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PARTITIONS AND SUMS AND PRODUCTS OF INTEGERS

BY

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ABSTRACT. The principal result of the paper is that, if $r < \omega$ and $\{A_i\}_{i<r}$ is a partition of ω, then there exist $i < r$ and infinite subsets B and C of ω such that $\Sigma F \in A_i$ and $\Pi G \in A_i$ whenever F and G are finite nonempty subsets of B and C respectively. Conditions on the partition are obtained which are sufficient to guarantee that B and C can be chosen equal in the above statement, and some related finite questions are investigated.

1. Introduction. The finite sum theorem [6, Theorem 3.1] states that if $r < \omega$ and $\{A_i\}_{i<r}$ is a partition of ω, then there are some $i < r$ and some infinite subset B of ω such that $\Sigma F \in A_i$ whenever F is a finite nonempty subset of B. It is a trivial consequence of that theorem that the corresponding statement holds with "ΣF" replaced by "ΠF". Indeed, let $C_i = \{x: 2^x \in A_i\}$ for each $i < r$. Pick $i < r$ and an infinite subset D of ω such that $\Sigma F \in C_i$ whenever F is a finite nonempty subset of D. Let $B = \{2^x: x \in D\}$. (To this author's knowledge, this consequence was first observed by R. Graham.)

A natural question thus arises. Namely, if $r < \omega$ and $\{A_i\}_{i<r}$ is a partition of ω, can one always find $i < r$ and an infinite subset B of ω such that $\Sigma F \in A_i$ and $\Pi F \in A_i$ whenever F is a finite nonempty subset of B? Numerous stronger versions also suggest themselves.

One of the strongest versions has as its conclusion that all iterated sums and products which can be written without repetition of terms lie in A_i. (This would include expressions like $(x_1 + x_6 + x_3)(x_7x_8 + x_2(x_4 + x_9))$, but not like x_1^2 or $x_1x_2 + x_1x_2x_3$. For an investigation of sums with repeated terms, see [7].) As far as this author is aware, no counterexample has been found for even this strong version.[2]

A natural candidate for a counterexample to the main question would be a partition of ω with the property that any infinite set with sums in one cell would have to come from a particular cell of that partition and any infinite

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[2]*Added in proof*. The author has recently answered the weaker question above in the negative.

set with products in one cell would have to come from some other cell. The principal result of this paper establishes that no such partition exists.

The proof of the main result borrows heavily from Glazer's proof of the finite sum theorem ([**5**], see also [**3**]). (The finite sum theorem has also been proved by Baumgartner [**1**]. See [**2**] for a presentation of Baumgartner's argument to the effect that this theorem is a theorem of ZF, even though all three known proofs use choice.)

§2 consists of a proof, using Glazer's methods, of the existence of a particular ultrafilter on ω, a derivation of the main result from the existence of that ultrafilter, and an investigation of questions related to the existence of special ultrafilters on ω. In §3 we present conditions on a partition of ω which are sufficient to guarantee the existence of one infinite set with all sums and products in one cell.

There are also several finite versions of the sums and products problem. The weakest of these has been established by R. Graham (and verified by the present author) with the aid of a computer. Namely, if $\{1, 2, 3, \ldots, 252\} = A_0 \cup A_1$, then there exist $i < 2$ and distinct x and y such that $\{x, y, x + y, x \cdot y\} \subseteq A_i$. This result is "sharp"; the statement is not true if 252 is replaced by 251. §4 consists of an investigation of related finite problems.

We write ω for the first infinite ordinal. Recall that each ordinal is the set of its predecessors so that, for example, $3 = \{0, 1, 2\}$. Unless otherwise stated lower case variables range over ω. By $\mathcal{P}_f(A)$ we mean the set of finite nonempty subset of A. Thus $\mathcal{P}_f(A) = [A]^{<\omega} \setminus \{\emptyset\}$.

2. Multiplicative idempotents in $\beta\omega$. By $\beta\omega$ we mean the Stone-Čech compactification of the discrete space ω. The points of $\beta\omega$ are the ultrafilters on ω (where the fixed ultrafilters–those with nonempty intersection–are identified with the points of ω). If $A \subseteq \omega$, then $\mathrm{cl}_{\beta\omega} A = \{p \in \beta\omega\colon A \in p\}$. If $p \in \beta\omega$, then a basic neighborhood system for p is $\{\mathrm{cl}_{\beta\omega} A\colon A \in p\}$. We will frequently use the fact that if $A \subseteq \mathcal{P}(\omega)$ and A has the finite intersection property then $A \subseteq p$ for some $p \in \beta\omega$. For further development of the Stone-Čech compactification see [**4**].

In Glazer's proof of the finite sum theorem, he introduced the notion of the sum of two ultrafilters p and q on ω, agreeing that $A \in p + q$ if and only if $\{x\colon A - x \in p\} \in q$, (where by $A - x$ is meant $\{y\colon y + x \in A\}$). He then showed that there exists an additive idempotent in $\beta\omega \setminus \omega$, thus answering affirmatively an unpublished question of Galvin.

From the existence of such an ultrafilter, that is an ultrafilter with the property that $\{x\colon A - x \in p\} \in p$ whenever $A \in p$, the finite sum theorem follows easily. Previously, the existence of such an ultrafilter had been a continuum hypothesis [**8**] or Martin's axiom [Alan Taylor, unpublished] consequence of the finite sum theorem.

We define here the analogous notion of the product of two ultrafilters, and show that there exists a multiplicative idempotent with the property that each member includes an infinite set and all of its finite sums.

2.1 DEFINITION. Let p and q be members of $\beta\omega$ and let $A \subseteq \omega$.

(a) For any x, $A/x = \{y: yx \in A\}$;

(b) $A \in p \cdot q$ if and only if $\{x: A/x \in p\} \in q$.

Lemma 2.2 is well known; its proof may be found in [3].

2.2 LEMMA. *Let X be a nonempty compact Hausdorff space and let $\cdot$ be an associative operation on X with the property that, for each x in X, the function L_x defined by the rule $L_x(y) = x \cdot y$ is continuous. Then there is some x in X such that $x \cdot x = x$.*

The following lemma is due to Glazer; its proof is included for completeness.

2.3 LEMMA (GLAZER). *The operation $\cdot$ is an associative operation on $\beta\omega$ and, for each p in $\beta\omega$, the function L_p, defined by $L_p(q) = p \cdot q$, is continuous.*

PROOF. The verification that $\cdot: \beta\omega \times \beta\omega \to \beta\omega$, i.e. that $p \cdot q$ is an ultrafilter on ω whenever p and q are, is routine. To see that $\cdot$ is associative, let p, q, and r be in $\beta\omega$ and let $A \subseteq \omega$. Note that, for any x, $\{z: A/z \in p\}/x = \{y: A/(xy) \in p\}$. Thus

$$\begin{aligned} A \in (p \cdot q) \cdot r &\leftrightarrow \{x: A/x \in p \cdot q\} \in r \\ &\leftrightarrow \{x: \{y: (A/x)/y \in p\} \in q\} \in r \\ &\leftrightarrow \{x: \{y: A/(xy) \in p\} \in q\} \in r \\ &\leftrightarrow \{x: \{z: A/z \in p\}/x \in q\} \in r \\ &\leftrightarrow \{z: A/z \in p\} \in q \cdot r \\ &\leftrightarrow A \in p \cdot (q \cdot r). \end{aligned}$$

Now let $p \in \beta\omega$. To see that L_p is continuous, let $q \in \beta\omega$ and let $A \in p \cdot q$ (so that $\mathrm{cl}_{\beta\omega} A$ is a basic neighborhood of $p \cdot q$). Let $B = \{x: A/x \in p\}$. By the definition of $p \cdot q$, $B \in q$ and consequently $\mathrm{cl}_{\beta\omega} B$ is a basic neighborhood of q. If $r \in \mathrm{cl}_{\beta\omega} B$, then $\{x: A/x \in p\} \in r$ so that $p \cdot r \in \mathrm{cl}_{\beta\omega} A$. Thus $L_p[\mathrm{cl}_{\beta\omega} B] \subseteq \mathrm{cl}_{\beta\omega} A$.

One gets immediately from Lemmas 2.2 and 2.3 the existence of a multiplicative idempotent in $\beta\omega \setminus \omega$ (using the easily established fact that $\cdot: (\beta\omega \setminus \omega) \times (\beta\omega \setminus \omega) \to \beta\omega \setminus \omega$) and hence a direct proof of the finite product theorem (without appeal to the finite sum theorem). The result which we now seek is the existence of a multiplicative idempotent in a particular subspace of $\beta\omega \setminus \omega$.

2.4 DEFINITION. (a) If $A \subseteq \omega$, then $FS(A) = \{\Sigma F\colon F \in \mathcal{P}_f(A)\}$ and $FP(A) = \{\Pi F\colon F \in \mathcal{P}_f(A)\}$;

(b) $\Gamma = \{A \subseteq \omega\colon$ There exists $B \in [A]^\omega$ such that $FS(B) \subseteq A\}$;

(c) $\bar{\Gamma} = \{p \in \beta\omega\colon p \subseteq \Gamma\}$.

2.5 LEMMA. *Γ is a closed nonempty subset of $\beta\omega \setminus \omega$ and* $\cdot\colon \bar{\Gamma} \times \bar{\Gamma} \to \bar{\Gamma}$.

PROOF. That $\bar{\Gamma} \neq \varnothing$ is a consequence of the finite sum theorem and Theorem 2.5 of [**8**]. To see that $\bar{\Gamma}$ is closed, let $p \in \beta\omega \setminus \bar{\Gamma}$ and pick $A \in p \setminus \Gamma$. Then $\mathrm{cl}_{\beta\omega} A \cap \bar{\Gamma} = \varnothing$. That $\bar{\Gamma} \subseteq \beta\omega \setminus \omega$ follows from the fact that every member of a member of $\bar{\Gamma}$ is infinite.

To see that $\cdot\colon \bar{\Gamma} \times \bar{\Gamma} \to \bar{\Gamma}$, let p and q be in $\bar{\Gamma}$ and let $A \in p \cdot q$. Then $\{x\colon A/x \in p\} \in q$ so $\{x > 0\colon A/x \in p\} \neq \varnothing$. Pick $x \neq 0$ such that $A/x \in p$. Since $p \subseteq \Gamma$, there exists $B \in [A/x]^\omega$ such that $FS(B) \subseteq A/x$. Let $C = \{xy\colon y \in B\}$. Then $FS(C) \subseteq A$ so that $A \in \Gamma$. Consequently $p \cdot q \in \bar{\Gamma}$ as desired. (Notice that we have in fact shown that $\cdot\colon \bar{\Gamma} \times (\beta\omega \setminus \{0\}) \to \bar{\Gamma}$.)

2.6 THEOREM. *Let $\{A_i\}_{i<r}$ be a partition of ω. There exist $i < r$ and sets B and C in $[A_i]^\omega$ such that $FS(B) \subseteq A_i$ and $FP(C) \subseteq A_i$.*

PROOF. By Lemmas 2.3 and 2.5, there exists p in $\bar{\Gamma}$ such that $p \cdot p = p$. Pick such p and pick $i < r$ such that $A_i \in p$. Since $p \subseteq \Gamma$, there exists $B \in [A_i]^\omega$ such that $FS(B) \subseteq A_i$.

Since $p \cdot p = p$, we have that whenever $A \in p$, $\{x\colon A/x \in p\} \in p$ and hence $A \cap \{x\colon A/x \in p\} \in p$. Let $x_0 \in A_i$ such that $A_i/x_0 \in p$ and let $D_0 = A_i$. Inductively, for $n > 0$, let $D_n = D_{n-1} \cap (D_{n-1}/x_{n-1})$ and pick $x_n \in D_n \cap \{x\colon D_n/x \in p\}$ such that $x_n > x_{n-1}$.

We claim that if $\min F \geqslant n$, then $\Pi_{k \in F}\, x_k \in D_n$. If $|F| = 1$, this is trivial since the sequence $\langle D_n \rangle_{n<\omega}$ is nested. Assume inductively that the result holds for sets smaller than F and let $m = \min F$. Let $G = F \setminus \{m\}$. Then $\min G \geqslant m + 1$ so $\Pi_{k \in G}\, x_k \in D_{m+1}$. Since $D_{m+1} \subseteq D_m/x_m$, we have $\Pi_{k \in F}\, x_k \in D_m \subseteq D_n$. Let $C = \{x_n\colon n < \omega\}$. We thus have that, if $F \in \mathcal{P}_f(C)$, then $\Pi F \in A_i$.

Note that if $p \in \beta\omega \setminus \omega$ and $p + p = p$, then we have as in Theorem 3.3 of [**8**] that $p \in \bar{\Gamma}$. In the event that the multiplicative idempotent obtained in the proof of Theorem 2.6 is also an additive idempotent, one has for any partition $\{A_i\}_{i<r}$ of ω an infinite subset of one cell all of whose sums and products are in that cell. (In fact a considerably stronger conclusion holds–see Theorem 2.13). Corollary 2.11 tells us in fact that each member of $\bar{\Gamma}$ is "close" to an additive idempotent, that is that $\bar{\Gamma} = \mathrm{cl}_{\beta\omega}\{p \in \beta\omega \setminus \omega\colon p + p = p\}$.

The author's original proof of Corollary 2.11 used the continuum hypothesis. We are grateful to K. Prikry and F. Galvin for permission to present their unpublished results (Theorem 2.8 and Corollary 2.9) which remove the need for an appeal to the continuum hypothesis for this result. (The proof of Theorem 2.8 presented here is this author's.)

2.7 DEFINITION. Let $\langle x_n \rangle_{n<\omega}$ be a sequence in ω such that, for each $n > 0$, $x_n > \Sigma_{k<n} x_k$. The natural map τ for $FS(\{x_n: n < \omega\})$ is defined by the rule $\tau(\Sigma_{n\in F} x_n) = \Sigma_{n\in F} 2^n$ whenever $F \in \mathcal{P}_f(\omega)$.

Note that, if $x_n > \Sigma_{k<n} x_k$ for $n > 0$, F and G are in $\mathcal{P}_f(\omega)$, and $\Sigma_{n\in F} x_n = \Sigma_{n\in G} x_n$, then $F = G$. Thus τ is well defined.

2.8 THEOREM (PRIKRY). *Let $\langle x_n \rangle_{n<\omega}$ be a sequence in ω such that for any m and n, if $2^m \leqslant x_n$, then $2^{m+1} | x_{n+1}$. Let τ be the natural map for $FS(\{x_n: n < \omega\})$. Let $p \in \beta\omega \setminus \omega$ such that $p + p = p$ and let $q = \{A \subseteq \omega$: There exists $B \in p$ such that $\tau^{-1}[B] \subseteq A\}$. Then $q \in \beta\omega \setminus \omega$ and $q + q = q$. (Thus, loosely, inverse images of idempotents are idempotents.)*

PROOF. (Note that the requirement that $p \in \beta\omega \setminus \omega$ serves only to eliminate the event that $p = \{A \subseteq \omega: 0 \in A\}$.) The verification that $q \in \beta\omega \setminus \omega$ is routine, and we omit it. (One needs to notice that τ is one-to-one and onto $\omega \setminus \{0\}$.)

To see that $q + q = q$, let $A \in q$. It suffices, by Lemma 3.1 of [**8**], to show that there is some $x \in A$ such that $A - x \in q$. Pick $B \in p$ such that $\tau^{-1}[B] \subseteq A$. Pick $y \in B$ such that $y > 0$ and $B - y \in p$. Let $x = \tau^{-1}(y)$.

Pick m such that $y < 2^m$, and let $C = \{2^m \cdot k: k < \omega\}$. Note that $C \in p$. (Otherwise, since $\omega = \bigcup_{t<2^m} C + t$, there is some t with $0 < t < 2^m$ such that $C + t \in p$. But if $z \in C + t$, then $(C + t) \cap ((C + t) - z) = \varnothing$.) Let $D = C \cap (B - y)$. Then $D \in p$. We claim that $\tau^{-1}[D] \subseteq A - x$. Let $z \in \tau^{-1}[D]$. Then

$$\tau(z) \in C \cap (B - y) = C \cap (B - \tau(x)).$$

Since $\tau(z) \in B - \tau(x)$, $\tau(z) + \tau(x) \in B$. Now $z = \Sigma_{n\in F} x_n$ and $x = \Sigma_{n\in G} x_n$ for some F and G in $\mathcal{P}_f(\omega)$. Since $\tau(z) \in C$ and $\tau(z) = \Sigma_{n\in F} 2^n$, we have $\min F \geqslant m$. Since $2^m > \tau(x) = \Sigma_{n\in G} 2^n$, we have $\max G < m$. Thus $\tau(x) + \tau(z) = \tau(x + z)$. Thus $\tau(x + z) \in B$, so $x + z \in A$. That is $z \in A - x$ as desired.

2.9 COROLLARY (GALVIN). *Let $A \in [\omega]^\omega$. There exists $p \in \beta\omega \setminus \omega$ such that $p + p = p$ and $FS(A) \in p$.*

PROOF. By Lemma 2.3 of [**8**], there is a sequence $\langle x_n \rangle_{n<\omega}$ such that $FS(\{x_n: n < \omega\}) \subseteq FS(A)$ and for any m and n, $2^{m+1} | x_{n+1}$ whenever $2^m \leqslant x_n$. By Theorem 2.8 there exists $q \in \beta\omega \setminus \omega$ such that $q + q = q$ and $FS(\{x_n: n < \omega\}) \in q$. Since $FS(\{x_n: n < \omega\}) \subseteq FS(A)$, we have $FS(A) \in q$.

2.10 COROLLARY. $|\{p \in \beta\omega: p + p = p\}| \geqslant c$.

PROOF. Choose inductively a sequence $\langle x_n \rangle_{n<\omega}$ in ω such that, for each $n > 0$, $x_n > \sum_{k<n} x_k$. By Theorem 7 of [9] we may choose a sequence $\langle A_\sigma \rangle_{\sigma<c}$ in $[\omega]^\omega$ such that, if $\sigma < \delta < c$, then $|A_\sigma \cap A_\delta| < \omega$. For each $\sigma < c$, let $B_\sigma = \{x_n : n \in A_\sigma\}$. Let $\sigma < \delta < c$. We claim that $|FS(B_\sigma) \cap FS(B_\delta)| < \omega$. Let $x \in FS(B_\delta) \cap FS(B_\delta)$ and pick $F \in \mathcal{P}_f(A_\sigma)$ and $G \in \mathcal{P}_f(A_\delta)$ such that

$$x = \sum_{n \in F} x_n \quad \text{and} \quad x = \sum_{n \in G} x_n.$$

Since, for $n > 0$, $x_n > \sum_{k<n} x_k$, we have $F = G$. Thus

$$|FS(B_\sigma) \cap FS(B_\delta)| = |\mathcal{P}_f(A_\sigma \cap A_\delta)| < \omega$$

as desired.

Pick, via Corollary 2.9, for each $\sigma < c$ some $p_\sigma \in \beta\omega \setminus \omega$ such that $p_\sigma + p_\sigma = p_\sigma$ and $FS(B_\sigma) \in p_\sigma$. Let $\sigma < \delta < c$. Since $|FS(B_\sigma) \cap FS(B_\delta)| < \omega$, we have $p_\sigma \neq p_\delta$.

2.11 COROLLARY. $\bar{\Gamma} = \mathrm{cl}_{\beta\omega}\{p \in \beta\omega \setminus \omega : p + p = p\}$.

PROOF. As we have already remarked, if $p \in \beta\omega \setminus \omega$ and $p + p = p$, then $p \in \bar{\Gamma}$. Now let $q \in \bar{\Gamma}$ and let $A \in q$. Then $A \in \Gamma$ so pick $B \in [A]^\omega$ such that $FS(B) \subseteq A$. Pick, via Corollary 2.9, $p \in \beta\omega \setminus \omega$ such that $p + p = p$ and $FS(B) \in p$. Then $A \in p$ so $p \in \mathrm{cl}_{\beta\omega} A$.

We now proceed to show, in Theorem 2.13, that in the event that one has some $p \in \beta\omega \setminus \omega$ such that $p + p = p$ and $p \cdot p = p$, a very strong conclusion holds.

We introduce now a notion of "finite sums and products", FSP_1. Another notion will be introduced later. If $A \subseteq \omega$, P is a polynomial in the variables $\langle y_n \rangle_{n \in A}$, and $\langle x_n \rangle_{n \in A}$ is a sequence in ω, we let $s(P) = \{n : y_n \text{ occurs in } P\}$ and let, as usual, $P(\langle x_n \rangle_{n \in A})$ denote the number obtained by replacing, for each $n \in A$, each occurence of y_n with x_n.

2.12 DEFINITION. Let $A \subseteq \omega$, let $\langle y_n \rangle_{n \in A}$ be a sequence of variables, and let $\langle x_n \rangle_{n \in A}$ be a sequence in ω.

(a) Let $R_0(A) = \{y_n : n \in A\}$. Define inductively, $R_{n+1}(A) = R_n(A) \cup \{y_m + P : P \in R_n(A) \text{ and } m < \min s(P)\} \cup \{y_m \cdot P : P \in R_n(A) \text{ and } m < \min s(P)\}$. Let $R(A) = \bigcup_{n<\omega} R_n(A)$.

(b) $FSP_1(\langle x_n \rangle_{n \in A}) = \{P(\langle x_n \rangle_{n \in A}) : P \in R(A)\}$.

An example of a member of $FSP_1(\langle x_n \rangle_{n<\omega})$ is

$$x_2 + x_5 + x_6(x_{10}(x_{11}(x_{13} + x_{15} + x_{20}))).$$

2.13 THEOREM. *If there exists $p \in \beta\omega \setminus \omega$ such that $p + p = p \cdot p = p$, then whenever $\{A_i\}_{i<r}$ is a partition of ω into finitely many cells, there exist $i < r$ and an increasing sequence $\langle x_n \rangle_{n<\omega}$ such that $FSP_1(\langle x_n \rangle_{n<\omega}) \subseteq A_i$.*

PROOF. Let $p \in \beta\omega \setminus \omega$ such that $p + p = p \cdot p = p$ and let $\{A_i\}_{i<r}$ be a partition of ω. Pick $i < r$ such that $A_i \in p$. Note that, for each $A \in p$,

$$A \cap \{x: A - x \in p\} \cap \{x: A/x \in p\} \in p.$$

Let $B_0 = A_i$ and pick $x_0 \in B_0$ such that $B_0/x_0 \in p$ and $B_0 - x_0 \in p$. Define inductively for $n > 0$,

$$B_n = B_{n-1} \cap (B_{n-1}/x_{n-1}) \cap (B_{n-1} - x_{n-1})$$

and pick $x_n > x_{n-1}$ such that $B_n/x_n \in p$ and $B_n - x_n \in p$. As in the proof of Theorem 2.6, one gets that $FSP_1(\langle x_n \rangle_{n<\omega}) \subseteq A_i$.

It is perhaps reasonable to expect that one might be able to show by invoking the continuum hypothesis, in a fashion similar to the proof of Theorem 3.3 of **[8]**, that the implication of Theorem 2.13 can be reversed. We have not in fact been able to accomplish this.

Since $\overline{\Gamma} = \mathrm{cl}_{\beta\omega}\{p \in \beta\omega \setminus \omega: p + p = p\}$ and there exists $p \in \Gamma$ such that $p \cdot p = p$, one might hope that any such p would also have the property that $p + p = p$. Corollary 2.16. shows that this is not the case. Part (a) of the following theorem may be of use in finding some simultaneous idempotent.

2.14 THEOREM. (a) *If $p \in \beta\omega \setminus \omega, p + p \neq p$, and $p \cdot p = p$, then there exists a sequence $\langle A_n \rangle_{n<\omega}$ in p such that*

(1) *for each $x \in A_0$, there exists m such that $(A_0 - x) \cap A_m = \varnothing$;*

(2) *for each n, $\phi \neq A_{n+1} \subseteq A_n$; and*

(3) *for each n there exists m such that, for each $x \in A_m$, there exists r with $A_r \subseteq A_n/x$.*

(b) *If $\langle A_n \rangle_{n<\omega}$ is a sequence in Γ satisfying conditions* (1), (2), *and* (3) *above, then there exists $p \in \overline{\Gamma}$ such that $p + p \neq p$, $p \cdot p = p$, and $\{A_n: n < \omega\} \subseteq p$.*

PROOF. (a) Since $p + p \neq p$, there exists $A \in p$ such that $\{x: A - x \in p\} \notin p$. Let $A_0 = A \cap \{x: A - x \notin p\}$. Then, for any $x \in A_0$, $A_0 - x \notin p$. Enumerate A_0 as $\{x_n: n < \omega\}$. Let $C_0 = \{y: A_0/y \in p\}$ and enumerate C_0 as $\{y_{0,n}: n < \omega\}$. Note that $A_0 \in p$ and $C_0 \in p$. Inductively let

$$A_{n+1} = \left(A_n \cap \bigcap_{k=0}^{n} (A_k/y_{k,n-k}) \cap C_n \right) \setminus (A_0 - x_n).$$

Note that $A_{n+1} \in p$, let $C_{n+1} = \{y: A_{n+1}/y \in p\}$, and enumerate C_{n+1} as $\{y_{n+1,k}: k < \omega\}$. Note that $C_{n+1} \in p$.

To verify condition (1), note that $(A_0 - x_n) \cap A_{n+1} = \varnothing$. Condition (2) is trivial. To verify condition (3), let n be given and let $m = n + 1$. Let $x \in A_m$ and note that $x \in C_n$ so that $x = y_{n,t}$ for some t. Let $r = t + n + 1$. Then $A_r \subseteq A_n/y_{n,r-1-n} = A_n/y_{n,t}$.

(b) With $\langle A_n \rangle_{n<\omega}$ as given in the hypothesis, let $\Delta = \{p \in \overline{\Gamma}: \{A_n: n < \omega\} \subseteq p\}$. The proof that $\Delta \neq \varnothing$ proceeds exactly like the proof of Theorem 2.5 in

[8]. If $q \in \overline{\Gamma} \setminus \Delta$, then for some n, $A_n \notin q$ and hence $\mathrm{cl}_{\beta\omega}(\omega \setminus A_n)$ is a neighborhood of q missing Δ. Thus Δ is closed in $\beta\omega$.

We claim that $\cdot: \Delta \times \Delta \to \Delta$. To this end, let p and q be in Δ and let n be given. Pick m such that, for all $x \in A_m$, there exist r with $A_r \subseteq A_n/x$. We claim that $A_m \subseteq \{x: A_n/x \in p\}$. Let $x \in A_m$ and pick A_r such that $A_r \subseteq A_n/x$. Since $A_r \in p$, $A_n/x \in p$. Since $A_m \subseteq \{x: A_n/x \in p\}$ and $A_m \in q$, we have $\{x: A_n/x \in p\} \in q$. Thus, $A_n \in p \cdot q$.

We thus have, by Lemmas 2.2 and 2.3, that there exists p in Δ with $p \cdot p = p$. By condition (1), we have $\{x: A_0 - x \in p\} \cap A_0 = \varnothing$ so that $A_0 \notin p + p$, and hence $p + p \neq p$.

The following notion of "finite sums and products" will be used in the proof of Corollary 2.16.

2.15 Definition. Let $A \subseteq \omega$ and let $\langle x_n \rangle_{n \in A}$ be a sequence in ω.

(a) $FSP_2(\langle x_n \rangle_{n \in A}) = \{\Sigma_{G \in \mathcal{G}} \Pi_{k \in G}\, x_k: \mathcal{G} \in \mathcal{P}_f(\mathcal{P}_f(A))\}$.

(b) $SP(\langle x_n \rangle_{n \in A}) = \{\Sigma_{G \in \mathcal{G}} \Pi_{k \in G}\, x_k: \mathcal{G} \in \mathcal{P}_f(\mathcal{P}_f(A))$ and whenever $\mathcal{G} = \mathcal{F} \cup \mathcal{H}$ with $\mathcal{F}$ and $\mathcal{H}$ nonempty $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{H}) \neq \varnothing\}$.

As an example, let $\mathcal{G} = \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}$. Then

$$\sum_{G \in \mathcal{G}} \prod_{k \in G} x_k = x_1 x_2 + x_1 x_3 + x_4 x_5.$$

Thus $x_1x_2 + x_1x_3 + x_4x_5 \in FSP_2(\langle x_n \rangle_{n<\omega})$. It is not (on its face) a member of $SP(\langle x_n \rangle_{n<\omega})$ since $(\bigcup\{\{1, 2\}, \{1, 3\}\}) \cap (\bigcup\{\{4, 5\}\}) = \varnothing$. (Of course it might happen that $x_1x_2 + x_1x_3 + x_4x_5 = x_1x_6$.)

The idea of the following proof is to start with a sequence $\langle x_n \rangle_{n<\omega}$ such that the expressions in the definition of $FSP_2(\langle x_n \rangle_{n<\omega})$ are unique. Then, using the notions of FSP_2 and SP, we obtain sets $\langle A_n \rangle_{n<\omega}$ satisfying the conditions of Theorem 2.14.

2.16 Corollary. *There exists p in $\overline{\Gamma}$ such that $p \cdot p = p$ and $p + p \neq p$.*

Proof. For each n, let $x_n = 2^{2^n}$, and note that the expressions in the definition of $FSP_2(\langle x_n \rangle_{n<\omega})$ are unique. That is, if $\Sigma_{G \in \mathcal{G}} \Pi_{k \in G}\, x_k = \Sigma_{F \in \mathcal{F}} \Pi_{k \in F}\, x_k$, then $\mathcal{G} = \mathcal{F}$. (To see this, note that $\Sigma_{G \in \mathcal{G}} \Pi_{k \in G}\, x_k = \Sigma_{t \in S} 2^t$, and the members G of $\mathcal{G}$ can be read off the binary expansions of the members t of S.)

Let $\{D_n\}_{n<\omega}$ be a partition of ω into infinite sets, and for each n, let $E_n = \bigcup_{k \geq n} D_k$. For each n, let $A_n = \bigcup_{k \geq n} FSP_2(\langle x_t \rangle_{t \in D_k}) \cup SP(\langle x_t \rangle_{t \in E_n})$. We claim that $\langle A_n \rangle_{n<\omega}$ is a sequence in Γ satisfying conditions (1), (2), and (3) of Theorem 2.14.

That each $A_n \in \Gamma$ follows from the fact that

$$FS(\{x_t: t \in D_n\}) \subseteq FSP_2(\langle x_t \rangle_{t \in D_n}).$$

That condition (2) holds is trivial.

To see that condition (3) holds, let n be given, let $m = n$, and let $x \in A_m$. Now

$$\bigcup_{k>n} FSP_2(\langle x_t \rangle_{t \in D_k}) \subseteq FSP_2(\langle x_t \rangle_{t \in E_n})$$

and

$$SP(\langle x_t \rangle_{t \in E_n}) \subseteq FSP_2(\langle x_t \rangle_{t \in E_n})$$

so $x \in FSP_2(\langle x_t \rangle_{t \in E_n})$. Pick $\mathcal{F} \in \mathcal{P}_f(\mathcal{P}_f(E_n))$ such that $x = \sum_{F \in \mathcal{F}} \prod_{k \in F} x_k$. Let $a = \max \bigcup \mathcal{F}$ and pick r such that $a < \min E_r$.

We claim that $A_r \subseteq A_n/x$. To this end let $y \in A_r$. Then, as above, we may pick $\mathcal{G} \in \mathcal{P}_f(\mathcal{P}_f(E_r))$ such that $y = \sum_{G \in \mathcal{G}} \prod_{k \in G} x_k$. Since $\min E_r > a$, we have that $F \cap G = \varnothing$ whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Thus we have that

$$x \cdot y = \sum_{(F,G) \in \mathcal{F} \times \mathcal{G}} \prod_{k \in (F \cup G)} x_k.$$

We also have that, if $\mathfrak{R}$ and $\mathfrak{S}$ are nonempty sets and $\mathcal{F} \times \mathcal{G} = \mathfrak{R} \cup \mathfrak{S}$, then

$$\left(\bigcup_{(F,G) \in \mathfrak{R}} (F \cup G) \right) \cap \left(\bigcup_{(F,G) \in \mathfrak{S}} (F \cup G) \right) \neq \varnothing.$$

Thus $x \cdot y \in SP(\langle x_k \rangle_{k \in E_n})$ so that $y \in A_n/x$ as desired.

Finally we verify that condition (1) of Theorem 2.14 holds. Let $x \in A_0$ and pick $\mathcal{F} \in \mathcal{P}_f(\mathcal{P}_f(E_0))$ such that $x = \sum_{F \in \mathcal{F}} \prod_{k \in F} x_k$. Let $a = \max \bigcup \mathcal{F}$ and pick n such that $a < \min E_n$. To see that $A_n \cap (A_0 - x) = \varnothing$, let $y \in A_n$. We claim that $y + x \notin A_0$.

Pick $\mathcal{G} \in \mathcal{P}_f(\mathcal{P}_f(E_n))$ such that $y = \sum_{G \in \mathcal{G}} \prod_{k \in G} x_k$. Then $x + y = \sum_{F \in \mathcal{F} \cup \mathcal{G}} \prod_{k \in F} x_k$. Since $\max \bigcup \mathcal{F} < \min \bigcup \mathcal{G}$, we have $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \varnothing$. Since, as remarked at the beginning of this proof, expressions of the form $\sum_{F \in \mathcal{F} \cup \mathcal{G}} \prod_{k \in F} x_k$ are unique, we have $x + y \notin SP(\langle x_k \rangle_{k \in E_0})$. Since $\min E_n > \max \bigcup \mathcal{F}$, we have that for no k is $\bigcup(\mathcal{F} \cup \mathcal{G}) \subseteq D_k$. Thus, again using the uniqueness of the expression of $x + y$, we have

$$x + y \notin \bigcup_{k>0} FSP_2(\langle x_t \rangle_{t \in D_k}).$$

Invoking Theorem 2.14 (b), the proof is complete.

We close this section with some results and remarks about the operations $+$ and $\cdot$ on $\beta\omega$. (For other information see [5].) This information is of interest in the context of this paper primarily because these operations provide what currently appears to be the most promising attack on the main problem.

Since $+$ and $\cdot$ extend the ordinary operations on ω, one might expect them to be commutative and expect the distributive law to hold. In fact none of these conclusions is valid, as we see in Theorems 2.17, 2.19, and 2.20.

The following theorem answers a question of Galvin (in a private communication). The idea of the second part of the proof is, given $p \in \beta\omega \setminus \omega$, to construct a countable subfamily $\{B_n: n < \omega\}$ of p and nearly complementary sets D and E such that, if r and q are in $\beta\omega$, $\{B_n: n < \omega\} \cup \{D\} \subseteq r$, and $\{B_n: n < \omega\} \cup \{E\} \subseteq q$, then $r + q \neq q + r$.

2.17 THEOREM. *The center of the semigroup* $(\beta\omega, +)$ *is* ω.

PROOF. Recall that we have identified each element n of ω with the ultrafilter $\{A \subseteq \omega: n \in A\}$. For the purposes of this proof, denote $\{A \subseteq \omega: n \in A\}$ by n^*.

Let $n \in \omega$ and let $p \in \beta\omega$. We show first that $n^* + p = p + n^*$. Since both $n^* + p$ and $p + n^*$ are ultrafilters it suffices to show that $n^* + p \subseteq p + n^*$. To this end, let $A \in n^* + p$. Then $\{x: A - x \in n^*\} \in p$. But, for any x, $A - x \in n^*$ if and only if $n + x \in A$. Thus $\{x: A - x \in n^*\} = A - n$. Thus $A - n \in p$ so $n \in \{x: A - x \in p\}$ and hence $A \in p + n^*$.

Now let $p \in \beta\omega \setminus \omega$ and, for each r, pick i_r with $0 \leqslant i_r < 2^r$ such that $\{2^r \cdot k + i_r: k < \omega\} \in p$. (We can do this since, for each r, $\omega = \bigcup_{i<2^r}\{2^r \cdot k + i: k < \omega\}$.) For each r, let $B_r = \{2^r \cdot k + i_r: k < \omega\}$ and note that $B_{r+1} \subseteq B_r$ (since otherwise $B_{r+1} \cap B_r = \varnothing$). Thus, for each r, either $i_{r+1} = i_r$ or $i_{r+1} = i_r + 2^r$.

We claim that, if $2^r(2k + 1) + 2i_r = 2^s(2t + 1) + 2i_s$, then $s = r$ and $k = t$. We assume that $r \leqslant s$. If $i_r = i_s$, then $2^r(2k + 1) = 2^s(2t + 1)$ and the claim is established. Otherwise $r < s$ and there is some $F \subseteq \{n: r \leqslant n < s\}$ such that $i_s = i_r + \sum_{n\in F} 2^n$. Then

$$2^r(2k + 1) + 2i_r = 2^s(2t + 1) + 2\left(i_r + \sum_{n\in F} 2^n\right)$$

so that $2^r(2k + 1) = 2^{r+1}(2^{s-1-r}(2t + 1) + \sum_{n\in F} 2^{n-r})$ so that 2^{r+1} divides $2^r(2k + 1)$, a contradiction.

Let $A = \{2^{2r}(2k + 1) + 2i_{2r}: \{r, k\} \subseteq \omega\}$ and note that, if $x = 2^{2r+1}(2k + 1) + 2i_{2r+1}$ for some r and k, then $x \notin A$. For each r, let $C_r = B_r \setminus B_{r+1}$, let $D = \bigcup_{r<\omega} C_{2r}$ and let $E = \bigcup_{r<\omega} C_{2r+1}$. In the event that there is some i such that, for all sufficiently large r, $i = i_r$ one has $\omega \setminus (D \cup E) = \{i\}$. Otherwise, $\omega = D \cup E$. In any event, since $p \in \beta\omega \setminus \omega$, either $D \in p$ or $E \in p$.

In the event that $D \in p$, pick $q \in \beta\omega$ such that $\{B_r: r < \omega\} \cup \{E\} \subseteq q$. In the event that $E \in p$, pick $q \in \beta\omega$ such that $\{B_r: r < \omega\} \cup \{D\} \subseteq q$.

We show now that, if p and q are in $\beta\omega$, $\{B_r: r < \omega\} \cup \{D\} \subseteq p$, and $\{B_r: r < \omega\} \cup \{E\} \subseteq q$, then $A \in (q + p) \setminus (p + q)$ so that $p + q \neq q + p$. Let $x \in D$ and pick r such that $x \in C_{2r}$. We claim that $B_{2r+1} \subseteq A - x$. To this end let $y \in B_{2r+1}$, so that $y = 2^{2r+1} \cdot k + i_{2r+1}$ for some k. Now $x \in B_{2r} \setminus$

B_{2r+1}. Thus, if $i_{2r+1} = i_{2r}$, then

$$x = 2^{2r+1} \cdot s + i_{2r} + 2^{2r}$$

for some s and $y = 2^{2r+1} \cdot k + i_{2r}$. If $i_{2r+1} = i_{2r} + 2^{2r}$, then $x = 2^{2r+1} \cdot s + i_{2r}$ for some s and $y = 2^{2r+1} \cdot k + i_{2r} + 2^{2r}$. In either event we have

$$x + y = 2^{2r}(2(k + s) + 1) + 2i_{2r}$$

so that $x + y \in A$. Thus we have, since $B_{2r+1} \in q$, that $A - x \in q$ whenever $x \in D$. So that $D \subseteq \{x: A - x \in q\}$ and hence $A \in q + p$.

By a similar argument we can show that, if $x \in C_{2r+1}$, then $B_{2r+2} \cap (A - x) = \varnothing$. Thus, $E \cap \{x: A - x \in p\} = \varnothing$ and hence $A \notin p + q$, as desired.

We abuse terminology slightly and present the following, which is a corollary to the proof of Theorem 2.17, rather than a corollary to the theorem itself.

2.18 Corollary. *There exist sets Δ_1, and Δ_2 such that $\overline{\Gamma} = \Delta_1 \cup \Delta_2$ and $p + q \neq q + p$ whenever $p \in \Delta_1$ and $q \in \Delta_2$.*

Proof. If $p \in \overline{\Gamma}$, then $\{2^r \cdot k: k < \omega\} \in p$ for each r. Let $A = \{2^{2r}(2k + 1): \{r, k\} \subseteq \omega\}$ and let $\Delta_1 = \{p \in \overline{\Gamma}: A \in p\}$ and $\Delta_2 = \overline{\Gamma} \setminus \Delta_1$. Let $p \in \Delta_1$. Then in the proof of Theorem 2.17, we have each $i_r = 0$ so that $A = \{2^{2r}(2k + 1) + 2i_{2r}: \{r, k\} \subseteq \omega\}$ as in that proof. The set D chosen in that proof is then equal to A. Thus $\Delta_2 = \{q \in \overline{\Gamma}: \{B_r: r < \omega\} \cup \{E\} \subseteq q\}$.

2.19 Theorem. *There is an isomorphism γ from $(\beta\omega, +)$ into $(\beta\omega, \cdot)$. (Consequently $(\beta\omega, \cdot)$ is not commutative.)*

Proof. Define γ by agreeing that for $p \in \beta\omega$, $A \in \gamma(p)$ if and only if $\{x: 2^x \in A\} \in p$. (The verification that $\gamma(p)$ is an ultrafilter is routine, as is the verification that γ is one-to-one.)

Let p and q be in $\beta\omega$. Since $\gamma(p + q)$ and $\gamma(p) \cdot \gamma(q)$ are ultrafilters, it suffices to show that $\gamma(p + q) \subseteq \gamma(p) \cdot \gamma(q)$. Let $A \in \gamma(p + q)$ and let $B = \{x: 2^x \in A\}$. Let $C = \{x: B - x \in p\}$. Since $B \in p + q$, $C \in q$. Let $D = \{2^x: x \in C\}$. Then $D \in \gamma(q)$. We claim that $D \subseteq \{x: A/x \in \gamma(p)\}$ and hence that $A \in \gamma(p) \cdot \gamma(q)$. Let $x \in D$ and pick $y \in C$ such that $x = 2^y$. Then $B - y \in p$. Let $E = \{2^z: z \in B - y\}$. Then $E \in \gamma(p)$. It thus suffices to show that $E \subseteq A/x$. Let $w \in E$ and pick $z \in B - y$ such that $w = 2^z$. Then $z + y \in B$ so $2^{z+y} \in A$. But $2^{z+y} = w \cdot x$ so $w \in A/x$.

2.20 Theorem. *There exist p and q in $\beta\omega$ such that $p \cdot (q + q) \neq p \cdot q + p \cdot q$ and $(q + q) \cdot p \neq q \cdot p + q \cdot p$.*

Proof. Let $A = \{2^{2n}(2k + 1): \{n, k\} \subseteq \omega\}$. Since $FS(\{2^{2n}: n < \omega\}) \subseteq A$, we may by Corollary 2.9, pick $p \in \beta\omega \setminus \omega$ such that $A \in p$ and $p + p = p$.

Let $q = \{B \subseteq \omega: 1 \in B\}$. Then $p \cdot q = q \cdot p = p$ so

$$p \cdot q + p \cdot q = q \cdot p + q \cdot p = p + p = p.$$

But $q + q = \{B \subseteq \omega: 2 \in B\}$, and hence $A \notin p \cdot (q + q)$. Since also $p \cdot (q + q) = (q + q) \cdot p$, we are done.

3. Infinite combinatorial results. We present here, in Theorems 3.1 and 3.4, two conditions on a partition $\{A_i\}_{i<r}$ of ω, each of which guarantees the existence of some infinite set with all of its finite sums and finite products in the same cell. Loosely speaking, the condition presented in Corollary 3.2 is that the partition include arbitrarily long blocks and that the gaps between blocks of a given length be bounded. The condition of Theorem 3.1 is the same except that reference is made to blocks of multiples of some fixed number c. The basic idea of the proof consists of obtaining a sequence, each of whose finite products is a long distance from the end of the block into which it falls. Then sums of those products must lie in the same block.

3.1 THEOREM. *Let $\{A_i\}_{i<r}$ be a partition of ω. Assume that there are some $c > 0$ and some function f from ω to ω such that, for any y and n, there exist $i < r$ and z, with $y \leqslant z \leqslant y + f(n)$, such that $\{(z + m)c: m \leqslant n\} \subseteq A_i$. Then there exist $i < r$ and $B \in [\omega]^\omega$ such that $FS(B) \subseteq A_i$ and $FP(B) \subseteq A_i$.*

PROOF. We assume that, for each $i < r$, $\{mc: m < \omega\} \cap A_i$ is infinite. (The set B which we will obtain will be contained in $\{mc: m < \omega\}$ so that cells including only finitely many multiples of c may be merged with other cells, still leaving an infinite subset of B with all finite sums and products in one of the orginal cells.)

For each $i < r$, let $a_{i,0} = \min\{m: mc \in A_i\}$ and let $b_{i,0} = \max\{m: a_{i,0} \leqslant m$ and $\{tc: a_{i,0} \leqslant t \leqslant m\} \subseteq A_i\}$. Inductively, let $a_{i,n+1} = \min\{m: b_{i,n} < m$ and $mc \in A_i\}$ and let $b_{i,n+1} = \max\{m: a_{i,n+1} \leqslant m$ and $\{tc: a_{i,n+1} \leqslant t \leqslant m\} \subseteq A_i\}$. Thus, for each i,

$$A_i \cap \{mc: m < \omega\} = \bigcup_{n<\omega} \{mc: a_{i,n} \leqslant m \leqslant b_{i,n}\}.$$

Let $y_0 = c$ and assume we have chosen $\langle y_k \rangle_{k<m}$ satisfying the following inductive hypotheses.

(1) $y_k \in \{nc: n < \omega\}$;

(2) if $k > 0$, then $y_k > \Pi_{t<k} y_t$; and

(3) if $F \in \mathcal{P}_f(k + 1)$ and $s = \max F$,

then there exist i and n such that $ca_{i,n} \leqslant \Pi_{t \in F} y_t \leqslant cb_{i,n} - s$.

(In hypothesis (3), recall that $k + 1 = \{t: t < k + 1\}$.) Let $d = 2^m$ and let $\{p_t: t < d\} = FP(\{y_t: t < m\}) \cup \{1\}$, with $p_0 = \Pi_{t<m} y_t$. Let $u_0 = 0$ and for $0 < t \leqslant d$, let $u_t = m + p_{d-t}(f(u_{t-1}) + u_{t-1} + 1)$.

We now do a subsidiary induction, choosing $\langle h_t \rangle_{t<d}$ and $\langle e_t \rangle_{t<d}$ such that,

for each $t < d$:

(a) if $t > 0$, then $p_0 < h_{t-1} \leqslant h_t$ and $e_t \leqslant e_{t-1}$;

(b) $e_t = h_t + f(u_{d-t-1}) + u_{d-t-1}$; and

(c) there exist $i < r$ and n such that $a_{i,n} \leqslant h_t p_t \leqslant e_t p_t \leqslant b_{i,n} - m$.

To ground this induction, pick $i < r$ and n such that $a_{i,n} > p_0^2$ and $b_{i,n} - a_{i,n} \geqslant u_d$, let $h_0 = \min\{h\colon a_{i,n} \leqslant hp_0\}$, and let $e_0 = h_0 + f(u_{d-1}) + u_{d-1}$. Note that $h_0 > p_0$. Condition (a) is satisfied vacuously and condition (b) and the first two inequalities of condition (c) are satisfied directly. Now

$$\begin{aligned} b_{i,n} - e_0 p_0 &= b_{i,n} - p_0(h_0 + f(u_{d-1}) + u_{d-1}) \\ &= b_{i,n} - p_0(h_0 - 1) - p_0(f(u_{d-1}) + u_{d-1} + 1) \\ &> b_{i,n} - a_{i,n} - \big(p_0(f(u_{d-1}) + u_{d-1} + 1) + m\big) + m \geqslant u_d - u_d + m, \end{aligned}$$

so that condition (c) holds.

Now let $0 < k < d$ and assume that we have chosen $\langle h_t \rangle_{t<k}$ and $\langle e_t \rangle_{t<k}$ satisfying conditions (a), (b), and (c). Pick $i < r$ and z such that $h_{k-1}p_k \leqslant z \leqslant h_{k-1}p_k + f(u_{d-k})$ and $\{(z + t)c\colon t \leqslant u_{d-k}\} \subseteq A_i$. Pick n such that $a_{i,n} \leqslant z \leqslant b_{i,n}$ and note that $b_{i,n} - z \geqslant u_{d-k}$. Let $h_k = \min\{h\colon z \leqslant hp_k\}$ and let $e_k = h_k + f(u_{d-k-1}) + u_{d-k-1}$. Condition (b) and the first two inequalities of condition (c) are satisfied directly. The verification of the third inequality of condition (c) is essentially the same as it was with $k = 0$ (except that $a_{i,n}$ is replaced by z).

Since $h_0 > p_0$, $h_k p_k \geqslant z \geqslant h_{k-1}p_k$, and $p_k > 0$, we have $p_0 < h_{k-1} \leqslant h_k$. Also, $p_k e_k = p_k h_k + p_k f(u_{d-k-1}) + p_k u_{d-k-1} < z + p_k + p_k f(u_{d-k-1}) + p_k u_{d-k-1} \leqslant h_{k-1}p_k + f(u_{d-k}) + p_k(f(u_{d-k-1}) + u_{d-k-1} + 1) = h_{k-1}p_k + f(u_{d-k}) + u_{d-k} - m < p_k(h_{k-1} + f(u_{d-k}) + u_{d-k}) = p_k e_{k-1}$. Thus, condition (a) is satisfied.

The subsidiary induction being complete, let $y_m = ch_{d-1}$. Since $h_{d-1} \geqslant h_0 > p_0 = \prod_{t<m} y_t$, we have that hypotheses (1) and (2) are satisfied. To verify hypothesis (3), it suffices to consider $F \in \mathcal{P}_f(m + 1)$ such that $m = \max F$. Let $p = 1$ if $F = \{m\}$ and otherwise let $p = \prod_{t \in F \setminus \{m\}} y_t$. Then $p = p_k$ for some $k < d$. Pick $i < r$ and n such that $a_{i,n} \leqslant h_k p_k \leqslant e_k p_k \leqslant b_{i,n} - m$. Since $h_k \leqslant h_{d-1} \leqslant e_{d-1} \leqslant e_k$ we have that $a_{i,n} \leqslant h_{d-1}p_k \leqslant b_{i,n} - m$. Thus

$$ca_{i,n} \leqslant ch_{d-1}p_k \leqslant cb_{i,n} - cm \leqslant cb_{i,n} - m.$$

Since $\prod_{t \in F} y_t = ch_{d-1}p_k$, hypothesis (3) is satisfied, and the induction is complete.

For each $i < r$, let $C_i = \{F \in \mathcal{P}_f(\omega)\colon \prod_{n \in F} y_n \in A_i\}$. Pick, by Corollary 3.3 of [6], $i < r$ and a sequence $\langle F_n \rangle_{n<\omega}$ of disjoint members of $\mathcal{P}_f(\omega)$ such that $\bigcup_{n \in G} F_n \in C_i$ whenever $G \in \mathcal{P}_f(\omega)$. We may presume, by suitably thinning the sequence $\langle F_n \rangle_{n<\omega}$, that if $d = \max F_n$, then $\min F_{n+1} > \prod_{k \leqslant d} y_k$.

For each n, let $x_n = \prod_{k \in F_n} y_k$, and let $B = \{x_n: n < \omega\}$. Since, for any $G \in \mathcal{P}_f(\omega), \prod_{n \in G} x_n = \prod_{k \in \cup_{n \in G} F_n} y_k$, we have immediately that $FP(B) \subseteq A_i$. To see that $FS(B) \subseteq A_i$, let $G \in \mathcal{P}_f(\omega)$ with $|G| \geqslant 2$. Let $s = \max G$, let $m = \max F_s$, and let $d = \max F_{s-1}$. Pick j and n such that $ca_{j,n} \leqslant \prod_{k \in F_s} y_k \leqslant cb_{j,n} - m$. Since $\prod_{k \in F_s} y_k \in A_i$, we have $j = i$.

Now $\sum_{t \in G} x_t = \sum_{t \in G \setminus \{s\}} \prod_{k \in F_t} y_k + \prod_{k \in F_s} y_k$. Thus $\sum_{t \in G} x_t \geqslant ca_{j,n}$. Also

$$\sum_{t \in G \setminus \{s\}} \prod_{k \in F_t} y_k \leqslant \prod_{k < d} y_k < \min F_s \leqslant m.$$

Thus $\sum_{t \in G} x_t \leqslant cb_{i,n}$. Since each x_t is a muliple of c, $\sum_{t \in G} x_t \in A_i$, as desired.

3.2 Corollary. *Let $\{A_i\}_{i<r}$ be a partition of ω. Assume that there is some function f from ω to ω such that, for any y and n, there exist $i < r$ and z with $y \leqslant z \leqslant y + f(n)$ and $\{z + m: m \leqslant n\} \subseteq A_i$. Then there exist $i < r$ and $B \in [\omega]^\omega$ such that $FS(B) \subseteq A_i$ and $FP(B) \subseteq A_i$.*

It is quite possible that, given a partition $\{A_i\}_{i<r}$ of ω, there is only one $i < r$ such that $FS(B) \subseteq A_i$ for some $B \in [\omega]^\omega$. (For example let $A_i = \{nr + i: n < \omega\}$.) Theorem 3.4 tells us that, in such a situation, we can get $i < r$ and $B \in [\omega]^\omega$ such that $FS(B) \subseteq A_i$ and $FP(B) \subseteq A_i$.

In the following proofs, when we write $a \equiv b \pmod{\{A_i\}_{i<r}}$ for a partition $\{A_i\}_{i<r}$ of ω, we mean a is congruent to b modulo the equivalence relation on ω which is induced by $\{A_i\}_{i<r}$.

3.3 Lemma. *Let $\{A_i\}_{i<r}$ be a partition of ω. There is an increasing sequence $\langle x_n \rangle_{n<\omega}$ in $\omega \setminus \{0\}$ such that, if $\{F, G\} \subseteq \mathcal{P}_f(\omega)$, $k = \min(F \cup G)$, and $t \leqslant \prod_{i<k} x_i$, then $t \cdot \sum_{n \in F} x_n \equiv t \cdot \sum_{n \in G} x_n \pmod{\{A_i\}_{i<r}}$.*

Proof. For each n, let $H_{0,n} = \{n + 1\}$. Let $q \geqslant 1$ and assume that for each $p < q$ we have chosen a sequence $\langle H_{p,n} \rangle_{n<\omega}$ in $\mathcal{P}_f(\omega)$ satisfying the following inductive hypotheses.

(1) For each n, $\max H_{p,n} < \min H_{p,n+1}$;

(2) if $p > 0$ and $n < p$, then $H_{p,n} = H_{p-1,n}$ and

(3) if $m \leqslant p$, $\{F, G\} \subseteq \mathcal{P}_f(\omega)$, $m \leqslant \min(F \cup G)$, and $t < m$ then

$$t \sum_{n \in F} \sum H_{p,n} \equiv t \sum_{n \in G} \sum H_{p,n} \pmod{\{A_i\}_{i<r}}.$$

Hypothesis (1) is trivially satisfied for $p = 0$ and hypotheses (2) and (3) are vacuous there.

For F and G in $\mathcal{P}_f(\omega)$, agree that $F \approx G$ if and only if, for all $t < q$, $t\sum_{n \in F} \sum H_{q-1,n} \equiv t\sum_{n \in G} \sum H_{q-1,n} \pmod{\{A_i\}_{i<r}}$. There are only finitely many equivalence classes mod $\approx$ (at most r^{q-1}). Pick, by Corollary 3.3 of **[6]**, a sequence $\langle F_n \rangle_{n<\omega}$ of pairwise disjoint members of $\mathcal{P}_f(\omega)$ such that, if

$\{H, K\} \subseteq \mathcal{P}_f(\omega)$, then $\bigcup_{n\in H} F_n \approx \bigcup_{n\in K} F_n$. We may presume that, for all n, $\max F_n < \min F_{n+1}$.

Now, for $n < q$ let $H_{q,n} = H_{q-1,n}$, and for $n \geqslant q$ let $H_{q,n} = \bigcup_{k\in F_n} H_{q-1,k}$. Hypotheses (1) and (2) can be easily verified. To verify hypothesis (3), let $m \leqslant q$, let $\{F, G\} \subseteq \mathcal{P}_f(\omega)$ with $m \leqslant \min(F \cup G)$, and let $t < m$. Let $K = \bigcup\{F_n\colon n \in F \text{ and } n \geqslant q\} \cup \{n \in F\colon n < q\}$ and let $L = \bigcup\{F_n\colon n \in G \text{ and } n \geqslant q\} \cup \{n \in G\colon n < q\}$. Then

$$\sum_{n\in F} \sum H_{q,n} = \sum_{n\in K} \sum H_{q-1,n} \quad \text{and} \quad \sum_{n\in G} \sum H_{q,n} = \sum_{n\in L} \sum H_{q-1,n}.$$

Assume first that $m < q$. Then $m \leqslant q - 1$ and $m \leqslant \min(K \cup L)$ so the conclusion follows from the fact that the hypothesis (3) holds at $q - 1$. (Notice that, since $t < m$, this case can hold only if $q \geqslant 2$.) Now assume $m = q$, and note that in this case $K = \bigcup_{n\in F} F_n$ and $L = \bigcup_{n\in G} F_n$. Since K and L are congruent mod $\approx$, the hypothesis is satisfied.

Let $f(0) = 2$ and let $x_0 = \sum H_{2,2}$. Inductively, given $\langle x_k\rangle_{k<n}$, let $f(n) = 1 + \prod_{k<n} x_k$ and let $x_n = \sum H_{f(n),f(n)}$. Now let $\{F, G\} \subseteq \mathcal{P}_f(\omega)$, let $k = \min(F \cup G)$, and let $t \leqslant \prod_{i<k} x_i$. Let $K = \{f(n)\colon n \in F\}$, let $L = \{f(n)\colon n \in G\}$ and let $q = \max(K \cup L)$. Note that, by hypothesis (2), for each $n \in F \cup G$, $x_n = \sum H_{q,f(n)}$. Thus

$$\sum_{n\in F} x_n = \sum_{n\in K} \sum H_{q,n} \quad \text{and} \quad \sum_{n\in G} x_n = \sum_{n\in L} \sum H_{q,n}.$$

Since $t \leqslant \prod_{i<k} x_i$, $t < f(k)$ and hence $t < \min(K \cup L)$. Thus, by hypothesis (3),

$$t \sum_{n\in F} x_n \equiv t \sum_{n\in G} x_n \pmod{\{A_i\}_{i<r}}.$$

3.4 THEOREM. *Let $\{A_i\}_{i<r}$ be a partition of ω, let $i < r$, and assume that, whenever $j < r$ and $B \in [\omega]^\omega$ with $FS(B) \subseteq A_j$, one has $j = i$. Then there is some $B \in [\omega]^\omega$ such that $FS(B) \subseteq A_i$ and $FP(B) \subseteq A_i$.*

PROOF. Let $\langle x_n\rangle_{n<\omega}$ be as guaranteed by Lemma 3.4. Let $B = \{x_n\colon n > 0\}$. Now, let $F \in \mathcal{P}_f(\omega)$ with $\min F > 0$. Then $1 = \min(F \cup \{1\})$ and $1 \leqslant \prod_{k<1} x_k$ so that $x_1 \equiv \sum_{n\in F} x_n \pmod{\{A_j\}_{j<r}}$. Thus, if $x_1 \in A_j$ we have that $FS(B) \subseteq A_j$ so that, by the hypothesis of the theorem we have $FS(B) \subseteq A_i$.

Now let $F \in \mathcal{P}_f(\omega)$ with $\min F > 0$ and pick $j < r$ such that $\prod_{n\in F} x_n \in A_j$. We show that $j = i$. If $|F| = 1$, this follows from the fact that $FS(B) \subseteq A_i$, so assume $|F| > 1$. Let $q = \max F$, let $G = F \setminus \{q\}$, and let $t = \prod_{n\in G} x_n$. Then $t \cdot x_q = \prod_{n\in F} x_n$ so $t \cdot x_q \in A_j$.

We claim that $FS(\{t \cdot x_n\colon n \geqslant q\}) \subseteq A_j$ and hence that $j = i$. To this end, let $H \in \mathcal{P}_f(\omega)$ with $\min H \geqslant q$. Then $q = \min(H \cup \{q\})$ and $t \leqslant \prod_{k<q} x_k$

so

$$tx_q \equiv t \sum_{n \in H} x_n \pmod{\{A_s\}_{s<r}}.$$

Thus $\Sigma_{n \in H} tx_n \in A_j$ as desired.

4. Finite results. We note first that the finite version of Theorem 2.6 holds. The proof, which we omit, is by a standard compactness argument, invoking Theorem 2.6.

4.1 THEOREM. *For each k and r, there is some n such that, whenever $\{A_i\}_{i<r}$ is a partition of n, there exist $i < r$ and B and C in $[n]^k$ such that $FS(B) \subseteq A_i$ and $FP(C) \subseteq A_i$.*

The natural finite version of the main sums and products problem seeks to have $B = C$ in Theorem 4.1. The simplest of the special cases (which has any substance at all) takes $k = 2$ and $r = 2$. In this case the result holds, letting $n = 4$. (Either the cell with 0 has some other member x, in which case $B = \{0, x\}$ will do, or one can take $B = \{1, 2\}$.) Wishing to be able to exclude such trivialities, we make the following definition.

4.2 DEFINITION. $p(a, r, k)$ is the supremum of the set of all n such that $\{x: a \leqslant x \leqslant n\}$ may be partitioned into r cells so that no k-element subset A of $\{x: a \leqslant x \leqslant n\}$ has $FS(A) \cup FP(A)$ contained in one cell.

The definition of $p(a, r, k)$ was phrased in this negative fashion since, with few exceptions, it is not known that $p(a, r, k) < \omega$. (We shall present here all of the nontrivial exceptions which we know.)

R. Graham first showed, with computer assistance (unpublished), that $p(1, 2, 2) = 251$. He noted that the presence of 1 was important since if x and 1 lie in the same cell, one cannot have either $x + 1$ or $x - 1$ (for $x > 2$) in that cell.

We proceeded to investigate $p(2, 2, 2)$ and independently verified (also with computer assistance) Graham's result. The partition presented in the proof of Theorem 4.3 is the one we obtained. It is essentially unique, there being a total of exactly 147,456 such partitions. (The partition is completely determined, except on 18 numbers. If one of them (namely 12) is assigned to the same cell as 1, then three other numbers from the set of 18 are forced into the other cell. Otherwise they may be assigned to either cell at will, giving a total of $2^{17} + 2^{14}$ different partitions with the desired property.)

We have been in something of a dilemma with respect to the proof that each two celled partition of $\{x: 1 \leqslant x \leqslant 252\}$ has some distinct x and y with $\{x, y, x + y, x \cdot y\}$ contained in one cell. The proof only required approximately two seconds of computer time, and involves only 27 cases. However, these cases required a minimum of 4 and maximum of 47 steps to settle (with an average around 20). We have thus tried to compromise between saying "Trust me and my computer" and presenting all details of the proof. We shall present the cases, and check one of them. The case we will check is unfortunately the longest, but it is the only case which involves the number 252. (In fact any 2-celled partition of $\{x: i \leqslant x \leqslant 247\}$ which has no distinct x and y with $\{x, y, x+y, x \cdot y\}$ in the same cell must fall under this case.)

4.3 THEOREM (GRAHAM). $p(1, 2, 2) = 251$.

PROOF. Let $A_0 = \{x: x \text{ is odd and } 1 \leqslant x \leqslant 5\} \cup (\{x: x \text{ is even and } 8 \leqslant x \leqslant 250\} \setminus \{10, 96, 112, 128, 144, 160, 168, 176, 192, 208, 216, 224, 240\})$ and let

$$A_1 = \{x: 1 \leqslant x \leqslant 251\} \setminus A_0.$$

(Thus $A_1 = \{x: x \text{ is even and } 2 \leqslant x \leqslant 6\} \cup \{x: x \text{ is odd and } 7 \leqslant x \leqslant 251\} \cup \{10, 96, 112, 128, 144, 160, 168, 176, 192, 208, 216, 224, 240\}$.) It is a routine matter to verify that neither A_0 nor A_1 contains $\{x, y, x+y, x \cdot y\}$ for any distinct x and y.

Suppose now that $\{A_0, A_1\}$ is a partition of $\{x: 1 \leqslant x \leqslant 252\}$ such that, whenever $x \neq y$ and $i < 2$, $\{x, y, x+y, x \cdot y\} \nsubseteq A_i$. Without loss of generality, $1 \in A_0$. Some one of the following 27 cases must hold: Case 1, $\{2, 4, 6, 9, 11\} \subseteq A_0$; case 2, $\{2, 4, 6, 9\} \subseteq A_0$ and $11 \in A_1$; case 3, $\{2, 4, 6, 10\} \subseteq A_0$ and $9 \in A_1$; case 4, $\{2, 4, 6\} \subseteq A_0$ and $\{9, 10\} \subseteq A_1$; case 5, $\{2, 4, 7, 9, 11\} \subseteq A_0$ and $6 \in A_1$; case 6, $\{2, 4, 7, 9\} \subseteq A_0$ and $\{6, 11\} \subseteq A_1$; case 7, $\{2, 4, 7\} \subseteq A_0$ and $\{6, 9\} \subseteq A_1$; case 8, $\{2, 4, 8, 10\} \subseteq A_0$ and $\{6, 7\} \subseteq A_1$; case 9, $\{2, 4, 8\} \subseteq A_0$ and $\{6, 7, 10\} \subseteq A_1$; case 10, $\{2, 4, 9\} \subseteq A_0$ and $\{6, 7, 8\} \subseteq A_1$; case 11, $\{2, 4\} \subseteq A_0$ and $\{6, 7, 8, 9\} \subseteq A_1$; case 12, $\{2, 5, 7, 9, 11\} \subseteq A_0$ and $4 \in A_1$; case 13, $\{2, 5, 7, 9\} \subseteq A_0$ and $\{4, 11\} \subseteq A_1$; case 14, $\{2, 5, 7\} \subseteq A_0$ and $\{4, 9\} \subseteq A_1$; case 15, $\{2, 5, 8\} \subseteq A_0$ and $\{4, 7\} \subseteq A_1$; case 16, $\{2, 5\} \subseteq A_0$ and $\{4, 7, 8\} \subseteq A_1$; case 17, $\{2, 6\} \subseteq A_0$ and $\{4, 5\} \subseteq A_1$; case 18, $\{2, 7, 9\} \subseteq A_0$ and $\{4, 5, 6\} \subseteq A_1$; case 19, $\{2, 7\} \subseteq A_0$ and $\{4, 5, 6, 9\} \subseteq A_1$; case 20, $\{2, 8\} \subseteq A_0$ and $\{4, 5, 6, 7\} \subseteq A_1$; case 21, $2 \in A_0$ and $\{4, 5, 6, 7, 8\} \subseteq A_1$; case 22, $\{3, 5\} \subseteq A_0$ and $2 \in A_1$; case 23, $\{3, 6\} \subseteq A_0$ and $\{2, 5\} \subseteq A_1$; case 24, $3 \in A_0$ and $\{2, 5, 6\} \subseteq A_1$; case 25, $4 \in A_0$ and $\{2, 3\} \subseteq A_1$; case 26, $5 \in A_0$ and $\{2, 3, 4\} \in A_1$; case 27, $\{2, 3, 4, 5\} \subseteq A_1$. We show here that case 22 cannot hold.

Assume, accordingly, that $\{3, 5\} \subseteq A_0$ and $2 \in A_1$. Since $\{1, 3\} \subseteq A_0$, $4 \in A_1$. Since $\{1, 5\} \subseteq A_0$, $6 \in A_1$. Since $\{2, 4, 6\} \subseteq A_1$, $8 \in A_0$. Since $\{1, 8\} \subseteq A_0$, $9 \in A_1$ and $7 \in A_1$. Since $\{3, 5, 8\} \subseteq A_0$, $15 \in A_1$. Since $\{2, 7, 9\} \subseteq A_1$, $14 \in A_0$. Since $\{6, 9, 15\} \subseteq A_1$, $54 \in A_0$. Since $\{1, 14\} \subseteq A_0$, $13 \in A_1$. Since $\{1, 54\} \subseteq A_0$, $53 \in A_1$ and $55 \in A_1$. Since $\{4, 9, 13\} \subseteq A_1$, $36 \in A_0$. Since $\{6, 7, 13\} \subseteq A_1$, $42 \in A_0$. Since $\{2, 53, 55\} \subseteq A_1$, $106 \in A_0$. Since

$\{1, 36\} \subseteq A_0$, $35 \in A_1$ and $37 \in A_1$. Since $\{3, 14, 42\} \subseteq A_0$, $17 \in A_1$. Since $\{1, 42\} \subseteq A_0$, $41 \in A_1$. Since $\{1, 106\} \subseteq A_0$, $105 \in A_1$. Since $\{2, 35, 37\} \subseteq A_1$, $70 \in A_0$. Since $\{4, 37, 41\} \subseteq A_1$, $148 \in A_0$. Since $\{2, 15, 17\} \subseteq A_1$, $30 \in A_0$. Since $\{7, 15, 105\} \subseteq A_1$, $22 \in A_0$. Since $\{5, 14, 70\} \subseteq A_0$, $19 \in A_1$. Since $\{1, 148\} \subseteq A_0$, $147 \in A_1$. Since $\{1, 30\} \subseteq A_0$, $31 \in A_1$. Since $\{1, 22\} \in A_0$, $21 \in A_1$. Since $\{2, 17, 19\} \subseteq A_1$, $34 \in A_0$. Since $\{4, 15, 19\} \subseteq A_1$, $60 \in A_0$. Since $\{2, 19, 21\} \subseteq A_1$, $38 \in A_0$. Since $\{7, 21, 147\} \subseteq A_1$, $28 \in A_0$. Since $\{4, 17, 21\} \subseteq A_1$, $68 \in A_0$, Since $\{1, 34\} \subseteq A_0$, $33 \in A_1$. Since $\{1, 60\} \subseteq A_0$, $61 \in A_1$ and $59 \in A_1$. Since $\{1, 38\} \subseteq A_0$, $39 \in A_1$. Since $\{8, 28, 36\} \subseteq A_0$, $224 \in A_1$. Since $\{1, 68\} \subseteq A_0$, $67 \in A_1$. Since $\{2, 31, 33\} \subseteq A_1$, $62 \in A_0$. Since $\{2, 59, 61\} \subseteq A_1$, $118 \in A_0$. Since $\{7, 39, 224\} \subseteq A_1$, $32 \in A_0$. Since $\{1, 62\} \subseteq A_0$, $63 \in A_1$. Since $\{1, 118\} \subseteq A_0$, $119 \in A_1$. Since $\{4, 63, 67\} \subseteq A_1$, $252 \in A_0$. Since $\{7, 17, 119\} \subseteq A_1$, $24 \in A_0$. Since $\{14, 32, 252\} \subseteq A_0$, $18 \in A_1$. Since $\{3, 8, 24\} \subseteq A_0$, $11 \in A_1$. But this means that $\{2, 9, 11, 18\} \subseteq A_1$, a contradiction.

We have no choice in the following result, except to just present it. It took about 4 minutes of computer time to show that each two celled partition of $\{x\colon 2 \leqslant x \leqslant 990\}$ has some distinct x and y with $\{x, y, x + y, x \cdot y\}$ contained in one cell.

4.4 THEOREM. $p(2, 2, 2) = 989$.

PROOF. As remarked above, we shall simply present a partition $\{A_0, A_1\}$ of $\{x\colon 2 \leqslant x \leqslant 989\}$ such that if $x \neq y$ and $i < 2$, then $\{x, y, x + y, x \cdot y\}$ is not contained in A_i.

Let $A_0 = (\{2, 4, 5, 7, 8, 11, 13, 14, 16, 17, 18, 19, 25, 26\} \cup \{3k\colon\ 27 \leqslant 3k \leqslant 483\} \cup \{3k + 1\colon\ 253 \leqslant 3k + 1 \leqslant 484\} \cup \{x\colon\ 487 < x < 989\} \cup \{166, 193, 218, 220, 245, 299, 326, 353, 380, 407, 434, 461, 764, 818, 860, 968\}) \setminus (\{6k + 2\colon 506 \leqslant 6k + 2 \leqslant 986\} \cup \{126, 144, 382, 409, 432, 436, 491, 494, 540, 594, 598, 648, 652, 702, 706, 756, 810, 864, 891, 918, 922, 965, 972, 976\})$. Let $A_1 = \{x\colon 2 \leqslant x \leqslant 989\} \setminus A_0$. (The curious reader will presumably test this partition on his own computer.)

It is conceivable that $p(3, 2, 2)$ is computer accessible. It is unlikely, in fact, that the time consumption would increase as dramatically as it did between $p(1, 2, 2)$ and $p(2, 2, 2)$. It is clear that $p(1, 2, 3)$ is huge—even if it is finite. (It is a trivial matter to write down a partition of $\{x\colon 1 \leqslant x \leqslant 10^{11}\}$ into 2 cells so that no 3 element subset has all sums and products in one cell.) We present the following lower bound for $p(1, 3, 2)$ only to show that it too is very large and almost certainly not computer accessible. In our opinion, the only reasonable hope for a proof that $p(n, r, k)$ is always finite is that the infinite version be proved.

4.5 THEOREM. $p(1, 3, 2) \geqslant 11,706,659$.

PROOF. Let $A_0 = (\{3k+1\colon 3k+1 \leqslant 16\} \cup \{9k\colon 18 \leqslant 9k \leqslant 45\} \cup \{3k\colon 54 \leqslant 3k \leqslant 3417\} \cup \{3k+2\colon 3422 \leqslant 3k+2 < 11,706,659\}) \setminus \{27k\colon 486 \leqslant 27k \leqslant 3402\}$, let $A_1 = (\{3k\colon 3 \leqslant 3k \leqslant 51\} \cup \{3k+1\colon 19 \leqslant 3k+1 \leqslant 11,706,658\} \cup \{27k\colon 486 \leqslant 27k \leqslant 3402\}) \setminus \{9k\colon 18 \leqslant 9k \leqslant 45\}$, and let $A_2 = \{3k+2\colon 3k+2 \leqslant 3419\} \cup \{3k\colon 3420 \leqslant 3k \leqslant 11,706,657\}$. It is a routine matter to verify that if $x \neq y$ and $i < 3$ then $\{x, y, x+y, x \cdot y\}$ is not contained in A_i.

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