# ON THE EXISTENCE OF GOOD MARKOV STRATEGIES

BY

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ABSTRACT. In contrast to the known fact that there are gambling problems based on a finite state space for which no stationary family of strategies is at all good, in every such problem there always exist $\varepsilon$-optimal Markov families (in which the strategy depends only on the current state and time) and also $\varepsilon$-optimal tracking families (in which the strategy depends only on the current state and the number of times that state has been previously visited). More generally, this result holds for all finite state gambling problems with a payoff which is shift and permutation invariant.

**1. Introduction.** Suppose to each element of a finite set $F$ is associated a nonempty collection $\Gamma(f)$ of transition probabilities on $F$. If one is at state $f \in F$, he selects a transition probability $\alpha$ from $\Gamma(f)$, and his next position is chosen according to the distribution of $\alpha$. During the course of the process, the selection of a transition probability (gamble) is based solely on the finite sequence of states already visited. A typical objective is to find a strategy which (nearly) maximizes the probability, under all available strategies, of hitting a particular subset $G$ of $F$ infinitely often.

The two main positive results for such a finite state problem, both due to Dubins and Savage, state that [3, Theorem 3.9.1] if each $\Gamma(f)$ is finite, then there is available a stationary family of optimal strategies, and [3, Theorem 3.9.2] if the gambler is permitted to stay at any state as long as he pleases (that is, the Dirac measure $\delta(f)$ is in $\Gamma(f)$ for each $f \in F$), then there is available a stationary family of strategies which is uniformly (nearly) optimal. The latter result was extended by Ornstein [8, Theorem B] to countable $F$, and by Sudderth [10, Theorem 2.3] to a much larger class of problems, including many with uncountable state space and finitely additive transition probabilities.

If, however, the sets $\Gamma(f)$ are completely arbitrary, it is known [3, Example 3.9.2] that even for finite state problems, stationary strategies may be worthless. Dubins has raised the question as to whether there always exist nearly

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optimal Markov strategies, that is, strategies for which the selection rule depends only on the time and the current state. As is the purpose of this paper to show, this question has an affirmative answer for finite state problems and leaves open the general question. This paper also shows that for finite state problems there also exist strategies in which the selection rule depends only on the current state, and the number of times that state has been previously visited. This result, and even persistent $\varepsilon$-optimality, holds for a larger class of utility functions than those treated by Dubins and Savage in [3].

An intermediate result (Proposition 3.2) yields a generalization of the classical decomposition theorem for finite Markov chains with stationary transition probabilities.

**2. Preliminaries.** Notation and terminology will generally follow that of Dubins and Savage [3]. In addition, for a set $F$, $|F|$ will denote the cardinality of $F$. $F^{\mathbf{N}}$ will denote the one-sided infinite sequences in $F$, a typical element of which is $h = (f_1 f_2 f_3 \ldots)$. $F^*$ will denote the free monoid generated by $F$ (that is, the set of finite sequences in $F$, including the empty sequence "$\varnothing$", with the binary operation concatenation). For an element $p = (f_1 f_2 \cdots f_n)$ of $F^*$, $\|p\|$ will denote the length of $p$. For a positive integer $k$, $F^{(k)}$ will denote the set of elements of $F^*$ with length less than $k$. For any subset $S$ of $F^*$, $(S - )$ will denote the subset $SF^{\mathbf{N}}$ of $F^{\mathbf{N}}$, that is, all one-sided infinite sequences in $F$ with initial segment in $S$. Thus, for $f \in F$ and $C \subset F$, the set $C^*f -$ is that collection of $h = (f_1 f_2 \ldots)$ such that there is an $n$ with $f_n = f$ and such that $f_k \in C$ for all $k < n$.

The same symbol $\sigma$ will be used to denote both a strategy (function from $F^*$ to probability measures on $F$) and the probability measure generated by $\sigma$ on the Borel subsets of $F^{\mathbf{N}}$ (with product topology generated by the discrete topology on $F$). If $F$ is finite, as is the case in the main results of this paper, then the measure generated by each strategy $\sigma$ is countably additive. For general $F$, the existence of a natural (finitely additive) measure determined by $\sigma$ on the Borel subsets of $F^{\mathbf{N}}$ is the main content of a paper by Purves and Sudderth [9].

A gambling house $(F, \Gamma)$ consists of a set $F$ and a function $\Gamma$ from $F$ to subsets of probability measures on (all subsets of) $F$. A strategy $\sigma$ in $\Gamma$ at $f$ is a strategy such that $\sigma(\varnothing) \in \Gamma(f)$, and $\sigma(pf') \in \Gamma(f')$ for all $p \in F^*$ and all $f' \in F$.

DEFINITION. In a gambling house $(F, \Gamma)$, $P_f$ and $P$ are the functions from the Borel subsets of $F^{\mathbf{N}}$ to $[0, 1]$ given by

$$P_f(B) = \sup\{\sigma(B): \sigma \text{ is in } \Gamma \text{ at } f\} \quad \text{and}$$
$$P(B) = \sup\{P_f(B): f \in F\}.$$

For a bounded function $u: F \to \mathbf{R}$, $u^*: F^{\mathbf{N}} \to \mathbf{R}$ is the function defined by $u^*(f_1 f_2 \ldots) = \lim \sup_{n \to \infty} u(f_n)$, and $V: F \to \mathbf{R}$ is the function $V(f) = \sup\{\int u^* \, d\sigma: \sigma \text{ is in } \Gamma \text{ at } f\}$. Considerable use will be made of the following result, which is an immediate consequence of [3, Theorem 3.9.2] and [11, Theorem 3.2].

PROPOSITION 2.1. *Let $(F, \Gamma)$ be a gambling house with $|F| < \infty$ and $\delta(f) \in \Gamma(f)$ for all $f \in F$ (i.e. $\Gamma$ is leavable). Then for each $u: F \to \mathbf{R}$, and each $\varepsilon > 0$, there exists a stationary family $\bar{\sigma}$ in $\Gamma$ such that $\int u^* \, d\bar{\sigma}(f) \geqslant V(f) - \varepsilon$ for all $f \in F$.*

**3. Decomposition of the state space.** The purpose of this section is to generalize, to include all gambling houses, the classical Markov chain notions of communicating states, transient states, communicating classes, and closed communicating classes, and then to prove (Proposition 3.2) a generalization of the classical decomposition of the state space of a finite Markov chain with stationary transition probabilities.

Let $(F, \Gamma)$ be any gambling house, and $f, \hat{f} \in F$.

DEFINITION. $f$ and $\hat{f}$ *communicate* (in $\Gamma$), written $f \equiv \hat{f}$, if $P_f(F^*\hat{f} -) = P_{\hat{f}}(F^*f -) = 1$.

LEMMA 3.1. *The following are equivalent*:
(i) $f \equiv \hat{f}$,
(ii) $P(\text{``}f\text{''} \ i.o. \cap \text{``}\hat{f}\text{''} \ i.o.) > 0$,
(iii) $P(\text{``}f\text{''} \ i.o. \cap \text{``}\hat{f}\text{''} \ i.o.) = 1$.

LEMMA 3.2. *If $f \equiv \hat{f}$ (in $\Gamma$), then for every classical gambler's problem $(\Gamma, u)$, $V(f) = V(\hat{f})$.*

The converse of Lemma 3.2 is not true, as is seen by the following example.
EXAMPLE 3.1. $F = \{a, b\}$, $\Gamma(a) = \Gamma(b) = \{\delta(a)\}$.
DEFINITION. $f$ is *transient* if $f \not\equiv \hat{f}$.
DEFINITION. $C \subset F$ is a *communicating class* if $f \equiv \hat{f}$ for all $f, \hat{f} \in C$.

PROPOSITION 3.1. *Let $(F, \Gamma)$ be any gambling house. Then there is a unique decomposition of $F$ into a transient class, and maximal communicating classes.*

PROOF. Let $F_0 = \{f \in F: f \text{ is transient}\}$. Then $\equiv$ is an equivalence relation on $F \setminus F_0$. $\square$

It is possible that the transient class is empty (e.g. any gambling house with $|F| = 1$) and, as the following easy example shows, it is possible that all states are transient.
EXAMPLE 3.2. $F = \{1, 2, 3, \ldots\}$, $\Gamma(n) = \{\delta(n + 1)\}$.
DEFINITION. A subset $C$ of $F$ is *closed* for $\Gamma$ if

$$P_f(C^*f' -) = 1 \quad \text{for all } f, f' \in C.$$

Plainly, only communicating classes can be closed.

A conditional form of the Borel-Cantelli Lemma [2, Corollary 2, p. 324] due to P. Lévy is stated here for easy reference:

LEMMA 3.3. *Let $M_1, M_2, \ldots$ be measurable sets and let $a_k$ be the conditional probability of $M_k$ given (the field generated by) $M_1, \ldots, M_{k-1}$. Then the set of points in infinitely many $M_j$'s, and the set of points of divergence of the series $\sum_1^\infty a_j$ differ by at most a set of measure zero.*

PROPOSITION 3.2. *Let $(F, \Gamma)$ be any gambling house with $|F| < \infty$. Then the maximal communicating classes are closed, and at least one is nonempty.*

PROOF. Let $C \subset F$ be a maximal communicating class, and let $f, \hat{f} \in C$. Since $f$ and $\hat{f}$ communicate, $P_f(F^*\hat{f} -) = 1$. An easy application of Lemma 3.3 shows, since $|F \setminus C| < \infty$, that $P_f(C^*\hat{f} -) = 1$, which implies that $C$ is closed.

Since $|F| < \infty$, there exists $f \in F$ with $P(\text{"}f\text{"}$ i.o.$) > 0$. Lemma 3.1 implies $f$ is not transient, that is, $f$ is contained in some (maximal) communicating class. ☐

A communicating class $C$ may fail to be closed if it is not maximal, or if $F$ is not finite, as the following examples show.

EXAMPLE 3.3. $F = \{a, b, c\}$, $\Gamma(a) = \{\delta(b)\}$, $\Gamma(b) = \{\delta(c)\}$, $\Gamma(c) = \{\delta(a)\}$. $C = \{a, b\}$ is a cofinite communicating class which is not closed.

EXAMPLE 3.4. $F = \{0, 1, 2, 3, \ldots\}$. $\Gamma(0) = \{\delta(0)\}$, $\Gamma(1) = \{\delta(0)/k + ((k - 1)/k)\delta(k): k > 1\}$, $\Gamma(n) = \{\delta(1)\}$ for $n > 1$. $C = \{1\}$ is a maximal communicating class which is not closed.

Proposition 3.2 is a generalization of the classical decomposition (classification of states) theorem for finite Markov chains with stationary transition probabilities, since such a Markov chain is simply a gambling house $(F, \Gamma)$ with $|\Gamma(f)| = 1$ for all $f \in F$, and hence only one strategy $\sigma$ in $\Gamma$.

**4. Stationary families of strategies.** A family of strategies $\bar{\sigma}$ is *stationary* [3] if $\bar{\sigma}(f)(\varnothing) = \bar{\sigma}(f')(pf)$ for all $f, f' \in F$ and all $p \in F^*$. Alternatively, there is one to one correspondence between stationary families and *Markov kernels*, that is, functions $\gamma$ from $F$ to probability measures on $F$, and the stationary family determined by $\gamma$ is written $\gamma^\infty$. If $\gamma(f) \in \Gamma(f)$ for all $f \in F$, then $\gamma$ is a $\Gamma$-*selector*. A stationary family corresponds to a Markov chain with stationary transition probabilities.

The first three lemmas of this section demonstrate several properties of stationary families; and the rest of the section demonstrates, for each gambling house, the existence of stationary families with certain properties.

LEMMA 4.1. *If $|F| < \infty$, and $\gamma_n$, $\gamma$ are Markov kernels on $F$ such that $\lim \gamma_n(f)(f') = \gamma(f)(f')$ for all $f, f' \in F$, then for every open subset $\mathcal{O}$ of $F^N$,*

$\lim \inf_{n \to \infty} \gamma_n^\infty(f)(\mathbb{O}) \geqslant \gamma^\infty(f)(\mathbb{O})$ *for all f in F.*

PROOF. Since $F^*$ is countable, each open set $\mathbb{O}$ is the countable disjoint union of sets of the form $(p_i - )$, where $p_i \in F^*$. Since $\lim_{n \to \infty} \gamma_n^\infty(f)(p_i - )$ $= \gamma^\infty(f)(p_i - )$ for all $f \in F$ and all $p_i \in F^*$, we have that

$$\gamma^\infty(f)(\mathbb{O}) = \sum_{i=1}^\infty \gamma^\infty(f)(p_i - ) = \sum_{i=1}^\infty \lim_{n \to \infty} \gamma_n^\infty(f)(p_i - )$$

$$\leqslant \lim_{n \to \infty} \inf \sum_{i=1}^\infty \gamma_n^\infty(p_i - ) = \lim_{n \to \infty} \inf \gamma_n^\infty(f)(\mathbb{O})$$

for all $f \in F$, where the inequality follows by Fatou's lemma. $\square$

As the following example shows, even for finite $F$, strict inequality may occur and lim inf may not be replaced by lim.

EXAMPLE 4.1. $F = \{a, b\}$. Let $\gamma(b) = \delta(b)$, and for $n = 1, 2, \ldots,$ let $\gamma_n(a) = \gamma(a) = \delta(a)$, and $\gamma_n(b) = \delta(a)/n + (n - 1)\delta(b)/n$. Then for $\mathbb{O} = \{(f_1 f_2 \ldots) \in F^N: f_i = a$ for some $i\}$, $\gamma_n^\infty(b)(\mathbb{O}) = 1 > 0 = \gamma^\infty(b)(\mathbb{O})$, for all $n > 1$. If, instead, $\gamma_n(b) = \delta(b)$ for $n$ odd, then $\lim_{n \to \infty} \gamma_n^\infty(b)(\mathbb{O})$ does not exist.

As in [5], a *stopping time* $t$ is a function from $F^N$ to $\mathbf{N} \cup \{\infty\}$ such that if $h'$ agrees with $h$ through the $t(h)$th coordinate, then $t(h) = t(h')$. For two stopping times $s$ and $t$, $t * s$ is the stopping time defined by $(t * s)(h) = \infty$ if $s(h) = \infty$, and otherwise $= s(h) + t(f_{s(h)+1} f_{s(h)+2} \ldots )$. For a stopping time $s$, and positive integer $n$, $s^n$ is the stopping time defined inductively by $s^1 = s$, $s^n = s * s^{n-1}$. As in [7], a *hitting time* $t$ is a stopping time such that, for some subset $D$ of $F$, $t(h) = \min\{m \geqslant 1: f_m \in D\}$ for all $h \in F^N$.

LEMMA 4.2. *Let s be a stopping time, and t a hitting time; then on $\{s < t\}$, $t = t * s$.*

PROOF. Let $D$ be the image of $f_t: F^N \to F$ (that is, $D$ is the set where $t$ stops), and let $h = (f_1 f_2 \ldots ) \in \{s < t\}$. Then

$$t(h) = \min\{m \geqslant 1: f_m \in D\}$$
$$= s(h) + \min\{m \geqslant 1: f_{s+m} \in D\} = t * s(h). \quad \square$$

[The conclusion of Lemma 4.2 may fail if $t$ is not a hitting time. For instance, let $s$ be identically 1, and $t$ identically 2. Then $s < t$ everywhere, but $t = 2 \neq 3 = t * s$.]

LEMMA 4.3. *Let r, s be stopping times, $\eta, \zeta > 0$, and $\gamma$ a Markov kernel.*
(i) *If t is a hitting time with $\gamma^\infty(f)(r < t) > \eta$ and $\gamma^\infty(f)(s < t) > \zeta$ for all $f \in F$, then $\gamma^\infty(f)(r * s < t) > \eta\zeta$, and consequently $\gamma^\infty(f)(s^n < t) > \zeta^n$, for all $f \in F$, and all $n \in \mathbf{N}$.*
(ii) *Let t be a constant stopping time. If $\gamma^\infty(f)(r < t) < \eta$ and $\gamma^\infty(f)(s < t)$*

$< \zeta$ for all $f \in F$, then $\gamma^\infty(f)(r * s < t) < \eta\zeta$ and $\gamma^\infty(f)(s^n < t) < \zeta^n$ for all $f \in F$ and all $n \in \mathbf{N}$.

PROOF. (i)

$$\gamma^\infty(f)(r * s < t) = \int_{s < t} \gamma^\infty(f)[p_s](r * sp_s < tp_s)d\gamma^\infty(f)(p_s)$$

$$= \int_{s < t} \gamma^\infty(f)[p_s](r * sp_s < t * sp_s)\, d\gamma^\infty(f)(p_s)$$

$$= \int_{s < t} \gamma^\infty(f_s)(r < t)\, d\gamma^\infty(f)(p_s) > \eta\gamma^\infty(f)(s < t) > \eta\zeta.$$

The first equality follows by 3.7.1 of [3], and since $\{r * s < t\} \subset \{s < t\} \subset \{s < \infty\}$; the second by Lemma 4.2 since $t$ is a hitting time; the third by the stationarity of $\gamma^\infty(f)$; and the inequalities by those in the hypotheses.

(ii)

$$\gamma^\infty(f)(r * s < t) = \int_{s < t} \gamma^\infty(f)[p_s](r * sp_s < tp_s)\, d\gamma^\infty(f)(p_s)$$

$$= \int_{s < t} \gamma^\infty(f)[p_s](r * sp_s < t)\, d\gamma^\infty(f)(p_s)$$

$$\leqslant \int_{s < t} \gamma^\infty(f_s)(r < t)\, d\gamma^\infty(f)(p_s) < \eta\zeta.$$

The first equality follows since $\{r * s < t\} \subset \{s < t < \infty\}$; the second since $tp_s = t$ for constant stopping times $t$; the first inequality by the stationarity of $\gamma^\infty(f)$, and the last inequality by those in the hypotheses. $\square$

The next two examples show, respectively, that (i) may fail if $t$ is not a hitting time, and (ii) may fail if $t$ is not constant.

EXAMPLE 4.2. Let $s$ be identically 1, and $t$ identically 2. Then $\gamma^\infty(f)(s < t) = 1$ for all $f$, but $\gamma^\infty(f)(s^2 < t) = 0$ for all $f$.

EXAMPLE 4.3. Let $F = \{a, b\}$, $\Gamma(a) = \Gamma(b) = \{\delta(a)/2 + \delta(b)/2\}$, let $s$ be identically 2, and define $t$ by $t(a - ) = 1$ and $t(b - ) = 5$. Then $\gamma^\infty(f)(s < t) = \frac{1}{2}$ for all $f \in F$, but $\gamma^\infty(f)(s^2 < t) = \frac{1}{2}$ for $f \in F$.

For the remainder of this section $(F, \Gamma)$ is a fixed gambling house with $|F| < \infty$, and $C \subset F$ is a maximal communicating class.

LEMMA 4.4. Let $g \in C$. Then for each $\varepsilon > 0$ there exists a $\Gamma$-selector $\gamma$ with $\gamma^\infty(f)(C^*g - ) > 1 - \varepsilon$ for all $f \in C$.

PROOF. Consider the classical gambler's problem $(\Gamma', u)$ on $F$ defined by $\Gamma'(f) = \Gamma(f) \cup \{\delta(f)\}$ for $f \in C$, $\Gamma'(f) = \{\delta(f)\}$ for $f \in F \setminus C$; and $u(f) = 1$ if $f = g$ and zero otherwise. Clearly $C$ is a closed maximal communicating class for $\Gamma'$. Since $F \setminus C$ is an absorbing class, Lemma 3.1 and Proposition 2.1 imply the existence for each $\varepsilon > 0$ of a $\Gamma'$-selector $\gamma'$ satisfying $\gamma'^\infty(f)(C^*g - )$

$> 1 - \varepsilon/2$ for all $f \in C$. But any $\Gamma$-selector $\gamma$ agreeing with $\gamma'$ on $C \setminus \{g\}$, and such that $\gamma(g)(C) > 1 - \varepsilon/2$, satisfies the conclusion of the lemma. $\square$

The next lemma is well known [6, p. 378, problem 18], and is recorded here for convenience.

LEMMA 4.5. *Let $\gamma$ be a Markov kernel, and $B$ a finite closed communicating class under $\gamma$. Then there exists $\zeta > 0$ such that $\gamma^{\infty}(f)(f_j = f'$ for some $j \in \{1, \ldots, |B|\}) > \zeta$ for all $f, f' \in B$.*

DEFINITION. A finite subset $B$ of $F$ is *positive recurrent* for $\Gamma$ if there exists $\zeta > 0$ with the following property: for each $\varepsilon > 0$ there is a $\Gamma$-selector $\gamma$ satisfying the following two conditions:
  (i) $\gamma^{\infty}(f)(f_n = f'$ for some $n \in \{1, \ldots, |B|\}) > \zeta$ for all $f, f' \in B$,
  (ii) $\gamma^{\infty}(f)(f_n \in B$ for all $n \in \{1, \ldots, |B|\}) > 1 - \varepsilon$ for all $f \in B$.

PROPOSITION 4.1. *Let $g \in C$. Then there exists $\zeta > 0$ with the following property: for each $\varepsilon > 0$ there is a positive recurrent set $B \subset C$, a $\Gamma$-selector $\gamma'$, and a positive integer $J$ satisfying the following two conditions:*

$$\gamma'^{\infty}(f)(f_j = g \text{ for some } j \in \{1, \ldots, J\}) > \zeta \quad \text{for all } f \in B, \quad (4.1)$$

$$\gamma'^{\infty}(f)(f_j \in C \text{ for all } j \in \{1, \ldots, J\}) > 1 - \varepsilon \quad \text{for all } f \in C. \quad (4.2)$$

PROOF. By Lemma 4.4, for each $n \geq 1$ there exists a $\Gamma$-selector $\gamma_n$ satisfying

$$\gamma_n^{\infty}(f)(C^*g -) > 1 - 1/n \quad \text{for all } f \in C. \quad (4.3)$$

Since the transition probability matrices on $F$ are sequentially compact (convergence entry-wise) if $|F| < \infty$, by considering subsequences it may be assumed that for some $\gamma$, not necessarily in $\Gamma$, $\lim_{n \to \infty} \gamma_n(f) = \gamma(f)$ in total variation. By (4.3), $\gamma(f)(C) = 1$ for all $f \in C$. Let $B_1, \ldots, B_k$ denote the $\gamma^{\infty}$ closed communicating classes in $C$, and let $\zeta_1, \ldots, \zeta_k$ be as in Lemma 4.5 for $B = B_j$. Let $2\zeta = \min \zeta_j$. The remainder of the proof will consist of showing that this $\zeta$ satisfies the conclusion of the proposition.

By Lemma 4.1, and the definition of $\zeta$, it may further be assumed that each $\gamma_n$ also satisfies

$$\gamma_n^{\infty}(f)\left(f_i = f' \text{ for some } i \in \{1, \ldots, |B_j|\}\right) > 2\zeta$$

$$\text{for all } j = 1, \ldots, k \text{ and all } f, f' \in B_j. \quad (4.4)$$

For each $j = 1, \ldots, k$, and $f \in B_j$, $\gamma(f)(B_j) = 1$, so by Lemmas 4.5 and 4.1 it follows that $B_j$ is a positive recurrent set. Let $R = B_1 \cup \cdots \cup B_k$, and let $T$ denote the set of $\gamma$ transient states in $C$, that is, $T$ is $C \setminus R$. Fix $\varepsilon > 0$.

  Case 1. $g \in B_j \subset R$. Let $B = B_j$, and $J = |B_j|$. Since $\gamma^{\infty}(f)(C^{\mathbb{N}}) = 1$ for all $f \in C$, and since $\gamma_n \to \gamma$, there exists an $n \geq 1$ such that $\gamma_n(f)(f_j \in C$ for all

$j \in \{1, \ldots, |B|\}) > 1 - \varepsilon$ for all $f \in C$. Let $\gamma' = \gamma_n$. (4.2) follows from choice of $n$, and (4.1) follows from (4.4), with $f' = g$, $\gamma' = \gamma_n$, and $B = B_j$.

*Case* 2. $g \in T$. Since $\gamma^\infty(f)(T^*R -) = 1$ for all $f \in T$, by Lemma 4.1 it may further be assumed that each $\gamma_n$ also satisfies

$$\gamma_n^\infty(f)(T^*R -) > 1 - 1/n \quad \text{for all } f \in T. \tag{4.5}$$

Let $c = |C|$, and find positive integers $k$ and $n > 2$ such that $(1/2)^k < \varepsilon/2$, $(1 - 1/n)^{3kc} > 1 - \varepsilon/2$, and $1/n < \varepsilon$.

By (4.3) there exists $\tilde{g} \in R$, and $m \geqslant 1$ such that

$$\gamma_n^\infty(\tilde{g})(C^{(m)}g -) \geqslant \tfrac{1}{2} > \gamma_n^\infty(f)(C^{(m-1)}g -) \quad \text{for all } f \in R. \tag{4.6}$$

(Recall that $C^{(m)}$ consists of all elements in $C^*$ with length less than $m$.)

Let $B = B_j \subset C$ be the $\gamma^\infty$ closed communicating class containing $\tilde{g}$, let $J = m + |B|$, and let $\gamma' = \gamma_n$. Then (4.1) follows from (4.4) (letting $f' = \tilde{g}$) and (4.6).

It remains only to show (4.2) is satisfied by this choice of $\gamma'$ and $J$. In other words, if $t$ is the hitting time of $F \setminus C$, then it must be shown that $\gamma'^\infty(f)(t \leqslant J) < \varepsilon$ for all $f \in C$.

To show this, let $s_1$ be the hitting time of $R$, $s_2$ the hitting time of $\{g\}$, and let $s = s_2 * s_1$. Then

$$\gamma'^\infty(f)(s^c < J) \leqslant \gamma'^\infty(f)(s + c - 1 < m + |B|) \leqslant \gamma'^\infty(f)(s < m) < \tfrac{1}{2}$$

for all $f \in F$, where the first inequality follows since $s^c \geqslant s + c - 1$; the second since $|B| \leqslant c - 1$; and the last by (4.6). By Lemma 4.3(ii) and choice of $k$, this implies that

$$\gamma'^\infty(f)(s^{kc} < J) < (1/2)^k < \varepsilon/2 \quad \text{for all } f \in C. \tag{4.7}$$

Without loss of generality $\Gamma(f) = \{\delta(g)\}$ for all $f \in F \setminus C$. By (4.3) and (4.5), since $g \in T$, it follows that $\gamma'^\infty(f)(s < t) > (1 - 1/n)^3$ for all $f \in F$. Lemma 4.3(i) and choice of $n$ imply that

$$\gamma'^\infty(f)(s^{kc} < t) > (1 - 1/n)^{3kc} > 1 - \varepsilon/2 \quad \text{for all } f \in C. \tag{4.8}$$

Thus for all $f \in C$, by (4.7) and (4.8) it follows that

$$\gamma'^\infty(f)(t \leqslant J) = \gamma'^\infty(f)(t \leqslant J \text{ and } s^{kc} < J) + \gamma'^\infty(f)(t < J \leqslant s^{kc})$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

DEFINITION. A subset $B$ of $F$ is *closable* for $\Gamma$ if for each $f \in B$ and each $\varepsilon > 0$ there exists $\alpha \in \Gamma(f)$ such that $\alpha(B) \geqslant 1 - \varepsilon$.

LEMMA 4.6. *Each of the following conditions implies, but is not implied by, its successor.*

(i) *$B$ is positive recurrent for $\Gamma$.*
(ii) *$B$ is closed for $\Gamma$.*
(iii) *$B$ is closable for $\Gamma$.*

PROPOSITION 4.2. *If* $B \subset C$ *is closable then for each* $\varepsilon > 0$ *there is a* $\Gamma$-*selector* $\gamma$, *and positive integer* $k$, *satisfying*

$$\gamma^{\infty}(f)(f_j \in C \text{ for all } j, 1 \leqslant j \leqslant k - 1, \text{ and } f_k \in B) \geqslant 1 - \varepsilon$$

$$\text{for all } f \in C. \quad (4.9)$$

PROOF. By Lemma 4.4 (take $g$ to be any element of $B$) there exists a $\Gamma$-selector $\bar{\gamma}$ such that $\bar{\gamma}^{\infty}(f)(C^* B -) \geqslant 1 - \varepsilon/3$ for all $f \in C$. Thus there exists a positive integer $k$ such that

$$\bar{\gamma}^{\infty}(f)(C^{(k)} B -) \geqslant 1 - \varepsilon/2 \quad \text{for all } f \in C. \quad (4.10)$$

Since $B$ is closable, there exists a $\Gamma$-selector $\gamma'$ such that $\gamma'(f)(B) \geqslant (1 - \varepsilon/2)^{1/k}$ for all $f \in B$. This implies

$$\gamma'^{\infty}(f)(f_j \in B \text{ for all } j \in \{1, \ldots, k\}) \geqslant 1 - \varepsilon/2 \quad \text{for all } f \in B. \quad (4.11)$$

Define $\gamma$ by $\gamma(f) = \gamma'(f)$ if $f \in B$, and $\gamma(f) = \bar{\gamma}(f)$ otherwise. Then (4.9) follows from (4.10) and (4.11), since $B \subset C$. $\square$

## 5. Markov strategies.

DEFINITION. If $\sigma(pf) = \sigma(p'f)$ whenever $\|p\| = \|p'\|$, for all $f \in F$, then the strategy $\sigma$ is *Markov*.

Alternatively, it is easy to see that each Markov strategy $\sigma$ determines, and is determined by, an initial gamble $\sigma(\varnothing)$ and a sequence of Markov kernels $\gamma_n$, by taking $\sigma(f_1 f_2 \cdots f_n) = \gamma_n(f_n)$ for all $f_1, \ldots, f_n \in F$.

The purpose of this section is to prove (Proposition 5.1) that if, for a subset $C$ of $F$, there is available some strategy under which there is positive probability of reaching $C$, remaining in $C$ forever, and visiting every state in $C$ infinitely often, then there is available a Markov strategy which guarantees that for a set of histories of arbitrarily high probability, once $C$ is entered, $C$ is never thereafter left, and every state in $C$ is visited infinitely often.

For the remainder of this section, let $(F, \Gamma)$ be any gambling house with $|F| < \infty$.

The proof of Proposition 5.1 will be based on the following lemma.

LEMMA 5.1. *Let* $C$ *be a closed communicating class, and let* $g \in C$. *Then there exists* $\zeta > 0$ *with the following property: for each* $\varepsilon > 0$ *there is a Markov strategy* $\sigma$ *in* $\Gamma$, *and a positive integer* $L$ *satisfying the following two conditions*:

$$\sigma[f](f_n = g \text{ for some } n \leqslant L) > \zeta \quad \text{for all } f \in C, \quad (5.1)$$

$$\sigma[pf](f_n \in C \text{ for all } n = 1, \ldots, L - \|p\|) \geqslant 1 - \varepsilon$$

$$\text{for all } f \in C \text{ and all } p \in F^* \text{ for which } \|p\| < L. \quad (5.2)$$

Condition (5.1) states that if one is anywhere in $C$ at time one, then by time $L + 1$ he will have visited $g$ with probability at least $\zeta$. Condition (5.2) states

that if $C$ is entered prior to time $L + 1$, then with probability at least $1 - \varepsilon$, $C$ is not left by time $L + 1$.

PROOF OF LEMMA. Without loss of generality (let $\Gamma(f) = \{\delta(f)\}$ for all $f \in F \setminus C$) $C$ is a maximal communicating class which is closed. Let $\zeta > 0$ be as in Proposition 4.1. Fix $\varepsilon > 0$, let $B$, $\gamma'$ and $J$ be as in Proposition 4.1, and let $\gamma$ and $k$ be as in Proposition 4.2. Let $L = k + J$, and define $\sigma$ as follows. The initial gamble $\sigma(\varnothing)$ is arbitrary (in $\Gamma$). For $\|p\| < k$, $\sigma(pf) = \gamma(f)$, and for $\|p\| \geqslant k$, $\sigma(pf) = \gamma'(f)$. For $\varepsilon$ sufficiently small, (5.1) follows from (4.1) and (4.9), and (5.2) follows from (4.2) and (4.9). $\square$

DEFINITION. For $C \subset F$, let $A_C$ be the set of all $h = (f_1 f_2 \ldots)$ such that $f_i \in C$ for all $i$ and such that for each $f \in C$ there are infinitely many $i$ for which $f_i = f$.

DEFINITION. For $C \subset F$, let $E_C$ be the set of all $h = (f_1 f_2 \ldots)$ such that for some $n \geqslant 1$, $f_i \in C$ for all $i \geqslant n$, and such that for each $f \in C$ there are infinitely many $i$ for which $f_i = f$.

LEMMA 5.2. *If $C$ and $D$ are distinct subsets of $F$, then $E_C \cap E_D = \varnothing$.*

The conclusion of the following lemma clearly fails for $|F| = \infty$.

LEMMA 5.3. *If $|F| < \infty$, then $F^N = \bigcup \{E_C: C \subset F\}$.*

PROPOSITION 5.1. *Let $(F, \Gamma)$ be a gambling house with $|F| < \infty$, and $C$ a subset of $F$ for which $P(E_C) > 0$. Then for each $\varepsilon > 0$ there exists a Markov strategy $\sigma$ in $\Gamma$ such that*

$$\sigma[pf](A_C) \geqslant 1 - \varepsilon \quad \text{for all } p \in F^* \text{ and all } f \in C. \tag{5.3}$$

PROOF. $P(E_C) > 0$ implies, by Lemma 3.1, that $C$ is a communicating class, and that $P_f(C^* f' -) = 1$ for all $f$ and $f'$ in $C$, that is, $C$ is also closed. Let $C = \{g_0, g_1, \ldots, g_m\}$, and let $\zeta_j > 0, j = 0, \ldots, m$ be as in Lemma 5.1 for $g = g_j$. Let $\zeta = \min \zeta_j$. Fix $\varepsilon > 0$, and for $i = 1, 2, 3, \ldots$ find the Markov strategy $\sigma_i$ and positive integer $L_i$ guaranteed by Lemma 5.1 for $g = g_{i \bmod n}$, and $\varepsilon$ replaced by $\varepsilon / 2^{i+1}$. Let $\sigma$ be the composition of the policies $(\sigma_1, L_1)$, $(\sigma_2, L_2), \ldots$ That is, $\sigma$ uses $\sigma_1$ through time $L_1$, $\sigma_2$ for the next $L_2$ steps, etc. For a formal definition, let $J_i = \sum_{j=1}^{i} L_j$, and define $\sigma$ by $\sigma(p) = \sigma_1(p)$ if $\|p\| \leqslant J_1$, and $\sigma(p_J p') = \sigma_i(p')$ if $J_i < \|p'\| \leqslant J_{i+1}$. That $\sigma$ satisfies (5.3) can now be verified thus.

Fix $p \in F^*$, and $f \in C$. By (5.2), $\sigma[pf](C^N) \geqslant 1 - \varepsilon$. Fix $g_j \in C$, and let $n_0 = \min\{j + km: k = 1, 2, \ldots \text{ and } j + km > \|p\|\}$. For $k = 1, 2, \ldots$ let $n_k = n_0 + km$, and define $M_k \subset F^N$ by $M_k = \{h: f_i = g_j \text{ for some } i, J_{n_k} < i + \|p\| \leqslant J_{n_k+1}\}$. By (5.1), $\sigma[pf](M_k | M_1, \ldots, M_{k-1}) > \zeta$ on $C^N$. Thus by Lemma 3.3, $\sigma[pf] (M_k \text{ i.o.} \cap C^N) = \sigma[pf](C^N)$. Since $\{M_k \text{ i.o.}\} \subset \{\text{``}g_j\text{''} \text{ i.o.}\}$, and $A_C = \bigcap_{j=0}^{m} \{\text{``}g_j\text{''} \text{ i.o.} \cap C^N\}$, $\sigma$ satisfies (5.3). $\square$

At this point it is easy to conclude, using Propositions 2.1 and 5.1, that in any finite state gambler's problem with single fixed goal, there always exist persistently $\varepsilon$-optimal Markov families. However, since a much more general result [Theorem 8.1] is possible utilizing the same techniques, a separate proof of this former result will not be given.

## 6. Tracking strategies.

DEFINITION. For $f \in F, p \in F^*$, let $N_f(p)$ be the number of $f$'s occurring in $p$.

DEFINITION. If $\sigma(pf) = \sigma(p'f)$ whenever $N_f(p) = N_f(p')$, for all $f \in F$, then the strategy $\sigma$ is *tracking*.

Intuitively, with a tracking strategy the selection of a transition probability during the course of the process depends only on the current state, and the number of times that state has been visited previously. Each tracking strategy $\sigma$ determines, and is determined by, an initial gamble $\sigma(\varnothing)$ and a sequence of Markov kernels $\gamma_n$, by taking $\sigma(pf) = \gamma_n(f)$, where $n = N_f(p)$.

The purpose of this section is to prove (Proposition 6.2) the analog of Proposition 5.1 for tracking strategies. It is easy to see that, in general, Markov strategies are not tracking, nor are tracking strategies in general Markov. However, the following relationship does exist among Markov, tracking and stationary strategies.

PROPOSITION 6.1. *A strategy is stationary if and only if it is both Markov and tracking.*

PROOF. "$\Rightarrow$" is clear.

"$\Leftarrow$". If $|F| = 1$, then there is only one gamble, $\delta(f)$, and the only strategy is stationary. Suppose $|F| > 1$, and let $\sigma$ be a strategy which is both Markov and tracking. Let $f, f' \in F, f \neq f'$, and let $\sigma(f) = \alpha \in \Gamma(f)$. Let $p \in F^*$, $\|p\| = n$. Since $\sigma$ is Markov, $\sigma(pf) = \sigma(p'f)$, where $p' \in \{f'\}^*$, $\|p'\| = n$. Since $\sigma$ is tracking, $\sigma(p'f) = \sigma(f) = \alpha$. Hence $\sigma$ is stationary. $\square$

The proof of Proposition 6.2 will depend on the following six lemmas, the first two of which are purely combinatorial in nature, and the third of which is purely analytical.

LEMMA 6.1. *Let $C$ be a finite set and $p = (f_1 f_2 \cdots f_n) \in fC^*f'$. Then there exists $\bar{p} = (f_1' \cdots f_k') \in fC^*f'$ with $k \leqslant |C| + 2$, and such that each adjacent pair of states in $\bar{p}$ also occurs as an adjacent pair in $p$, that is, for each $i = 1, \ldots, k - 1$, there exists a $j$, $1 \leqslant j \leqslant n - 1$, with $f_i' f_{i+1}' = f_j f_{j+1}$.*

PROOF (INDUCTION ON $n$). If $n \leqslant |C| + 2$, let $\bar{p} = p$. Suppose $n > |C| + 2$. Then there exist $i$ and $j$, $1 < i < j < n$, for which $f_i = f_j$. For $p' = f_1 \cdots f_i f_{j+1} \cdots f_n$ we have $\|p'\| < n$, and that every adjacent pair in $p'$ also

occurs as an adjacent pair in $p$. Applying the induction hypothesis completes the proof. $\square$

LEMMA 6.2. *Let $C$ be a finite set and $h = (f_1 f_2 \dots) \in A_C$. Then for each pair of states $f$, $f'$ in $C$ there exists $p = f_1' \cdots f_k' \in fC^*f'$, with $k \leqslant |C| + 2$, and such that $h \in \{\text{``}f_j' f_{j+1}'\text{''} i.o.\}$ for each $j = 1, \dots, k - 1$.*

PROOF. Any $h \in A_C$ can be written in the form $h = p_1 p_2 p_3 \dots$ where $p_{2n} \in fC^*f'$ for each $n \geqslant 1$. For each $p_{2n}$, let $\bar{p}_{2n}$ be as in Lemma 6.1. Since $|C| < \infty$, there are only a finite number of such partial histories $\bar{p}_{2n}$, so at least one, call it $p$, must occur infinitely often. $\square$

The conclusion of Lemma 6.2 need not hold if $C$ is countable, as the following example shows.

EXAMPLE 6.1. Let $C = \{0, 1, 2, \dots\}$, let $h = (10120123012340 \dots) \in A_C$, and let $f = 1$, $f' = 0$.

LEMMA 6.3. *Let $\{a_n\} \in (0, 1]$, $\{b_n\} \in [0, 1]$ for $n = 1, 2, \dots$, and suppose $\lim_{n \to \infty} b_n / a_n = 0$. Then for each $\varepsilon > 0$ there exist positive integers $n(1) < n(2) < \dots$, and a sequence $\{r_k\}$ of positive integers, satisfying the following two conditions:*

$$\sum_{k=1}^{\infty} r_k a_{n(k)} = \infty, \tag{6.1}$$

$$\sum_{k=1}^{\infty} r_k b_{n(k)} < \varepsilon. \tag{6.2}$$

PROOF. Fix $\varepsilon > 0$. For each $k \geqslant 1$ choose $n(k)$ such that $b_{n(k)} / a_{n(k)} < \varepsilon / 2^{k+1}$. Let $r_k = \min\{m \in \mathbb{N}: m \geqslant 1/a_{n(k)}\}$.

For each $k \geqslant 1$, $r_k a_{n(k)} \geqslant 1$ and, since $0 \leqslant a_{n(k)} \leqslant 1$, it follows that $r_k < 1/a_{n(k)} + 1 \leqslant 2/a_{n(k)}$, and that $r_k b_{n(k)} < \varepsilon/2^k$. $\square$

For the remainder of this section, let $(F, \Gamma)$ be a gambling house with $|F| < \infty$, and $C$ an arbitrary subset of $F$.

LEMMA 6.4. *Let $f, \hat{f} \in C$. If there exists a $d > 0$ such that $\alpha(F \setminus C) \geqslant d\alpha(\hat{f})$ for all $\alpha \in \Gamma(f)$, then $P(\text{``}f\hat{f}\text{''} i.o. \cap C^{\mathbb{N}}) = 0$.*

PROOF. Define the stopping time $t$ by $t(h) = \min\{n \geqslant 2: f_{n-1} = f, f_n \in \{\hat{f}\} \cup F \setminus C\}$. Let $D_m = \{h \in F^{\mathbb{N}}: t^m(h) < \infty \text{ and } f_{t^m(h)+1} = \hat{f}\}$ for $m \geqslant 1$. Since $\alpha(F \setminus C) \geqslant d\alpha(\hat{f})$ for all $\alpha \in \Gamma(f)$, if $\sigma$ is any strategy in $\Gamma$, and $k$ is any positive integer, then $\sigma(\cap\{D_m: m = 1, \dots, k\}) \leqslant (1 + d)^{-k}$. But since $d > 0$, and since $\cap\{D_m: m = 1, \dots, k\} \supset \{\text{``}f\hat{f}\text{''} i.o. \cap C^{\mathbb{N}}\}$ for all $k \geqslant 1$, it follows that $\sigma\{\text{``}f\hat{f}\text{''} i.o. \cap C^{\mathbb{N}}\} = 0$. $\square$

LEMMA 6.5. *Let $f, \hat{f} \in C$ and suppose $P(\text{``}f\hat{f}\text{''} i.o. \cap C^{\mathbb{N}}) > 0$. Then for each $\varepsilon > 0$ there exists a sequence of gambles $\{\alpha_k\} \in \Gamma(f)$, not necessarily distinct,*

*satisfying the following two conditions*:

$$\sum_{j=1}^{\infty} \alpha_j(\hat{f}) = \infty, \tag{6.3}$$

$$\sum_{j=1}^{\infty} \alpha_j(F \setminus C) < \varepsilon. \tag{6.4}$$

PROOF. By Lemma 6.4, for each $n \geq 1$ there exists $\beta_n \in \Gamma(f)$ such that $\beta_n(F \setminus C) < \beta_n(\hat{f})/n$. By Lemma 6.3 there exists a subsequence $\{\beta_{n(k)}\}$, and sequence $\{r_k\}$, of positive integers satisfying (6.1) and (6.2) with $a_{n(k)} = \beta_{n(k)}(\hat{f})$, and $b_{n(k)} = \beta_{n(k)}(F \setminus C)$. Define $\{\alpha_j\}$ by $\alpha_j = \beta_{n(i)}$, where $i$ is the unique solution (letting $r_0 = 0$) of $\sum_{k=0}^{i-1} r_k < j \leq \sum_{k=0}^{i} r_k$. Then

$$\sum_{j=1}^{\infty} \alpha_j(\hat{f}) = \sum_{k=1}^{\infty} r_k \beta_{n(k)}(\hat{f}) = \infty,$$

and

$$\sum_{j=1}^{\infty} \alpha_j(F \setminus C) = \sum_{k=1}^{\infty} r_k \beta_{n(k)}(F \setminus C) < \varepsilon. \quad \square$$

LEMMA 6.6. *Let* $f \in C$, *and let* $B$ *be a subset of* $C$ *such that* $P(``\hat{f}\hat{f}"$ *i.o.* $\cap C^N) > 0$ *for each* $\hat{f} \in B$. *Then for each* $\varepsilon > 0$ *there exists a sequence of gambles* $\{\alpha_k\} \in \Gamma(f)$ *such that if* $\sigma$ *is any strategy, not necessarily in* $\Gamma$, *with* $\sigma(pf) = \alpha_k$ *whenever* $N_f(p) = k - 1$, *then* $\sigma$ *satisfies the following two conditions*:

$$\sigma[p](``\hat{f}\hat{f}" \text{ i.o.}) = \sigma[p](``\hat{f}" \text{ i.o.}) \quad \text{for all } \hat{f} \in B, \text{ and all } p \in F^*; \tag{6.5}$$

$$\sigma[p](f_n f_{n+1} = ff' \text{ for some } n \geq 1 \text{ and some } f' \in F \setminus C) < \varepsilon$$
$$\text{for all } p \in F^*. \tag{6.6}$$

PROOF. Fix $\varepsilon > 0$ and $p \in F^*$. Let $N_f(p) = i$.

*Case* 1. $|B| = 1$. Let $B = \{\hat{f}\}$, and find $\alpha_1, \alpha_2, \ldots \in \Gamma(f)$ as in Lemma 6.5. Let $\sigma$ be any strategy with $\sigma(pf) = \alpha_k$ whenever $N_f(p) = k - 1$.

To prove (6.5), let $t$ be the hitting time of $f$, that is, $t(h) = \min\{m \geq 1: f_m = f\}$, and for each $n \geq 1$, let $M_n = \{h \in F^N: t^n(h) < \infty$ and $f_{t^n + 1} = \hat{f}\}$. Since $N_f(pp') = N_f(p) + N_f(p')$ for all $p, p' \in F^*$, and since $N_f(p_{t^n}) = n$, $\sigma$ satisfies

$$\sigma(pp_{t^n}) = \alpha_{i+n} \quad \text{for all } n = 1, 2, \ldots. \tag{6.7}$$

On $\{t^n < \infty\}$,

$$\sigma[p](M_n | M_1, \ldots, M_{n-1})$$

$$= \int_{\{t^n < \infty\}} \sigma[pp_{t^n}](\hat{f}-) \, d\sigma[p](p_{t^n} | M_1, \ldots, M_{n-1}) = \alpha_{i+n}(\hat{f}),$$

where the first equality follows by the definition of the $M_j$ and the second follows by (6.7). Thus on $\{$"$f$" i.o.$\}$, it follows from (6.3) that $\sum_{n=1}^{\infty}\alpha_{i+n}(\hat{f}) = \infty$. Since $\{M_j$ i.o.$\} = \{$"$\hat{f}f$" i.o.$\}$, (6.5) follows from Lemma 3.3.

To prove (6.6), let $D_n = \{h: t^n(h) < \infty \text{ and } f_{t^n+1} \in F\setminus C\}$. Then

$$\sigma[p](f_n f_{n+1} = ff' \text{ for some } n \geq 1 \text{ and some } f' \in F\setminus C)$$

$$= \sigma[p]\left(\bigcup_{n=1}^{\infty} D_n\right) \leq \sum_{n=1}^{\infty} \sigma[p](D_n)$$

$$= \sum_{n=1}^{\infty} \int_{t^n < \infty} \sigma[pp_{t^n}](F\setminus C - ) \, d\sigma[p](p_{t^n})$$

$$\leq \sum_{n=1}^{\infty} \alpha_{i+n}(F\setminus C) < \varepsilon,$$

where the first equality follows by the definition of the $D_j$; the first inequality by the countable additivity of $\sigma[p]$; the second equality by the definition of $D_n$; the second inequality by (6.7); and the last inequality by (6.4), which concludes Case 1.

*Case 2.* $|B| = 2$. Let $B = \{\check{f}, \hat{f}\}$, and find $\alpha_1', \alpha_2', \ldots$ and $\alpha_1'', \alpha_2'', \ldots$ as in Lemma 6.5 for $\hat{f} = \check{f}$ and $\hat{f}$, respectively, and $\varepsilon = \varepsilon/2$. Define $\alpha_1, \alpha_2, \ldots$ by $\alpha_{2n-1} = \alpha_n'$, and $\alpha_{2n} = \alpha_n''$ for $n \geq 1$. To prove (6.6) for $\hat{f} = \check{f}$, let $M_n' = \{h: t^{2n-1}(h) < \infty \text{ and } f_{t^{2n-1}+1} = \check{f}\}$ and argue as in Case 1. Similarly for $\hat{f} = \hat{f}$. To prove (6.5), argue, as in Case 1, that $\sigma[p](\bigcup_{n=1}^{\infty}D_n) \leq \sum_{n=1}^{\infty}\alpha_{i+n}(F\setminus C)$. But by definition of $\alpha_i$, $\sum_{n=1}^{\infty}\alpha_{i+n}(F\setminus C) \leq \sum_{n=1}^{\infty}\alpha_n'(F\setminus C) + \sum_{n=1}^{\infty}\alpha_n''(F\setminus C) < \varepsilon$. This concludes Case 2.

*General case.* Let $B = \{b_1, b_2, \ldots, b_n\}$, and for $i = 1, 2, \ldots, n$ find $\alpha_{i,1}, \alpha_{i,2}, \ldots \in \Gamma(f)$ as in Lemma 6.5 for $\hat{f} = b_i$ and $\varepsilon$ replaced by $\varepsilon/2^n$. Define $\{\alpha_i\}$ "diagonally" by $\alpha_1 = \alpha_{1,1}, \alpha_2 = \alpha_{1,2}, \alpha_3 = \alpha_{2,1}, \alpha_4 = \alpha_{3,1}$, etc. Then (6.5) and (6.6) follow easily as in Case 2. $\square$

PROPOSITION 6.2. *Let* $(F, \Gamma)$ *be a gambling house with* $|F| < \infty$, *and* $C$ *a subset of* $F$ *for which* $P(E_C) > 0$. *Then for each* $\varepsilon > 0$ *there exists a tracking strategy* $\sigma$ *in* $\Gamma$ *such that* $\sigma[pf](A_C) \geq 1 - \varepsilon$ *for all* $p \in F^*$ *and all* $f \in C$.

PROOF. Since $F^*$ is countable, and $E_C = \bigcup\{pA_C: p \in F^*\}$, it follows that $P(A_C) > 0$. Fix $\varepsilon > 0$.

*Case 1.* $C = \{f\}$. Then $A_C = (fff\ldots)$, and $P(A_C) > 0$ implies that for each $k \geq 1$ there exists $\alpha_k \in \Gamma(f)$ with $\alpha_k(f) \geq 1 - \varepsilon/2^k$. Let $\sigma$ be any tracking strategy in $\Gamma$ satisfying $\sigma(pf) = \alpha_k$ whenever $N_f(p) = k - 1$. Then $\sigma[pf](A_C) \geq 1 - \varepsilon$ for all $p \in F^*$.

*Case 2.* $C = \{f, \hat{f}\}$. Since $P(\text{"}\hat{f}f\text{" i.o.} \cap \text{"}\hat{f}f\text{" i.o.} \cap C^N) = P(A_C) > 0$, there exists a sequence $\{\alpha(f, k)\} \in \Gamma(f)$ satisfying the conclusion of Lemma 6.6, with $B = \{\hat{f}\}$, $\alpha_k$ replaced by $\alpha(f, k)$, and $\varepsilon$ replaced by $\varepsilon/2$. Similarly, there

exists a sequence $\{\alpha(\hat{f}, k)\} \in \Gamma(\hat{f})$ satisfying the conclusion of Lemma 6.6, with $B = \{f\}$, $\alpha_k$ replaced by $\alpha(\hat{f}, k)$, $\varepsilon$ by $\varepsilon/2$, and with $f$ and $\hat{f}$ interchanged. Let $\sigma$ be any tracking strategy in $\Gamma$ with $\sigma(pf) = \alpha(f, k)$ whenever $N_f(p) = k - 1$, and with $\sigma(p\hat{f}) = \alpha(\hat{f}, k)$ whenever $N_{\hat{f}}(p) = k - 1$.

For each $p \in F^*$, (6.6) implies that $\sigma[pf](C^N) \geqslant 1 - \varepsilon$ and $\sigma[p\hat{f}](C^N) \geqslant 1 - \varepsilon$. Since every $h \in C^N$ contains either $f$ or $\hat{f}$ infinitely often, this, with (6.5), implies that $\sigma[pf](A_C) \geqslant 1 - \varepsilon$ and $\sigma[p\hat{f}](A_C) \geqslant 1 - \varepsilon$.

*General case.* Since $|C| < \infty$ and $P(A_C) > 0$, applying Lemma 6.2 successively to each of the states in $C$ guarantees the existence of a partial history $p' = (f_1 \cdots f_m) \in C^*$ containing all the elements of $C$, and such that $f_1 = f_m$, $m \leqslant |C|^2 + 2|C|$, and for which $P(\text{``} f_1 f_2 \text{''} \text{ i.o. } \cap \text{``} f_2 f_3 \text{''}$ i.o.$\cap \cdots \cap \text{``} f_{m-1} f_m \text{''}$ i.o.$\cap C^N) > 0$. Now proceed as in Case 2. For each $f \in C$, let $B(f)$ be the successors of $f$ in $p'$, that is, $B(f) = \{\tilde{f} \in C : f_i f_{i+1} = f\tilde{f}$ for some $i$, $1 \leqslant i \leqslant m - 1\}$. Find a sequence $\{\alpha(f, k)\} \in \Gamma(f)$ as in Lemma 6.6 with $\alpha_k$ replaced by $\alpha(f, k)$, $B$ by $B(f)$, and $\varepsilon$ by $\varepsilon/|C|$. Let $\sigma$ be any tracking strategy in $\Gamma$ satisfying $\sigma(pf) = \alpha(f, k)$ whenever $N_f(p) = k - 1$, for each $f \in C$. The conclusion follows easily as in Case 2, since each $h \in C^N$ contains some element of $C$ infinitely often, and since $p'$ contains each element of $C$. $\square$

## 7. Shift and permutation invariant payoffs.

The purpose of this section is to introduce a class of payoff functions (real-valued functions on $F^N$), called shift-and-permutation invariant, for which (as will be shown in the next section) both Markov and tracking strategies are adequate, and to prove two lemmas needed in the next section.

DEFINITION. A function $w: F^N \to \mathbf{R}$ is *shift invariant* if $w(f_1 f_2 f_3 \ldots) = w(f_2 f_3 \ldots)$ for every $h = (f_1 f_2 f_3 \ldots) \in F^N$.

DEFINITION. Let $w: F^N \to \mathbf{R}$ be bounded and integrable (with respect to all strategies in $(F, \Gamma)$). Then $W: F \to \mathbf{R}$ is defined by $W(f) = \sup\{\int w \, d\sigma : \sigma$ is in $\Gamma$ at $f\}$.

The following lemma is a generalization of Lemma 3.2.

LEMMA 7.1. *If $f \equiv f'$ then for every bounded, integrable, shift invariant function $w: F^N \to \mathbf{R}$, $W(f) = W(f')$.*

PROOF. Without loss of generality, $0 \leqslant w \leqslant 1$ (otherwise add/multiply by suitable constants). Fix $\varepsilon > 0$, and let $\sigma'$ be in $\Gamma$ at $f'$ such that $\int w \, d\sigma' \geqslant W(f') - \varepsilon$. Since $f \equiv f'$, there exists a strategy $\hat{\sigma}$ in $\Gamma$ at $f$ such that $\hat{\sigma}(F^*f' - ) \geqslant 1 - \varepsilon$. Let $t: F^N \to \mathbf{N}$ be the hitting time of $f'$ and define $\sigma$ in $\Gamma$ at $f$ by $\sigma = \hat{\sigma}$ prior to time $t$ and $\sigma[p_t] = \sigma'$.

Since $w$ is shift invariant, and $0 \leqslant w \leqslant 1$, $\int w \, d\sigma \geqslant \int_{t<\infty} w \, d\sigma \geqslant \int w \, d\sigma' - \varepsilon > W(f') - 2\varepsilon$. Since $\varepsilon$ was arbitrary, this implies $W(f) \geqslant W(f')$. By symmetry, $W(f') \geqslant W(f)$. $\square$

DEFINITION. A function $w: F^N \to \mathbf{R}$ is *permutation invariant* if $w(h) = w(h')$ for every history $h' \in F^N$ which is obtained from $h$ by an arbitrary permutation of the coordinates.

LEMMA 7.2. *Suppose $|F| < \infty$ and $w: F^N \to \mathbf{R}$. Then $w$ is shift-and-permutation invariant if and only if $w$ is constant on $E_C$ for each $C \subset F$.*

PROOF. "$\Rightarrow$". Suppose $w$ is shift-and-permutation invariant. Let $h = (f_1 f_2 \ldots) \in E_C$, and $h' = (f_1' f_2' \ldots) \in E_C$ for some $C \subset F$. Then there exists $n \geqslant 1$ with $(f_n f_{n+1} \ldots) \in A_C$ and $(f_n' f_{n+1}' \ldots) \in A_C$. This implies that $(f_n' f_{n+1}' \ldots)$ can be obtained from $(f_n f_{n+1} \ldots)$ by a permutation of the coordinates. Since $w$ is permutation invariant, we have $w(f_n f_{n+1} \ldots) = w(f_n' f_{n+1}' \ldots)$. Since $w$ is shift invariant, this implies $w(h) = w(h')$.

"$\Leftarrow$". Suppose $w$ is constant on $E_C$ for all $C \subset F$, and let $h \in F^N$. By Lemma 5.3, $h \in E_C$ for some $C \subset F$. Since $h = (f_1 f_2 \ldots) \in E_C$ iff $(f_n f_{n+1} \ldots) \in E_C$ for all $n \geqslant 1$, we have $w(f_1 f_2 \ldots) = w(f_n f_{n+1} \ldots)$. Similarly, if $h'$ is any history obtained from $h$ by a permutation of the coordinates, then $h' \in E_C$ iff $h \in E_C$, and therefore $w(h) = w(h')$.  $\square$

If $F$ is infinite, $w$ may be constant on each $E_C$ but neither shift nor permutation invariant, as the following example shows:

EXAMPLE 7.1. $F = 1, 2, 3, \ldots$. Then $\bar{h} = (123 \ldots)$ is not in $E_C$ for any $C \subset F$. Let $w(\bar{h}) = 1$ and $w(h) = 0$ for $h \neq \bar{h}$.

Finite state gambling houses with shift-and-permutation invariant payoffs include the classical finite fortune gambling problems of Dubins and Savage, as well as others such as multiple goal problems (for example, the payoff is $+1$ if two goals are both hit infinitely often, otherwise payoff is 0) and avoidance problems (payoff is $+1$ if a "bad" set is visited only finitely often, otherwise the payoff is 0).

**8. Persistent $\varepsilon$-optimality.** As in [10], a family $\bar{\sigma}$ of strategies is *Markov* if $\bar{\sigma}(f')(pf) = \bar{\sigma}(f'')(p'f)$ whenever $\|p\| = \|p'\|$, for all $f, f', f''$ in $F$.

DEFINITION. A family $\bar{\sigma}$ of strategies is *tracking* if $\bar{\sigma}(f')(pf) = \bar{\sigma}(f'')(p'f)$ whenever $N_f(p) = N_f(p')$, for all $f, f', f''$ in $F$.

DEFINITION. Let $(F, \Gamma)$ be a gambling house, $\varepsilon > 0$, and $w: F^N \to \mathbf{R}$ a bounded, integrable function. After [5], a family of strategies $\bar{\sigma}$ is *persistently $\varepsilon$-optimal for $w$* if

$$W(f) - \int w \, d\bar{\sigma}(f) < \varepsilon \quad \text{and} \quad W(f') - \int w \, d\bar{\sigma}(f)[pf'] < \varepsilon$$

$$\text{for all } p \in F^* \text{ and all } f, f' \in F.$$

THEOREM 8.1. *Let $(F, \Gamma)$ be any gambling house with $|F| < \infty$. If $w: F^N \to \mathbf{R}$ is shift-and-permutation invariant, then for each $\varepsilon > 0$, there exists a Markov (respectively, tracking) family of strategies in $\Gamma$ which is persistently $\varepsilon$-optimal for $w$.*

PROOF. Fix $\varepsilon > 0$. It is enough to find a single Markov (respectively, tracking) strategy $\sigma$ in $\Gamma$ which, although it may be initially bad, is conditionally $\varepsilon$-optimal along every nonempty partial history. Then define $\bar{\sigma}$ by $\bar{\sigma}(f) = \sigma[f]$ for each $f$ in $F$.

By Lemma 7.2, $w$ is bounded and is constant on $E_C$ for all $C \subset F$. Let $w(C)$ denote this common value, and assume, without loss of generality, that $0 < w \leqslant 1$. Let $F = \bigcup_{j=0}^{n} F_j$ be the decomposition of $F$ guaranteed by Proposition 3.2, with $F_0$ the transient class and $n \geqslant 1$. For each $j = 1, \ldots, n$, let $C_j \neq \varnothing$ be a subset of $F_j$ such that $w(C_j) = \max\{w(C): C \subset F_j$ and $P(E_C) > 0\}$. For each $j \in \{1, \ldots, n\}$, let $f_j \in C_j$. Define the classical gambler's problem $(\Gamma', u)$ on $F$ by: $\Gamma'(f) = \Gamma(f) \cup \{\delta(f)\}$ for all $f \in F$; $u(f) = w(C_j)$ if $f \in F_j$, $j > 0$, and $u(f) = 0$ otherwise. If $f \equiv f'$ in $\Gamma$, then clearly $f \equiv f'$ in $\Gamma'$ as well. Let $V$ be that for the problem $(\Gamma', u)$. By Lemma 3.2, it follows that

$$V \text{ is constant on } F_j \text{ for all } j > 0. \tag{8.1}$$

Next it shall be shown that

$$W(f) \leqslant V(f) \quad \text{for all } f \in F. \tag{8.2}$$

[Note: Actually $W = V$, as will be evident later in the proof.]

PROOF OF (8.2). Fix $f \in F$, and $\sigma$ in $\Gamma$ at $f$. Then

$$\int w \, d\sigma = \sum_{j=1}^{n} \left( \sum_{C \subset F_j} w(C) \sigma(E_C) \right)$$

$$\leqslant \sum_{j=1}^{n} u(f_j) \left( \sum_{C \subset F_j} \sigma(E_C) \right) = \int u^* \, d\sigma.$$

The first equality follows from Lemma 3.1 ($\sigma(E_C) = 0$ unless $C \subset F_j$ for some $j > 0$) and from Lemma 7.2; the inequality follows by the definitions of $C_j$, $u$, and $f_j$; and the last equality follows by Lemmas 5.2 and 5.3, and the fact that $u$ vanishes on $F_0$ and is constant on $F_j$ for each $j > 0$.

Thus $\int w \, d\sigma \leqslant \int u^* \, d\sigma$ for all $\sigma$ in $\Gamma$ at $f$, and since $\Gamma' \supset \Gamma$, this establishes (8.2).

By Proposition 5.1 (respectively Proposition 6.2), for each $j \in \{1, \ldots, n\}$ there exists a Markov (respectively, tracking) strategy $\sigma_j$ in $\Gamma$ satisfying

$$\sigma_j[pf](A_{C_j}) \geqslant 1 - \varepsilon \quad \text{for all } p \in F^* \text{ and all } f \in C_j. \tag{8.3}$$

By Lemma 4.4 (letting $C = F_j$, $g = f_j$), for each $j \in \{1, \ldots, n\}$ there exists a $\Gamma$-selector $\gamma_j$ satisfying

$$\gamma_j^\infty(f)(F_j^* f_j -) \geqslant 1 - \varepsilon \quad \text{for all } f \in F_j. \tag{8.4}$$

Since $\Gamma'$ is leavable, by Proposition 2.1 there exists a $\Gamma'$-selector $\gamma'$ satisfying

$$\int u^* \, d\gamma'^\infty(f) \geqslant V(f) - \varepsilon \quad \text{for all } f \in F. \tag{8.5}$$

Since $u$ vanishes on $F_0$, it may be assumed, without loss of generality, that $\gamma'(f) \neq \delta(f)$ for all $f \in F_0$.

Let $R = \bigcup \{F_j: \exists f \in F_j, \text{ with } f \text{ recurrent under } \gamma'^\infty\}$. Then for $f \in F \setminus R$, $\gamma'(f) \in \Gamma(f)$.

Define the Markov (respectively tracking) strategy $\sigma$ in $\Gamma$ as follows: $\sigma(\varnothing)$ is arbitrary (in $\Gamma$), and

$$\sigma(pf) = \begin{cases} \gamma'(f), & f \in F \setminus R, \\ \gamma_j(f), & f \in R \cap (F_j \setminus C_j), \\ \sigma_j(pf), & f \in R \cap C_j, \end{cases}$$

By (8.2), we will be done once we show that

$$\int w \, d\sigma[pf] \geqslant V(f) - 4\varepsilon \quad \text{for all } p \in F^*, \text{ all } f \in F. \tag{8.6}$$

PROOF OF (8.6). Fix $p \in F^*$.

*Case 1.* $f \in R \cap F_j$ for some $j \in \{1, \ldots, n\}$. By the definition of $R$, there exists $f' \in F_j$ such that $f'$ is a $\gamma'^\infty$ recurrent state. Since $F_j$ is a maximal communicating class (in $\Gamma'$ as well as in $\Gamma$), this implies that $\gamma'^\infty(f')(F_j^N) = 1$. Since $u = w(C_j)$ on $F_j$, by (8.1) and (8.5) this implies that $w(C_j) = u(f') \geqslant V(f') - \varepsilon = V(f) - \varepsilon$. By (8.3) and (8.4), and the definition of $\sigma$, it follows that $\sigma[pf](E_{C_j}) \geqslant 1 - 2\varepsilon$. Thus (recall $0 \leqslant w \leqslant 1$)

$$\int w \, d\sigma[pf] \geqslant w(C_j) - 2\varepsilon \geqslant V(f) - 3\varepsilon.$$

*Case 2.* $f \in F \setminus R$. Let $t$ be the hitting time of $R$.

Since $\sigma[pf]$ and $\gamma'^\infty(f)$ agree prior to time $t$, we have that $\sigma[pf](\{t < \infty\}) = \gamma'^\infty(f)(\{t < \infty\}) = 1$, and that $\sigma[pf](\{t < \infty \text{ and } f_t = f'\}) = \gamma'^\infty(\{t < \infty \text{ and } f_t = f'\})$ for all $f' \in R$.

Let $\sigma' = \sigma[pf]$. Then

$$\sigma'w = \int \sigma'[p_t] w \, d\sigma'$$

$$> \int (V(f_t) - 3\varepsilon) \, d\sigma' \quad \text{(by Case 1)}$$

$$= \int V(f_t) \, d\gamma'^\infty(f) - 3\varepsilon$$

$$> \int u^* \, d\gamma'^\infty(f) - 3\varepsilon$$

(by [3, Corollary 3.3.4] and [11, Theorem 3.2])

$$> V(f) - 4\varepsilon \quad \text{(by (8.5))}. \qquad \square$$

Even for finite state problems, both Markov and tracking strategies may

fail to be adequate if the payoff function fails to be either shift or permutation invariant, as the following two examples show.

EXAMPLE 8.1. $F = \{a, b, g\}$, $\Gamma(b) = \Gamma(g) = \{\delta(a)\}$, $\Gamma(a) = \{\delta(g)/2 + \delta(b)/2, \delta(a)\}$, and $w(h) = 1$ or $0$ according as $|\{m: f_m = g\}| = 1$ or $\neq 1$.

The payoff $w$ is permutation invariant, and $W(a) = 1$, but for $\sigma$ a Markov or tracking strategy in $\Gamma$ at $a$, $\int w \, d\sigma \leqslant 3/4$.

EXAMPLE 8.2. $F = \{a, b, c, d, e, f\}$, $\Gamma(a) = \{\delta(b)/2 + \delta(c)/2\}$, $\Gamma(b) = \Gamma(c) = \{\delta(d)\}$, $\Gamma(d) = \{\delta(e), \delta(f)\}$, $\Gamma(e) = \Gamma(f) = \{\delta(a)\}$,

$$
w(h) = \begin{cases} 1 & \text{if } \exists\, k > 1 \text{ such that if } m, n > k, \text{ and if} \\ & \quad f_m = b, f_n = c, \text{ then } f_{m+2} = e \text{ and } f_{n+2} = f, \\ 0 & \text{otherwise.} \end{cases}
$$

The payoff $w$ is shift invariant, and $W \equiv 1$, but for any Markov or tracking strategy $\sigma$ in $\Gamma$, $\int w \, d\sigma = 0$.

Shift-and-permutation invariant payoffs are not the only ones for which Markov and tracking strategies are persistently adequate for all $\Gamma$ on a finite state space $F$. Let $r$ be a real-valued function on $F$, let $0 < \beta < 1$, and define $w(f_1 f_2 f_3 \ldots) = \Sigma_n \beta^n r(f_n)$. Then $w$ need not be permutation or shift invariant. However, there are even good stationary families available for all $\Gamma$ as follows from Blackwell's article [1].

## 9. Gambling problems with infinite state space.

Ornstein gives an example [8, Theorem A] of a gambling problem with an (uncountably) infinite state space in which stationary families are not uniformly adequate (terminology as in [4]). That example has the property that no state (other than the two absorbing states) can be visited more than once by any strategy, and thus it can be seen that tracking families are no better than stationary families, that is, in his example it is even true that tracking families are not uniformly adequate.

Markov families, however, are uniformly adequate in Ornstein's example, and it is not known if this is always the case, even if $F$ is countable. The following lemma is of some use in identifying gambling problems with infinite state space in which Markov (and tracking) families are adequate.

As in [4], in a gambling house $(F, \Gamma)$ with bounded utility $u$, let $S(f)$ be the most that is achievable in $\Gamma$ with stationary families. Similarly, let $M(f)$ and $T(f)$ be the most that is achievable with Markov and tracking families, respectively. Let $S^L$ be the $S$ for the leavable closure $\Gamma^L$ of $\Gamma$.

DEFINITION. A gambling house $(F, \Gamma)$ is *almost leavable* if for all $f \in F$, and all $\varepsilon > 0$, there exists $\alpha \in \Gamma(f)$ with $\alpha(f) \geqslant 1 - \varepsilon$.

LEMMA 9.1. *If $(\Gamma, u)$ is an almost-leavable gambling problem with bounded utility $u$, then $M \geqslant S^L$, and $T \geqslant S^L$.*

PROOF. Without loss of generality, $0 \leqslant u \leqslant 1$. Fix $\varepsilon > 0$. Since $\Gamma$ is almost leavable, for each $f \in F$ and each $n \in \mathbf{N}$ there exists a gamble $\alpha_{f,n} \in \Gamma(f)$ with $\alpha_{f,n}(f) \geqslant 1 - \varepsilon/2^{n+2}$. Let $\gamma$ be a $\Gamma^L$-selector, and let $D = \{f \in F: \gamma(f) = \delta(f)\}$. Define the Markov family $\bar{\sigma}$ in $\Gamma$ as follows. For $f \in F \setminus D$, $\bar{\sigma}(f)(\varnothing) = \bar{\sigma}(f')(pf) = \gamma(f)$ for all $f' \in F$, and all $p \in F^*$. For $f \in D$, $\bar{\sigma}(f)(\varnothing) = \alpha_{f,1}$, and for all $p \in F^*$ and $f' \in F$, $\bar{\sigma}(f')(pf) = \alpha_{f,\|pf\|}$.

For $f \in D$, clearly $u(\bar{\sigma}(f)) \geqslant u(\gamma^\infty(f)) - \varepsilon$. Let $t$ be the hitting time of $D$. Then for $f \in F \setminus D$, $\bar{\sigma}(f)$ and $\gamma^\infty(f)$ agree prior to time $t$, and for $t(h) < \infty$, $u(\bar{\sigma}(f)[p_t]) \geqslant u(f_t) - \varepsilon = u(\gamma^\infty(f)[p_t]) - \varepsilon$. Thus by [5, Lemma 2.3], $u(\bar{\sigma}(f)) \geqslant u(\gamma^\infty(f)) - \varepsilon$.

This proves that $M \geqslant S^L$. Proof that $T \geqslant S^L$ is similar. $\square$

An example of an application of Lemma 9.1 is

PROPOSITION 9.1. *Let $\Gamma$ be an almost-leavable house in which every gamble is discrete and let $u$ be bounded. Then Markov (and tracking) families are adequate.*

PROOF. Immediate from [4, Proposition 1] and Lemma 9.1. $\square$

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