^\dagger) \\ &> M_1 j\big(t^\dagger - a_2\big)^{3/2}\Phi(0, t^\dagger) > e_1 j^{3/8}/4, \end{aligned}$$

and hence it follows from (3.15)–(3.16), (4.1)–(4.2) that as $j \to \infty$ (throughout this proof this limit symbol is to be interpreted as: when $j$ exceeds a certain fixed positive number and $\eta_{j,k}$ satisfies the given supposition),

$$\begin{aligned} \phi_2(x_2) &= P^{-1/4}(0)P^{-1/4}(x_2) \\ &\quad\times\left[\sin\theta(x_2, 0)\left\{1 + O(j^{-3/4})\right\} + \cos\theta(x_2, 0)\left\{O(j^{-3/8})\right\}\right], \\ \phi_2'(x_2) &= P^{-1/4}(0)P^{1/4}(x_2) \\ &\quad\times\left[\cos\theta(x_2, 0)\left\{1 + O(j^{-3/4})\right\} + \sin\theta(x_2, 0)\left\{O(j^{-3/8})\right\}\right], \end{aligned}$$

for $0 \leqslant x_2 \leqslant 1$, where here (and in the sequel) $\phi_2(x_2) = \phi_2(x_2, \lambda_{j,k}, \mu_{j,k})$ and all other expressions are defined in §3. A standard argument now shows that as $j \to \infty$,

$$\int_0^1 P_2^{1/2}(x_2, \lambda_{j,k}, \mu_{j,k})\, dx_2 = (k + 1)\pi + O(j^{-3/8}),$$

while from Lemma 4.1 and [6, Theorem 3.4] we also have

$$\int_0^1 P_1^{1/2}(x_1, \lambda_{j,k}, \mu_{j,k})\, dx_1 = (j + 1)\pi + O(j^{-1}), \tag{4.4}$$

and thus in light of (4.1)–(4.2), it follows that $g(\eta_{j,k}) = k/j + O(j^{-1})$. Since $g(\eta_{j,k}) \geqslant g(t^\dagger)$, we therefore conclude that there exists the positive integer $j_1$

such that $k/j > (g(a_2) + 2^{-1}a\beta j^{-5/12})$ when $\eta_{j,k} > (a_2 + \beta j^{-5/12})$ and $j > j_1$.

Suppose next that $|\eta_{j,k} - a_2| \leqslant \beta j^{-5/12}$. Then as above we can show that $|w(0)| < 4e_2 j^{3/8}$ and hence it follows from (3.17), (4.1)–(4.2) that

$$0 = \phi_2(1) = \kappa\rho_0(z'(0))^{-1/2}P^{-1/4}(0)$$
$$\times\left[\sin W_{12}^*\{1 + O(j^{-1})\} + \cos W_{12}^*\{O(j^{-1})\}\right] \quad \text{as } j \to \infty,$$

where now $W_{12}^* = (w(1) + \gamma_0 - \pi/4)$, $\rho_0 = \rho(z(0))$, and $\gamma_0 = \gamma(z(0))$ (see §3 for terminology). From the results given in part (b) of subsection 3.4 it is a simple matter to deduce that

$$w(1) = \left((k+1)\pi - \gamma_0 + \pi/4 + O(j^{-1})\right) \quad \text{as } j \to \infty,$$

and thus in light of (4.1)–(4.2), the bound for $\gamma_0$ given in the statements preceding (3.21), and (4.4), we conclude that

$$w(1) = \lambda_{j,k}^{1/2} H(\eta_{j,k}) = (k+1)\pi\{1 + O(j^{-5/8})\}$$

and

$$G(\eta_{j,k}) = k/j + O(j^{-5/8})$$

as $j \to \infty$. Hence it follows that there exists the positive integer $j_2$ such that

$$|g(a_2) - k/j| < 2a\beta j^{-5/12}$$

when $|a_2 - \eta_{j,k}| \leqslant \beta j^{-5/12}$ and $j \geqslant j_2$.

Consider now the case where

$$\left(a_2 - \beta j^{-1/6}\right) \leqslant \eta_{j,k} \leqslant \left(a_2 - \beta j^{-5/12}\right).$$

Then for this case we may show that

$$e_1 j^{3/8}/4 < |w(0)| < 4e_2 j^{3/4}, \quad \arg w(0) = 3\pi/2,$$

and hence it follows from (3.11), (3.19), and (4.1)–(4.2) that

$$0 = \phi_2(1) = |P(0)|^{-1/4}P^{-1/4}(1)e^{|w(0)|}$$
$$\times\left[\sin W_1\{1 + O(j^{-3/8})\} + \cos W_1\{O(j^{-1})\}\right]$$

as $j \to \infty$, where $W_1 = w(1) + \pi/4$. From the results given in part (a) of subsection 3.4, it is now easy to see that as $j \to \infty$,

$$w(1) = \lambda_{j,k}^{1/2} H(\eta_{j,k}) = (k + 3/4)\pi + O(j^{-1}),$$

and hence (see (4.4))

$$G(\eta_{j,k}) = k/j + O(j^{-1}).$$

We therefore conclude that there exists the positive integer $j_3$ such that

$$2^{-1}j^{-5/12} < (g(a_2) - k/j)/a\beta < 2j^{-1/6}$$

whenever $\eta_{j,k}$ satisfies the restrictions given above and $j \geq j_3$.

Turning to the case where

$$(a_1 + \beta j^{-5/12}) \leq \eta_{j,k} \leq (a_2 - \beta j^{-1/6}),$$

we may show that here

$$|w(0)| > e_1 j^{3/4}/4, \quad \arg w(0) = 3\pi/2, \quad \text{and} \quad w(1) > e_1 j^{3/8}/4.$$

By arguing as in the previous case it is readily verified that $G(\eta_{j,k}) = k/j + O(j^{-1})$ as $j \to \infty$, and hence it follows that there exists the positive integer $j_4$ such that

$$2^{-1} a^* \beta^{3/2} j^{-5/8} < k/j < \left(g(a_2) - 2^{-1} a \beta j^{-1/6}\right)$$

whenever $\eta_{j,k}$ satisfies the above bounds and $j \geq j_4$.

Finally, suppose that

$$\eta_{j,k} \leq (a_1 + \beta j^{-5/12}).$$

Then it is not difficult to verify that

$$w(1) < 4e_2 j^{3/8}, \quad |w(0)| > e_1\left(2^{-1}(a_2 - a_1)\right)^{3/2} j, \quad \arg w(0) = 3\pi/2,$$

and hence it follows from (3.19), (4.1)–(4.2) that as $j \to \infty$,

$$0 = \phi_2(1) = \kappa |P(0)|^{-1/4} (z'(1))^{-1/2} e^{|w(0)|}$$
$$\times \left[ T(z(1))\{1 + O(j^{-1})\} + S(z(1))\{O(j^{-1})\} \right].$$

From the results of subsection 3.4 (see in particular part (a)) it now follows that as $j \to \infty$,

$$z(1) = \lambda_{j,k}^{1/3} (3H(\eta_{j,k})/2)^{2/3} = \tau_{k+1} + O(j^{-1}),$$

and hence

$$G_1(\eta_{j,k}) = \tau_{k+1} (j\pi)^{-2/3} + O(j^{-1}).$$

Thus we conclude that there exists the positive integer $j_5$ such that

$$2\tau_{k+1}^{3/2}/3j\pi < 2a^* \beta^{3/2} j^{-5/8}$$

when $\eta_{j,k} \leq (a_1 + \beta j^{-5/12})$ and $j \geq j_5$.

By taking $\beta = \frac{1}{2}$ in the above calculations and assuming that $j > \max\{j_i\}$, $i = 1, \ldots, 5$, we see that

$$k/j < \left(g(a_2) + a j^{-5/12}\right) \quad \text{whenever } |\eta_{j,k} - a_2| \leq 2^{-1} j^{-5/12},$$

$$k/j < \left(g(a_2) - a j^{-5/12}/4\right)$$
$$\text{whenever } (a_1 + 2^{-1} j^{-5/12}) \leq \eta_{j,k} < (a_2 - 2^{-1} j^{-5/12}),$$

and in light of the definition of $\Omega_1^*$ and the results given in subsection 3.4 concerning the zeros of $T(z)$, we also see that there exists the positive number

$j_0$ such that $(k + \frac{1}{2})/j < a^* j^{-5/8}$ whenever $\eta_{j,k} < (a_1 + 2^{-1} j^{-5/12})$, $(j, k) \in \Omega_1^*$, and $j \geq j_0$. The assertion of the theorem concerning the values of $\eta_{j,k}$ for $(j, k) \in \Omega_1^*$ follows immediately from these results and we may argue in the same way with each of the sets $\Omega_i^*$, $i = 2, \ldots, 5$, to complete the proof of the theorem.

**5. Asymptotic formulae.**

5.0. *Introduction*. We are now going to employ the foregoing results to derive asymptotic formulae for the eigenvalues and eigenfunctions of the system (1.1)–(1.4) for $(j, k) \in \Omega_1$ (see subsections 5.1–5.5). Accordingly, in this section we will suppose that $(j, k) \in \bigcup_{i=1}^{5} \Omega_i^*$ and that $j$ is large enough to ensure that (4.1) and the assertions of Lemma 4.1 and Theorem 4.1 are valid. Furthermore, it will always be assumed henceforth that fractional powers of positive quantities have their positive values. Finally, in the sequel careful attention must be paid to the notation given in §§2, 4, and subsection 3.0.

*Notation*. (1) Let $\chi_i$ $(i = 1, 2)$ denote the solution of $(1.2i - 1)$ satisfying $\chi_i(1, \lambda, \mu) = \sin \beta_i$, $\chi_i'(1, \lambda, \mu) = \cos \beta_i$, and put

$$\psi_{j,k}^{\dagger}(x_1, x_2) = \prod_{i=1}^{2} \chi_i(x_i, \lambda_{j,k}, \mu_{j,k}),$$

$$\psi_{j,k}^{*}(x_1, x_2) = \prod_{i=1}^{2} \phi_i(x_1, \lambda_{j,k}, \mu_{j,k})$$

(see §4). Also let

$$\|F\| = \left( \iint_{I^2} \Delta(x_1, x_2) |F(x_1, x_2)|^2 \, dx_1 \, dx_2 \right)^{1/2}$$

for any function $F$ that is square-integrable in $I^2$.

(2) For $a_1 < t \leq a_2 + \delta$, $0 \leq \xi_1 \leq x_1 \leq 1$, $c \leq \xi_2 \leq x_2 \leq 1$, and $0 < \omega_i < \pi$, $i = 1, 2$, let

$$X_1(x_1, \xi_1, t) = \int_{\xi_1}^{x_1} P_1^{1/2}(s_1, 1, t) \, ds_1,$$

$$X_2(x_2, \xi_2, t) = \int_{\xi_2}^{x_2} \left(|t - r(s_2)| B_2(s_2)\right)^{1/2} ds_2,$$

$$X_1^*(x_1, t) = \int_{x_1}^{1} B_1(s_1) P_1^{-1/2}(s_1, 1, t) \, ds_1,$$

$$X_2^*(x_2, t) = \int_{x_2}^{1} B_2(s_2) \left(|s_2 - x(t)| \Psi_2(s_2, t)\right)^{-1/2} ds_2,$$

$$X_1^\dagger(x_1, \omega_1, t) = P_1^{-1/2}(x_1, 1, t) \times\left[\cot\omega_1 + P_1'(x_1, 1, t)/4P_1(x_1, 1, t)\right], \qquad ' = d/dx_1,$$

$$X_2^\dagger(x_2, \omega_2, t) = \left[\cot\omega_2 + \Phi_1'(x_2, t)/2\Phi_1(x_2, t)\right], \qquad ' = d/dx_2,$$

$$Q_1(x_1, t) = q_1(x_1) + P_1^{1/4}(x_1, 1, t)\left[d^2P_1^{-1/4}(x_1, 1, t)/dx_1^2\right],$$

$$Q_2(x_2, t) = q_2(x_2) + \Phi_1^{1/2}(x_2, t)\left[d^2\Phi_1^{-1/2}(x_2, t)/dx_2^2\right],$$

$$X_1^\#(x_1, \xi_1, t) = -2^{-1}\int_{\xi_1}^{x_1} Q_1(s_1, t)P_1^{-1/2}(s_1, 1, t)\, ds_1,$$

$$X_2^\#(x_2, \xi_2, t) = -2^{-1}\int_{\xi_2}^{x_2} Q_2(s_2, t)\left(|s_2 - x(t)|\Psi_2(s_2, t)\right)^{-1/2} ds_2,$$

$$\begin{aligned} c_1^*(t) &= X_1^\dagger(0, \alpha_1, t) - X_1^\dagger(1, \beta_1, t) + X_1^\#(1, 0, t) \quad \text{if } \alpha_1 \neq 0 \text{ and } \beta_1 \neq \pi, \\ &= X_1^\dagger(0, \alpha_1, t) + X_1^\#(1, 0, t) \quad \text{if } \alpha_1 \neq 0 \text{ and } \beta_1 = \pi, \\ &= -X_1^\dagger(1, \beta_1, t) + X_1^\#(1, 0, t) \quad \text{if } \alpha_1 = 0 \text{ and } \beta_1 \neq \pi, \\ &= X_1^\#(1, 0, t) \quad \text{if } \alpha_1 = 0 \text{ and } \beta_1 = \pi. \end{aligned}$$

$$\begin{aligned} c_2^*(x_2, t) &= -7\left(72(t - a_1)^{3/2}\Phi(1, t)\right)^{-1} \\ &\quad - X_2^\dagger(1, \beta_2, t)\left(B_2(1)(t - a_1)\right)^{-1/2} + X_2^\#(1, x_2, t) \quad \text{if } \beta_2 \neq \pi, \\ &= 5\left(72(t - a_1)^{3/2}\Phi(1, t)\right)^{-1} + X_2^\#(1, x_2, t) \quad \text{if } \beta_2 = \pi, \end{aligned}$$

$$Y_i^*(x_i, t) = \left(c_1^*(t)/h_1(t)\right)X_i(1, x_i, t) \quad \text{for } i = 1, 2,$$

(see §4 for the definition of $h_1$),

$$\begin{aligned} Y_1^\dagger(x_1, t) &= -Y_1^*(x_1, t) - X_1^\dagger(1, \beta_1, t) + X_1^\#(1, x_1, t) \quad \text{if } \beta_1 \neq \pi, \\ &= Y_1^*(x_1, t) - X_1^\#(1, x_1, t) \quad \text{if } \beta_1 = \pi, \end{aligned}$$

$$Y_2^\dagger(x_2, t) = \mp Y_2^*(x_2, t) \pm c_2^*(x_2, t), \quad Y^\#(x_2, t) = \mp 5/72\Phi(x_2, t),$$

where the upper signs are taken if $\beta_2 \neq \pi$ and the lower signs if $\beta_2 = \pi$,

$$Y_1(x_1, j, k) = N_{j,k}^{-1} Y_1^\dagger(x_1, \eta_{j,k})$$

and

$$Z_i(x_i, j, k) = N_{j,k} X_i(1, x_i, \eta_{j,k}) \quad \text{for } i = 1, 2,$$

where

$$N_{j,k} = (j + \nu_1)\pi/h_1(\eta_{j,k}),$$

$\nu_i = 0$ if $\alpha_i \neq 0$ and $\beta_i \neq \pi$, $\nu_i = \frac{1}{2}$ if $\alpha_i \neq 0$ and $\beta_i = \pi$ or if $\alpha_i = 0$ and $\beta_i \neq \pi$, $\nu_i = 1$ if $\alpha_i = 0$ and $\beta_i = \pi$, and lastly, when $x_2 > x(\eta_{j,k})$ we shall also put

$$Y_2(x_2, j, k) = N_{j,k}^{-1}\Big[\big(\eta_{j,k} - r(x_2)\big)^{-3/2}Y^{\#}(x_2, \eta_{j,k}) + Y_2^{\dagger}(x_2, \eta_{j,k})\Big].$$

(3) For $a_1 < t \leqslant a_2 + \delta, 0 \leqslant x_1 \leqslant 1$, and $c \leqslant x_2 \leqslant 1$, let

$$c_{21}^{\dagger}(t) = -7/72\Phi(0, t) \quad \text{and} \quad c_{22}^{\dagger}(t) = X_2^{\dagger}(0, \alpha_2, t)/B_2^{1/2}(0) \quad \text{if } \alpha_2 \neq 0,$$

$$c_{21}^{\dagger}(t) = 5/72\Phi(0, t) \quad \text{and} \quad c_{22}^{\dagger}(t) = 0 \quad \text{if } \alpha_2 = 0,$$

$$y_1^*(x_1, t) = C^*(t)X_1(x_1, 0, t),$$

$$y_2^*(x_2, t) = C^*(t)|r(x_2) - t|^{3/2}\Phi(x_2, t),$$

where $C^*(t) = c_1^*(t)/h_1(t)$,

$$y_1^{\dagger}(x_1, t) = -y_1^*(x_1, t) + X_1^{\dagger}(0, \alpha_1, t) + X_1^{\#}(x_1, 0, t) \quad \text{if } \alpha_1 \neq 0,$$

$$= y_1^*(x_1, t) - X_1^{\#}(x_1, 0, t) \quad \text{if } \alpha_1 = 0,$$

$$y_1(x_1, j, k) = N_{j,k}^{-1}y_1^{\dagger}(x_1, \eta_{j,k}),$$

$$Z_1^*(x_1, j, k) = N_{j,k}X_1(x_1, 0, \eta_{j,k}),$$

$$y^{\#}(x_2, t) = -5/72\Phi(x_2, t),$$

$$\nu^* = 1/4 \quad \text{if } \beta_2 \neq \pi, \qquad \nu^* = 3/4 \quad \text{if } \beta_2 = \pi,$$

and lastly, when $x_2 < x(\eta_{j,k})$ we shall also put

$$Z_2^*(x_2, j, k) = N_{j,k}X_2\big(x(\eta_{j,k}), x_2, \eta_{j,k}\big)$$

$$= N_{j,k}\big(r(x_2) - \eta_{j,k}\big)^{3/2}\Phi(x_2, \eta_{j,k}).$$

Finally, we remark that the notation given in part (3) above will only be utilized in subsections 5.3–5.5.

5.1. *Formulae in* $\Omega_1^*$. In this subsection we suppose that $(j, k) \in \Omega_1^*$ (see Theorem 4.1). Then it follows from the definitions of $\Omega_1^*$ and $g(t)$ given in §4 that

$$\big(g(a_2) + 2^{-1}aj^{-5/12}\big) < (k + \nu_2)/(j + \nu_1) < g(a_2 + \delta)$$

when $j$ is sufficiently large. Hence assuming $j$ large, we shall denote by $t_{j,k}$ the solution of the equation $g(t) = (k + \nu_2)/(j + \nu_1)$; and it is easy to see that $(a_2 + 4^{-1}j^{-5/12}) < t_{j,k} < (a_2 + \delta)$.

For $a_2 \leqslant t \leqslant a_2 + \delta$, let

$$c_{21}^*(t) = 7/72\Phi(0, t) \quad \text{and} \quad c_{22}^*(t) = X_2^{\dagger}(0, \alpha_2, t)/B_2^{1/2}(0) \quad \text{if } \alpha_2 \neq 0,$$

$$c_{21}^*(t) = -5/72\Phi(0, t) \quad \text{and} \quad c_{22}^*(t) = 0 \quad \text{if } \alpha_2 = 0,$$

$$c_{23}^*(t) = c_2^*(0, t), \quad d_i^*(t) = L(t)X_i^*(0, t)$$

and

$$d_i^\dagger(t) = td_i^*(t) + (-1)^{i-1}L(t)h_i(t) \quad \text{for } i = 1, 2$$

(where $L(t) = 1/h_1^2(t)g'(t)$),

$$c_1(t) = c_1^*(t)d_2^*(t), \quad d_1(t) = c_1^*(t)d_2^\dagger(t),$$

$$c_{2m}(t) = c_{2m}^*(t)d_1^*(t), \quad d_{2m}(t) = c_{2m}^*(t)d_1^\dagger(t) \quad \text{for } m = 1, 2, 3,$$

$$c(t) = c_1(t) + c_{23}(t), \quad d(t) = d_1(t) + d_{23}(t),$$

and

$$D(t) = \iint_{I^2} \Delta(x_1, x_2)\left(|x_2 - x(t)|P_1(x_1, 1, t)\Psi_2(x_2, t)\right)^{-1/2} dx_1\, dx_2.$$

For $0 < x_i < 1$, $i = 1, 2$, let us define the expressions $E_m$ $(= E_m(x_1, x_2, j, k))$, $\sigma_m$, $\sigma_m^*$, $m = 1, \ldots, 4$, by putting:

$$\begin{aligned} &E_1 = 1, \quad E_2 = Y_2(x_2, j, k), \quad E_3 = Y_1(x_1, j, k), \quad E_4 = 0,\\ &\sigma_1 = 1, \quad \sigma_1^* = 0, \quad \text{and} \quad \sigma_m = 0,\\ &\sigma_m^* = 1 \quad \text{for } m \neq 1 \text{ if } \beta_i \neq \pi \text{ for } i = 1, 2; \end{aligned} \tag{1}$$

$$\begin{aligned} &E_1 = Y_2(x_2, j, k), \quad E_2 = 1, \quad E_3 = 0, \quad E_4 = Y_1(x_1, j, k),\\ &\sigma_2 = 1, \quad \sigma_2^* = 0, \quad \text{and} \quad \sigma_m = 0,\\ &\sigma_m^* = 1 \quad \text{for } m \neq 2 \text{ if } \beta_1 \neq \pi \text{ and } \beta_2 = \pi; \end{aligned} \tag{2}$$

$$\begin{aligned} &E_1 = Y_1(x_1, j, k), \quad E_2 = 0, \quad E_3 = 1, \quad E_4 = Y_2(x_2, j, k),\\ &\sigma_3 = 1, \quad \sigma_3^* = 0, \quad \text{and} \quad \sigma_m = 0,\\ &\sigma_m^* = 1 \quad \text{for } m \neq 3 \text{ if } \beta_1 = \pi \text{ and } \beta_2 \neq \pi; \end{aligned} \tag{3}$$

$$\begin{aligned} &E_1 = 0, \quad E_2 = Y(x_1, j, k), \quad E_3 = Y_2(x_2, j, k), \quad E_4 = 1,\\ &\sigma_4 = 1, \quad \sigma_4^* = 0, \quad \text{and} \quad \sigma_m = 0,\\ &\sigma_m^* = 1 \quad \text{for } m \neq 4 \text{ if } \beta_i = \pi \text{ for } i = 1, 2. \end{aligned} \tag{4}$$

Then putting $D_{j,k} = D(\eta_{j,k})$ and $\psi_{j,k}(x_1, x_2) = \psi_{j,k}^\dagger(x_1, x_2)/\|\psi_{j,k}^\dagger\|$, we have

THEOREM 5.1. *It is the case that*

$$\lambda_{j,k} = M_{j,k}^2\Big[1 + M_{j,k}^{-2}\Big\langle (t_{j,k} - a_2)^{-3/2}c_{21}(t_{j,k}) + (t_{j,k} - a_2)^{-1/2}c_{22}(t_{j,k}) + c(t_{j,k})\Big\rangle o(j^{-2})\Big],$$

$$\mu_{j,k} = M_{j,k}^2\Big[t_{j,k} + M_{j,k}^{-2}\Big\langle (t_{j,k} - a_2)^{-3/2}d_{21}(t_{j,k}) + (t_{j,k} - a_2)^{-1/2}d_{22}(t_{j,k}) + d(t_{j,k})\Big\rangle + o(j^{-2})\Big],$$

$$\psi_{j,k}(x_1, x_2) = 2D_{j,k}^{-1/2}\big[(\eta_{j,k} - r(x_2))P_1(x_1, 1, \eta_{j,k})B_2(x_2)\big]^{-1/4}$$
$$\times\Bigg[\prod_{i=1}^{2} \cos Z_i(x_i, j, k)\{E_1(x_1, x_2, j, k) + \sigma_1 O(j^{-3/4}) + \sigma_1^* o(j^{-1})\}$$
$$+ \cos Z_1(x_1, j, k)\sin Z_2(x_2, j, k)$$
$$\times \{E_2(x_1, x_2, j, k) + \sigma_2 O(j^{-3/4}) + \sigma_2^* o(j^{-1})\}$$
$$+ \sin Z_1(x_1, j, k)\cos Z_2(x_2, j, k)$$
$$\times \{E_3(x_1, x_2, j, k) + \sigma_3 O(j^{-3/4}) + \sigma_3^* o(j^{-1})\}$$
$$+ \prod_{i=1}^{2} \sin Z_i(x_i, j, k)\{E_4(x_1, x_2, j, k) + \sigma_4 O(j^{-3/4}) + \sigma_4^* o(j^{-1})\}\Bigg] \tag{5.1}$$

*as* $j \to \infty$, $(j, k) \in \Omega_1^*$, *where* $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k})$. *This last result holds uniformly for* $0 \leqslant x_i \leqslant 1$, $i = 1, 2$.

PROOF. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then we know from the proof of Theorem 4.1 that when $j$ is sufficiently large (3.15)–(3.16) apply in full force, and consequently, by arguing with these equations as we did in the proof just cited, it is easy to see that as $j \to \infty$,

$$\lambda_{j,k}^{1/2}h_2(\eta_{j,k}) = (k + 1)\pi + \lambda_{j,k}^{-1/2}C(\eta_{j,k}) + o(j^{-1}), \tag{5.2}$$

where

$$C(t) = (t - a_2)^{-3/2}c_{21}^*(t) + c_{23}^*(t),$$

while from [6, Theorem 3.4] we also have

$$\lambda_{j,k}^{1/2}h_1(\eta_{j,k}) = (j + 1)\pi + \lambda_{j,k}^{-1/2}c_1^*(\eta_{j,k}) + o(j^{-1}), \tag{5.3}$$

and hence

$$g(\eta_{j,k}) = (k+1)/(j+1) + O(j^{-11/8}).$$

From this last result and (5.3) it follows that

$$\eta_{j,k} = t_{j,k} + O(j^{-11/8}) \quad \text{and} \quad \lambda_{j,k} = M_{j,k}^2\left(1 + O(j^{-11/8})\right) \quad \text{as } j \to \infty.$$

Substituting these expressions for $\lambda_{j,k}$ and $\eta_{j,k}$ in (5.2)–(5.3), we obtain

$$g(\eta_{j,k}) = (k+1)/(j+1) + C_1(t_{j,k})/((j+1)\pi)^2 + o(j^{-2}),$$

where

$$C_1(t) = C(t)h_1(t) - c_1^*(t)h_2(t),$$

and hence

$$\eta_{j,k} = t_{j,k} + C_1(t_{j,k})/((j+1)\pi)^2 g'(t_{j,k}) + o(j^{-2})$$

as $j \to \infty$. The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows immediately from this result, (5.3), and the relation $d_1^*(t)h_2(t) + d_2^*(t)h_1(t) = 2$.

From (5.3) and (3.15) it is easy to see that

$$\begin{aligned}\chi_2(x_2, \lambda_{j,k}, \mu_{j,k}) = K_2(j,k)P_2^{-1/4}(x_2, 1, \eta_{j,k}) \\ \times\Big[\sin Z_2(x_2, j, k)\{1 + O(j^{-3/4})\} \\ + \cos Z_2(x_2, j, k)\{Y_2(x_2, j, k) + o(j^{-1})\}\Big] \quad (5.4)\end{aligned}$$

as $j \to \infty$, uniformly in $0 \leqslant x_2 \leqslant 1$, where

$$K_i(j,k) = 1/N_{j,k}P_i^{1/4}(1, 1, \eta_{j,k}) \quad \text{for} \quad i = 1, 2,$$

while by arguing in a manner similar to that in [6, Theorem 3.4] we can also verify that

$$\begin{aligned}\chi_1(x_1, \lambda_{j,k}, \mu_{j,k}) = K_1(j,k)P_1^{-1/4}(x_1, 1, \eta_{j,k}) \\ \times\Big[\sin Z_1(x_1, j, k)\{1 + o(j^{-1})\} \\ + \cos Z_1(x_1, j, k)\{Y_1(x_1, j, k) + o(j^{-1})\}\Big] \quad (5.5)\end{aligned}$$

as $j \to \infty$, uniformly in $0 \leqslant x_1 \leqslant 1$. A simple argument now shows that

$$\|\psi_{j,k}^\dagger\|^{-1} = \left(2/D_{j,k}^{1/2}\prod_{i=1}^{2} K_i(j,k)\right)\{1 + O(j^{-3/4})\}$$

as $j \to \infty$, and hence the assertion of our theorem concerning $\psi_{j,k}$ follows immediately.

5.2. *Formulae in* $\Omega_2^*$. In this subsection we suppose that $(j, k) \in \Omega_2^*$ (see Theorem 4.1) and for $c \leqslant x_2 \leqslant d$, $r(d) \leqslant t \leqslant r(c)$, and $s > 0$, let

$$\zeta_1 = \zeta_1(x_2, s, t) = s^{1/3}(t - r(x_2))\Phi_2(x_2, t),$$

$$\zeta_2 = \zeta_2(x_2, s, t) = ((s + \nu_1)\pi/h_1(t))^{2/3}(t - r(x_2))\Phi_2(x_2, t),$$

$$R(x_2, s, t) = \kappa\rho^*(\zeta_2),$$

$$J_i(x_2, s, t) = \gamma(\zeta_i) \quad \text{and} \quad J_i^*(x_2, s, t) = \gamma^*(\zeta_i) \quad \text{for } i = 1, 2,$$

where all terms are defined in §§2 and 3.

Let $\beta^\dagger = 4[3\pi N_2/h_1((b_1 + a_2)/2)]^{2/3}$, where $N_2$ is defined in §2. Then we observe from the properties of $\gamma(z)$, $\gamma^*(z)$ given in subsection 3.4 and from (3.11)–(3.12) that there exists a positive constant $\beta$ such that both $|\gamma(z)|$ and $|\gamma^*(z)|$ do not exceed $\beta j^{3/8}$ when $|z| \leqslant \beta^\dagger j^{1/4}$. Furthermore, it is clear from the definitions of $\Omega_2^*$ and $G(t)$ given in §4 that there exists the positive integer $j_1^\dagger$ such that when $j \geqslant j_1^\dagger$,

$$G((a_1 + a_2)/2) < f_-(j, k) < f_+(j, k) < G(a_2 + 3\delta/4)$$

and

$$|t_\pm(j, k) - a_2| < 4j^{-5/12},$$

where

$$f_\pm(j, k) = \big((k + \nu_2)\pi \pm (\beta j^{3/8} + \pi/4)\big)/(j + \nu_1)\pi$$

and $G(t_\pm(j, k)) = f_\pm(j, k)$. Hence if we assume that $j \geqslant j_1^\dagger$ and, for $i = 1, \ldots, 6$, define $t_i(j, k)$ recursively by means of the formulae

$$G(t_i(j, k)) = ((k + \nu_2)\pi + F_{i-1}(j, k))/(j + \nu_1)\pi,$$

where $F_0(j, k) = 0$ and for $i \geqslant 1$, $F_i(j, k)$ equals $-(J_2^*(0, j, t_i(j, k)) + \pi/4)$ if $\alpha_2 \neq 0$ and $-(J_2(0, j, t_i(j, k)) - \pi/4)$ if $\alpha_2 = 0$, then it is readily verified that each $t_i(j, k)$ exists,

$$G((a_1 + a_2)/2) < G(t_i(j, k)) < G(a_2 + 3\delta/4),$$

and

$$|t_i(j, k) - a_2| < 4j^{-5/12}.$$

Next put

$$F(j, k) = ((j + \nu_1)\pi/h_1(t_6))^{-2/3}X_2^\dagger(0, \alpha_2, t_6)/\Phi_1(0, t_6)R^2(0, j, t_6) \qquad \text{if } \alpha_2 \neq 0,$$

$$= 0 \quad \text{if } \alpha_2 = 0,$$

where $t_6 = t_6(j, k)$. Then it is clear from (3.11)–(3.12) that $F(j, k) = O(j^{-2/3})$ as $j \to \infty$, and hence it follows that there exists the integer $j_1 \geqslant j_1^\dagger$

such that when $j \geqslant j_1$,

$$G((a_1 + a_2)/2) < f^{\dagger}_{-}(j, k) < f^{\dagger}_{+}(j, k) < G(a_2 + 3\delta/4)$$

and

$$|t^{\dagger}_{\pm}(j, k) - a_2| < 4j^{-5/12},$$

where

$$f^{\dagger}_{\pm}(j, k) = f_{\pm}(j, k) \pm |F(j, k)|/(j + \nu_1)\pi \quad \text{and} \quad G\left(t^{\dagger}_{\pm}(j, k)\right) = f^{\dagger}_{\pm}(j, k)$$

Assuming henceforth that $j \geqslant j_1$ and putting

$$f(j, k) = ((k + \nu_2)\pi + F_6(j, k) + F(j, k))/(j + \nu_1)\pi,$$

it now follows that

$$G((a_1 + a_2)/2) < f(j, k) < G(a_2 + 3\delta/4)$$

and

$$|t_7(j, k) - a_2| < 4j^{-5/12}$$

(where $G(t_7(j, k)) = f(j, k)$), while if we define $F_7(j, k)$ to be that quantity which is obtained by putting $t_i(j, k) = t_7(j, k)$ in the above formula for $F_i$ and let

$$f^*(j, k) = ((k + \nu_2)\pi + F_7(j, k) + F(j, k))/(j + \nu_1)\pi,$$

then

$$G((a_1 + a_2)/2) < f^*(j, k) < G(a_2 + 3\pi/4).$$

*Finally, we shall denote by* $t^*_{j,k}$ *the solution of the equation* $G(t) = f^*(j, k)$; observe that $|t^*_{j,k} - a_2| < 4j^{-5/12}$.

Referring to the notation given at the beginning of this and subsection 5.0, respectively, let

$$c^*_{21}(t) = c^*_2(x(t), t),$$

$$d^*_1(t) = L(t)X^*_1(0, t),$$

$$d^*_2(t) = L(t)X^*_2(x(t), t),$$

$$d^{\dagger}_1(t) = td^*_1(t) + L(t)h_1(t), \quad d^{\dagger}_2(t) = td^*_2(t) - L(t)H(t)$$

(where $L(t) = 1/h_1^2(t)G'(t)$),

$$c(t) = c^*_1(t)d^*_2(t) + c^*_{21}(t)d^*_1(t),$$

and

$$d(t) = c^*_1(t)d^{\dagger}_2(t) + c^*_{21}(t)d^{\dagger}_1(t).$$

Putting

$$A_{j,k} = X^{\dagger}_2(0, \alpha_2, \eta_{j,k})/\Phi_1(0, \eta_{j,k})R^2(0, j, \eta_{j,k}),$$

we shall also let

$$J_{j,k} = J_2^*(0, j, \eta_{j,k}),$$
$$a_{j,k} = -A_{j,k}\cos J_{j,k}, \text{ and } b_{j,k} = A_{j,k}\sin J_{j,k} \quad \text{if } \alpha_2 \neq 0,$$
$$J_{j,k} = J_2(0, j, \eta_{j,k}) - \pi \text{ and } a_{j,k} = b_{j,k} = 0 \quad \text{if } \alpha_2 = 0,$$
$$\zeta(x_2, j, k) = \zeta_2(x_2, j, \eta_{j,k}),$$
$$T_{j,k}(x_2) = T(\zeta(x_2, j, k)), \quad S_{j,k}(x_2) = S(\zeta(x_2, j, k)),$$

and

$$D_i(j, k) = N_{j,k}^{-1/3} \iint_{I^2} \Delta(x_1, x_2)\zeta'(x_2, j, k) W_i(x_2, j, k) \times P_1^{-1/2}(x_1, 1, \eta_{j,k})\Phi_1^{-2}(x_2, \eta_{j,k})\, dx_1\, dx_2,$$

for $i = 1, 2$, where $T$ and $S$ are defined in §3, $' = d/dx_2$, $W_1(x_2, j, k) = W^2(x_2, j, k)$,

$$W_2(x_2, j, k) = W(x_2, j, k)\big(a_{j,k}T_{j,k}(x_2) - b_{j,k}S_{j,k}(x_2)\big),$$

and

$$W(x_2, j, k) = \big(T_{j,k}(x_2)\sin J_{j,k} - S_{j,k}(x_2)\cos J_{j,k}\big).$$

It is easy to see from (3.11)–(3.12) and [4, p. 86] that $D_1(j, k)$, $1/D_1(j, k)$, and $|D_2(j,k)|$ all remain less than some positive number independent of $j$ and $k$ for all large $j$.

Lastly, for large $j$ we shall put

$$B_{j,k} = N_{j,k}^{-2/3} D_1^{-1}(j, k) D_2(j, k),$$
$$B_{j,k}^* = N_{j,k}^{-2/3}\big(a_{j,k} - D_1^{-1}(j, k)D_2(j, k)\sin J_{j,k}\big),$$
$$B_{j,k}^\dagger = N_{j,k}^{-2/3}\big(b_{j,k} - D_1^{-1}(j, k)D_2(j, k)\cos J_{j,k}\big),$$

and for $0 \leqslant x_1 \leqslant 1$, $x(\eta_{j,k}) < x_2 \leqslant 1$, define the expressions $E_m$ $(= E_m(x_1, x_2, j, k))$, $\sigma_m$, $\sigma_m^*$, $E_m^*$ $(= E_m^*(x_1, j, k))$, $E_m^\dagger$ $(= E_m^\dagger(x_1, j, k))$, $\varepsilon_m$, and $\varepsilon_m^*$, $m = 1, \ldots, 4$, by putting:

$$E_1 = 1 - B_{j,k}, \quad E_2 = Y_2(x_2, j, k), \quad E_3 = Y_1(x_1, j, k), \quad E_4 = 0,$$
$$\sigma_1 = 1, \quad \sigma_1^* = 0, \quad \text{and} \quad \sigma_m = 0,$$
$$\sigma_m^* = 1 \quad \text{for } m \neq 1 \text{ if } \beta_i \neq \pi \text{ for } i = 1, 2; \tag{1}$$
$$E_1 = Y_2(x_2, j, k), \quad E_2 = 1 - B_{j,k}, \quad E_3 = 0, \quad E_4 = Y_1(x_1, j, k),$$
$$\sigma_2 = 1, \quad \sigma_2^* = 0, \quad \text{and} \quad \sigma_m = 0,$$
$$\sigma_m^* = 1 \quad \text{for } m \neq 2 \text{ if } \beta_1 \neq \pi \text{ and } \beta_2 = \pi; \tag{2}$$

$$E_1 = Y_1(x_1, j, k), \quad E_2 = 0, \quad E_3 = 1 - B_{j,k}, \quad E_4 = Y_2(x_2, j, k),$$
$$\sigma_3 = 1, \quad \sigma_3^* = 0, \quad \text{and} \quad \sigma_m = 0,$$
$$\sigma_m^* = 1 \quad \text{for } m \neq 3 \text{ if } \beta_1 = \pi \text{ and } \beta_2 \neq \pi; \tag{3}$$
$$E_1 = 0, \quad E_2 = Y_1(x_1, j, k), \quad E_3 = Y_2(x_2, j, k), \quad E_4 = 1 - B_{j,k},$$
$$\sigma_4 = 1, \quad \sigma_4^* = 0, \quad \text{and} \quad \sigma_m = 0,$$
$$\sigma_m^* = 1 \quad \text{for } m \neq 4 \text{ if } \beta_i = \pi \text{ for } i = 1, 2; \tag{4}$$
$$E_1^* = \sin J_{j,k} + B_{j,k}^*, \quad E_2^* = -\cos J_{j,k} - B_{j,k}^\dagger,$$
$$E_3^* = Y_1(x_1, j, k)\sin J_{j,k}, \quad E_4^* = -Y_1(x_1, j, k)\cos J_{j,k},$$
$$E_1^\dagger = -E_2^\dagger = 1, \quad E_3^\dagger = -E_4^\dagger = Y_1(x_1, j, k),$$
$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 1, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 0 \quad \text{if } \beta_1 \neq \pi; \tag{5}$$

and

$$E_1^* = Y_1(x_1, j, k)\sin J_{j,k}, \quad E_2^* = -Y_1(x_1, j, k)\cos J_{j,k},$$
$$E_3^* = \sin J_{j,k} + B_{j,k}^*, \quad E_4^* = -\cos J_{j,k} - B_{j,k}^\dagger,$$
$$E_1^\dagger = -E_2^\dagger = Y_1(x_1, j, k), \quad E_3^\dagger = -E_4^\dagger = 1,$$
$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 0, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 1 \quad \text{if } \beta_1 = \pi. \tag{6}$$

It is clear from the proof of Theorem 4.1 and (4.2) that if in §3 we take $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, and suppose that $j$ is sufficiently large, then

$$|w(0)| < 4e_2 j^{3/8}, \quad w(1) > e_1((a_2 - a_1)/2)^{3/2} j,$$

where the $e_i$ are defined in (4.3). Assuming $j$ large, we shall now denote by $\xi_{j,k}$ the point of $[c, d]$ at which $r(x_2) = \eta_{j,k} - (4e_2/e_1)^{2/3} j^{-5/12}$; and by arguing with (4.2) it is not difficult to verify that

$$w(x_2) \geqslant 4e_2 j^{3/8} \quad \text{for } x_2 \geqslant \xi_{j,k}$$

(from which it follows that $\xi_{j,k} > 0$) and that

$$|w(x_2)| < 4e_1^{-1} e_2^2 j^{3/8} \quad \text{for } 0 \leqslant x_2 < \xi_{j,k}.$$

Then putting

$$D_{j,k}^* = D_1(j, k), \quad D_{j,k} = 2\kappa^2 D_{j,k}^*,$$

and

$$\psi_{j,k}(x_1, x_2) = \psi_{j,k}^\dagger(x_1, x_2)/\|\psi_{j,k}^\dagger\|,$$

we have

**THEOREM 5.2.** *It is the case that*

$$\lambda_{j,k} = M_{j,k}^2\left[1 + M_{j,k}^{-2}c(t_{j,k}^*) + o(j^{-2})\right],$$
$$\mu_{j,k} = M_{j,k}^{-2}\left[t_{j,k}^* + M_{j,k}^{-2}d(t_{j,k}^*) + o(j^{-2})\right], \tag{5.6}$$

$\psi_{j,k}(x_1, x_2)$ *is given by* (5.1) *for* $0 \leqslant x_1 \leqslant 1$, $\xi_{j,k} \leqslant x_2 \leqslant 1$, *provided that throughout* (5.1) *we replace the expression* $O(j^{-3/4})$ *by* $o(j^{-2/3})$,

$$\begin{aligned}\psi_{j,k}(x_1, x_2) = {} & E(x_1, x_2, j, k)\\ & \times\Big[\cos Z_1(x_1, j, k)T_{j,k}(x_2)\{E_1^*(x_1, j, k) + \varepsilon_1 o(j^{-2/3}) + \varepsilon_1^* o(j^{-1})\}\\ & + \cos Z_1(x_1, j, k)S_{j,k}(x_2)\{E_2^*(x_1, j, k) + \varepsilon_2 o(j^{-2/3}) + \varepsilon_2^* o(j^{-1})\}\\ & + \sin Z_1(x_1, j, k)T_{j,k}(x_2)\{E_3^*(x_1, j, k) + \varepsilon_3 o(j^{-2/3}) + \varepsilon_3^* o(j^{-1})\}\\ & + \sin Z_1(x_1, j, k)S_{j,k}(x_2)\{E_4^*(x_1, j, k) + \varepsilon_4 o(j^{-2/3}) + \varepsilon_4^* o(j^{-1})\}\Big]\\ & \qquad \textit{for } 0 \leqslant x_1 \leqslant 1,\ \max\{0, x(\eta_{j,k})\} \leqslant x_2 < \xi_{j,k},\end{aligned}$$

$$\begin{aligned}\psi_{j,k}(x_1, x_2) = {} & E(x_1, x_2, j, k)\\ & \times\Big[\cos Z_1(x_1, j, k)T_{j,k}(x_2)\sin J_{j,k}\\ & \times\{E_1^\dagger(x_1, j, k) + \varepsilon_1 o(j^{-1/2}) + \varepsilon_1^* o(j^{-1})\}\\ & + \cos Z_1(x_1, j, k)S_{j,k}(x_2)\cos J_{j,k}\\ & \times\{E_2^\dagger(x_1, j, k) + \varepsilon_2 o(j^{-1/2}) + \varepsilon_2^* o(j^{-1})\}\\ & + \sin Z_1(x_1, j, k)T_{j,k}(x_2)\sin J_{j,k}\\ & \times\{E_3^\dagger(x_1, j, k) + \varepsilon_3 o(j^{-1/2}) + \varepsilon_3^* o(j^{-1})\}\\ & + \sin Z_1(x_1, j, k)S_{j,k}(x_2)\cos J_{j,k}\\ & \times\{E_4^\dagger(x_1, j, k) + \varepsilon_4 o(j^{-1/2}) + \varepsilon_4^* o(j^{-1})\}\Big]\\ & \qquad \textit{for } 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 < x(\eta_{j,k}) \textit{ if } x(\eta_{j,k}) > 0,\end{aligned} \tag{5.7}$$

*as* $j \to \infty$, $(j, k) \in \Omega_2^*$, *where* $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k}^*)$ *and*

$$\begin{aligned}(-1)^{k+1}&E(x_1, x_2, j, k)\\ &= N_{j,k}^{1/6}\left(2/D_{j,k}^*\right)^{1/2}P_1^{-1/4}(x_1, 1, \eta_{j,k})\Phi_1^{-1/2}(x_2, \eta_{j,k}).\end{aligned} \tag{5.8}$$

*These results for* $\psi_{j,k}(x_1, x_2)$ *hold uniformly in* $x_1$ *and* $x_2$.

**REMARK.** Before proving the theorem, let us make the following clarifications. By arguing as we did in the proof of Theorem 4.1, it is easily seen that $x(\eta_{j,k}) > 0$ for at least one (but not all) $k$ when $j$ is sufficiently large.

(5.7) is, of course, only intended to apply to that subset of $\Omega_2^*$ composed of points $(j, k)$ for which $j$ exceeds a certain fixed positive number and $x(\eta_{j,k}) > 0$. Writing $\zeta$ for $\zeta(x_2, j, k)$, $\eta$ for $\eta_{j,k}$ and $x^*$ for $x(\eta)$, observe also that $\zeta = N_{j,k}^{2/3}(\eta - r(x_2))\Phi_2(x_2, \eta)$, while from (4.2) we have $(\eta - r(x_2)) = (x_2 - x^*)\Psi_1(x_2, \eta)$. These equations together with the convention described in the statements preceding (3.3) determine $(2/3)\zeta^{3/2}$. By taking $z_2 = \zeta$ and $w_2 = (2/3)\zeta^{3/2}$ in (3.11)–(3.12), the behaviour of $T_{j,k}$ and $S_{j,k}$ may be ascertained. In particular, it is important to observe in (5.7) that the absolute value of $S_{j,k}(x_2)\cos J_{j,k}$ remains less than some bound independent of $x_2$, $j$, and $k$. This observation follows immediately from the definitions of the terms involved.

PROOF. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then from the discussion immediately preceding this theorem it is clear that (3.17) applies in full force, and by arguing as we did in the proof of Thoerem 4.1 it may readily be verified that

$$\lambda_{j,k}^{1/2}H(\eta_{j,k}) = (k + 1)\pi + c^*(\lambda_{j,k}, \eta_{j,k}) + \lambda_{j,k}^{-1/2}c_{21}^*(\eta_{j,k}) + o(j^{-1}) \quad (5.9)$$

as $j \to \infty$, where $c^*(s, t) = -J_1(0, s, t) + \pi/4$ (see the beginning of this subsection). Thus in light of (5.3) and the arguments given in the proof of Theorem 4.1, we therefore conclude that

$$G(\eta_{j,k}) = ((k + 1)/(j + 1)) + O(j^{-5/8}) \quad \text{as } j \to \infty.$$

From this result and (5.3) it now follows that as $j \to \infty$,

$$\eta_{j,k} = t_1 + O(j^{-5/8}), \quad \lambda_{j,k} = ((j + 1)\pi/h_1(t_1))^2\{1 + O(j^{-5/8})\}$$

(where $t_1 = t_1(j,k)$), and a simple argument also shows that

$$c^*(\lambda_{j,k}, \eta_{j,k}) = F_1(j, k) + O(j^{1/6})$$

(we refer to the discussion concerning $t_{j,k}^*$ given at the beginning of this subsection for terminology). If we substitute this latter expression for $c^*$ in (5.9) and make use of (5.3), then we conclude that

$$G(\eta_{j,k}) = ((k + 1)\pi + F_1(j, k))/(j + 1)\pi + O(j^{-5/6})$$

as $j \to \infty$, and hence it follows that

$$\eta_{j,k} = t_2 + O(j^{-5/6}), \quad \lambda_{j,k} = ((j + 1)\pi/h_1(t_2))^2\{1 + O(j^{-5/6})\},$$

and

$$c^*(\lambda_{j,k}, \eta_{j,k}) = F_2(j, k) + O(j^{-1/24})$$

as $j \to \infty$, where $t_2 = t_2(j, k)$. By substituting this latter expression for $c^*$ in (5.9) and repeating the above arguments, it is now easy to deduce that

$$\eta_{j,k} = t_i + O(j^{-m_i}), \quad \lambda_{j,k} = ((j + 1)\pi/h_1(t_i))^2\{1 + O(j^{-m_i})\},$$

and

$$c^*(\lambda_{j,k}, \eta_{j,k}) = F_i(j, k) + O\left(j^{-(m_i - 19/24)}\right)$$

as $j \to \infty$ for $i = 1, \ldots, 7$, where $t_i = t_i(j, k)$ and $m_i = (5i + 10)/24$. Thus, as $j \to \infty$, $G(\eta_{j,k}) = f^*(j, k) + O(j^{-2})$, and hence

$$\eta_{j,k} = t^*_{j,k} + O(j^{-2}), \quad \lambda_{j,k} = M^2_{j,k}\left(1 + O(j^{-2})\right).$$

A substitution of these expressions for $\eta_{j,k}$ and $\lambda_{j,k}$ in (5.3) and (5.9) shows that

$$G(\eta_{j,k}) = f^*(j, k) + ((j + 1)\pi)^{-2} C(t^*_{j,k}) + o(j^{-2}),$$

and hence

$$\eta_{j,k} = t^*_{j,k} + C(t^*_{j,k}) / ((j + 1)\pi)^2 G'(t^*_{j,k}) + o(j^{-2})$$

as $j \to \infty$, where $C(t) = c^*_{21}(t)h_1(t) - c^*_1(t)H(t)$. The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows immediately from this latter result, (5.3), and the relation $d^*_1(t)H(t) + d^*_2(t)h_1(t) = 2$.

Turning next to $\chi_{j,k}(x_2) = \chi_2(x_2, \lambda_{j,k}, \mu_{j,k})$, it follows from (5.3) and (3.15) that (5.4) remains valid as $j \to \infty$ for $\xi_{j,k} \leqslant x_2 \leqslant 1$, and with this result holding uniformly in $x_2$. To obtain approximations for $\chi_{j,k}$ when $x_2 < \xi_{j,k}$ we must turn again to the work of §3, and taking now $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, we shall indicate the dependence on $j$ and $k$ of the functions $P$, $w$, $z$, $\theta^*$ defined there by writing $\omega_{j,k}(x_2)$ for $\omega(x_2)$ and $\theta^*_{j,k}(x_2, x_1)$ for $\theta^*(x_2, x_1)$, where $\omega$ is any one of the functions $P$, $w$, or $z$. Then putting

$$L_{j,k}(x_2) = (-1)^k \kappa \left(z'_{j,k}(x_2)\right)^{-1/2} P_{j,k}^{-1/4}(1),$$

$$T^*_{j,k}(x_2) = T\left(z_{j,k}(x_2)\right), \quad S^*_{j,k}(x_2) = S\left(z_{j,k}(x_2)\right),$$

$$\Gamma_{j,k} = \gamma\left(z_{j,k}(0)\right), \quad \text{and} \quad I_{j,k}(x_2) = \theta^*_{j,k}\left(x_2, x(\eta_{j,k})\right),$$

we may argue with (3.17) and (5.9) to show that as $j \to \infty$,

$$\chi_{j,k}(x_2) = L_{j,k}(x_2)\Big[T^*_{j,k}(x_2)\left\{\sin \Gamma_{j,k} - I_{j,k}(x_2)\cos \Gamma_{j,k} + o\left(\lambda_{j,k}^{-1/2}\right)\right\}$$
$$- S^*_{j,k}(x_2)\left\{\cos \Gamma_{j,k} + I_{j,k}(x_2) \sin \Gamma_{j,k} + o\left(\lambda_{j,k}^{-1/2}\right)\right\}\Big]$$
$$\text{for } \max\left\{0, x(\eta_{j,k})\right\} \leqslant x_2 \leqslant 1, \tag{5.10}$$

$$= L_{j,k}(x_2)\Big[T^*_{j,k}(x_2)\left\{\sin \Gamma_{j,k} + \lambda_{j,k}^{-1/2} I^*_{j,k}(x_2)\right\}$$
$$- S^*_{j,k}(x_2)\left\{\cos \Gamma_{j,k} + \lambda_{j,k}^{-1/2} I^\dagger_{j,k}(x_2)\right\}\Big]$$
$$\text{for } 0 \leqslant x_2 < x(\eta_{j,k}) \quad \text{if } x(\eta_{j,k}) > 0, \tag{5.11}$$

where (5.10) holds uniformly in $x_2$ and both $I^*_{j,k}$ and $I^\dagger_{j,k}$ are $o(1)$ as $j \to \infty$, uniformly in $x_2$ (see the remark immediately preceding the statement of this theorem).

We now assert that as $j \to \infty$,

$$\chi_{j,k}(x_2) = L_{j,k}(x_2)\Big[T^*_{j,k}(x_2)\sin\Gamma_{j,k}\big\{1 + o\big(\lambda_{j,k}^{-1/4}\big)\big\} - S^*_{j,k}(x_2)\cos\Gamma_{j,k}\big\{1 + o\big(\lambda_{j,k}^{-1/4}\big)\big\}\Big] \quad (5.12)$$

for $0 \leqslant x_2 < x(\eta_{j,k})$ if $x(\eta_{j,k}) > 0$, with this result holding uniformly in $x_2$. To prove this assertion let $0 < \varepsilon < 1$, choose the positive integer $j_0$ large enough so that $|I^*_{j,k}(x_2)| + |I^\dagger_{j,k}(x_2)| < \varepsilon$ in $0 \leqslant x_2 < x(\eta_{j,k})$ whenever $x(\eta_{j,k}) > 0$ and $j \geqslant j_0$, and under these latter conditions consider each of the cases: (i) $|w_{j,k}(0)| \leqslant \omega_{j,k}(\varepsilon)$ $(= 4^{-1}\log\{\lambda_{j,k}^{1/2}/\varepsilon\})$, (ii) $|w_{j,k}(0)| > \omega_{j,k}(\varepsilon)$. For case (i) we may argue with (3.12) and (5.11) to show that when $j$ is sufficiently large, (5.12) is valid in $0 \leqslant x_2 < x(\eta_{j,k})$, provided that in (5.12) each of the expressions $o(\lambda_{j,k}^{-1/4})$ is replaced by an expression of the form $C_{j,k}(x_2)$, where $|C_{j,k}(x_2)| < \beta\varepsilon^{1/2}\lambda_{j,k}^{-1/4}$ and $\beta$ denotes a positive constant. A similar result also holds for case (ii), which can easily be seen by arguing with (3.12), (5.11) and the relation

$$\chi_{j,k}(x_2) = \chi_{j,k}(x_1^\#)\phi_{j,k}(x_2)/\phi_{j,k}(x_1^\#),$$

where $x_1^\#$ $(= x_1^\#(j, k))$ denotes the point of $[c, d]$ defined in the statements preceding (3.15),

$$\phi_{j,k}(x_2) = \phi_2(x_2, \lambda_{j,k}, \mu_{j,k}) = v(x_2, 0),$$

and $v(x_2, 0)$ is given by (3.18). Since $\varepsilon$ is arbitrary, the assertion follows.

By arguing with (5.3), (5.10), and (5.12) we can next show that as $j \to \infty$,

$$\begin{aligned}
\chi_{j,k}(x_2) &= L^*_{j,k}(x_2)\Big[T_{j,k}(x_2)\big\{\sin J_{j,k} + O(j^{-1})\big\} - S_{j,k}(x_2)\big\{\cos J_{j,k} + O(j^{-1})\big\}\Big] \\
&\qquad \text{for } \xi_{j,k} \leqslant x_2 \leqslant 1, \\
&= L^*_{j,k}(x_2)\Big[T_{j,k}(x_2)\big\{\sin J_{j,k} + o(j^{-1})\big\} - S_{j,k}(x_2)\big\{\cos J_{j,k} + o(j^{-1})\big\}\Big] \\
&\qquad \text{for } \max\{0, x(\eta_{j,k})\} \leqslant x_2 < \xi_{j,k}, \\
&= L^*_{j,k}(x_2)\Big[T_{j,k}(x_2)\sin J_{j,k}\big\{1 + o(j^{-1/2})\big\} - S_{j,k}(x_2)\cos J_{j,k}\big\{1 + o(j^{-1/2})\big\}\Big] \\
&\qquad \text{for } 0 \leqslant x_2 < x(\eta_{j,k}) \text{ if } x(\eta_{j,k}) > 0, \quad (5.13)
\end{aligned}$$

uniformly in $x_2$, where

$$L^*_{j,k}(x_2) = K^*_2(j, k)\Phi_1^{-1/2}(x_2, \eta_{j,k}),$$

$$K^*_2(j, k) = (-1)^{k+1}\kappa N_{j,k}^{-5/6}P_2^{-1/4}(1, 1, \eta_{j,k}),$$

and all other terms are defined in the statements preceding this theorem. From these results and (5.5) it may readily be verified that

$$\|\psi^\dagger_{j,k}\|^{-1} = N_{j,k}^{1/6}K_1^{-1}(j, k)|K^*_2(j, k)|^{-1}(2/D^*_{j,k})^{1/2}\{1 + o(j^{-2/3})\}$$

as $j \to \infty$ (where $K_1$ is defined in the statements preceding (5.5)), and the assertion of our theorem concerning $\psi_{j,k}$ now follows from (5.4), (5.5), and (5.13).

5.3. *Formulae in* $\Omega^*_3$. Throughout this subsection we assume that $(j, k) \in \Omega^*_3$ (see Theorem 4.1). Then referring to the notation of subsection 5.0 (see, in particular, part (3)), it follows from the definition of $\Omega^*_3$ that

$$(G(a_2) - 2aj^{-1/6}) < (k + \nu^*)/(j + \nu_1) < (G(a_2) - 2^{-1}aj^{-5/12})$$

when $j$ is sufficiently large. Assuming $j$ large, we shall now denote by $t^*_{j,k}$ the solution of the equation $G(t) = (k + \nu^*)/(j + \nu_1)$; and it is not difficult to verify that

$$(a_2 - 3j^{-1/6}) < t^*_{j,k} < (a_2 - 4^{-1}j^{-5/12})$$

(see part (6) of the notation of §4 for the properties of $G$).

In this subsection we shall define the functions $c^*_{21}(t)$, $d^*_i(t)$ and $d^\dagger_i(t)$ $(i = 1, 2)$, $c(t)$, $d(t)$, $\zeta(x_2, j, k)$, $T_{j,k}(x_2)$, and $S_{j,k}(x_2)$ precisely as we did in subsection 5.2, put $D^*_{j,k} = N_{j,k}^{-1/3}I_{j,k}$, where

$$I_{j,k} = \iint_{I^2} \Delta(x_1, x_2)\zeta'(x_2, j, k)T^2_{j,k}(x_2)P_1^{-1/2}(x_1, 1, \eta_{j,k}) \times \Phi_1^{-2}(x_2, \eta_{j,k})\, dx_1\, dx_2, \quad (5.14)$$

$(' = d/dx_2)$, and for $t < a_2$ and $x_2 < x(t)$ (here and below special attention must be paid to part (3) of the notation of subsection 5.0) let

$$c^\dagger(t) = c^\dagger_{21}(t)(a_2 - t)^{-3/2},$$

$$y^\dagger_2(x_2, t) = (y^{\#}(x_2, t)(r(x_2) - t)^{-3/2} + c^\dagger(t)),$$

$$y^\dagger_{21}(x_2, t) = y^\dagger_2(x_2, t) - c^\dagger(t), \quad y^\dagger_{22}(x_2, t) = y^\dagger_2(x_2, t) + c^\dagger(t),$$

and

$$y_{2i}(x_2, j, k) = N_{j,k}^{-1}y^\dagger_{2i}(x_2, \eta_{j,k}) \quad \text{for } i = 1, 2.$$

It is easy to see from (3.11)–(3.12) and the results given in [4, p. 86] that $D^*_{j,k}$ and $1/D^*_{j,k}$ remain less than some positive number independent of $j$ or $k$ for all large $j$. We shall also define the expressions $E_m(x_1, x_2, j, k)$ (for $0 \leqslant x_1 \leqslant 1$, $x(\eta_{j,k}) < x_2 \leqslant 1$), $\sigma_m$, and $\sigma^*_m$ $(m = 1, \ldots, 4)$ precisely as we did in subsection 5.1 and for $0 \leqslant x_1 \leqslant 1$, $0 \leqslant x_2 \leqslant x(\eta_{j,k})$ define the expressions $E^\dagger(x_1, j, k)$, $E^\#(x_1, j, k)$, $E_m^*(x_1, x_2, j, k)$, $\varepsilon_m$, $\varepsilon_m^*$ $(m = 1, \ldots, 4)$, and $\varepsilon$ by putting:

$$\begin{aligned} &E^\dagger = 1, \quad E^\# = y_1(x_1, j, k), \quad E_1^* = 1 + y_{21}(x_2, j, k), \\ &E_2^* = -1 + y_{22}(x_2, j, k), \quad E_3^* = -E_4^* = y_1(x_1, j, k), \\ &\varepsilon_1 = \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 1, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 0 \quad \text{if } \alpha_1 \neq 0, \end{aligned} \tag{1}$$

$$\begin{aligned} &E^\dagger = y_1(x_1, j, k), \quad E^\# = 1, \quad E_1^* = -E_2^* = y_1(x_1, j, k), \\ &E_3^* = 1 + y_{21}(x_2, j, k), \quad E_4^* = -1 + y_{22}(x_2, j, k), \\ &\varepsilon_1 = \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 0, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 1 \quad \text{if } \alpha_1 = 0, \end{aligned} \tag{2}$$

and

$$\varepsilon = 1 \quad \text{if } \alpha_2 \neq 0, \quad \varepsilon = 0 \quad \text{if } \alpha_2 = 0. \tag{3}$$

Finally, if in §3 we take $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, and assume that $j$ is sufficiently large, then we may argue as we did in the proof of Theorem 4.1 to show that

$$4^{-1}e_1 j^{3/8} < |w(0)| < 4e_2 j^{3/4} \quad \text{and} \quad w(1) > e_1((a_2 - a_1)/2)^{3/2} j,$$

where the $e_i$ are defined in (4.3). Assuming $j$ large, we shall now denote by $\xi_{j,k}^-$ and $\xi_{j,k}^+$ the points of $[c, d]$ at which

$$r(x_2) = \eta_{j,k} + e^* j^{-5/12} \quad \text{and} \quad r(x_2) = \eta_{j,k} - e^* j^{-5/12},$$

respectively, where $e^* = (e_1/4e_2)^{2/3}$; it is clear that

$$0 < \xi_{j,k}^- < x(\eta_{j,k}) < \xi_{j,k}^+ < 1, \quad |w(x_2)| < 4^{-1}e_1 j^{3/8}$$

for $\xi_{j,k}^- < x_2 < \xi_{j,k}^+$, and $|w(x_2)| > (e_1^2/4e_2) j^{3/8}$ for $x_2 \leqslant \xi_{j,k}^-$ and $x_2 \geqslant \xi_{j,k}^+$. Then putting

$$D_{j,k} = 2\kappa^2 D_{j,k}^*, \quad \psi_{j,k} = \psi_{j,k}^*(x_1, x_2)/\|\psi_{j,k}^*\|,$$

and recalling the definition of $M_{j,k}$ given in Theorem 5.2, we have

THEOREM 5.3. *It is the case that* $\lambda_{j,k}$, $\mu_{j,k}$ *are given by* (5.6), $(-1)^{j+k}\psi_{j,k}(x_1, x_2)$ *is given by the right-hand side of* (5.1) *for* $0 \leqslant x_1 \leqslant 1$, $\xi_{j,k}^+ \leqslant x_2 \leqslant 1$,

$$\begin{aligned} \psi_{j,k}(x_1, x_2) = {} & E(x_1, x_2, j, k) \\ & \times \Big[\cos Z_1^*(x_1, j, k) T_{j,k}(x_2)\{E^\dagger(x_1, j, k) + \varepsilon_1 O(j^{-3/4}) + \varepsilon_1^* o(j^{-1})\} \\ & + \cos Z_1^*(x_1, j, k) S_{j,k}(x_2) f_1(x_1, x_2, j, k) \\ & + \sin Z_1^*(x_1, j, k) T_{j,k}(x_2)\{E^\#(x_1, j, k) + \varepsilon_3 O(j^{-3/4}) + \varepsilon_3^* o(j^{-1})\} \\ & + \sin Z_1^*(x_1, j, k) S_{j,k}(x_2) f_2(x_1, x_2, j, k)\Big] \\ & \text{for } 0 \leqslant x_1 \leqslant 1, \quad \xi_{j,k}^- < x_2 < \xi_{j,k}^+, \end{aligned} \tag{5.15}$$

$$\psi_{j,k}(x_1, x_2) = D_{j,k}^{-1/2}\left[(r(x_2) - \eta_{j,k})P_1(x_1, 1, \eta_{j,k})B_2(x_2)\right]^{-1/4}$$
$$\times\Big[\cos Z_1^*(x_1, j, k)\exp\{-Z_2^*(x_2, j, k)\}$$
$$\times\{E_1^*(x_1, x_2, j, k) + \varepsilon_1 O(j^{-3/4}) + \varepsilon_1^* o(j^{-1})\}$$
$$+ (-1)^{\varepsilon}\cos Z_1^*(x_1, j, k)\exp\{-Z_2^\dagger(x_2, j, k)\}$$
$$\times\{E_2^*(x_1, x_2, j, k) + \varepsilon_2 O(j^{-3/4}) + \varepsilon_2^* o(j^{-1})\}$$
$$+\sin Z_1^*(x_1, j, k)\exp\{-Z_2^*(x_2, j, k)\}$$
$$\times\{E_3^*(x_1, x_2, j, k) + \varepsilon_3 O(j^{-3/4}) + \varepsilon_3^* o(j^{-1})\}$$
$$+ (-1)^{\varepsilon}\sin Z_1^*(x_1, j, k)\exp\{-Z_2^\dagger(x_2, j, k)\}$$
$$\times\{E_4^*(x_1, x_2, j, k) + \varepsilon_4 O(j^{-3/4}) + \varepsilon_4^* o(j^{-1})\}\Big]$$
$$\text{for } 0 \leqslant x_1 \leqslant 1, \quad 0 \leqslant x_2 \leqslant \xi_{j,k}^-, \tag{5.16}$$

*as* $j \to \infty$, $(j, k) \in \Omega_3^*$, *where* $E(x_1, x_2, j, k)$ *is given by the right-hand side of* (5.8), $f_p(x_1, x_2, j, k)$ *equals* $o(j^{-1})$ *for* $x_2 > x(\eta_{j,k})$ *and* $\exp\{-2Z_2^*(0, j, k)\}O(1)$ *for* $x_2 < x(\eta_{j,k})$, $p = 1, 2$, *and* $Z_2^\dagger(x_2, j, k) = 2Z_2^*(0, j, k) - Z_2^*(x_2, j, k)$. *These results for* $\psi_{j,k}(x_1, x_2)$ *hold uniformly in* $x_1$ *and* $x_2$.

Remark. It is clear from the remark immediately following the statement of Theorem 5.2 and the definitions of the terms involved that

$$(2/3)\zeta^{3/2}(x_2, j, k) = -iZ_2^*(x_2, j, k) \quad \text{for } 0 \leqslant x_2 < x(\eta_{j,k}).$$

Observe also that $Z_2^*$ strictly decreases in this interval. Hence it follows from (3.12) that the expression $S_{j,k}(x_2)f_p(x_1, x_2, j, k)$ given in (5.15) is $o(\exp\{-Z_2^*(0, j, k)\})$ as $j \to \infty$ uniformly in $\xi_{j,k}^- < x_2 < x(\eta_{j,k})$ for $p = 1, 2$.

Proof. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then by arguing with (3.11), (3.19) as we did in the proof of Theorem 4.1, it is easy to see that

$$\lambda_{j,k}^{1/2}H(\eta_{j,k}) = (k + 3/4)\pi + \lambda_{j,k}^{-1/2}c_{21}^*(\eta_{j,k}) + o(j^{-1}) \tag{5.17}$$

as $j \to \infty$. In light of (5.3), it follows that as $j \to \infty$,

$$G(\eta_{j,k}) = ((k + 3/4)/(j + 1)) + O(j^{-2}),$$

and hence

$$\eta_{j,k} = t_{j,k}^* + O(j^{-2}), \quad \lambda_{j,k} = M_{j,k}^2\{1 + O(j^{-2})\}.$$

Substituting these expressions for $\eta_{j,k}$ and $\lambda_{j,k}$ in (5.3) and (5.17), we obtain

$$G(\eta_{j,k}) = (k + 3/4)/(j + 1) + C(t_{j,k}^*)/((j + 1)\pi)^2 + o(j^{-2}),$$

where $C(t) = c_{21}^*(t)h_1(t) - c_1^*(t)H(t)$, and hence

$$\eta_{j,k} = t_{j,k}^* + C(t_{j,k}^*)/((j+1)\pi)^2 G'(t_{j,k}^*) + o(j^{-2})$$

as $j \to \infty$. The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows from this latter result, (5.3), and the relation $d_1^*(t)H(t) + d_2^*(t)h_1(t) = 2$.

For convenience of notation we shall henceforth put

$$\phi_i^\#(x_i, j, k) = \phi_i(x_i, \lambda_{j,k}, \mu_{j,k}), \quad \chi_i^\#(x_i, j, k) = \chi_i(x_i, \lambda_{j,k}, \mu_{j,k})$$

for $i = 1, 2$. Then by arguing with (3.19) and (5.3) it is easy to see that as $j \to \infty$,

$$\phi_2^\#(x_2, j, k) = K_2^*(j,k)\Phi_1^{-1/2}(x_2, \eta_{j,k}) \times \left[T_{j,k}(x_2)\left\{1 + N_{j,k}^{-1}c^\dagger(\eta_{j,k}) + F_{j,k}(x_2)\right\} + S_{j,k}(x_2)F_{j,k}^*(x_2)\right]$$

for $0 \leqslant x_2 \leqslant 1$, where

$$K_2^*(j,k) = \kappa N_{j,k}^{-5/6}|P_2(0, 1, \eta_{j,k})|^{-1/4}\exp\{Z_2^*(0, j, k)\},$$

$F_{j,k}(x_2) = O(j^{-3/4})$, and $F_{j,k}^*(x_2)$ equals $O(j^{-1})$ for $\xi_{j,k}^+ \leqslant x_2 \leqslant 1$, $o(j^{-1})$ for $x(\eta_{j,k}) \leqslant x_2 < \xi_{j,k}^+$, and $O(\exp\{-2Z_2^*(0, j, k)\})$ for $0 \leqslant x_2 < x(\eta_{j,k})$, with these results for $F_{j,k}$ and $F_{j,k}^*$ holding uniformly in $x_2$. Arguments similar to those used in [**6**, Theorem 3.4] show that

$$\phi_1^\#(x_1, j, k) = K_1^*(j,k)P_1^{-1/4}(x_1, 1, \eta_{j,k}) \times \Big[\sin Z_1^*(x_1, j, k)\{1 + o(j^{-1})\} + \cos Z_1^*(x_1, j, k)\{y_1(x_1, j, k) + o(j^{-1})\}\Big]$$

as $j \to \infty$, uniformly in $0 \leqslant x_1 \leqslant 1$, where $K_1^*(j,k) = 1/N_{j,k}P_1^{1/4}(0, 1, \eta_{j,k})$, and hence it is now not difficult to verify that

$$\|\psi_{j,k}^*\|^{-1} = N_{j,k}^{1/6}(2/D_{j,k}^*)^{1/2}\left(\prod_{i=1}^{2} K_i^*(j,k)\right)^{-1}\{1 - N_{j,k}^{-1}c^\dagger(\eta_{j,k}) + O(j^{-3/4})\}$$

as $j \to \infty$. By arguing with (3.15), (5.3), we may next deduce that as $j \to \infty$,

$$\phi_2^\#(x_2, j, k) = K_2^\#(j,k)|P_2(x_2, 1, \eta_{j,k})|^{-1/4} \times \Big[\exp\{-Z_2^*(x_2, j, k)\}\{1 + N_{j,k}^{-1}y_2^\ddagger(x_2, \eta_{j,k}) + O(j^{-3/4})\} + \exp\{-Z_2^\dagger(x_2, j, k)\}\{-1 + N_{j,k}^{-1}y_2^\ddagger(x_2, \eta_{j,k}) + O(j^{-3/4})\}\Big]$$

for $0 \leqslant x_2 \leqslant \xi_{j,k}^-$, uniformly in $x_2$, where

$$K_2^\#(j,k) = \left(2N_{j,k}|P_2(0, 1, \eta_{j,k})|^{1/4}\right)^{-1}\exp\{Z_2^*(0, j, k)\}.$$

Finally, we also know that for $i = 1, 2$,

$$\phi_i^\#(x_i, j, k) = k_i(j,k)\chi_i^\#(x_i, j, k) \quad \text{for } 0 \leqslant x_i \leqslant 1,$$

where $k_i(j,k) = -\,d\phi_i^{\#}(1,j,k)/dx_i$ and $\chi_2^{\#}$ is given by (5.4) for $\xi_{j,k}^{+} \leqslant x_2 \leqslant 1$ and $\chi_1^{\#}$ by (5.5) for $0 \leqslant x_1 \leqslant 1$ (with the first formula holding uniformly in $x_2$ and the second uniformly in $x_1$). Moreover, by arguing with (3.11), (3.20), and (5.17), it is not difficult to deduce that

$$k_2(j,k) = (-1)^k |P_2(0,1,\eta_{j,k})|^{-1/4} P_2^{1/4}(1,1,\eta_{j,k}) \times \exp\{Z_2^*(0,j,k)\}\{1 + N_{j,k}^{-1} c^{\dagger}(\eta_{j,k}) + O(j^{-3/4})\}$$

as $j \to \infty$, while a standard argument shows that

$$k_1(j,k) = (-1)^j P_1^{-1/4}(0,1,\eta_{j,k}) P_1^{1/4}(1,1,\eta_{j,k})\{1 + o(j^{-1})\}$$

as $j \to \infty$. With these results it is now a simple matter to verify the assertion of the theorem concerning $\psi_{j,k}$.

5.4. *Formulae in* $\Omega_4^*$. In this subsection we assume that $(j,k) \in \Omega_4^*$ (see Theorem 4.1). Then it may readily be verified that

$$2^{-1} a^* j^{-5/8} < (k + \nu^*)/(j + \nu_1) < \left(G(a_2) - 2^{-1} a j^{-1/6}\right)$$

when $j$ is sufficiently large (see subsection 5.0 for terminology). Assuming $j$ large, we shall denote by $t_{j,k}^*$ the solution of the equation $G(t) = (k + \nu^*)/(j + \nu_1)$; and it is not difficult to show that $(a_1 + 4^{-1} j^{-5/12}) < t_{j,k}^* < (a_2 - 4^{-1} j^{-1/6})$.

Referring to the notation of §§2, 4, and subsection 5.0 (see part (2)), we shall now put

$$c_{21}^*(t) = -7/72\Phi(1,t), \quad c_{22}^*(t) = -B_2^{-1/2}(1) X_2^{\dagger}(1,\beta_2,t) \quad \text{if } \beta_2 \neq \pi,$$
$$c_{21}^*(t) = 5/72\Phi(1,t) \quad \text{and} \quad c_{22}^*(t) = 0 \quad \text{if } \beta_2 = \pi,$$
$$c_{23}^*(t) = X_2^{\#}(1, x(t), t),$$
$$G^*(t) = G'(t)(1 - x(t))^{-1/2} \quad \text{for } a_1 < t \leqslant a_2 + \delta,$$
$$G^*(a_1) = -x'(a_1) h_1^{-1}(a_1) \Psi(1,a_1)$$

(where $' = d/dt$; observe also from the definition of $G(t)$ in §4 that $G^*(t)$ is continuous and positive in $[a_1, a_2 + \delta]$),

$$d_1^*(t) = L(t) X_1^*(0,t) \Psi_1^{1/2}(1,t), \quad d_2^*(t) = L(t) X_2^*(x(t),t) \Psi_1^{1/2}(1,t),$$
$$d_1^{\dagger}(t) = t d_1^*(t) + L(t) h_1(t) \Psi_1^{1/2}(1,t),$$
$$d_2^{\dagger}(t) = t d_2^*(t) - L(t) H(t) \Psi_1^{1/2}(1,t)$$

(where $L(t) = 1/h_1^2(t) G^*(t)$),

$$c_1(t) = c_1^*(t) d_2^*(t), \quad d_1(t) = c_1^*(t) d_2^{\dagger}(t),$$
$$c_{2m}(t) = c_{2m}^*(t) d_1^*(t), \quad d_{2m}(t) = c_{2m}^*(t) d_1^{\dagger}(t) \quad \text{for } m = 1, 2, 3,$$
$$c(t) = c_1(t) + c_{23}(t) \quad \text{and} \quad d(t) = d_1(t) + d_{23}(t).$$

Referring next to the terminology of (3.11), let

$$\kappa^{\#} = -\beta^{\dagger} + \nu^{\dagger}\alpha^{\dagger} + 3^{-1}(\nu^{\dagger})^{3} \quad \text{and} \quad \kappa^{\dagger} = \nu^{\dagger} \quad \text{if} \quad \beta_2 \neq \pi,$$

$$\kappa^{\#} = \beta - \nu\alpha - 3^{-1}\nu^{3} \quad \text{and} \quad \kappa^{\dagger} = -\nu \quad \text{if } \beta_2 = \pi,$$

and in the interval $a_1 < t \leqslant a_2 + \delta$ put

$$G^{\dagger}(t) = 2^{-1}(1 - x(t))^{1/2}G''(t)$$

(observe from the definition of $G$ in §4 that $G^{\dagger}$ is bounded in this interval),

$$c_1^{\#}(t) = d_{21}(t) - tc_{21}(t),$$

$$c_2^{\#}(t, s) = \kappa^{\#}\Phi^{-3}(1, t) - \left(c_1^{\#}(t)\right)^2\Psi_1^{1/2}(1, t)h_1(t)G^{\dagger}(s) + 3\kappa^{\dagger}c_1^{\#}(t)/2\Phi(1, t),$$

$$c^{\#}(t, s) = d_1^{*}(t)c_2^{\#}(t, s) \quad \text{and} \quad d^{\#}(t, s) = d_1^{\dagger}(t)c_2^{\#}(t, s),$$

where $a_1 < s \leqslant a_2 + \delta$ and the terminology of the previous paragraph is utilized.

Lastly, paying special attention to part (3) of the notation of subsection 5.0, let

$$c^{\dagger}(t) = c_{21}^{\dagger}(t)(a_2 - t)^{-3/2} + c_{22}^{\dagger}(t)(a_2 - t)^{-1/2} + X_2^{\#}(x(t), 0, t) + y_2^{*}(0, t)$$

for $t < a_2$, and when $t < a_2$ and $x_2 < x(t)$, put

$$y_2^{\dagger}(x_2, t) = y^{\#}(x_2, t)(r(x_2) - t)^{-3/2} - X_2^{\#}(x(t), x_2, t) - y_2^{*}(x_2, t) + c^{\dagger}(t),$$

$$y_{21}^{\dagger}(x_2, t) = y_2^{\dagger}(x_2, t) - c^{\dagger}(t), \quad y_{22}^{\dagger}(x_2, t) = y_2^{\dagger}(x_2, t) + c^{\dagger}(t),$$

and

$$y_{2i}(x_2, j, k) = N_{j,k}^{-1}y_{2i}^{\dagger}(x_2, \eta_{j,k}) \quad \text{for } i = 1, 2.$$

Then in this subsection we shall define $\zeta(x_2, j, k)$, $T_{j,k}(x_2)$, $S_{j,k}(x_2)$, $E_m(x_1, x_2, j, k)$, $\sigma_m$, $\sigma_m^{*}$ $(m = 1, \ldots, 4)$, $E^{\dagger}(x_1, j, k)$, $E^{\#}(x_1, j, k)$, and $\varepsilon$ precisely as we did in subsection 5.3. We shall also define the $E_m^{*}$, $m = 1, \ldots, 4$, in terms of the $y_{2i}$ and $y_1$ precisely as we did in subsection 5.3, except we are now to use our new definition of the $y_{2i}$ here; and as for the $\varepsilon_m$ and $\varepsilon_m^{*}$ defined in that subsection, we shall henceforth take $\varepsilon_m = 0$, $\varepsilon_m^{*} = 1$, $m = 1, \ldots, 4$, regardless of whether $\alpha_1$ is zero or not. Finally, let

$$D_{j,k}^{*} = N_{j,k}^{-1/3}(\eta_{j,k} - a_1)^{-1/2}I_{j,k},$$

where $I_{j,k}$ is given by (5.14). It may readily be verified that $D_{j,k}^{*}$ and $1/D_{j,k}^{*}$ remain less than some positive number independent of $j$ and $k$ for all large $j$.

If in §3 we take $\lambda = \lambda_{j,k}$ and $t = \eta_{j,k}$, then we may argue as we did in the proof of Theorem 4.1 to show that

$$|w(0)| > 4^{-1}e_1 j^{3/4} \quad \text{and} \quad w(1) > 4^{-1}e_1 j^{3/8},$$

where the $e_i$ are defined in (4.3). We shall now denote by $\xi_{j,k}^-$ and $\xi_{j,k}^+$ the points of $[c, d]$ at which

$$r(x_2) = \eta_{j,k} + e^* j^{-1/6} \quad \text{and} \quad r(x_2) = \eta_{j,k} - e^* j^{-5/12},$$

respectively, where $e^* = (e_1/4e_2)^{2/3}$; and it is clear that $0 < \xi_{j,k}^- < x(\eta_{j,k}) < \xi_{j,k}^+ < 1$. Then putting

$$D_{j,k} = 2\kappa^2 D_{j,k}^*, \quad \psi_{j,k}(x_1, x_2) = \psi_{j,k}^*(x_1, x_2)/\|\psi_{j,k}^*\|,$$

and recalling the definitions of $E(x_1, x_2, j, k)$, the $f_i(x_1, x_2, j, k)$, and $Z_2^\dagger(x_2, j, k)$ given in Theorem 5.3, we may argue as we did in that theorem to show that

THEOREM 5.4. *It is the case that*

$$\begin{aligned}\lambda_{j,k} = M_{j,k}^2\Big[1 &+ M_{j,k}^{-2}\langle (t_{j,k}^* - a_1)^{-2} c_{21}(t_{j,k}^*) \\ &\quad + (t_{j,k}^* - a_1)^{-1} c_{22}(t_{j,k}^*) + (t_{j,k}^* - a_1)^{-1/2} c(t_{j,k}^*)\rangle \\ &+ (t_{j,k}^* - a_1)^{-5} M_{j,k}^{-4} c^{\#}(t_{j,k}^*, s_{j,k}) \\ &\quad + (t_{j,k}^* - a_1)^{-1/2} O(j^{-53/24}) + o(j^{-2})\Big],\end{aligned}$$

$$\begin{aligned}\mu_{j,k} = M_{j,k}^2\Big[t_{j,k}^* &+ M_{j,k}^{-2}\langle (t_{j,k}^* - a_1)^{-2} d_{21}(t_{j,k}^*) \\ &\quad + (t_{j,k}^* - a_1)^{-1} d_{22}(t_{j,k}^*) + (t_{j,k}^* - a_1)^{-1/2} d(t_{j,k}^*)\rangle \\ &+ (t_{j,k}^* - a_1)^{-5} M_{j,k}^{-4} d^{\#}(t_{j,k}^*, s_{j,k}) \\ &\quad + (t_{j,k}^* - a_1)^{-1/2} O(j^{-53/24}) + o(j^{-2})\Big],\end{aligned}$$

$(-1)^{j+k}(\eta_{j,k} - a_1)^{1/4}\psi_{j,k}(x_1, x_2)$ *is given by the right-hand side of* (5.1) *for* $0 < x_1 < 1$, $\xi_{j,k}^+ \leq x_2 < 1$, $(\eta_{j,k} - a_1)^{1/4}\psi_{j,k}(x_1, x_2)$ *is given by the right-hand side of* (5.15) *for* $0 \leq x_1 \leq 1$, $\xi_{j,k}^- < x_2 < \xi_{j,k}^+$, *and by the right-hand side of* (5.16) *for* $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq \xi_{j,k}^-$, *as* $j \to \infty$, $(j, k) \in \Omega_4^*$, *where* $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k}^*)$ *and* $s_{j,k} = t_{j,k}^* + O(j^{-7/6})$. *These results for* $\psi_{j,k}(x_1, x_2)$ *hold uniformly in* $x_1$ *and* $x_2$.

REMARK. It is important to observe from the definitions of terms involved that in the above formulae for $\lambda_{j,k}$ and $\mu_{j,k}$, the expressions $(t_{j,k}^* - a_1)^{-1/2}c(t_{j,k}^*)$ and $(t_{j,k}^* - a_1)^{-1/2}d(t_{j,k}^*)$, which depend upon the $q_i$, are both $O(1)$ as $j \to \infty$. These expressions, when multiplied by $M_{j,k}^{-2}$, are, of course, absorbed into the error terms when $k/j$ is small.

5.5. *Formulae in* $\Omega_5^*$. In this subsection we assume that $(j, k) \in \Omega_5^*$ (see Theorem 4.1). Then recalling that in subsection 3.4 we denoted the zeros of $T(z)$ and $T'(z)$ by $\tau_n$ and $\tau_n'$, respectively, $n \geq 1$, we shall now put

$$c_{21}^*(t) = -X_2^\dagger(1, \beta_2, t)/\Phi_1(1, t) \quad \text{and} \quad \Lambda_k = \tau_{k+1}' \quad \text{if} \quad \beta_2 \neq \pi,$$
$$c_{21}^*(t) = 0 \quad \text{and} \quad \Lambda_k = \tau_{k+1} \quad \text{if } \beta_2 = \pi$$

(see subsection 5.0). In light of the properties of $\Lambda_k$ given in subsection 3.4, it is clear that $j^{-1/4}\Lambda_k$ remains less than some positive number independent of $j$ and $k$ for all large $j$. Assuming $j$ large, we shall denote by $t_{j,k}^\dagger$ the solution of the equation $G_1(t) = \Lambda_k/((j + \nu_1)\pi)^{2/3}$; and it is not difficult to verify that $a_1 < t_{j,k}^\dagger < a_1 + 3j^{-5/12}$ (see part (7) of the notation of §4 for the properties of $G_1$).

In what follows we let

$$c(t) = c_{21}^*(t)L(t)X_1^*(0, t), \quad d(t) = tc(t) + c_{21}^*(t)L(t)h_1(t),$$

where $L(t) = 1/h_1^{5/3}(t)G_1'(t)$, $' = d/dt$, and define $\zeta(x_2, j, k)$, $T_{j,k}(x_2)$, $S_{j,k}(x_2)$, $E^\dagger(x_1, j, k)$, $E^\#(x_1, j, k)$, $\varepsilon$, $E_m^*(x_1, x_2, j, k)$, $\varepsilon_m$, $\varepsilon_m^*$, $m = 1, \ldots, 4$, precisely as we did in subsection 5.4. We shall also put $D_{j,k}^* = \Lambda_k^{-1/2}I_{j,k}$, where $I_{j,k}$ is given by (5.14). It is clear from (3.11)–(3.12) and [4, p. 86] that $D_{j,k}^*$ and $1/D_{j,k}^*$ remain less than some positive number independent of $j$ and $k$ for all large $j$.

Finally, if in §3 we take $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, and assume that $j$ is sufficiently large, then we may argue as we did in the proof of Theorem 4.1 to show that

$$w(1) < 4e_2 j^{3/8} \quad \text{and} \quad |w(0)| > e_1((a_2 - a_1)/2)^{3/2} j,$$

where the $e_i$ are defined in (4.3). Assuming $j$ large, we shall now denote by $\xi_{j,k}$ the point of $[c, d]$ at which $r(x_2) = \eta_{j,k} + e_1^{-2/3} j^{-1/6}$; and it is clear that $0 < \xi_{j,k} < x(\eta_{j,k})$. Then putting

$$D_{j,k} = 2\kappa^2 D_{j,k}^*, \quad \psi_{j,k}(x_1, x_2) = \psi_{j,k}^*(x_1, x_2)/\|\psi_{j,k}^*\|,$$

and recalling the definitions of $E(x_1, x_2, j, k)$, the $f_i(x_1, x_2, j, k)$, and $Z_2^\dagger(x_2, j, k)$ given in Theorem 5.3, we have

THEOREM 5.5. *It is the case that*

$$\lambda_{j,k} = M_{j,k}^2\left[1 + M_{j,k}^{-4/3}\Lambda_k^{-1}c\left(t_{j,k}^\dagger\right) + O(j^{-2})\right],$$
$$\mu_{j,k} = M_{j,k}^2\left[t_{j,k}^\dagger + M_{j,k}^{-4/3}\Lambda_k^{-1}d\left(t_{j,k}^\dagger\right) + O(j^{-2})\right],$$

$\Lambda_k^{1/4}N_{j,k}^{-1/6}\psi_{j,k}(x_1, x_2)$ *is given by the right-hand side of* (5.15) *for* $0 \leq x_1 \leq 1$, $\xi_{j,k} < x_2 \leq 1$, *and by the right-hand side of* (5.16) *for* $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq \xi_{j,k}$ *as* $j \to \infty$, $(j, k) \in \Omega_5^*$, *where* $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k}^\dagger)$. *These results for* $\psi_{j,k}(x_1, x_2)$ *hold uniformly in* $x_1$ *and* $x_2$.

PROOF. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then by arguing with (3.19) as we did in the proof of Theorem 4.1, it is easy to see that

$$\lambda_{j,k}^{1/3}(3H(\eta_{j,k})/2)^{2/3} = \Lambda_k + O(j^{-4/3}) \quad \text{as } j \to \infty.$$

In light of (5.3) it follows that as $j \to \infty$,

$$G_1(\eta_{j,k}) = \Lambda_k((j+1)\pi)^{-2/3} + O(j^{-2}),$$

and hence

$$\eta_{j,k} = t_{j,k}^{\dagger} + O(j^{-2}).$$

The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows from this result and (5.3), while the assertion concerning $\psi_{j,k}$ can be proved by using arguments similar to those used in the proof of Theorem 5.3.

**6. Conclusions and final remarks.** We are now in a position to furnish a negative answer to the question regarding the uniform boundedness of the eigenfunctions of the system (1.1)–(1.4) (see §1); for it is clear from the foregoing results that the eigenfunctions, $\psi_{j,k}(x_1, x_2)$, are not uniformly bounded in $I^2$ for $(j, k) \in \Omega_1$. We do note however that

THEOREM 6.1. *The absolute values of the* $j^{-1/3}\psi_{j,k}(x_1, x_2)$ *for* $(j, k) \in \Omega_1$ *and* $(x_1, x_2) \in I^2$ *remain less than some bound independent of* $x_1, x_2, j$, *and* $k$.

It is important to observe from Theorem 5.5 that this result is false if $1/3$ is replaced by $1/3 - \varepsilon$, $\varepsilon > 0$.

A second point worth mentioning deals with the dependence of the eigenvalues on the $q_i$. Indeed, it is clear from the above results that if in $(1.2i - 1)$, $1 \leq i \leq 2$, we replace $q_i(x_i)$ by $q_i^*(x_i)$ and denote the eigenvalues of the system (1.1)–(1.4), with the $q_i$ replaced by the $q_i^*$, by $(\lambda_{j,k}^*, \mu_{j,k}^*)$, $j, k = 0, 1, \ldots$, then $\lambda_{j,k}^* - \lambda_{j,k}$ and $\mu_{j,k}^* - \mu_{j,k}$ are $O(1)$ as $j \to \infty$, $(j, k) \in \Omega_1$.

We wish also to state that further developments of the formulae given in Theorems 5.1–5.5 are possible if the coefficients in the differential equations (1.1), (1.3) are suitably defined; we refer to the remark given at the end of §3 in [6] and to the work in [4, pp. 6–11] for further clarification.

As stated in §2, Theorems 5.1–5.5 are valid only for the case $r'(x_2) < 0$ in $c \leq x_2 \leq d$. However this is no restriction; for if $r'(x_2) > 0$ in this interval, then the substitutions $x_2 = 1 - s_2$, $\alpha_2^* = \pi - \beta_2$, $\beta_2^* = \pi - \alpha_2$, and a relabelling of terms in (1.3)–(1.4) reduces the system (1.1)–(1.4) to a form where the results of Theorems 5.1–5.5 may again be utilized.

Finally, recalling the definition of $\theta_2$ given in §4, let $\Omega_2$ denote the sector in the $(x, y)$-plane defined by the inequalities $\theta_2 < \theta \leq \pi/2$. Then it is not difficult to verify that results analogous to those above hold for the case

$(j, k) \in \Omega_2$ if we choose $\theta_2$ appropriately and assume that $A_1$, $B_1$ are of class $C^3$ in some interval containing the interval $0 \leqslant x_1 \leqslant 1$ in its interior and that $A_1'B_1 - A_1B_1' \neq 0$ in $0 \leqslant x_1 \leqslant 1$.

## BIBLIOGRAPHY

1. F. V. Atkinson, *Multiparameter eigenvalue problems*, Vol. 1, Academic Press, New York, 1972.

2. ______, *Multiparameter spectral theory*, Bull. Amer. Math. Soc. **74** (1968), 1–27.

3. P. J. Browne, *A multi-parameter eigenvalue problem*, J. Math. Anal. Appl. **38** (1972), 553–568.

4. A. A. Dorodnicyn, *Asymptotic laws of distribution of the characteristic values for certain special forms of differential equations of the second order*, Amer. Math. Soc. Transl. (2) **16** (1960), 1–101.

5. M. Faierman, *The completeness and expansion theorems associated with the multi-parameter eigenvalue problem in ordinary differential equations*, J. Differential Equations **5** (1969), 197–213.

6. ______, *On the distribution of the eigenvalues of a two-parameter system of ordinary differential equations of the second order*, SIAM J. Math. Anal. **8** (1977), 854–870.

7. ______, *An eigenfunction expansion associated with a two-parameter system of ordinary differential equations of the second order*, preprint 125, Ben Gurion University of the Negev, Beer Sheva, 1975.

8. ______, *Bounds for the eigenfunctions of a two-parameter system of ordinary differential equations of the second order* (submitted).

9. B. D. Sleeman, *Completeness and expansion theorems for a two-parameter eigenvalue problem in ordinary differential equations using variational principles*, J. London Math. Soc. (2) **6** (1973), 705–712.

10. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., University Press, Cambridge, 1966.

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