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IRRATIONAL CONNECTED SUMS AND THE TOPOLOGY OF ALGEBRAIC SURFACES

BY

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ABSTRACT. Suppose W is an irreducible nonsingular projective algebraic 3-fold and V a nonsingular hypersurface section of W. Denote by V_m a nonsingular element of $|mV|$. Let V_1, V_m, V_{m+1} be generic elements of $|V|$, $|mV|$, $|(m+1)V|$ respectively such that they have normal crossing in W. Let $S_{1m} = V_1 \cap V_m$ and $C = V_1 \cap V_m \cap V_{m+1}$. Then S_{1m} is a nonsingular curve of genus g_m and C is a collection of $N = m(m+1)V_1^3$ points on S_{1m}. By [MM2] we find that (*) V_{m+1} is diffeomorphic to $\overline{V_m - T(S_{1m})} \cup_\eta \overline{V_1' - T(S_{1m}')}$ where $T(S_{1m})$ is a tubular neighborhood of S_{1m} in V_m, V_1' is V_1 blown up along C, S_{1m}' is the strict image of S_{1m} in V_1', $T(S_{1m}')$ is a tubular neighborhood of S_{1m}' in V_1' and η: $\partial T(S_{1m}) \to \partial T(S_m')$ is a bundle diffeomorphism.

Now V_1' is well known to be diffeomorphic to $V_1 \# N(-CP^2)$ (the connected sum of V_1 and N copies of CP^2 with opposite orientation from the usual). Thus in order to be able to inductively reduce questions about the structure of V_m to ones about V_1 we must simplify the "irrational sum" (*) above.

The general question we can ask is then the following:

Suppose M_1 and M_2 are compact smooth 4-manifolds and K is a connected q-complex embedded in M_i. Let T_i be a regular neighborhood of K in M_i and let η: $\partial T_1 \to \partial T_2$ be a diffeomorphism:

Set $V = \overline{M_1 - T_1} \cup \overline{M_2 - T_2}$. How can the topology of V be described more simply in terms of those of M_1 and M_2.

In this paper we show how surgery can be used to simplify the structure of V in the case $q = 1, 2$ and indicate some applications to the topology of algebraic surfaces.

Introduction. Suppose M_1, M_2 are smooth compact 4-manifolds. Then the connected sum $M_1 \# M_2$ is defined (see [KM, §2]) by removing 4-discs D_1 from M_1 and D_2 from M_2 and identifying $\partial \overline{M_1 - D_1} \approx S^3$ with $\partial \overline{M_2 - D_2}$ by means of an orientation-reversing diffeomorphism h. As noted in [KM] $M_1 \# M_2 = \overline{M_1 - D_1} \cup_h \overline{M_2 - D_2}$ is in fact independent of the particular diffeomorphism used. Now a 4-disc D_1 in M_1 is in fact a regular neighborhood of a zero-cell "p" in M_1. Suppose we "connect" two such manifolds along the boundary, not of a regular neighborhood of a point, but of a regular neighborhood of an embedded n-complex? In particular, suppose K is

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a wedge of k 1-spheres embedded in M_1 and M_2 and $V = \overline{M_1 - N_1(K)} \cup_\eta \overline{M_2 - N_2(K)}$, (where $N_i(K)$ is a regular neighborhood of K in M_i and η: $\partial N_1 \to \partial N_2$ is some diffeomorphism). What can be said about the topology of V? In the general case this depends, of course, on the homotopy class of K in M_i. If this is zero, that is, if K is homotopic to zero in both M_1 and M_2 we can reduce V to a connected sum.

In particular let $P = CP^2$, $Q = -CP^2$ (where $-$ will denote reversed orientation) and let $kX = X \# \cdots \# X$ (k-times) for a manifold X. Then in the above situation, under some mild hypothesis on η we have

THEOREM A (SEE PROPOSITION 2.2 AND THEOREM 2.3). *Suppose $V = \overline{M_1 - N_1} \cup_\eta \overline{M_2 - N_2}$ as above and η is the restriction of an orientation reversing diffeomorphism ϕ: $N_1 \to N_2$ which induces the identity on $\pi_1(K)$. Then either*

(a) $V \approx M_1 \# M_2 \# k(S^2 \times S^2)$ *or*

(b) $V \approx M_1 \# M_2 \# k(P \# Q)$

(*where "$\approx$" is read "is diffeomorphic to"*).

Now suppose we consider 2-complexes. In particular, replace K by an embedded 2-manifold S in M_i with tubular neighborhood T_i and let

$$X = \overline{M_1 - T_1} \cup_\phi \overline{M_2 - T_2}$$

where ϕ is a bundle isomorphism of ∂T_1 and ∂T_2.

Now as opposed to the case of 1-complexes in 4-manifolds, even if the M_i are simply-connected the nature of X will, a priori, depend on the placement of K in M_i.

Using either surgery or in some cases a combination of σ and $\bar{\sigma}$-processes we can still obtain a good deal of information about X even without assuming anything about the placement of K. In fact we have

THEOREM B (SEE THEOREMS 3.4 AND 3.5). *If $X = \overline{M_1 - T_1} \cup_\phi \overline{M_2 - T_2}$ as above with the M_i simply-connected and the ϕ orientation reversing, then*:

(1) *If a fiber F of $\partial T_1 \to S$ considered as a loop in X is homotopic to zero, then either*

$$X \# (S^2 \times S^2) \approx M_1 \# M_2 \# k(S^2 \times S^2)$$

or

$$X \# (P \# Q) \approx M_1 \# M_2 \# k(P \# Q)$$

where k = minimal number of generators of $H_1(S, \mathbf{Z})$.

(2) *If M_2 is obtained by blowing up a manifold N by a σ-process at a point of a submanifold Σ whose strict image in M_2 is S, then*

$$X \# P \approx M_1 \# N \# k(P \# Q).$$

In our applications of Theorems A and B it is rare that we begin with manifolds M_1, M_2 and patch them together to get X (or V) as above. The more usual situation is that we are given some manifold X and by various means we demonstrate the existence of manifolds M_1, M_2 such that X is obtained by patching the M_i along some neighborhoods T_i. A very useful method for obtaining such 'irrational connected sum' representations is provided by the theorem of §2 of **[MM2]**. In particular, we have

THEOREM C (SEE THEOREMS 3.7 AND 3.8). *Suppose W is a compact complex 3-manifold and V, X_1, X_2 are compact complex submanifolds of W with normal intersection such that as divisors on W, V is linearly equivalent to $X_1 + X_2$. Suppose $S = X_1 \cap X_2$ is connected, $V \cdot X_1 \cdot X_2 = n > 0$ and the X_i are simply connected. Then*

(1) $V \approx \overline{X_1 - T_1} \cup \overline{X_2' - T_2'}$

where X_2' is X_2 blown up at the n-points of $V \cap X_1 \cap X_2$ and T_1, T_2' are tubular neighborhoods of S, respectively S' (S' = the strict image of S), in X_1, respectively X_2'.

(2) $V \# P \approx X_1 \# X_2 \# (n-1)Q \# 2g(P \# Q)$

where g = genus of S.

In §4 we put the above results to work in investigating the structure of simply connected algebraic surfaces.

It may be recalled that if M is a simply-connected compact 4-manifold then [see **Mi**] the homotopy type of M is completely determined by the congruence class of the quadratic form q_M given by the cup product (or dually by homology-intersection) on $H^2(M, \mathbf{Z})$ (or $H_2(M, \mathbf{Z})$). In **[W2]** Wall shows that q_M in fact determines M up to h-cobordism and, by a modification of Smale's results in higher-dimensions, that M is determined by q_M up to diffeomorphism modulo connected sums of $S^2 \times S^2$'s.

That is, by **[W2]** if M_1, M_2 are simply-connected compact 4-manifolds with q_{M_1} congruent to q_{M_2}, then for some integer k, $M_1 \# k(S^2 \times S^2) \approx M_2 \# k(S^2 \times S^2)$.

If M is a 4-manifold diffeomorphic to $aP \# bQ$ for some integers a, b we say that M is completely decomposable. As noted in **[MM1**, §0**]** a consequence of Wall's result is that for any simply-connected 4-manifold there exists an integer k such that $M \# (k+1)P \# kQ$ is completely decomposable.

If $M \# P$ is completely decomposable then we shall say that M is almost completely decomposable (abbreviated as M is A.C.D.). Note that if M is A.C.D. it is of course simply-connected.

The main result of **[MM1]** was that any nonsingular complex hypersurface of CP^3 was almost completely decomposable.

In §4 we generalize this result to other classes of simply-connected algebraic surfaces. Among other results we have

THEOREM D (SEE THEOREM 4.2 AND COROLLARIES 4.3, 4.5). *Suppose X is an almost completely decomposable algebraic surface and C is an irreducible nonsingular hypersurface section of X. Let $W \xrightarrow{\pi} X$ be the projective bundle over X obtained by compactifying the line bundle $E = [kC]$ $(k > 0)$. Then*:

(1) *If V is an irreducible nonsingular hypersurface section of W then V is A.C.D.*

(2) *If V is a nonsingular irreducible subvariety of W such that $\pi|V\colon V \to X$ is a pure n-fold branched cover of X then V is A.C.D.*

(3) *If V is a nonsingular k-fold cyclic branched cover of X with branch locus M linearly equivalent to kC, for some $k > 0$ then V is A.C.D.*

In particular, the nonsingular 'double planes' are all A.C.D.

In the remainder of this article we will adopt the following conventions and notations.

All of our manifolds will be smooth. Furthermore when we construct manifolds by a 'cutting and pasting' technique we shall assume the resulting 'manifolds with corners' are smoothed to manifolds or manifolds with boundary by the process noted in **[KM]** or **[Mz]**. Thus for example if D^q is a closed q-disc then it will make sense in terms of our smoothing, to say $D^n \approx D^p \times D^{n-p}$.

When we speak of regular neighborhoods or complexes in a manifold we shall be tacitly assuming that the manifold is given with a triangulation making such terms meaningful. Similarly tubular neighborhoods will be understood as referring to some underlying Riemannian metric.

We use the definition of σ and $\bar{\sigma}$-process on an arbitrary 4-manifold enunciated in **[MM1**, Introduction] and elaborated on in **[MM2]**. For the convenience of the reader we recall that definition here:

Suppose M is a 4-manifold and $p \in M$. Let U be a small neighborhood of p and take local coordinates on U making it into a domain in $\mathbf{C}^2$. Perform a classical σ-process (blowing up, quadratic transformation, see **[Sh]** for definition) on $p \in U \subset \mathbf{C}^2$ and call the resulting manifold M'. Then if M is oriented and the complex orientation on U coincides with the induced orientation U inherits from M then we say M' is obtained by a σ-process of $p \in M$. If the two orientations are opposite we shall say M is obtained by a $\bar{\sigma}$ process from M. (If M is not orientable then the two notions coincide.) We note that a σ-operation changes M into $M \# Q$ by introducing a new 2-sphere L with self-intersection -1 while a $\bar{\sigma}$-operation changes M into

$M \# P$ with the resulting 2-sphere thereby introduced having self-intersection $+1$. We sometimes will speak of the projective closure of the complex vector bundle $E \to M$ over some manifold M. By this we refer to the projective bundle $\mathcal{P}(E \oplus 1)$.

The first result of §3 of [**W1**] will be used repeatedly in our work and we single it out as

THEOREM W (SEE [**W1**, §§1–3]). *Let N be a compact four manifold and suppose C is a loop in N homotopic to zero. Let $f\colon S^1 \times D^3 \to N$ be an embedding with $f(S^1 \times \{0\}) = C$ and set $T = \mathrm{Im}(f)$.*

Suppose N' is the manifold obtained by surgering N along T. Then

(1) *either $N' \approx N \# (S^2 \times S^2)$ or $N' \approx N \# (P \# Q)$ and if N is of odd type, then*

(2)

$$N \# (S^2 \times S^2) \approx N \# (P \# Q) \approx N'.$$

As the proof of the first part of the above result is short and introduces other notions used later we give a brief sketch of it here.

By [**W1**, Lemma 4] and [**RS**, Theorem 7.10] since C is homotopic to zero we may assume that it spans an embedded 2-disc, and lies in the interior of a 4-disc D of N. Thus $N' = N - \{\text{4-disc}\} \cup D - T \cup D^2 \times S^2$. But $D - T \approx D^2 \times S^2 - \{\text{4-disc}\}$ so that $D - T \cup D^2 \times S^2$ is diffeomorphic to either $S^2 \times S^2 - \{\text{4-disc}\}$ or to $S^2 \times S^2 - \{\text{4-disc}\}$ (where $S^2 \times S^2$ is the unique nontrivial S^2-bundle over S^2). Thus $N' \approx N \# (S^2 \times S^2)$ or $N' \approx N \# (S^2 \times S^2)$. But as in [**W1**, Lemma 1] and [**S**, pp. 135,136] $S^2 \times S^2 \approx P \# Q$. This concludes part (1).

Part (2) is just Corollary 1 of §3 of [**W1**]. We recall that the statement 'N is of odd type' is equivalent to the existence of some homology 2-class D in N with D^2 odd.

Our general references for notions from complex analysis and algebraic geometry are [**GR**], [**Sh**] and for PL-topology [**RS**], [**H**].

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Lastly by way of explanation of the title we can think of manifolds patched together via neighborhoods of things other than points as being 'irrational connected sums' of such manifolds.

2. Connected sums along 1-complexes. We now turn to the first question raised in the introduction, namely what is the structure of the manifold obtained by joining two manifolds together via the boundary of regular neighborhoods of 1-complexes in each of them. We begin with a preliminary lemma.

LEMMA 2.1. *Suppose M is a 4-manifold and K is the image of an embedding of $\bigvee_{\alpha=1}^{n} S_\alpha^1$ in M. Let T be a closed regular neighborhood of K and set $H = \partial T$.*

Let $\{e_\alpha\}$, $\alpha = 1, \ldots, n$, be a collection of disjointly embedded 1-spheres in H with e_α homotopic to $\operatorname{Im}(S_\alpha^1 \hookrightarrow M)$.

Let $\{B_\alpha\}$, $\alpha = 1, \ldots, n$, be a collection of disjoint 2-handles $\approx D^2 \times D^2$ and suppose for each α, $\phi_\alpha: (S^1 \times D^2)_\alpha \subset \partial B_\alpha \to H$ is a diffeomorphism into H with $\phi_\alpha(S^1 \times 0) = e_\alpha$ such that $T \cup_\phi B$ is a 4-ball, where $\phi = \bigcup \phi_\alpha$ and $B = \bigcup B_\alpha$.

Now suppose $\omega_\alpha: (S^1 \times D^3)_\alpha \to M$ are diffeomorphisms, $\alpha = 1, \ldots, n$, with $\omega_\alpha | (S^1 \times D^2)_\alpha = \phi_\alpha$ [where $D^2 \hookrightarrow \partial D^3$ is the upper hemisphere] and with disjoint images. Let $\chi(M)$ be the manifold obtained by doing surgery on M along the e_α with framing equivalent to ω_α. Let $M' = \overline{M - T} \cup_\phi B$. Then

$$M' \approx \overline{\chi(M) - \{4\text{-ball}\}}.$$

PROOF. Let $T_\alpha = \omega_\alpha(S^1 \times D^3)$. By isotoping T slightly we may suppose that $T_\alpha \subsetneq T$ with $\overline{T - T_\alpha} \approx T$.

Now

$$\chi(M) = \overline{M - \Big(\bigcup_\alpha T_\alpha\Big)} \cup_\omega \bigcup_\alpha (D^2 \times S^2)_\alpha,$$

$$\omega = \bigcup \omega_\alpha \Big| \bigcup \partial (S^1 \times D^3)_\alpha = \bigcup \partial (D^2 \times S^2)_\alpha \to M.$$

Identify B_α with $(D^2 \times S_+^2)_\alpha$, where S_+^2, S_-^2 are the closed upper and lower hemisphere of S^2, respectively, and note that $\omega_\alpha | (S^1 \times S_+^2)_\alpha = \phi_\alpha$. Let $\tilde{\phi}_\alpha = \omega_\alpha | (S^1 \times S_-^2)_\alpha$, $\tilde{\phi} = \bigcup \tilde{\phi}_\alpha$, $\tilde{B}_\alpha = (D^2 \times S_-^2)_\alpha$, $\tilde{B} = \bigcup \tilde{B}_\alpha$. Let

$$\chi(T) = \overline{T - \Big(\bigcup_\alpha T_\alpha\Big)} \cup_{\tilde{\phi}} \tilde{B}.$$

It is clear that

$$\chi(T) \approx T \cup_\phi B \approx \{4\text{-ball}\}$$

and therefore

$$\overline{\chi(M) - \{4\text{-ball}\}} = \overline{\chi(M) - \chi(T)} = \overline{M - T} \cup_\phi B = M'$$

as desired. We now obtain

PROPOSITION 2.2. *Let M_1, M_2 be compact 4-manifolds, set $K = \bigvee_{\alpha=1}^{n} S_\alpha^1$ and let $K_i \subset M_i$ be the image of an embedding of K in M_i, $i = 1, 2$. Let T_i be a regular neighborhood of K_i, set $H_i = \partial T_i$ and suppose Φ: $T_1 \to T_2$ is a diffeomorphism which induces the identity on $\pi_1(K)$. Let $\phi = \Phi | H_1$: $H_1 \to H_2$ and set*

$$V = \overline{M_1 - T_1} \cup_\phi \overline{M_2 - T_2}.$$

Let $\chi(M_2)$ be the result of a surgery on M_2 along disjointly embedded circles in M_2 homotopic to the images of the S_α^1. (The framings to be used for this surgery are described in the proof.)

Then if $\pi_1(K_1) \to \pi_1(M_1)$ is trivial, V is diffeomorphic to $M_1 \# (\varepsilon\chi(M_2))$ where $\varepsilon = \pm 1$ depending on orientations.

Furthermore: (1) *If M_1, M_2 are oriented with H_1, H_2 having induced orientation and ϕ is orientation preserving then $\varepsilon = -1$. Otherwise $\varepsilon = +1$.*

(2) *If, in addition, $\pi_1(K_2) \to \pi_1(M_2)$ is trivial then $\chi(M_2) = M_2 \# W$ where $W = n(S^2 \times S^2)$ or $W = n(P \# Q)$.*

PROOF. Since K_1 is homotopic to zero and of codimension 3 we may without loss of generality [W1], [RS] suppose K_1 is embedded in some 2-disc D_α lying in M_1.

Then it is clear that we can find 2-discs D_α in M, $\alpha = 1, \ldots, n$, with disjoint interiors satisfying (i) $D_\alpha \cap K_1 = \partial D_\alpha = \operatorname{Im}(S_\alpha^1 \hookrightarrow M_1)$, (ii) $K_1 \cup (\bigcup_\alpha D_\alpha)$ collapses to a point in M_1, and (iii) $\bigcup_\alpha D_\alpha$ intersects H_1 transversely in n disjoint 1-spheres. Set $B_\alpha = \overline{T(D_\alpha) - T(D_\alpha) \cap T_1}$, (where $T(X)$ will denote a small tubular neighborhood of $X \subset M$). Then

$$T_1 \cup \left(\bigcup_\alpha B_\alpha \right) \approx T\left(K_1 \cup \left(\bigcup_\alpha D_\alpha \right) \right) \approx T(\mathrm{pt}) = \text{4-ball}.$$

Set $D = T_1 \cup (\bigcup_\alpha B_\alpha)$ so that

$$\overline{M_1 - T_1 - \bigcup_\alpha B_\alpha} = \overline{M_1 - D}.$$

Now set

$$N_\alpha = B_\alpha \cap H_1 \quad \text{and} \quad D' = T_2 \cup_{\phi^{-1}|\phi(\cup_\alpha N_\alpha)} \left(\bigcup_\alpha B_\alpha \right).$$

Then clearly

$$D' = \Phi(T_1) \cup_{\phi^{-1}|\phi(\cup_\alpha N_\alpha)} \left(\bigcup_\alpha B_\alpha \right) = \tilde{\Phi}\left(T_1 \cup \left(\bigcup_\alpha B_\alpha \right) \right) = \tilde{\Phi}(D),$$

where $\tilde{\Phi}$ is a diffeomorphism obtained by extending Φ by the identity on $\bigcup B_\alpha$. Clearly then D' is a 4-ball.

Now identify B_α with $(D^2 \times D^2)_\alpha$ and note that this induces a framing $\phi_\alpha | (S^1 \times D^2)_\alpha \to N_\alpha$.

Extend each ϕ_α to a diffeomorphism ω_α: $(S^1 \times D^3)_\alpha \to T_1$ by 'pushing' N_α 'into' T_1 slightly, while keeping the various Im ω_α disjoint. Set $\tilde{\omega}_\alpha = \Phi \circ \omega_\alpha$ and $e_\alpha = \tilde{\omega}_\alpha(S^1 \times 0)_\alpha$. Note that since Φ is the identity on $\pi_1(K)$ we see that e_α is homotopic to $\operatorname{Im}(S^1_\alpha \hookrightarrow M_2)$.

Then we surger M_2 along the e_α with framing $\tilde{\omega}_\alpha$ to obtain $\chi(M_2)$.

But

$$V = \overline{M_1 - T_1} \cup_\phi \overline{M_2 - T_2}$$

$$= \overline{M_1 - T_1 - \bigcup_\alpha B_\alpha} \cup_\phi \overline{M_2 - T_2 \cup_{\phi'} \left(\bigcup_\alpha B_\alpha \right)}$$

where

$$\phi' = \phi^{-1}|\phi\left(\bigcup_\alpha N_\alpha \right)$$

and applying Lemma 2.1 we obtain

$$V = \overline{M_1 - D} \cup_\phi \overline{\chi(M) - D^1} = M_1 \# (\varepsilon\chi(M))$$

with ε depending on the orientation conditions in (1).

We extract the case of simply-connected M_i as a theorem.

THEOREM 2.3. *Suppose M_1, M_2 are oriented simply-connected compact 4-manifolds and let $K_i \subset M_i$ be the image of an embedding of $\bigvee_{\alpha=1}^n S^1_\alpha$ in M_i. Let T_i be a regular neighborhood of K_i and set $H_i = \partial T_i$. Suppose Φ: $T_1 \to T_2$ is a diffeomorphism which induces the identity on $\pi_1(K)$ and such that $\phi = \Phi|H_1$: $H_1 \to H_2$ is orientation-reversing relative to the induced orientations on the H_i.*

Let $V = \overline{M_1 - T_1} \cup_\phi \overline{M_2 - T_2}$.

Then:

(a) *V is simply connected and diffeomorphic to either $M_1 \# M_2 \# n(S^2 \times S^2)$ or to $M_1 \# M_2 \# n(P \# Q)$.*

(b) *If $M_1 \# M_2$ has nonzero 2nd Steifel-Whitney class (i.e. is not spin) then $V \approx M_1 \# M_2 \# n(P \# Q)$.*

3. Connected sums along 2-manifolds. We now consider 4-manifolds joined along tubular neighborhoods of 2-submanifolds. We recall that if S is a 2-submanifold of a 4-manifold M with tubular neighborhood T in M and if $S^* = S - \{\text{open 2-disc}\}$ then S^* has a wedge of 1-spheres $K \subset M$ as deformation retract and $T|_{S^*} = T^*$ is a regular neighborhood of K in M.

To facilitate the statement of successive statements we introduce the following definition.

DEFINITION 3.1. Let $i = 1, 2$. Let S be a 2-manifold and suppose S_i is the image of S under an embedding $S \hookrightarrow M_i$ into a 4-manifold M_i. Let T_i be a tubular neighborhood of S_i in M_i. Let $H_i = \partial T_i$.

Then we say $\phi: H_1 \to H_2$ is an identity-like diffeomorphism iff there exists a diffeomorphism $\Phi: T_1 \to T_2$ which induces the identity on $\pi_1(S)$ such that (i) there exist 2-discs $d_i \subset S_i$ such that $\Phi|T_1|_{d_1}: T_1|_{d_1} \to T_2|_{d_2}$ is a fiber-preserving diffeomorphism and (ii) $\phi = \Phi|H_1$.

By a slight abuse of notation if S is simply a 1-complex and T_i is a regular neighborhood of S_i in M_i we will continue to say ϕ is an identity-like diffeomorphism provided it is simply the restriction of a diffeomorphism Φ: $T_1 \to T_2$ inducing the identity on $\pi_1(S)$.

Our main point of interest will be diffeomorphisms $\phi: H_1 \to H_2$ which are fiber-bundle maps of the H_i as S^1-bundles. Clearly such maps will always be identity-like. Furthermore, if d_i is a disc in S_i with $\phi|H_{1|d_1}: H_{1|d_1} \to H_{2|d_2}$ a diffeomorphism, then $\phi|H_1|_{\overline{S_1 - d_1}}$ always extends to an identity-like diffeomorphism $\tilde{\phi}: H_1 \to H_2$, where $H_i^* = \partial(T_i|_{\overline{S_i - d_i}})$, and we think of $T_i|_{\overline{S_i - d_i}}$ as a regular neighborhood of an embedded 1-complex K_i = image of the 1-skeleton of (S-disc).

We can now state

Lemma 3.2. *Suppose M_1, M_2' are compact 4-manifolds with compact 2-submanifolds S_1, S_2', respectively. Let T_1, T_2' be tubular neighborhoods of S_1, S_2' with projection maps π, π' and set $H_1 = \partial T_1$, $H_2' = \partial T_2'$.*

Suppose $\eta: H_1 \to H_2$ is an identity-like diffeomorphism and $V = \overline{M_1 - T_1} \cup_\eta \overline{M_2' - T_2'}$. Suppose also some fiber C of H_1 considered as a loop in V is homotopic to zero in V. Then there exist 2-discs d_1 in S_1 and d_2 in S_2' such that if $S_1^* = \overline{S_1 - d_1}$, $S_2'^* = \overline{S_2' - d_2}$, $\tilde{H}_1 = H_1|_{S_1^*}$, $\tilde{H}_2' = H_2'|_{S_2'^*}$, $T_1^* = T_1|_{S_1^*}$, $T_2'^* = T_2'|_{S_2'^*}$, $H_1^* = \partial T_1^*$ and $H_2'^* = \partial T_2'^*$ then $\tilde{\eta} = \eta|H_1$ is a diffeomorphism of $\tilde{H}_1$ and $\tilde{H}_2'$ which extends to an identity-like diffeomorphism $\eta^*: H_1^* \to H_2'^*$ such that if

$$V' = \overline{M_1 - T_1^*} \cup_{\eta^*} \overline{M_2' - T_2'^*}$$

then V' is diffeomorphic to either $V \# (P \# Q)$ or to $V \# (S^2 \times S^2)$.

Proof. Let $\pi(C) = p \in S$, $C' = \eta(C)$ is a fiber of H' and we can without loss of generality suppose that for some discs $d_1 \ni p$ in S_1 and $d_2' \ni p' = \pi'(C') \in S'$ we have that $\eta|\tilde{H}_1$ is a diffeomorphism of $\tilde{H}_1$ onto $\tilde{H}_2'$ which extends to an identity-like diffeomorphism η^* of H_1^* onto $H_2'^*$. Let $N_1 = \pi^{-1}(d_1)$ and $N_2 = (\pi')^{-1}(d_2)$. Push N_1, respectively N_2, 'into' $\overline{M_1 - T_1}$, respectively $\overline{M_2' - T_2'}$, using some inward pointing vector field to get submanifolds $\tilde{N}_1 \approx (S^1 \times D^2) \times I$, $\tilde{N}_2 \approx (S' \times D^2) \times I$ of $\overline{M_1 - T_1}$, respectively $\overline{M_2' - T_2'}$. Note that $N_1 \subset \partial\tilde{N}_1$ and $N_2 \subset \partial\tilde{N}_2$ and using η we can identify N_1 with N_2 in V to obtain a neighborhood $N(C) = \tilde{N}_1 \cup_{N_1 = \eta(N_2)} \tilde{N}_2$ of C in V. Note that $N(C) \approx C \times D^3$. Now surger V along $N(C)$ using the induced framing of C as a fiber of H to obtain $\chi(V) = \overline{V - N(C)} \cup_f D^2 \times$

S^2 where f: $\partial N(C) \to S^1 \times S^2$ is the identifying diffeomorphism along the boundary. $\overline{\qquad\qquad}$

Let $f_1 = f|\partial\tilde{N}_1 - N_1$ and $f_2 = f|\partial\tilde{N}_2 - N_2$. Then we can write

$$\chi(V) = \overline{M_1 - T_1 - N_1} \cup_{f_1} D^2 \times S^2_+ \cup_\eta \overline{M'_2 - T'_2 - \tilde{N}_2} \cup_{f_2} D^2 \times S^2_-.$$

But

$$\overline{M_1 - T_1 - \tilde{N}_1} \cup_{f_1} D^2 \times S^2_+ \simeq M_1 - T_1 \cup_{\tilde{f}_1} D^2 \times D^2$$

and

$$\overline{M'_2 - T'_2 - \tilde{N}_1} \cup_{f_1} D^2 \times S^2_- \simeq \overline{M'_2 - T'_1} \cup_{\tilde{f}_2} D^2 \times D^2$$

where $\tilde{f}_1$, $\tilde{f}_2$ are diffeomorphisms of N_1, respectively N_2, onto $S^1 \times D^2$ induced by f_1, f_2.

However the framing on $D^2 \times S^1$ was induced by the framing of C as a fiber so that we can conclude that

$$\overline{M_1 - T_1} \cup_{\tilde{f}_1} D^2 \times D^2 \approx \overline{M_1 - T_1}\,|_{\overline{S_1 - d_1}} = \overline{M_1 - T_1^*}$$

and similarly

$$\overline{M'_2 - T'_2} \cup_{\tilde{f}_2} D^2 \times D^2 \approx \overline{M'_2 - T'^*_2}\,.$$

Thus $\chi(V) \approx V'$. But C is homotopic to zero in V. Thus by Theorem W we have that $\chi(V)$ is either $V \# (S^2 \times S^2)$ or $V \# (P \# Q)$ and so our theorem is proven.

In almost all cases of interest when we encounter manifolds V admitting the above decomposition it turns out that M'_2 arises from some manifold M_2 by means of a σ-process at some point on a compact 2-submanifold $S_2 \subset M_2$. We then have

LEMMA 3.3. *Suppose M_1, M_2, M'_2 are compact 4-manifolds containing 2-submanifolds S_1, S_2, S'_2, respectively. Suppose $p \in S_2 \subset M_2$ and σ: $M'_2 \to M_2$ (resp. $\bar{\sigma}$: $M'_2 \to M_2$) is a smooth map of M'_2 onto M_2 such that* (i) *$L = \sigma^{-1}(p)$ is an embedded 2-sphere in M'_2 and $L \cdot L = -1$ (resp. $L = \bar{\sigma}^{-1}(p)$ an embedded 2-sphere with $L \cdot L = +1$),* (ii) *σ (resp. $\bar{\sigma}$) restricts to a diffeomorphism of $M'_2 - L$ onto $M_2 - p$ and S'_2 onto S_2. (The above hypotheses on σ are equivalent to assuming that σ: $M'_2 \to M_2$ is M_2 blown up by a σ-process at $p \in S_2 \subset M_2$ and S'_2 is the strict image of S_2 in M'_2.) Let T_1, T_2, T'_2 be tubular neighborhoods of S_1, S_2, S'_2, respectively, and set $H_1 = \partial T_1$, $H_2 = \partial T_2$, $H'_2 = \partial T'_2$. Suppose η: $H_1 \to H'_2$ is identity-like and that $V = \overline{M_1 - T_1} \cup_\eta \overline{M'_2 - T'_2}$. Then there exist a 2-disc d_2 about p in S_2 and a 2-disc d_1 in S_1, such that, denoting $S_1^* = \overline{S_1 - d_1}$, $S_2^* = \overline{S_2 - d_2}$, $T_i^* = T_i|_{S_i^*}$, $H_i^* = \partial T_i^*$, $\tilde{H}_i = H_i|_{S_i^*}$ $\eta|\tilde{H}_1$ is a diffeomorphism of $\tilde{H}_1$ onto $\tilde{H}_2$ which extends to an identity-like diffeomorphism η^*: $H_1^* \to H_2^*$, such that if $V^\# = \overline{M_1 - T_1^*} \cup_{\eta^*} \overline{M_2 - T_2^*}$, then $V^\#$ is diffeomorphic to $V \# P$ (respectively, $V \# Q$).*

Furthermore, if F is a fiber of H_1 in V then F is homotopic to zero in V.

REMARK. We shall indicate two proofs of the lemma. First we shall show how Lemma 3.3 is a consequence of Lemma 3.2. We then shall sketch an independent proof of Lemma 3.3 not involving Lemma 3.2 or surgery theory but rather more heavily algebraic-geometric.

PROOF. (1) Let p' be the unique point in $S_2' \cap L$. Without loss of generality we can suppose that $\overline{L \cap T_2'}$ is just the fiber of T_2' over p'. Let $C' = \overline{L \cap H_2'}$. Without loss of generality we may also suppose that $C = \eta^{-1}(C')$ is a fiber of H_1 over some point $q \in S_1$. Now $L^* = \overline{L \cap M_2' - T_2'}$ can be considered as an embedded 2-disc in V bounding C so that C is homotopic to zero in V. We can thus use Lemma 3.2 to obtain that $\chi(V) \approx \overline{M_1 - T_1^*} \cup \overline{M_2' - T_2'^*}$ (using the notations of Lemma 3.2). But $L \subset \overline{M_2' - T_2^{*\prime}}$ by construction so that $L \subset \chi(V)$. Now L is an embedded 2-sphere with self-intersection -1 (resp. $+1$). Thus if N is a tubular neighborhood of L we can easily verify that $\partial N = S^3$ and $N \approx Q - \{4\text{-ball}\}$. Taking N sufficiently small we note that

$$\chi(V) - N \approx \overline{M_1 - T_1} \cup \overline{(M_2' - T_2' - N)}$$

and

$$M_2' - T_2' - N \approx M_2' - T_2^* - \{4\text{-ball}\}.$$

Thus $\chi(V)$ is diffeomorphic to $V^{\#} \# Q$ (resp. $V^{\#} \# P$ if $L^2 = +1$). However using Lemma 3.2 and noting that $L^2 \not\equiv 0 \pmod 2$ we see that $\chi(V) \approx V \# P \# Q$. In addition, analyzing the surgery we performed on C in Lemma 3.2 we can obtain that our tubular neighborhood N above can be identified with the factor $\# Q$ in $V \# P \# Q$. (Resp. the factor $\# P$ if $L^2 = +1$.) Thus splitting off N we find $V^{\#} \approx V \# P$ (resp. $V^{\#} \approx V \# Q$ if $L^2 = +1$).

(2) Let p, p', L be as in (1). Let $\zeta \in L - \overline{L \cap T_2'} \subset V$ and blow up B at ζ by a $\bar{\sigma}$-process. Let M_2'' denote M_2' blown up at ζ and let A'' be the strict image of any $A' \subset M_2'$. We thus have that

$$V \# P \approx \overline{M_1 - T_1} \cup_{\eta'} \overline{M_2'' - T_2''},$$

where η' is the obvious induced diffeomorphism. We note that L', the strict image of L in M_2'', is a sphere with $L' \cdot L' = 0$. Thus we can find a disc d about $p'' \in S_2''$ and a collection $G \approx d \times L'$ of 2-spheres in M_2'' such that $G \cap S_2'' = d$ and $G \cap T_2'' = (\pi_2'')^{-1}(d)$ where $\pi_2''\colon T_2'' \to S_2''$ is the obvious projection map. Let $B = \overline{G - G \cap T_2''}$ so that $B \approx D^2 \times D^2$ with $B \cap H_2'' = \partial B \cap H_2' \approx S^1 \times D^2$.

We can write

$$V \# P \approx \overline{M_1 - T_1 \bigcup_{\eta_1'} B} \cup_{\tilde{\eta}'} \overline{M_2'' - T_2'' - B}$$

where $\eta'_1 = \eta'|\eta'^{-1}(B \cap H''_2)$ and $\tilde{\eta}' = \eta'|H_1 - \eta'^{-1}(B \cap H''_2)$. It is clear that $\overline{M_1 - T_1 \cup_{\eta'_1} B} \approx \overline{M_1 - T_1^*}$ and to conclude our proof it suffices to show that

$$\overline{M''_2 - T''_2 - B} \approx \overline{M_2 - T_2^*} \qquad (*)$$

(where d_2 is the image of some disc D in S_2 and d_1 is obtained in the obvious fashion in S_1 and T_1^*, T_2^* are defined accordingly).

To demonstrate (*) first denote the preimage of ζ in M''_2 by Γ and let Z be a regular neighborhood of the wedge of 2-spheres $L' + \Gamma$ in M''_2. Let $D = Z \cap S''_2$. Thus D is a disc on S''_2 about p'' and we may assume without loss of generality that $D = d$ and that $Z \cap T''_2 = G \cap T''_2$.

Then it is readily verifiable that

$$T''_2 \cap (M''_2 - Z) \approx T''_2|_{\overline{S''_2 - D}} = T''^*_2$$

and since $B \subset Z$ if d_2 is the image of D in M then

$$\overline{M''_2 - T''_2 - B} \approx \overline{M_2 - T_2|_{\overline{S_2 - d}} - T_2|_d} \bigcup_{\partial(T_2|d) = \partial Z} \overline{Z - T''_2 \cap Z - B}$$

where the identification is that induced by the identification of $\partial(T_2|d)$ with $\partial Z \approx S^3$.

It is thus sufficient to show that $\overline{Z - T''_2 \cap Z - B}$ is a 4-ball to conclude that $M''_2 - T''_2 - B \approx M_1 - T_2^*$ where $T_2^* = T_2|_{\overline{S_2 - d_2}}$. But

$$Z - T''_2 \cap Z - B = Z - G.$$

But Z is just a regular neighborhood of the wedge of 2-spheres $L + \Gamma$ and so $Z \approx P \# Q - R$ where R is a 4-ball. G is just a tubular neighborhood of a fiber of Z. We can thus write $Z - G \approx S^2 \times D^2 - R$. Let ρ: $S^2 \times D^2 \to D^2$ be the projection map. Then it is seen that

$$R \cap \partial(S^2 \times D^2) = \partial R \cap \partial(S^2 \times D^2)$$

$$= \text{tubular neighborhood of a cross-section of } \partial(S^2 \times D^2) \overset{\rho|\partial(S^2 \times D^2)}{\longrightarrow} S^1.$$

But then by Sublemma 3.3a below $Z - G$ is a 4-ball as desired.

SUBLEMMA 3.3a (COMPARE [MM1]). *Suppose π: $S^2 \times D^2 \to D^2$ is the obvious projection and ρ: $S^2 \times S^1 \to S^1$ is just $\pi|S^2 \times S^1$. Let B be a 4-ball embedded in $S^2 \times D^2$ such that*

$$R = B \cap \partial(S^2 \times D^2) = B \cap (S^2 \times S^1) = \partial B \cap (S^2 \times S^1)$$

is a tubular neighborhood in $S^2 \times S^1$ of a cross-section of ρ.

Then $\overline{S^2 \times D^2 - B}$ is a 4-ball.

PROOF. Let $Y = \overline{S^2 \times D^2 - B}$. Let $W = D^3 \times S^1$ and attach W to $S^2 \times D^2$ above so as to get $S^2 \times D^2 \cup W \approx S''$. Note that $R \subset \partial(S^2 \times D^2) = \partial W$ and that $W \approx R \times I$. Thus since $R \subset \partial B$ we have $B \cup W \approx B$. But then

$$Y \approx \overline{S^2 \times D^2 \cup W - B \cup W} \approx \overline{S^4 - (B \cup W)} \approx \overline{S^4 - B}.$$

But it is well known (see **[H]**, **[RS]** for example) that $\overline{S^4 - B}$ is a 4-ball.

Combining the results of §2 with the above lemmas we are led to

THEOREM 3.4. *Suppose M_1, M_2' are oriented compact 4-manifolds and suppose S_1, S_2' are compact 2-submanifolds of M_1, M_2', respectively, with tubular neighborhoods T_1, T_2', respectively. Set $H_1 = \partial T_1$, $H_2' = \partial T_2'$, respectively, and suppose η: $\underline{H_1 \to H_2}$ is an identity-like orientation reversing diffeomorphism.*

Let $V = \overline{M_1 - T_1} \cup_\eta \overline{M_2' - T_2'}$. Suppose K_1, K_2 are 1-skeletons of S_1, S_2' and K_1, K_2 are homotopic to zero in M_1, respectively M_2'. Then:

(1) *If C is a fiber of H_1 in V which is homotopic to zero in V, $k = rkH_1(K_1, Z)$, then either*

(a) *$V \# (S^2 \times S^2)$ is diffeomorphic to $M_1 \# M_2' \# k(S^2 \times S^2)$, or*

(b) *$V \# (P \# Q)$ is diffeomorphic to $M_1 \# M_2' \# k(P \# Q)$.*

If either M_1 or M_2' is of odd-type then alternatives (a) *and* (b) *both hold.*

(2) *If M_2' is obtained by blowing up M_2 by means of a σ or $\bar{\sigma}$ process as in Lemma* 2.3, *then in the former case*

$$V \# P \approx M_1 \# M_2 \# k(P \# Q)$$

and in the latter

$$V \# Q \approx M_1 \# M_2 \# k(P \# Q).$$

PROOF. (1) Using Lemma 3.2 and Proposition 2.2, and noting that if X is of odd type with $X \approx W \# S^2 \times S^2$, then also $X \approx W \# P \# Q$, we obtain that either

$$V \# P \# Q \approx M_1 \# M_2' \# k(P \# Q)$$

or

$$V \# (S^2 \times S^2) \approx M_1 \# M_2' \# k(S^2 \times S^2).$$

Clearly if $M_1 \# M_2'$ is of odd type the first alternative above will hold.

(2) If M_2' is obtained by a blowing up we can apply Lemma 3.3 and Proposition 2.2 to obtain the desired result.

We extract a particularly simple case of our theorem for separate mention.

COROLLARY 3.5. *Suppose M_1, M_2 are oriented simply-connected compact 4-manifolds and S_1, S_2 are oriented compact 2-submanifolds of genus g. Let M_2' be M_2 blown up by a σ-process at some point of S_2 and denote the strict image of S_2 in M_2' by S_2'. Let T_1, T_2' be tubular neighborhoods of S_1, S_2' in M_1, M_2', respectively, and set $H_1 = \partial T_1$, $H_2' = \partial T_2'$.*

Suppose η: $H_1 \to H_2'$ is a fiber bundle diffeomorphism of H_1 onto H_2' reversing orientation.

Let $V = \overline{M_1 - T_1} \cup_\eta \overline{M_2' - T_2'}$. *Then* $V \# P$ *is diffeomorphic to* $M_1 \# M_2 \# 2g(P \# Q)$.

In order to apply our results on irrational connected sums to determine the structure of algebraic surfaces we recall two results from our work in [**MM2**, §2].

THEOREM 3.6 (SEE COROLLARY 2.4 OF [**MM2**]). *Let* W *be a complex manifold. Suppose* $f\colon W \to \mathbf{C}$ *is a nonconstant proper holomorphic mapping of* W *onto a disc* Δ *about the origin such that* 0 *is a critical value of* f. *Suppose the zero divisor* Z *of* f *consists of two nonsingular irreducible components* A_1, A_2 *of multiplicity* 1 *crossing normally in a nonsingular irreducible subvariety* S.

Suppose $\Gamma \to S$ *is a tubular neighborhood of* S *in* W *such that* $T_1 = \Gamma \cap A_1$ *and* $T_2 = \Gamma \cap A_2$ *are tubular neighborhoods of* S *in* A_1, *respectively* A_2. *Set* $H_i = \partial T_i \to S$.

Then there exists an orientation-reversing bundle isomorphism $\eta\colon H_1 \to H_2$ *such that for any regular value* $\lambda \in \Delta$ *of* f *if* $V_\lambda = f^{-1}(\lambda)$ *then*

$$V \approx \overline{A_1 - T_1} \cup_\eta \overline{A_2 - T_2}.$$

THEOREM 3.7 (SEE COROLLARY 2.5 OF [**MM2**]). *Let* W *be a compact complex manifold. Suppose* V, X_1, X_2 *are compact complex submanifolds of* W *intersecting normally and denote* $X_1 \cap X_2$ *by* S *and* $V \cap X_1 \cap X_2$ *by* C. *Suppose as divisors on* W, V *is linearly equivalent to* $X_1 + X_2$.

Let $\sigma\colon X_2' \to X_2$ *be the monoidal transformation of* X_2 *with center* C *and let* S' *be the strict image of* S *in* X_2'.

Let $T_2' \to S$, $T_1 \to S$ *be tubular neighborhoods of* S' *in* X_2' *and* S *in* X_1, *respectively, with* $H_2' = \partial T_2'$ *and* $H_1 = \partial T_1$.

Then there exists a bundle-isomorphism $\eta\colon H_2' \to H_1$ *which is orientation reversing such that*

$$V \approx \overline{X_2' - T_2'} \cup_\eta \overline{X_1 - T_1}.$$

PROOF. We can reduce Theorem 3.7 to 3.6 by first blowing up W along $V \cap X_1$ and then blowing up the resultant manifold W' along the strict image of $V \cap X_2$ in W'. This then separates the strict image of V from that of $X_1 + X_2$ and allows us to apply 3.6. For more details see the above reference. For our work we need a special case of 3.7, combined with 3.5.

COROLLARY 3.8. *Suppose in Theorem* 3.7, W *is of complex dimension* 3 *and* $V \cdot X_1 \cdot X_2 = n > 0$. *Then if* S *is connected of genus* g *and* X_1, X_2 *are simply connected, we have*

$$V \# P \approx X_1 \# X_2 \# (n-1)Q \# 2g(P \# Q).$$

PROOF. By 3.7 we have $V \approx \overline{X_2' - T_2'} \cup_\eta \overline{X_1 - T_1}$. But C in our case is just n distinct points and X_2 blown up along C is then topologically $X_2 \# nQ$. Applying Corollary 3.5 we obtain

$$V \# P \simeq (X_2 \# (n-1)Q) \# X_1 \# 2g(P \# Q)$$

as desired.

4. Applications to algebraic surfaces.

LEMMA 4.1. *Suppose X is a nonsingular algebraic surface which is almost completely decomposable. Let $W \xrightarrow{\pi} X$ be an analytic CP^1-bundle over X and let F denote a fiber of $W \xrightarrow{\pi} X$.*

Suppose V is an irreducible nonsingular subvariety of W with $V \cdot F = k > 0$ and suppose there are given divisors E_j on $W, j = 1, \ldots, k$, such that $E_j \cdot F = 1$ and there exist irreducible nonsingular representatives $V_i \in |\Sigma_{j=1}^i E_j|$, $W_j \in |E_j|$ such that:

(1) V_i, V_{i+1}, W_{i+1} have normal crossing in W with $V_i \cdot V_{i+1} \cdot W_{i+1} \neq 0$ for $i = 1, \ldots, k-1$.

(2) $V_i \cap W_{i+1}$ has genus greater than zero or either W_{i+1} is rational or V_i is rational. Then if $V \in |D_k|$ then V is almost completely decomposable.

PROOF. We use induction on $k = V \cdot F$. If $k = 1$ then $\pi | V$ is a morphism of degree 1 of V onto X. Thus V is simply X blown up at a finite number m of points and so $V \approx X \# mQ$. Thus since X was A.C.D. so is V. Now suppose the lemma is true whenever $V \cdot F \leqslant l - 1$ and suppose that $V \cdot F = l > 1$.

Set $D_i = \Sigma_1^i E_j$ and let V_{l-1}, V_l, W_l be irreducible nonsingular elements of $|D_{l-1}|, |D_l|, |E_l|$, respectively, satisfying the hypothesis of our lemma. Let $t = V_{l-1} \cdot V_l \cdot W_l > 0$ and $g = \text{genus}\,(V_{l-1} \cap W_l) \geqslant 0$.

Then by Corollary 3.8 we have that

$$V_l \# P \approx V_{l-1} \# W_l \# (t-1)Q \# 2g(P \# Q). \qquad (*)$$

Suppose $g > 0$. Then by our inductive hypothesis we find V_{l-1}, W are A.C.D. and since $g > 0$ the right side of $(*)$ is completely decomposable. Thus V is A.C.D. If $g = 0$ then either V_{l-1} or W_l is rational. However since the left side of $(*)$ is of odd type we can conclude that the right side is diffeomorphic to either $V_{l-1} \# P \# (v + t - 1)Q$ or to $W_l \# P \# (w + t - 1)Q$ for some integers v, w. In either case use of the inductive hypothesis shows that V_l is A.C.D. Since V is linearly equivalent to V_l and both are nonsingular they are diffeomorphic. Thus V is A.C.D. as desired.

THEOREM 4.2. *Suppose X is an almost completely decomposable nonsingular algebraic surface. Let C be some irreducible nonsingular hypersurface section of X in some embedding $X \hookrightarrow CP^N$. Let $E = [kC]$ be the line bundle over X associated to the divisor kC, $k \geqslant 0$, and let $W \xrightarrow{\pi} X$ be the projective closure of*

$E \to X$ so that W is a CP^1-bundle over X. Let $S = \pi^{-1}(C) \subset W$ and let $X \hookrightarrow W$ also denote X embedded as the 'zero-section' of W.

Then for any $m > 0$ and $l > 0$ there exists an irreducible nonsingular $V \in |mX + lS|$. If $V_{(m,l)}$ is any irreducible nonsingular element of $|mX + lS|$ then $V_{(m,l)}$ is almost completely decomposable.

Furthermore if $k \neq 0$ then the above conclusion is also true for $l = 0$.

PROOF. Let H_c be a hypersurface of CP^N cutting out $C \subset X$ and let $d = \deg H_c$. Now fix $m > 0$ and $l \geqslant 0$ and let H_{mk+l}, J_l be hypersurfaces of degree $d(mk + l)$, dl, respectively, cutting out nonsingular hypersurface sections C_{mk+l}, S_l of X, respectively, such that C_{mk+l} intersects S_l transversely in X. Clearly using Bertini's theorem we can always produce such H_{mk+l}, J_l.

Now let $\{U_\alpha\}$ be a coordinate cover of X trivializing E and thus W. Let C_{mk+l}, S_l have local equations $e_\alpha = 0$, $d_\alpha = 0$, respectively, in U_α and suppose W has fiber coordinate $(\zeta_{0\alpha}, \zeta_{1\alpha})$. Let $V_{(m,l)} \subset W$ be the subset of W given locally by $d_\alpha \zeta_{1\alpha}^m - e_\alpha \zeta_{0\alpha}^m = 0$ (where $X \hookrightarrow W$ is given locally by $\zeta_{1\alpha} = 0$). Then an explicit computation shows that $V_{(m,l)}$ is a nonsingular subvariety of W, irreducible if $l > 0$ or $l = 0$ and $k > 0$. Furthermore we find $V_{(m,l)} \cdot F = m$ and $V_{(m,l)} \cdot C = (km + l)C^2$ so that $V_{(m,l)} \in |mX + lS|$.

Now suppose $l > 0$ or $l = 0$ and $k > 0$. Let $V \in |mX + lS|$ be an irreducible nonsingular subvariety of W. Let $D_i = iX + lS$ for $1 \leqslant i \leqslant m$ and $E_j = X$ for $2 \leqslant j \leqslant m$. It is clear by our construction of the $V_{(m,l)}$ above that we can always choose appropriate hypersurfaces in CP^N such that there exist irreducible nonsingular $V_i \in |D_i|$ and $W_j \in |E_j|$ so that V_i, V_{i+1}, W_{i+1} cross normally in W for $1 \leqslant i \leqslant m - 1$. Furthermore

$$V_i \cdot V_{i+1} \cdot W_{i+1} = i(i + 1)X^3 + 2(i + 1)lX^2S + l^2XS^2 > 0.$$

Then by the adjunction formula we have that

$$g_i = \text{genus}(V_i \cap W_{i+1}) = \text{genus}(r_iC) = \big(r_i(r_i - 1)C^2 - r_i\chi_0\big)/2$$

where $\chi_0 = 2 - 2g(C)$ and $r_i = ki + l$. Then if $g(C) > 0$ we have $g_i > 0$ all i. Suppose $g(C) = 0$. Then X has a rational hypersurface section and so must be rational. But W_j is linearly equivalent to X for all j and so is also rational.

We can thus apply Lemma 4.1 to conclude that V is A.C.D.

We can apply our theorem to analytic or branched covers (see **[GR**,III.B], a branched cover is an analytic cover with nonempty critical set).

So suppose $V \xrightarrow{\pi} X$ is as above with $k > 0$. Let V be a nonsingular irreducible subvariety of W such that $\pi|V$ exhibits V as an m-fold branched cover of X. Thus as a cycle in W we have $V \sim mX + Z$ where $Z \in \rho(H_2(X))$ in the natural splitting $H_4(W) \approx \rho(H_2(X)) \oplus H_4(X)$ induced by the portion

$$0 \to H_2(X) \xrightarrow{\rho} H_4(W) \xrightarrow{\pi} H_4(X) \to 0$$

of the Thom-Gysin sequence of the 2-sphere bundle $W \to X$. If $V \sim mX$ we will call V a pure m-fold branched cover. But in this case $V \sim V_{(m,0)}$ and so is A.C.D. by Theorem 4.2. We thus have

COROLLARY 4.3. *Suppose V is a nonsingular subvariety of W above, with π: $V \to X$ a pure m-fold branched cover of X.*
Then V is A.C.D.

We recall the following definition [**Wk**, Definition 15].

DEFINITION 4.4. Let (M, π, V) be a k-sheeted branched covering of V where M and V are complex manifolds. Let R be the branch locus.

We shall call M a cyclic covering of V if the following conditions are satisfied:

(i) For each $x \in R$, $\pi^{-1}(x)$ consists of one point.

(ii) The group of covering transformations of $M - \pi^{-1}(R)$ over $V - R$ is cyclic of order k.

(iii) If $k \neq 2$ then R is connected.

We note that as a consequence of [**Wk**, Appendix] we have that R will be nonsingular.

We now state

COROLLARY 4.5. *Suppose X is an almost completely decomposable nonsingular algebraic surface and C is an irreducible nonsingular hypersurface section of X in some embedding $X \hookrightarrow CP^N$. Suppose M is a k-fold cyclic branched covering of X with branch locus $R \in |kC|$.*
Then M is almost completely decomposable.

PROOF. By Theorem 1.2 of [**Wk**] there exists a line bundle E on X and a covering $\{V_i\}$ of X over which E is trivialized such that M is diffeomorphic to the submanifold of E defined by the equation $\xi_i^k = \phi_i$ where ξ_i is the fiber coordinate of E over V_i and $\phi_i = 0$ is the equation of R in V_i. Furthermore using the corollary to Theorem 1.2 we see that E is precisely the bundle $[C]$ over X. Let W be the projectivization of E. Then it is clear that $V \in |kX|$ on W so that by our theorem it is A.C.D.

We recall that a 'double plane' is a 2-sheeted analytic branched cover of CP^2. Corollary 4.5 now gives us

COROLLARY 4.6. *Suppose V is an irreducible nonsingular double plane. Then V is almost completely decomposable.*

PROOF. Since V is nonsingular it is in fact a 2-fold cyclic cover. But then by Corollary 4.5 it is A.C.D.

REMARK. In [**MM1**] we proved that the nonsingular hypersurfaces in CP^3 were all A.C.D. The hypersurface of degree n, V_n, thus gives us a sequence of

A.C.D. algebraic surfaces whose 'vital statistics' (i.e., geometric genus $p_g(V)$ and signature $\sigma(V)$) vary with n^3. (Specifically $p_g(V_n) = 1/6(n-1)(n-2)(n-3)$ and $\sigma(V) = -1/3n(n^2-4)$.) The sequence of double-planes with ramification locus a curve R of degree $2r$ in CP^3 give rise to a sequence of A.C.D. algebraic surfaces M_r with vital statistics varying with r^2. (In particular, $p_g(M_r) = 1/2(r-1)(r-2)$ and $\sigma(M_r) = 2(1-r^2)$.)

We also note that Corollary 4.5 provides a new (and different) proof that all hypersurfaces of degree n in CP^3 are $A \subset D$. We extract this as a separate corollary.

COROLLARY 4.7 (COMPARE [**MM1**, 5]). *Suppose V_n is a nonsingular hypersurface of CP^3 of degree n.*

Then V is almost completely decomposable.

PROOF. Since all nonsingular hypersurfaces of fixed degree are diffeomorphic we may assume V_n is given by the equation $Z_0^n + Z_1^n + Z_2^n + Z_3^n = 0$ in CP^3. Projecting this surface onto the hyperplane $Z_3 = 0$ from the point (0, 0, 0, 1) makes V_r into a cyclic n-sheeted cover of CP^2 with branch locus a nonsingular curve of degree n. Thus by Corollary 4.5 V_n is A.C.D.

Returning to Theorem 4.2 let us examine the divisors $mX + lS$ more closely. If k is positive we note the following facts.

(1) If $D_{(m,l)} = mX + lS$ then if $m > 0$ and $l > 0$ then $D_{(m,l)}$ is a very ample divisor.

This follows almost immediately from the local form

$$d_\alpha \zeta_{1\alpha}^n - e_\alpha \zeta_{0\alpha}^m = 0 \tag{*}$$

for a nonsingular $V_{(m,l)} \in |D_{(m,l)}|$. This local representation shows us that $|D_{(m,l)}|$ separates points and it is easy to verify that it also separates infinitely near points and so as in [**Ht**] $|D_{(m,l)}|$ is very ample.

(2) The map Φ associated to the linear system $|X|$ maps W onto a projective cone $\overline{CX} \hookrightarrow CP^{N+1}$ over $X \hookrightarrow CP^N$ with Φ: $W \hookrightarrow \overline{CX}$ blowing down the 'section at ∞' of W to the vertex of $\overline{CX}$.

This can also be seen by an explicit computation as above using the fact that if $V \in |X|$ and X_∞ is the '∞-section' of W then $Y \cap X_\infty = \varnothing$.

These observations then motivate the following:

THEOREM 4.8. *Suppose W is a compact analytic variety of complex dimension 3 and V is an almost completely decomposable complex submanifold of W. Suppose $|V|$ is base point free and induces a holomorphic map Φ: $W \to CP^N$ of W onto some algebraic 3-fold W' such that Φ embeds V onto a nonsingular hypersurface section V' of W' and is a diffeomorphism on some tubular neighborhood $T(V)$ of V onto $T'(V')$.*

Then for any $n \geqslant 1$

(1) *there exists a nonsingular element* $V_n \in |nV|$,

(2) *any nonsingular element of* $|nV|$ *is almost completely decomposable.*

PROOF. Since V' is a hypersurface section of W' it is clear that we can always find a nonsingular $V_n' \in |nV'|$ with $V_n' \subset T'(V')$. But then setting $\Phi^{-1}(V_n') = V_n$ we get the desired nonsingular element of $|nV|$. Furthermore since V' is a hypersurface section of W' it is clear that for any $n \geqslant 1$ we can choose nonsingular $V_n' \in |nV'|$, $V_{n+1}' \in |(n+1)V'|$, $X_{n+1}' \in |V'|$, with normal crossing in $T'(V')$. Clearly

$$V_n' \cdot V_{n+1}' \cdot X_{n+1}' = n(n+1)(V')^3 \geqslant n(n+1)$$

and either genus $(V_n' \cap X_{n+1}') > 0$ or X_{n+1}' is rational. Then the same argument used to prove Lemma 4.1 works here also and we can conclude that V_n' is A.C.D. for all n. But then clearly so is V_n and since nonsingular linearly equivalent divisors of W are diffeomorphic any nonsingular element of $|nV|$ is also A.C.D.

An immediate corollary is, of course,

COROLLARY 4.9. *Suppose W is a compact analytic variety of complex dimension 3 and V is an almost completely decomposable complex submanifold of W which defines a very ample line bundle $[V]$ on W. Then W is projective algebraic and for any $n \geqslant 1$*

(1) *there exists a nonsingular element of* $V_n \in |nV|$,

(2) *any nonsingular element of* $V_n \in |nV|$ *is almost completely decomposable.*

PROOF. Since very ample line bundles induce embeddings and V is nonsingular it is easy to see that all the conditions of our theorem are satisfied.

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