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DIFFERENTIAL ALGEBRAIC LIE ALGEBRAS

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ABSTRACT. A class of infinite-dimensional Lie algebras over the field $\mathcal{K}$ of constants of a universal differential field $\mathcal{U}$ is studied. The simplest case, defined by homogeneous linear differential equations, is analyzed in detail, and those with underlying set $\mathcal{U} \times \mathcal{U}$ are classified.

Introduction. Let $\mathcal{U}$ be a universal differential field of characteristic zero with set Δ of commuting derivation operators and field $\mathcal{K}$ of constants. We study a class of Lie algebras over $\mathcal{K}$ called *differential algebraic*. A differential algebraic Lie algebra g is, in general, infinite-dimensional but has defined on it an additional structure that gives it a tractability it might not otherwise have. We require the additive group g^+ to be a differential algebraic group relative to the universe $\mathcal{U}$ (roughly speaking, a group object in the category of differential algebraic sets in the sense of Kolchin and Ritt). We also require of the Lie product and scalar multiplication operations that they be morphisms of differential algebraic sets. The Lie algebra g thus inherits, in particular, the finite differential dimensionality of its additive group.

Throughout, we will use the prefix "Δ-" (or "δ-" if $\mathcal{U}$ is an ordinary differential field with derivation operator δ) in place of "differential algebraic" and "differential rational." The primed letter a' will always stand for δa. If $i \geqslant 0$, $a^{(i)}$ denotes $\delta^i a$.

Δ-Lie algebras arise naturally in the development of a suitable Lie theory for Δ-groups. If G is a connected Δ-group and $\mathcal{U}\langle G\rangle$ is the differential field of Δ-functions on G, a *differential derivation on G* is a derivation D of $\mathcal{U}\langle G\rangle$ over $\mathcal{U}$ such that $D \circ \delta = \delta \circ D$ ($\delta \in \Delta$). G acts (through right translations) on the set of differential derivations. The set of differential derivations on G that are invariant under this action is readily observed to be a Lie algebra over the field $\mathcal{K}$ of constants. We call it the *Lie algebra of G*. In a work in preparation **[8]**, in which he defines "Δ-group" intrinsically, Kolchin shows that the Lie algebra g of a connected Δ-group can be given a structure of

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affine Δ-Lie algebra, i.e., the additive group of g is isomorphic to a differential algebraic subgroup of the additive group $\mathbf{G}_a^n$ of the finite-dimensional $\mathcal{U}$-vector space $\mathcal{U}^n$. In this case, $\mathfrak{g}^+$ can be identified with the set of solutions in $\mathcal{U}^n$ of finitely many homogeneous linear differential equations.

In Chapter I we show that every affine δ-Lie algebra is an extension of a finite-dimensional Lie algebra over $\mathcal{K}$ by a δ-Lie algebra whose additive group is isomorphic to $\mathbf{G}_a^n$ where n is its differential dimension (a so-called vector group). Unfortunately, the assumption that the coefficient field is ordinary is difficult to remove since the proof uses the fact that the ring of differential operators in a single derivation operator with coefficients in a differential field is a left and right principal ideal domain. We also show in Chapter I that if $\mathcal{U}$ is ordinary and g has the property that $\mathfrak{g}^+$ is a vector group, then every homomorphic image of g has this property, as does the smallest δ-subalgebra containing the derived algebra. However, an example in Chapter IV shows that the additive group of the center is not necessarily a vector group.

The appearance of affine Δ-Lie algebras in the Lie theory of Δ-groups, together with the extension theorem in the ordinary case, encourages us to study δ-Lie algebra structures on $\mathbf{G}_a^n$. (Of course, every finite-dimensional Lie algebra over $\mathcal{U}$ is a special case.) An impetus from another direction comes from the equivalence of the category of δ-Lie algebras whose additive groups are vector groups with the category of formal differential groups studied by J. F. Ritt just before his death. These remarkable formal groups are generalizations of formal Lie groups, which were derived from classical Lie theory by Bochner in 1946 and which have since been studied extensively in the foundational papers of Cartier, Dieudonné, Gabriel, and Lazard. An *n-dimensional formal differential group* is simply an n-tuple $\mathbf{f}$ of formal differential power series subject to conditions expressing associativity and the fact that the origin is the identity element. By antisymmetrizing the homogeneous part of $\mathbf{f}$ of degree 2 we define a δ-Lie algebra structure on $\mathbf{G}_a^n$, which we call the *Lie algebra of* $\mathbf{f}$. In Chapter III we show that although Ritt never explicitly defines the Lie algebra of a formal differential group, his object in **[12]** is to show that every δ-Lie algebra whose additive group is $\mathbf{G}_a^n$ is the Lie algebra of a formal differential group and two such Lie algebras are isomorphic if and only if their associated groups are equivalent.

In Chapter IV we assume throughout that $\mathcal{U}$ is ordinary and use Ritt's amazing classification of formal differential groups of dimension $\leqslant 2$ to list the δ-Lie algebra structures on $\mathbf{G}_a$ and on $\mathbf{G}_a \times \mathbf{G}_a$ up to isomorphism. There are precisely two δ-Lie algebra structures on the line: the abelian Lie algebra $\mathfrak{g}_a$ and the so-called substitution Lie algebra $\mathfrak{g}_s$. The Lie product on $\mathfrak{g}_s$ is given by the simple differential polynomial $xy' - yx'$. $\mathfrak{g}_s$ has no counterpart in classical Lie theory since it is a nonabelian 1-parameter Lie algebra. In fact, $\mathfrak{g}_s$ is highly nonabelian since it has trivial center and equals its derived algebra. Also, in contrast to the classical case of 2-dimensional Lie algebras, there is an infinity of δ-Lie algebra structures on the plane. They are of thirteen types, three finite types and ten substitutional types. The three finite types include an abelian type, a nilpotent nonabelian type and a solvable nonnilpotent type. The nilpotent δ-Lie algebras are all central extensions of $\mathfrak{g}_a$ by $\mathfrak{g}_a$. The solvable δ-Lie algebras are all split extensions of $\mathfrak{g}_a$ by $\mathfrak{g}_a$ relative to natural actions. We prove in Chapter IV that a δ-Lie algebra structure on the plane is solvable if and only if it has finite type.

In [3], which we shall refer to as DAG, we showed that if a Δ-group G is linear, then so is its Lie algebra. In fact, if G is isomorphic to a Δ-subgroup of $GL_{\mathfrak{U}}(n)$ then the Lie algebra of G is isomorphic to a Δ-subalgebra of $\mathfrak{gl}_{\mathfrak{U}}(n)$ whose defining homogeneous linear differential equations are derived in a natural way from the defining differential equations of G. It is an open question whether the Lie algebra of an arbitrary Δ-group is linear. It would be reasonable to expect that a Δ-Lie algebra structure on $\mathbf{G}_a^n$ is linear since Ado's theorem states that a classical Lie algebra structure on $\mathbf{G}_a^n$ has a faithful representation as a Lie algebra of matrices. However, we show in Chapter IV that no Lie algebra of substitutional type is linear. That $\mathfrak{g}_s$ is not linear follows immediately from a necessary condition for linearity established in Chapter II, namely that the automorphism group not consist of the identity automorphism alone.

It follows immediately from the results of Chapter IV that a linear δ-group whose Lie algebra has additive group a vector group of dimension 1 is abelian. If the additive group of the Lie algebra of G is a vector group of dimension 2 then G is solvable.

Notation. The multiplicative monoid generated by Δ is denoted by Θ. The additive and multiplicative groups of $\mathfrak{U}$ are denoted by $\mathbf{G}_a$ and $\mathbf{G}_m$, respectively. The Δ-subgroups of these groups, consisting of the additive and multiplicative groups of $\mathcal{K}$, are denoted by $(\mathbf{G}_a)_{\mathcal{K}}$ and $(\mathbf{G}_m)_{\mathcal{K}}$, respectively. The differential polynomial algebra over $\mathfrak{U}$ in n differential indeterminates $y_1, \ldots, y_n$ is denoted by $\mathfrak{U}\{y_1, \ldots, y_n\}$. If S is a subset of a Δ-set the differential Zariski closure of S is denoted by $D(S)$. If S is a subset of an algebraic set, the Zariski closure of S is denoted by $A(S)$.

Chapter I. Differential algebraic Lie algebras

1. Basic notions. A Lie algebra $\mathfrak{g}$ over the field $\mathcal{K}$ of constants of $\mathfrak{U}$ is *differential algebraic* if the following conditions are met:

(1) The additive group $\mathfrak{g}^+$ of $\mathfrak{g}$ is a differential algebraic group.

(2). The Lie product map κ: $\mathfrak{g}^+ \times \mathfrak{g}^+ \to \mathfrak{g}^+$, defined by the formula

$\kappa(u, v) = [u, v]$, is an everywhere defined differential rational map.

(3). The scalar multiplication map σ: $(\mathbf{G}_a)_{\mathcal{K}} \times \mathrm{g}^+ \to \mathrm{g}^+$, defined by the formula $\sigma(c, u) = cu$, is an everywhere defined differential rational map.

A differential algebraic Lie algebra g is called a Δ-$\mathcal{F}$-Lie algebra if g^+ is a Δ-$\mathcal{F}$-group and κ and σ are Δ-$\mathcal{F}$-maps.

It follows immediately that if u is an element of the Δ-$\mathcal{F}$-Lie algebra g, then the endomorphism ad u: $\mathrm{g}^+ \to \mathrm{g}^+$, which maps v onto $[u, v]$, and the homomorphism σ_u: $(\mathbf{G}_a)_{\mathcal{K}} \to \mathrm{g}^+$, which maps c onto cu, are Δ-$\mathcal{F}\langle u\rangle$-homomorphisms. If $c \in \mathcal{K}$, then the homomorphism σ_c: $\mathrm{g}^+ \to \mathrm{g}^+$, which maps u onto cu, is a Δ-$\mathcal{F}\langle c\rangle$-homomorphism.

If h is a subalgebra of the Lie algebra g and h^+ is a Δ-subgroup of g^+, then we call h a Δ-*subalgebra* of g. If, in addition, h is an ideal of g, we call h a Δ-ideal of g. A homomorphism of Lie algebras is a Δ-*homomorphism* if it is a Δ-homomorphism of the additive groups. The kernel of a Δ-homomorphism is evidently a Δ-ideal and the image is a Δ-subalgebra.

Let g and h be a Δ-$\mathcal{F}$-Lie algebras. Kolchin has shown in **[8]** that the direct product $\mathrm{g}^+ \times \mathrm{h}^+$ of the additive groups can be given a structure of Δ-$\mathcal{F}$-group in such a way that the projection maps are Δ-$\mathcal{F}$-homomorphisms. $\mathrm{g}^+ \times \mathrm{h}^+$ is, moreover, the additive group of the direct product $\mathrm{g} \times \mathrm{h}$ of the Lie algebras. Thus, $\mathrm{g} \times \mathrm{h}$ is easily seen to be a Δ-$\mathcal{F}$-Lie algebra and the projection maps are homomorphisms of Δ-$\mathcal{F}$-Lie algebras. We say that g *acts* on h if the following conditions are satisfied:

(1) The Lie algebra g acts on the Lie algebra h, i.e., there is a homomorphism from g into the Lie algebra of derivations of h.

(2) The map α: $\mathrm{h} \times \mathrm{g} \to \mathrm{h}$, defined by the formula $\alpha(u_1, u_2) = D_{u_2}u_1$ (where D_{u_2} is the derivation associated with u_2 under the action) is an everywhere defined Δ-$\mathcal{F}$-map.

If g acts on h we can define on the direct product $\mathrm{h}^+ \times \mathrm{g}^+$ a Lie product by means of the formula $[(u_1, u_2),(v_1, v_2)] = ([u_1, v_1] + D_{u_2}v_1 - D_{v_2}u_1, [u_2, v_2])$. We call the resulting Δ-$\mathcal{F}$-Lie algebra the *split extension of* g *by* h. $\mathrm{h} \times 0$ is a Δ-$\mathcal{F}$-ideal and $0 \times \mathrm{g}$ is a Δ-$\mathcal{F}$-subalgebra.

EXAMPLES OF Δ-LIE ALGEBRAS. (1) Let G be a connected differential algebraic group. Let g be the Lie algebra over $\mathcal{K}$ of right-invariant derivations D of the differential rational function field $\mathcal{U}\langle G\rangle$ over $\mathcal{U}$ such that $D \circ \delta = \delta \circ D (\delta \in \Delta)$. g can be given a structure of Δ-Lie algebra (Kolchin **[8]**).

(2) The Lie algebra $\mathrm{gl}_{\mathcal{U}}(n)$ over $\mathcal{U}$ of $n \times n$ matrices with entries in $\mathcal{U}$ is an infinite-dimensional Lie algebra over the field $\mathcal{K}$ of constants of $\mathcal{U}$ and is a Δ-$\mathcal{Q}$-Lie algebra. The Δ-subalgebras of $\mathrm{gl}_{\mathcal{U}}(n)$ are a subclass of the class of $\mathcal{K}$-subalgebras, viz. those that are also the solution sets of finitely many homogeneous linear differential equations.

PROPOSITION 1. *Let* g *be a* Δ*-Lie algebra. Then* g^+ *is connected. If* h *is a* Δ*-subgroup of* g^+*, then* h *is a subspace of the vector space* g*. If* φ *is a homomorphism from* g^+ *into* g_1^+*, where* g_1 *is a* Δ*-Lie algebra, then* φ *is a homomorphism of vector spaces.*

PROOF. Let $u \in g$. The abelian subalgebra $\mathcal{K} \cdot u$ is equal to $\sigma_u(\mathbf{G}_a)_{\mathcal{K}}$, hence is a connected Δ-subgroup of g^+. Since every element of g is thus contained in a connected Δ-subgroup, g is itself connected. Suppose $u \in h$. $\sigma_u(\mathbf{G}_a)_{\mathbf{Z}} \subset h$, whence $D(\sigma_u(\mathbf{G}_a)_{\mathbf{Z}}) \subset h$. Since σ_u is a Δ-homomorphism, $D(\sigma_u(\mathbf{G}_a)_{\mathbf{Z}}) = \sigma_u(D(\mathbf{G}_a)_{\mathbf{Z}}) = \sigma_u(\mathbf{G}_a)_{\mathcal{K}} = \mathcal{K} \cdot u$. The last statement is clear. For, $\varphi(nu) = n\varphi(u)$ $(n \in \mathbf{Z}, u \in g)$ implies that $\varphi(cu) = c\varphi(u)$ $(c \in \mathcal{K}, u \in g)$.

PROPOSITION 2. *Let* g *be a* Δ*-Lie algebra.*

(1) *Let* S *be a subset and* T *a* Δ*-closed subset of* g.

(a) *The set* Tran(S, T) *consisting of all* $u \in g$ *such that* $[u, v] \in T$ *for all* $v \in S$ *is* Δ*-closed.*

(b) *The normalizer of* T *is* Δ*-closed. The normalizer of a* Δ*-subgroup* h *of* g *is a* Δ*-subalgebra of* g.

(c) *The centralizer of* S *in* g *is a* Δ*-subalgebra of* g*, and is equal to the centralizer of* $D(S)$.

(2) *Let* h *be a subalgebra of* g*. Then* D(h) *is a* Δ*-subalgebra of* g*. If* h *is an ideal, so is* D(h).

PROOF. Tran$(S, T) = \bigcap_{v \in S} \operatorname{ad}(-v)^{-1}(T)$, hence is Δ-closed. The normalizer of T equals Tran(T, T). Suppose h is a Δ-subgroup of g. Then the noramlizer of h is a Δ-subgroup of g^+. Let u_1 and u_2 be in the normalizer of h and let $v \in h$. $[[u_1, u_2], v] = [u_1, [u_2, v]] - [u_2, [u_1, v]]$. Therefore, $[u_1, u_2]$ is in the normalizer of h, and the normalizer of h in g is a Δ-subalgebra of g. The centralizer in g of S is Tran$(S, 0)$, hence is Δ-closed. Furthermore, u is in the centralizer of S if and only if $S \subset \ker(\operatorname{ad} u)$. Hence, u is in the centralizer of S if and only if u is in the centralizer of $D(S)$. Let h be a subalgebra of g. We know that $D(h^+)$ is a Δ-subgroup of g^+, and, thus, is also a subspace of the vector space g (Proposition 1). Let $u \in h$. Then $\operatorname{ad} u(h) \subset h$. Therefore, $\operatorname{ad} u(D(h)) = D(\operatorname{ad} u(h)) \subset D(h)$. The set of $u \in g$ such that $\operatorname{ad} u(D(h)) \subset D(h)$ is Δ-closed (Proposition 2.(1)(a)), and contains h, hence contains D(h). Thus, D(h) is a Δ-subalgebra of g. If h is an ideal of g, $\operatorname{ad} u(h) \subset h$ for all $u \in g$, whence $\operatorname{ad} u(D(h)) \subset D(h)$ for all $u \in g$.

COROLLARY. *The center* Z(g) *is* Δ*-closed.*

The following example shows that it is not always true, if the cardinality of Δ is > 1, that the derived algebra of a Δ-Lie algebra is Δ-closed.

Let $\Delta = \{\delta_1, \delta_2\}$ and let $\mathcal{K}_i$ be the field of constants of δ_i, $i = 1, 2$. Let g be the Δ-subalgebra of $\mathrm{gl}_{\mathfrak{U}}(3)$ consisting of all matrices

$$m(c_1, c_2, u) = \begin{bmatrix} 0 & c_1 & u \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix},$$

where $c_i \in \mathcal{K}_i$, $i = 1, 2$, and $u \in \mathfrak{U}$. The commutator $[m(c_1, c_2, u), m(d_1, d_2, v)] = m(0, 0, c_1d_2 - c_2d_1)$. Thus, $[\mathfrak{g}, \mathfrak{g}]$ can be identified with the ring compositum $\mathcal{K}_1[\mathcal{K}_2]$, which is clearly not Δ-closed in $\mathbf{G}_a$.

It follows from Proposition 2, however, that $D[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$. The following lemma is useful in our discussion of solvable and nilpotent Δ-Lie algebras.

LEMMA 1. *Let* $\mathfrak{g}$ *be a* Δ*-Lie algebra and let* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *be subalgebras of* $\mathfrak{g}$. *Then* $D[\mathfrak{h}_1, \mathfrak{h}_2] = D[D(\mathfrak{h}_1), D(\mathfrak{h}_2)]$.

PROOF. Since $[\mathfrak{h}_1, \mathfrak{h}_2] \subset [D(\mathfrak{h}_1), D(\mathfrak{h}_2)]$, $D[\mathfrak{h}_1, \mathfrak{h}_2] \subset D[D(\mathfrak{h}_1), D(\mathfrak{h}_2)]$. Let $u \in \mathfrak{h}_1$. $\mathrm{ad}\, u(D(\mathfrak{h}_2)) = D(\mathrm{ad}\, u(\mathfrak{h}_2)) \subset D[\mathfrak{h}_1, \mathfrak{h}_2]$. The set of $u \in \mathfrak{g}$ such that $\mathrm{ad}\, u(D(\mathfrak{h}_2)) \subset D[\mathfrak{h}_1, \mathfrak{h}_2]$ is Δ-closed. Since it contains $\mathfrak{h}_1$ it contains $D(\mathfrak{h}_1)$. Thus, for all $u \in D(\mathfrak{h}_1)$, $\mathrm{ad}\, u(D(\mathfrak{h}_2)) \subset D[\mathfrak{h}_1, \mathfrak{h}_2]$. Therefore, since $D[\mathfrak{h}_1, \mathfrak{h}_2]$ is Δ-closed, $D[D(\mathfrak{h}_1), D(\mathfrak{h}_2)] \subset D[\mathfrak{h}_1, \mathfrak{h}_2]$.

Let $\mathfrak{g}$ be a Δ-$\mathcal{F}$-Lie algebra and $\mathfrak{a}$ be a Δ-$\mathcal{F}$-ideal of $\mathfrak{g}$. The additive group $\mathfrak{g}^+/\mathfrak{a}^+$ of cosets $\bar{u} = u + \mathfrak{a}$, $u \in \mathfrak{g}$, is equipped with two additional structures–that of Lie algebra over $\mathcal{K}$ and that of Δ-$\mathcal{F}$-group (Kolchin **[8]**). The Lie algebra structure is easy to describe. $\bar{\sigma}(c, \bar{u}) = \overline{cu}$ $(c \in \mathcal{K}, u \in \mathfrak{g})$. $\bar{\kappa}(\bar{u}, \bar{v}) = \overline{[u, v]}$ $(u, v \in \mathfrak{g})$. The Δ-$\mathcal{F}$-group structure is more complicated and is described in **[8]** (for an analogous treatment for algebraic groups, see Kolchin [7, p. 269]).

THEOREM 1. *Let* $\mathfrak{g}$ *be a* Δ*-*$\mathcal{F}$*-Lie algebra and let* $\mathfrak{a}$ *be a* Δ*-*$\mathcal{F}$*-ideal of* $\mathfrak{g}$. *The Lie algebra* $\mathfrak{g}/\mathfrak{a}$ *is a* Δ*-*$\mathcal{F}$*-Lie algebra and the quotient map* π *is a* Δ*-*$\mathcal{F}$*-homomorphism.*

PROOF. To show that the Δ-$\mathcal{F}$-group and Lie algebra structures on $\mathfrak{g}/\mathfrak{a}$ are compatible, we must show that $\bar{\sigma}$ and $\bar{\kappa}$ are everywhere defined Δ-$\mathcal{F}$-maps. We cite a theorem of Kolchin **[8]**. *Let* G, H, *and* I *be connected* Δ*-*$\mathcal{F}$*-groups and let* φ: $G \times H \to I$ *be a map such that for fixed* $g \in G$ *the map* φ_g: $H \to I$, *which maps* h *onto* $\varphi(g, h)$ *is a* Δ*-*$\mathcal{F}\langle g\rangle$*-homomorphism, and for fixed* $h \in H$ *the map* φ_h: $G \to I$, *which maps* g *onto* $\varphi(g, h)$ *is a* Δ*-*$\mathcal{F}\langle h\rangle$*-homomorphism. Then* φ *is an everywhere defined* Δ*-*$\mathcal{F}$*-map.*

It follows that we need only show that for $u \in \mathfrak{g}$, $\mathrm{ad}\, \bar{u}$ is a Δ-$\mathcal{F}\langle\bar{u}\rangle$-endomorphism of $\mathfrak{g}^+/\mathfrak{a}^+$ and $\bar{\sigma}_{\bar{u}}$ is a Δ-$\mathcal{F}\langle\bar{u}\rangle$-homomorphism from $(\mathbf{G}_a)_{\mathcal{K}}$ into $\mathfrak{g}^+/\mathfrak{a}^+$, and for $c \in \mathcal{K}$, $\bar{\sigma}_c$ is a Δ-$\mathcal{F}\langle c\rangle$-endomorphism of $\mathfrak{g}^+/\mathfrak{a}^+$. As all the proofs are similar, we shall consider only the first assertion. Let $\alpha_u = \pi \circ \operatorname{ad} u$. Kolchin proved in [8] that π is a Δ-$\mathfrak{F}$-homomorphism of Δ-$\mathfrak{F}$-groups. Therefore, α_u is a Δ-$\mathfrak{F}\langle u\rangle$-homomorphism from $\mathfrak{g}^+$ to $\mathfrak{g}^+/\mathfrak{a}^+$, whose kernel contains $\mathfrak{a}$. Furthermore, ad $\bar{u}$ is the unique Δ-$\mathfrak{F}\langle u\rangle$-endomorphism of $\mathfrak{g}^+/\mathfrak{a}^+$ such that $\alpha_u = \operatorname{ad} \bar{u} \circ \pi$. We must now show that ad $\bar{u}$ is defined over $\mathfrak{F}\langle \bar{u}\rangle$, the smallest differential field of definition for the coset $\bar{u}$. Let v be generic for $\mathfrak{g}$ over $\mathfrak{F}\langle u\rangle$. Since π is defined over $\mathfrak{F}$, $\mathfrak{F}\langle \bar{u}\rangle \subset \mathfrak{F}\langle u\rangle$. Let σ be a Δ-isomorphism over $\mathfrak{F}\langle \bar{u}\rangle$ of any extension in $\mathfrak{U}$ of $\mathfrak{F}\langle u\rangle\mathfrak{F}\langle v\rangle$. Then

$$\begin{aligned}\sigma(\operatorname{ad} \bar{u}(\bar{v})) &= \sigma[\bar{u}, \bar{v}] = \sigma\,\overline{[u, v]} = \overline{[\sigma u, \sigma v]}\\ &= [\,\overline{\sigma u}\,,\ \overline{\sigma v}\,] = [\sigma\bar{u}, \sigma\bar{v}] = [\bar{u}, \sigma\bar{v}]\end{aligned}$$

(since σ leaves fixed the elements of $F\langle\bar{u}\rangle$). Therefore, $\sigma(\operatorname{ad} \bar{u}(\bar{v})) = \operatorname{ad} \bar{u}(\sigma\bar{v})$. Now, π is defined over $\mathfrak{F}\langle\bar{u}\rangle$. Therefore, $\pi v = \bar{v}$ is generic for $\mathfrak{g}^+/\mathfrak{a}^+$ over $\mathfrak{F}\langle\bar{u}\rangle$. Thus, $\sigma(\operatorname{ad} \bar{u}) = \operatorname{ad} \bar{u}$, whence ad $\bar{u}$ is a Δ-$\mathfrak{F}\langle\bar{u}\rangle$-map.

Let $\mathfrak{g}$ be a Δ-$\mathfrak{F}$-Lie algebra, let $\mathfrak{h}_1$ be a Δ-$\mathfrak{F}$-ideal of $\mathfrak{g}$ and let $\mathfrak{h}_2$ be a Δ-$\mathfrak{F}$-subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}_1 + \mathfrak{h}_2$ is a Δ-$\mathfrak{F}$-subalgebra of $\mathfrak{g}$. There is a unique isomorphism φ of Lie algebras over $\mathcal{K}$ from $\mathfrak{h}_2/(\mathfrak{h}_1 \cap \mathfrak{h}_2)$ onto $(\mathfrak{h}_1 + \mathfrak{h}_2)/\mathfrak{h}_1$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{h}_2 & \xrightarrow{\text{inclusion}} & \mathfrak{h}_1 + \mathfrak{h}_2 \\ {\scriptstyle\text{natural}}\downarrow & & \downarrow{\scriptstyle\text{natural}} \\ \mathfrak{h}_2/\mathfrak{h}_1 \cap \mathfrak{h}_2 & \xrightarrow[\varphi]{} & \mathfrak{h}_1 + \mathfrak{h}_2/\mathfrak{h}_1 \end{array}$$

Kolchin has shown in [8] that φ is a Δ-$\mathfrak{F}$-isomorphism of Δ-$\mathfrak{F}$-groups. Therefore, φ is an isomorphism of Δ-$\mathfrak{F}$-Lie algebras.

Suppose $\mathfrak{h}_2$ is, in addition, an ideal of $\mathfrak{g}$ and that $\mathfrak{h}_1 \subset \mathfrak{h}_2$. There is a unique isomorphism φ of Lie algebras over $\mathcal{K}$ from $(\mathfrak{g}/\mathfrak{h}_1)/(\mathfrak{h}_2/\mathfrak{h}_1)$ onto $\mathfrak{g}/\mathfrak{h}_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h}_1 & \xrightarrow{\text{natural}} & \mathfrak{g}/\mathfrak{h}_2 \\ {\scriptstyle\text{natural}}\searrow & & \nearrow{\scriptstyle\varphi} \\ & (\mathfrak{g}/\mathfrak{h}_1)/(\mathfrak{h}_1/\mathfrak{h}_2) & \end{array}$$

As above, φ is an isomorphism of Δ-$\mathfrak{F}$-Lie algebras.

It follows immediately that $\mathfrak{a}$ is a Δ-ideal of $\mathfrak{g}$ containing $D[\mathfrak{g}, \mathfrak{g}]$ if and only if $\mathfrak{g}/\mathfrak{a}$ is abelian.

2. Affine differential algebraic Lie algebras. Let A and B be Δ-$\mathfrak{F}$-groups, with B commutative and written additively. We say that *A acts on B* if there is an everywhere defined Δ-$\mathfrak{F}$-map $\alpha\colon A \times B \to B$, sending $(a, b) \to a * b$, such that

(1) $(a_1 a_2) * b = a_1 * (a_2 * b)$,

(2) $1 * b = b$,

(3) $a * (b_1 + b_2) = (a * b_1) + (a * b_2)$.

We call B an *A-module defined over $\mathfrak{F}$*. If B and B' are A-modules defined over $\mathfrak{F}$, a Δ-$\mathfrak{F}$-homomorphism $\varphi\colon B \to B'$ is called an *A-homomorphism* defined over $\mathfrak{F}$ if $\varphi(a * b) = a * \varphi(b)$, $a \in A$, $b \in B$.

The additive group of a Δ-$\mathfrak{F}$-Lie algebra g is a $(\mathbf{G}_m)_{\mathfrak{K}}$-module defined over $\mathfrak{F}$. The action of $(\mathbf{G}_m)_{\mathfrak{K}}$ on g is by scalar multplication.

Since $\mathfrak{U}^n$ is a vector space over $\mathfrak{U}$, its additive group $\mathbf{G}_a^n$ is a $\mathbf{G}_m$-module defined over $\mathbf{Q}$, where the action is induced by scalar multiplication. We call this the natural action of $\mathbf{G}_m$ on $\mathbf{G}_a^n$. A Δ-$\mathfrak{F}$-group G (commutative and written additively) is called a *vector group defined over* $\mathfrak{F}$ if G is a $\mathbf{G}_m$-module defined over $\mathfrak{F}$ and there exists a $\mathbf{G}_m$-isomorphism φ defined over $\mathfrak{F}$ from G onto the $\mathbf{G}_m$-module $\mathbf{G}_a^n$ relative to the natural action (where, of course, n is the differential dimension of G). $\mathbf{G}_a^n$ is a vector group relative to the natural action. This is not, however, the only vector group structure on $\mathbf{G}_a^n$. For example, another vector group structure on $\mathbf{G}_a^2$ is defined by the action given by the formula

$$a * (u_1, u_2) = (au_1, au_2 - a'u_1), \qquad a \in \mathbf{G}_m, u_1, u_2 \in \mathbf{G}_a.$$

If G is a vector group then it is evident that every proper Δ-closed subset of G has smaller differential dimension, a phenomenon useful but rare in differential algebra.

In [4], a δ-subgroup of $\mathbf{G}_a^n$ was said to be wound over $\mathbf{G}_a$ if there is no nontrivial δ-homomorphism $\mathbf{G}_a \to G$. Using the fact that the ring of linear differential operators in a single derivation operator δ is a left and right principal ideal domain, we showed that if G is a δ-subgroup of $\mathbf{G}_a^n$ then $G = G_v \times W$, where G_v is a vector group defined over $\mathfrak{F}$ and W is a δ-$\mathfrak{F}$-subgroup of G that is wound over $\mathbf{G}_a$. Furthermore, there is a finitely generated Picard-Vessiot extension of $\mathfrak{F}$ such that over this extension W is δ-isomorphic to $(\mathbf{G}_a)^r_{\mathfrak{K}}$, for some $r \in \mathbf{N}$. The subgroup G_v is the unique maximal δ-subgroup of G that is a vector group (G_v contains all δ-subgroups of G that are vector groups). We call G_v the *vector component* of G. Using this decomposition, we showed that if G is a vector group and $\varphi\colon G \to G'$ is a δ-homomorphism, then $\varphi(G)$ is also a vector group.

Let g be an affine δ-Lie algebra. We denote the vector component of g by $\mathfrak{g}_v$.

PROPOSITION 3. *Let* $\mathfrak{g}$ *be an* $\mathcal{F}$*-affine* δ-$\mathcal{F}$*-Lie algebra. Then the vector component* $\mathfrak{g}_v$ *is a* δ-$\mathcal{F}$*-ideal of* $\mathfrak{g}$ *and* $\mathfrak{g}/\mathfrak{g}_v$ *is an* $\mathcal{F}$*-affine* δ-$\mathcal{F}$*-Lie algebra whose additive group is wound over* $\mathbf{G}_a$.

PROOF. We know by Proposition 1 that $\mathfrak{g}_v$ is a subspace of $\mathfrak{g}$. Let $u \in \mathfrak{g}$. Since ad u is a δ-endomorphism of $\mathfrak{g}^+$, ad $u(\mathfrak{g}_v^+)$ is a δ-subgroup of $\mathfrak{g}^+$ that is a vector group. Therefore, ad $u(\mathfrak{g}_v^+) \subset \mathfrak{g}_v$. Thus, for every $u \in \mathfrak{g}$ and $v \in \mathfrak{g}_v$, $[u, v] \in \mathfrak{g}_v$.

Now, $\mathfrak{g}^+ = \mathfrak{g}_v \times W$, where W is a δ-$\mathcal{F}$-subgroup of $\mathfrak{g}^+$ that is wound over $\mathbf{G}_a$. Since $\mathfrak{g}^+/\mathfrak{g}_v$ is δ-$\mathcal{F}$-isomorphic to W, it follows that $\mathfrak{g}/\mathfrak{g}_v$ is $\mathcal{F}$-affine and $\mathfrak{g}^+/\mathfrak{g}_v^+$ is wound over $\mathbf{G}_a$.

Thus, if $\mathfrak{g}$ is $\mathcal{F}$-affine, then over a finitely generated Picard-Vessiot extension of $\mathcal{F}$, $\mathfrak{g}$ is an extension of a Lie algebra whose underlying vector space is $\mathcal{K}^r$ (a finite-dimensional Lie algebra over $\mathcal{K}$) by a δ-Lie algebra whose underlying vector space is $\mathcal{U}^d$.

If $\mathfrak{g}$ is a Δ-Lie algebra, then $\mathfrak{g}$ may have many Δ-subalgebras of differential dimension 0. However, if $\mathfrak{g}^+$ is a vector group and the differential dimension of $D[\mathfrak{g}, \mathfrak{g}]$ is 0, then $D[\mathfrak{g}, \mathfrak{g}] = 0$ and $\mathfrak{g}$ is abelian.

PROPOSITION 4. *Let* $\mathfrak{g}$ *be a* Δ*-Lie algebra and let* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *be* Δ*-subalgebras of* $\mathfrak{g}$, *with* $(\mathfrak{h}_2)^+$ *a vector group. If* $D[\mathfrak{h}_1, \mathfrak{h}_2]$ *has differential dimension* 0, *then* $D[\mathfrak{h}_1, \mathfrak{h}_2] = 0$ *and* $\mathfrak{h}_1$ *centralizes* $\mathfrak{h}_2$.

PROOF. If the dimension of $\mathfrak{h}_2$ is 0, then $\mathfrak{h}_2 = 0$ since $\mathfrak{h}_2^+$ is a vector group. So, we may assume that $\mathfrak{h}_2$ has positive dimension. Let $u \in \mathfrak{h}_1$. Let $\alpha =$ ad $u|\mathfrak{h}_2$. α is a Δ-homomorphism from $(\mathfrak{h}_2)^+$ to $D[\mathfrak{h}_1, \mathfrak{h}_2]$. Now,

$$\text{diff dim}(\ker \alpha) + \text{diff dim}(\alpha(\mathfrak{h}_2)) = \text{diff dim}(\mathfrak{h}_2)$$

(Kolchin [8]). Since $\alpha(\mathfrak{h}_2) \subset D[\mathfrak{h}_1, \mathfrak{h}_2]$, the differential dimension of $\alpha(\mathfrak{h}_2)$ is 0. Therefore, the differential dimension of $\ker(\alpha)$ equals the differential dimension of $\mathfrak{h}_2$. Since $\mathfrak{h}_2$ is a vector group, $\mathfrak{h}_2 = \ker(\alpha)$. Therefore, u centralizes $\mathfrak{h}_2$.

COROLLARY 1. *Let* $\mathfrak{g}$ *be a* Δ*-Lie algebra such that* $\mathfrak{g}^+$ *is a vector group. If the differential dimension of* $D[\mathfrak{g}, \mathfrak{g}]$ *is* 0, *then* $\mathfrak{g}$ *is abelian.*

COROLLARY 2. *Let* $\mathfrak{g}$ *be a* Δ*-Lie algebra such that* $\mathfrak{g}^+$ *is a vector group of dimension* 1. *Either* $D[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ *or* $\mathfrak{g}$ *is abelian.*

PROPOSITION 5. *Let* $\mathfrak{g}$ *be a* δ*-Lie algebra such that* $\mathfrak{g}^+$ *is a vector group. Then the additive group of* $D[\mathfrak{g}, \mathfrak{g}]$ *is a vector group.*

PROOF. We may assume that $\mathfrak{g}$ has positive dimension. Let $\mathfrak{g}' = D[\mathfrak{g}, \mathfrak{g}]$ and let $u \in \mathfrak{g}$. For every $x \in \mathfrak{g}'_v$, $[u, x] \in \mathfrak{g}'$ since $\mathfrak{g}' \supset [\mathfrak{g}, \mathfrak{g}]$. Therefore, ad $u(\mathfrak{g}'_v)$

$\subset \mathfrak{S}'_v$. Therefore, $\mathfrak{g}'_v$ is a δ-ideal of $\mathfrak{g}$ (Proposition 1). Now, $\mathfrak{g}/\mathfrak{g}'$ is δ-isomorphic to $(\mathfrak{g}/\mathfrak{g}'_v)/(\mathfrak{g}'/\mathfrak{g}'_v)$. $(\mathfrak{g}/\mathfrak{g}'_v)^+$ is a vector group. $\mathfrak{g}/\mathfrak{g}'$ is abelian, whence $\mathfrak{g}'/\mathfrak{g}'_v \supset D[\mathfrak{g}/\mathfrak{g}'_v, \mathfrak{g}/\mathfrak{g}'_v]$. $\mathfrak{g}'/\mathfrak{g}'_v$ has differential dimension 0 (Proposition 3). Therefore, $\mathfrak{g}/\mathfrak{g}'_v$ is abelian (Corollary 1 of Proposition 4). Thus, $\mathfrak{g}'_v \supset \mathfrak{g}'$.

3. Nilpotent and solvable differential algebraic Lie algebras. If $\mathfrak{g}$ is a Lie algebra over $\mathcal{K}$, the *derived series* of ideals of g is defined as follows: $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}^{i-1}]$, $i \geqslant 2$. The *central descending series* of ideals of $\mathfrak{g}$ is defined as follows: $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}]$, $i \geqslant 2$. $\mathfrak{g}$ is *solvable* (resp. *nilpotent*) if there is a natural number r such that $\mathfrak{g}^r = 0$ (resp. $\mathfrak{g}_r = 0$). If $\mathfrak{g}$ is solvable (resp. nilpotent) the smallest such r is called the *solv class* (resp. *nil class*) of g.

It follows immediately from the definition that every nilpotent Lie algebra over $\mathcal{K}$ has nontrivial center. Every abelian Lie algebra is nilpotent and every nilpotent Lie algebra is solvable. Furthermore, $\mathfrak{g}$ is solvable if and only if we can find ideals $\mathfrak{h}^0 = \mathfrak{g}, \mathfrak{h}^1, \ldots, \mathfrak{h}^s = 0$, such that $\mathfrak{h}^i \supset \mathfrak{h}^{i+1}$ and $\mathfrak{h}^i/\mathfrak{h}^{i+1}$ is abelian, for $0 \leqslant i \leqslant s-1$.

Suppose $\mathfrak{g}$ is a Δ-Lie algebra. Then, for every natural number i $D(\mathfrak{g}^i)$ and $D(\mathfrak{g}_i)$ are Δ-ideals of $\mathfrak{g}$. Clearly $D(\mathfrak{g}^i) \supset D(\mathfrak{g}^{i+1})$ and $D(\mathfrak{g}_i) \supset D(\mathfrak{g}_{i+1})$. Furthermore, it is evident that $\mathfrak{g}$ is solvable (resp. nilpotent) if and only if there is a natural number r such that $D(\mathfrak{g}^r) = 0$ (resp. $D(\mathfrak{g}_r) = 0$). Thus, $\mathfrak{g}$ is solvable if and only if we can find Δ-ideals $\mathfrak{h}^0 = \mathfrak{g}, \mathfrak{h}^1, \ldots, \mathfrak{h}^s = 0$ such that $\mathfrak{h}^i \supset \mathfrak{h}^{i+1}$ and $\mathfrak{h}^i/\mathfrak{h}^{i+1}$ is abelian, for $0 \leqslant i \leqslant s-1$.

The following two propositions are obvious:

PROPOSITION 6. *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathcal{K}$.

(a) *If* $\mathfrak{g}$ *is solvable, then so are all subalgebras and homomorphic images of* $\mathfrak{g}$.

(b) *If* $\mathfrak{a}$ *is a solvable ideal of* $\mathfrak{g}$ *such that* $\mathfrak{g}/\mathfrak{a}$ *is solvable, then* $\mathfrak{g}$ *is solvable.*

(c) *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are solvable ideals of* $\mathfrak{g}$, *then so is* $\mathfrak{a} + \mathfrak{b}$.

PROPOSITION 7. *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathcal{K}$.

(a) *If* $\mathfrak{g}$ *is nilpotent, then so are all subalgebras and homomorphic images of* $\mathfrak{g}$.

(b) *If* $\mathfrak{g}/Z(\mathfrak{g})$ *is nilpotent, then so is* $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Δ-Lie algebra and let $u \in \mathfrak{g}$. We call u *ad-nilpotent* if ad u is a nilpotent endomorphism of $\mathfrak{g}^+$, i.e., there is a natural number r such that $(\text{ad } u)^r = 0$. If $\mathfrak{g}$ is nilpotent, then every element of $\mathfrak{g}$ is ad-nilpotent.

In addition to the differential Zariski topology on $\mathfrak{gl}_{\mathcal{U}}(n)$ we have the Zariski topology. The Zariski closed subalgebras of $\mathfrak{gl}_{\mathcal{U}}(n)$ are defined by homogeneous linear equations and thus are the subalgebras over $\mathcal{U}$ of $\mathfrak{gl}_{\mathcal{U}}(n)$. As we remarked earlier, the Δ-subalgebras of $\mathfrak{gl}_{\mathcal{U}}(n)$ are those subalgebras over $\mathcal{K}$ that are defined by homogeneous linear differential equations (for a full discussion see Chapter III of DAG). Thus, every Zariski closed subalgebra of $\mathrm{gl}_{\mathcal{U}}(n)$ is, in particular, a Δ-subalgebra. If g is any subalgebra over $\mathcal{K}$ of $\mathrm{gl}_{\mathcal{U}}(n)$, then the Zariski closure $A(\mathrm{g})$ is a subalgebra over $\mathcal{U}$ of $\mathrm{gl}_{\mathcal{U}}(n)$ and is, in fact, the subspace $\mathcal{U} \cdot g$ over $\mathcal{U}$ generated by g. If a is an ideal of g, then $A(\mathrm{a})$ is an ideal of $A(\mathrm{g})$. Clearly, $A(D(\mathrm{g})) = A(\mathrm{g})$.

Note. If g_1 and g_2 are subalgebras of $\mathrm{gl}_{\mathcal{U}}(n)$, we shall reserve the notation $[\mathrm{g}_1, \mathrm{g}_2]$ for the subspace over K generated by the set of commutators $[u_1, u_2]$, $u_i \in \mathrm{g}_i$, $i = 1, 2$.

PROPOSITION 8. *Let* g *be a subalgebra over* $\mathcal{K}$ *of* $\mathrm{gl}_{\mathcal{U}}(n)$. *Then* $A(\mathrm{g}^i) = \mathcal{U}(A(\mathrm{g})^i)$ *and* $A(\mathrm{g}_i) = \mathcal{U}(A(\mathrm{g})^i)$.

PROOF. We first state a lemma whose proof parallels that of Lemma 1.

LEMMA 2. *Let* h_1 *and* h_2 *be subalgebras over* $\mathcal{K}$ *of* $\mathrm{gl}_{\mathcal{U}}(n)$. *Then* $A[\mathrm{g}_1, \mathrm{g}_2] = A[A(\mathrm{g}_1), A(\mathrm{g}_2)]$.

To prove the proposition, we use induction on i. $A(\mathrm{g}^1) = A[\mathrm{g}, \mathrm{g}] = A[A(\mathrm{g}), A(\mathrm{g})] = \mathcal{U}(A(g)^1)$. $A(g^{i+1}) = A[\mathrm{g}^i, \mathrm{g}^i] = A[A(\mathrm{g}^i), A(\mathrm{g}^i)] = A[\mathcal{U} \cdot A(\mathrm{g})^i, \mathcal{U} \cdot A(\mathrm{g})^i] = \mathcal{U}[A(\mathrm{g})^i, A(\mathrm{g})^i] = \mathcal{U}(A(\mathrm{g})^{i+1})$ (since the Lie product on $\mathrm{gl}_{\mathcal{U}}(n)$ is $\mathcal{U}$-linear).

The proof for the lower descending series is parallel.

COROLLARY. *Let* g *be a subalgebra over* $\mathcal{K}$ *of* $\mathrm{gl}_{\mathcal{U}}(n)$. *Then* g *is solvable of solv class* r (*resp. nilpotent of nil class* r) *if and only if* $A(\mathrm{g})$ *is solvable of solv class* r (*resp. nilpotent of nil class* r).

In particular, if g is a Zariski closed subalgebra over $\mathcal{K}$ of $\mathrm{gl}_{\mathcal{U}}(n)$, then g is solvable (resp. nilpotent) as a Lie algebra over $\mathcal{U}$ if and only if g is solvable (resp. nilpotent) as a Lie algebra over $\mathcal{K}$.

Let g be a subalgebra over $\mathcal{K}$ of $\mathrm{gl}_{\mathcal{U}}(n)$ consisting of nilpotent matrices. u is a nilpotent matrix if and only if 0 is the only eigenvalue of u. Clearly, $A(\mathrm{g})$ consists of nilpotent matrices. Since $A(\mathrm{g})$ is a finite-dimensional Lie algebra over $\mathcal{U}$, Engel's theorem (Humphreys [5, p. 12]) implies that $A(\mathrm{g})$ is a nilpotent Lie algebra over $\mathcal{U}$, hence also over $\mathcal{K}$ (corollary to Proposition 8). Therefore, g is a nilpotent Lie algebra over $\mathcal{K}$ (Proposition 7). So, we have proved the following proposition.

PROPOSITION 9. *Let* g *be a subalgebra over* $\mathcal{K}$ *of* $\mathrm{gl}_{\mathcal{U}}(n)$ *consisting of nilpotent matrices. Then* g *is nilpotent.*

Engel's theorem also says that under these circumstances there is a matrix $s \in \mathrm{GL}_{\mathcal{U}}(n)$ such that the matrices in $sA(\mathrm{g})s^{-1}$ are upper triangular with zeros on the diagonal.

Proposition 10. *Let* g *be a subalgebra over* $\mathcal{K}$ *consisting of nilpotent matrices. Then there is a matrix s in* $\mathrm{GL}_{\mathcal{U}}(n)$ *such that every matrix in* sgs^{-1} *is upper triangular with zeros on the diagonal.*

Lie's theorem applied to the Zariski closure of a solvable subalgebra g over $\mathcal{K}$ of $\mathrm{gl}_{\mathcal{U}}(n)$ implies the next proposition.

Proposition 11. *Let* g *be a solvable subalgebra over* $\mathcal{K}$ *of* $\mathrm{gl}_{\mathcal{U}}(n)$. *Then there is a matrix s in* $\mathrm{GL}_{\mathcal{U}}(n)$ *such that every matrix in* sgs^{-1} *is upper triangular.*

Corollary 1. *Let* g *be a solvable subalgebra over* $\mathcal{K}$ *of* $\mathrm{gl}_{\mathcal{U}}(n)$. *Then* [g, g] *consists of nilpotent matrices and, in particular, is nilpotent.*

Corollary 2. *Let* g *be a solvable* Δ*-subalgebra of* $\mathrm{gl}_{\mathcal{U}}(n)$. *Then* D[g, g] *consists of nilpotent matrices and, in particular, is nilpotent.*

If g is a solvable Lie algebra of solv class r or a nilpotent Lie algebra of nil class r then for every i such that $0 \leqslant i \leqslant r-1$, $g^i \neq g^{i+1}$ (resp. $g_i \neq g_{i+1}$).

Proposition 12. *Let* g *be a solvable* (resp. *nilpotent*) Δ*-Lie algebra of solv class r* (*resp. nil class r*). *Then* $D(g^i) \neq D(g^{i+1})$ (*resp.* $D(g_i) \neq D(g_{i+1})$), *if* $0 \leqslant i \leqslant r-1$.

Corollary. *Let* g *be a solvable* δ*-Lie algebra whose additive group is a vector group of dimension* 1. *Then* g *is abelian.*

Proof. $D[g, g]^+$ is a vector group (Proposition 5). $D[g, g] \neq g$ (Proposition 12). Therefore, $D[g, g] = [g, g] = 0$.

We can relax the condition on the coefficient field if we strengthen the condition on g.

Proposition 13. *Let* g *be a nilpotent* Δ*-Lie algebra whose additive group is a vector group of dimension* 1. *Then* g *is abelian.*

Proof. Let $u \in g$. Then u is ad-nilpotent. Therefore, there is a natural number k such that $(\mathrm{ad}\, u)^k = 0$. Let $\sigma\colon g^+ \to \mathbf{G}_a$ be a Δ-isomorphism. Then $\sigma \circ \mathrm{ad}\, u \circ \sigma^{-1}$ is a Δ-endomorphism of $\mathbf{G}_a$ such that $(\sigma \circ \mathrm{ad}\, u \circ \sigma^{-1})^k = 0$. There exists a linear differential operator L such that $\sigma \circ \mathrm{ad}\, u \circ \sigma^{-1}(u) = L(u)$. Since there are no nonzero nilpotent linear differential operators, $\sigma \circ \mathrm{ad}\, u \circ \sigma^{-1} = 0$, whence $\mathrm{ad}\, u = 0$ and g is abelian.

Now, suppose $\mathcal{U}$ is an ordinary differential field. Suppose g is a solvable δ-Lie algebra whose additive group is a vector group of dimension 2. The additive group of D[g, g] is a vector group of dimension $\leqslant 1$ (Propositions 5 and 12). If the dimension is 0, g is abelian, and is isomorphic to $g_a \times g_a$. If the dimension is 1, D[g, g] is isomorphic to g_a (corollary to Proposition 12). $g/D[g, g]$ is solvable and its additive group is a vector group (see the remarks at the beginning of this section) of dimension 1 (the additivity of differential dimension), hence is isomorphic to $\mathfrak{g}_a$. So, the solv class of $\mathfrak{g}$ is $\leqslant 2$ and $\mathfrak{g}$ is an extension of $\mathfrak{g}_a$ by $\mathfrak{g}_a$. A parallel argument shows that if $\mathfrak{g}$ is nilpotent, the nil class of $\mathfrak{g}$ is $\leqslant 2$ and $\mathfrak{g}$ is a central extension of $\mathfrak{g}_a$ by $\mathfrak{g}_a$.

Chapter II. Linear and integrable Δ-Lie algebras

A Δ-$\mathcal{F}$-Lie algebra $\mathfrak{g}$ is *$\mathcal{F}$-linear* (or simply *linear*) if $\mathfrak{g}$ is Δ-$\mathcal{F}$-isomorphic to a Δ-$\mathcal{F}$-subalgebra of $\mathfrak{gl}_{\mathcal{U}}(n)$. If, in addition, the image of $\mathfrak{g}$ is the Lie algebra of matrices of a Δ-$\mathcal{F}$-subgroup of $\mathrm{GL}_{\mathcal{U}}(n)$, we call $\mathfrak{g}$ *$\mathcal{F}$-integrable* (or *integrable*) in $\mathrm{GL}_{\mathcal{U}}(n)$. In this chapter we investigate the integrability in $\mathrm{GL}_{\mathcal{U}}(n)$ of certain Δ-subalgebras of $\mathfrak{gl}_{\mathcal{U}}(n)$. This work, which is the differential algebraic analog of Chevalley's work on algebraic Lie algebras, was begun in DAG, Chapter III.

Recall that a matrix $x \in \mathfrak{gl}_{\mathcal{U}}(n)$ is the Lie algebra $\mathfrak{l}(G)$ of matrices of a differential algebraic subgroup G of $GL_{\mathcal{U}}(n)$ if and only if $D_x(\mathfrak{a}) \subset \mathfrak{a}$, where $\mathfrak{a}$ is the defining differential ideal of G in the differential polynomial ring $\mathcal{U}\{(y_{ij})\}$ and D_x is the unique derivation of $\mathcal{U}\{(y_{ij})\}$ such that $D_x(y_{ij}) = (xy)_{ij}$ and $D_x \circ \delta = \delta \circ D_x$ ($\delta \in \Delta$). The Lie algebra of matrices of G is equal to that of its identity component G^0. We can relate the defining differential ideal of $\mathfrak{l}(G)$ to the defining differential ideal $\mathfrak{a}$ of G in a very natural way. We define a $\mathcal{U}$-linear map D of $\mathcal{U}\{(y_{ij})\}$ such that $D \circ \delta = \delta \circ D$ ($\delta \in \Delta$) and $D(PQ) = P(1)D(Q) + D(P)Q(1)$ ($P, Q \in \mathcal{U}\{(y_{ij})\}$) by the formula $D(P) = \sum \partial P/\partial\theta y_{ij}(1)\theta y_{ij}$, summed over $\theta \in \Theta$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. The defining differential ideal of $\mathfrak{l}(G)$ is then the differential ideal generated by the homogeneous linear differential polynomials $D(P)$ ($P \in \mathfrak{a}$).

We recall some results from DAG (which also hold for algebraic groups defined over fields of characteristic 0) that will be useful in the sequel. *If $(G_i)_{i\in I}$ is a family of Δ-subgroups of $GL_{\mathcal{U}}(n)$ then* $\mathfrak{l}(\bigcap_{i\in I} G_i) = \bigcap_{i\in I} \mathfrak{l}(G_i)$ (*Proposition* 28, *p*. 933). *Thus, the intersection of a family of integrable Δ-subalgebras of $\mathfrak{gl}_{\mathcal{U}}(n)$ is integrable. If G is a Δ-subgroup of $GL_{\mathcal{U}}(n)$ and ρ: $G \to GL_{\mathcal{U}}(r)$ is a Δ-homomorphism, then the differential $\rho^{\#}: \mathfrak{l}(G) \to \mathfrak{gl}_{\mathcal{U}}(r)$ is a Δ-homomorphism and* $\rho^{\#}(\mathfrak{l}(G)) = \mathfrak{l}(\rho(G))$ *and* $\ker(\rho^{\#}) = \mathfrak{l}(\ker(\rho))$ (Propositions 22, p. 930 and 29, 30, p. 934). The following proposition, whose proof we omit since it uses the same techniques as that of Proposition 29, has Proposition 29 as a corollary.

Proposition 14. *Let G be a Δ-subgroup of $GL_{\mathcal{U}}(n)$ and let ρ: $G \to GL_{\mathcal{U}}(r)$ be a Δ-homomorphism. Let H be a Δ-subgroup of $GL_{\mathcal{U}}(r)$ and let $N = \rho^{-1}(H)$. Then* $\mathfrak{l}(N) = \rho^{\#-1}(\mathfrak{l}(H))$.

We identify $\mathcal{U}^n$ with the vector space over $\mathcal{U}$ of column matrices. Let a be a Δ-subgroup of $\mathbf{G}_a^n$ (and hence, in particular, a $\mathcal{K}$-subspace of $\mathcal{U}^n$) and let b be any subset of $\mathcal{U}^n$ containing a. We denote by T(b, a) the set of all

$x \in \mathrm{gl}_{\mathcal{U}}(n)$ such that $x\mathrm{b} \subset \mathrm{a}$. It is easy to see that $T(\mathrm{b}, \mathrm{a})$ is a $\mathcal{K}$-subalgebra of $\mathrm{gl}_{\mathcal{U}}(n)$. The next proposition shows that $T(\mathrm{b}, \mathrm{a})$ is, in fact, an integrable Δ-subalgebra of $\mathrm{gl}_{\mathcal{U}}(n)$.

PROPOSITION 15. *Let* a *be a* Δ-*subgroup of* $\mathbf{G}_a^n$ *and let* b *be a subset of* $\mathbf{G}_a^n$, *with* $\mathrm{a} \subset \mathrm{b}$. *The set* G *of* $s \in GL_{\mathcal{U}}(n)$ *such that* $s\mathbf{u} \equiv \mathbf{u} \bmod \mathrm{a}$ *for all* $\mathbf{u} \in \mathrm{b}$, *is a connected* Δ-*subgroup of* $GL_{\mathcal{U}}(n)$ *such that* $\mathfrak{l}(G) = T(\mathrm{b}, \mathrm{a})$.

PROOF. G is nonempty since the identity matrix 1 satisfies the defining condition, and $GG \subset G$. Since a is a Δ-subgroup of $\mathbf{G}_a^n$, a is the set of zeros of a homogeneous linear differential ideal $\mathfrak{a}$ in $\mathcal{U}\{y_1, \ldots, y_n\}$. Let L be a homogeneous linear differential polynomial in $\mathfrak{a}$ and let $\mathbf{u} \in \mathrm{b}$. The function that sends a matrix s onto $L(s\mathbf{u})$ is a homogeneous linear differential polynomial function. G is the set of all $s \in GL_{\mathcal{U}}(n)$ such that $L(s\mathbf{u} - \mathbf{u}) = 0$ (L a homogeneous linear differential polynomial in $\mathfrak{a}$, $\mathbf{u} \in \mathrm{b}$). Thus, G is a Δ-subgroup of $GL_{\mathcal{U}}(n)$. Let $\mathfrak{p}$ be the differential ideal generated by the differential polynomials $L(y\mathbf{u}) - L(\mathbf{u})$, $\mathbf{u} \in \mathrm{b}$. $\mathfrak{p}$ is a proper linear differential ideal and thus is prime. Therefore, $\mathfrak{p}$ is the defining differential ideal of G (DAG, p. 894). In particular, G is connected. As we remarked earlier, the defining differential ideal of $\mathfrak{l}(G)$ is generated by the set DP, $P \in \mathfrak{p}$. Suppose $P = \Sigma Q(y)(L(y\mathbf{u}) - L(\mathbf{u}))$, where L is a homogeneous linear differential polynomial in $\mathfrak{a}$, $\mathbf{u} \in \mathrm{b}$, and $Q(y) = 0$ for all but finitely many L and $\mathbf{u}$. $DP = \Sigma Q(1)(D(L(y\mathbf{u}) - L(\mathbf{u}))) + DQ(L(1 \cdot \mathbf{u}) - L(\mathbf{u})) = \Sigma Q(1)L(y\mathbf{u})$ (since $L(\mathbf{u}) \in \mathcal{U}$ and $L(y\mathbf{u})$ is linear). Therefore, the defining differential ideal of $\mathfrak{l}(G)$ is generated differentially by the $L(y\mathbf{u})$. Thus, $\mathfrak{l}(G)$ is the set of all $x \in \mathrm{gl}_{\mathcal{U}}(n)$ such that $L(x\mathbf{u}) = 0$ (L a homogeneous linear differential polynomial vanishing on a, $\mathbf{u} \in \mathrm{b}$). It follows that $\mathfrak{l}(G) = T(\mathrm{b}, \mathrm{a})$.

Let $s \in GL_{\mathcal{U}}(n)$ and let $\mathrm{Ad}\, s\colon \mathrm{gl}_{\mathcal{U}}(n) \to \mathrm{gl}_{\mathcal{U}}(n)$ be the $\mathcal{U}$-Lie algebra automorphism defined by the formula $\mathrm{Ad}\, s(u) = sus^{-1} (u \in \mathrm{gl}_{\mathcal{U}}(n))$. Let $\mathrm{Ad}\colon GL_{\mathcal{U}}(n) \to GL_{\mathcal{U}}(n^2)$ be the Δ-homomorphism that assigns to s the matrix of $\mathrm{Ad}\, s$ relative to the canonical basis. The differential $\mathrm{Ad}^{\#}$ of Ad is the Δ-homomorphism $\mathrm{ad}\colon \mathrm{gl}_{\mathcal{U}}(n) \to \mathrm{gl}_{\mathcal{U}}(n^2)$ that sends u onto the matrix of $\mathrm{ad}\, u$ relative to the canonical basis. If a is a Δ-subgroup of the additive group of $\mathrm{gl}_{\mathcal{U}}(n)$ and b is a subset of $\mathrm{gl}_{\mathcal{U}}(n)$, with $\mathrm{a} \subset \mathrm{b}$, then by Proposition 15, $T(\mathrm{b}, \mathrm{a})$ is the Lie algebra of the Δ-subgroup H of $GL_{\mathcal{U}}(n^2)$ consisting of all t such that $tv - v \in \mathrm{a}$ for all $v \in \mathrm{b}$. If G is the set of all $s \in \mathrm{GL}_{\mathcal{U}}(n)$ such that $svs^{-1} - v \in \mathrm{a}$ for all $v \in \mathrm{b}$, then $G = \mathrm{Ad}^{-1}(H)$. By Proposition 14, $\mathfrak{l}(G) = \mathrm{Ad}^{\#-1}(\mathfrak{l}(H)) = \mathrm{Ad}^{\#-1}(T(\mathrm{b}, \mathrm{a})) = \mathrm{Tran}(\mathrm{b}, \mathrm{a})$. So, we have the following result:

PROPOSITION 16. *Let* a *be a* Δ-*subgroup of the additive group of* $\mathrm{gl}_{\mathcal{U}}(n)$, *and let* b *be a subset of* $\mathrm{gl}_{\mathcal{U}}(n)$, *with* $\mathrm{a} \subset \mathrm{b}$. *The set* $\mathrm{Tran}(\mathrm{b}, \mathrm{a})$ *of elements* $u \in \mathrm{gl}_{\mathcal{U}}(n)$ *such that* $[u, v] \in \mathrm{a}$ *for all* $v \in \mathrm{b}$ *is the Lie algebra of the connected*

Δ-subgroup G of $GL_{\mathcal{U}}(n)$ consisting of all $s \in GL_{\mathcal{U}}(n)$ such that $svs^{-1} - v \in$ a *for all $v \in$* b.

COROLLARY 1. *Let G be a Δ-subgroup of $GL_{\mathcal{U}}(n)$ and let* g = l(G).

(1). *Let* b *be a subset of* g *containing the zero matrix. The centralizer in* g *of* b *is the Lie algebra of the Δ-subgroup H of G consisting of all $s \in G$ such that $sus^{-1} = u$ for all $u \in$* b.

(2). *Let* a *be a Δ-subgroup of* g. *The normalizer in* g *of* a *is the Lie algebra of the Δ-subgroup H of G consisting of all $s \in G$ such that $sus^{-1} \in$* a *for all $u \in$* a.

PROOF. The centralizer in $\mathrm{gl}_{\mathcal{U}}(n)$ of b is Tran(b, 0). By Proposition 16, Tran(b, 0) = l(Z), where Z is the Δ-subgroup of $GL_{\mathcal{U}}(n)$ consisting of all s such that $sus^{-1} - u = 0$ for all $u \in$ b. $H = Z \cap G$. By our earlier remarks, l(H) = l(Z) $\cap$ l(G) = Tran(b, 0) $\cap$ g, which is the centralizer in g of b. The normalizer in $\mathrm{gl}_{\mathcal{U}}(n)$ of a is Tran(a, a). By Proposition 16, Tran(a, a) is the Lie algebra of the Δ-subgroup N of $GL_{\mathcal{U}}(n)$ consisting of all s such that $sus^{-1} - u \in$ a for all $u \in$ a. Clearly, N is the set of all $s \in GL_{\mathcal{U}}(n)$ such that $sus^{-1} \in$ a for all $u \in$ a. As above, the normalizer of a in g is the Lie algebra of $N \cap G$, which gives us 2.

COROLLARY 2. *Let G be a connected Δ-subgroup of $GL_{\mathcal{U}}(n)$. The Lie algebra of the center of G is the center of the Lie algebra* g *of G.*

PROOF. By Corollary 1, Z(g) is the Lie algebra of the Δ-subgroup H of G consisting of all $s \in G$ such that $sus^{-1} = u$ for all $u \in$ g. We must show that H is the center of G.

Let $s \in GL_{\mathcal{U}}(n)$. We first observe that Ad s is the differential of the rational automorphism $\alpha_s = \mathrm{Ad}\, s | GL_{\mathcal{U}}(n)$. We next observe that if $C(s)$ denotes the set of all $t \in GL_{\mathcal{U}}(n)$ such that $ts = st$ and c(s) denotes the set of all $u \in \mathrm{gl}_{\mathcal{U}}(n)$ such that $us = su$, then l($C(s)$) = c(s) (Humphreys [**6**, p. 76]). If $s \in G$, then $(\alpha_s | G)^{\#} = \mathrm{Ad}\, s |$g. Thus, the center of G is a subset of H. Also, if $s \in G$, l($C(s) \cap G$) = c(s) $\cap$ g. Now, suppose $s \in H$. Then c(s) $\cap$ g = g = l(G). Therefore, since G is connected, $C(s) \cap G = G$ (DAG, Proposition 26, p. 933). Thus, H is a subset of the center of G.

COROLLARY 3. *Let G be a connected Δ-subgroup of $GL_{\mathcal{U}}(n)$ and let N be a connected Δ-subgroup of G such that* a = l(N) *is an ideal of* l(G). *Then N is normal in G.*

PROOF. l(G) is the normalizer in l(G) of a. Let H be the Δ-subgroup of G consisting of all $s \in G$ such that $sus^{-1} \in$ a for all $u \in$ a. Then l(H) = l(G), whence $H = G$. So, if $s \in G$, l(sNs^{-1}) = Ad s(l(N)) = Ad s(a) = a. So, l(sNs^{-1}) = l(N), whence, $N = sNs^{-1}$ and N is normal in G.

Proposition 17. *The group of* Δ*-automorphisms of a nonzero linear* Δ*-Lie algebra* $\mathfrak{g}$ *is nontrivial.*

Proof. We may assume that $\mathfrak{g} \subset \mathfrak{gl}_{\mathfrak{U}}(n)$ for some n. Suppose the only Δ-automorphism of $\mathfrak{g}$ is the identity automorphism. Let Z (resp. N) be the set of all $s \in GL_{\mathfrak{U}}(n)$ such that $sus^{-1} = u$ (resp. $sus^{-1} \in g$) for all $u \in \mathfrak{g}$. Clearly, $Z \subset N$. Let $s \in N$. Ad $s|\mathfrak{g}$ is a Δ-automorphism of $\mathfrak{g}$, whence the hypothesis implies that $s \in Z$. So, $Z \supset N$, and therefore, $\mathfrak{l}(Z) = \mathfrak{l}(N)$. By Corollary 1 of Proposition 16, $\mathfrak{l}(Z)$ is the centralizer and $\mathfrak{l}(N)$ is the normalizer of $\mathfrak{g}$ in $\mathfrak{gl}_{\mathfrak{U}}(n)$. In particular, since $\mathfrak{g}$ is contained in its normalizer, $\mathfrak{g}$ is abelian. But, then, scalar multiplication by any nonzero element of $\mathcal{K}$ is a Δ-automorphism of $\mathfrak{g}$, which contradicts the hypothesis.

Chapter III. Ritt's theory of formal differential groups and δ-Lie algebra structures on $\mathbf{G}_a^n$

J. F. Ritt's study of formal differential groups began in **[11]** with an examination of the so-called "substitutional group in one parameter." He says: "The operation of substituting one function of x into another is associative, and confers to an extent upon the functions of x, the status of a group." If we formally substitute $x + u(x)$ in $x + v(x)$ the result is the function $x + u(x) + v(x + u(x))$. If we expand $u + v(x + u)$ in powers of u by "Taylor's theorem" we obtain the formal power series $u + v + \sum_{k=1}^{\infty}(v^{(k)}u^k/k!)$, where $v^{(k)}$ is the kth derivative of v. Since substitution is an associative operation, this formal power series is a formal group. Ritt defines on his category of formal differential groups a natural equivalence. He proves that up to equivalence, the operations of addition and substitution furnish the only 1-dimensional formal differential groups. He remarks that "the substitution operation, with its quality of noncommutativity, has no counterpart in 1-parameter Lie groups."

If the underlying coefficient field is ordinary, we attach to an n-dimensional formal differential group a δ-Lie algebra whose additive group is $\mathbf{G}_a^n$. Although he never defines the Lie algebra of a formal differential group explicitly, Ritt's object in **[12]** is to show that every δ-Lie algebra whose additive group is $\mathbf{G}_a^n$ is the Lie algebra of an n-dimensional differential group and two such groups are equivalent if and only if their Lie algebras are δ-isomorphic.

In this penetrating but little known study, carried out when the work on formal groups was in its infancy, Ritt used structure constants relentlessly in his proofs. This makes his papers difficult for the modern reader. Ritt's approach to formal groups seems to be close to that of Lazard and Cartier (Cartier **[1]** and **[2]**; Lazard **[9]**, Lubin **[10]**), and his work richly deserves to be expressed in their language. We apologize for not having done so. Our main interest is in the classification up to equivalance of formal differential groups of dimension $\leqslant 2$.

In the theory of formal differential groups two differential algebras of power series arise that are somewhat dual in nature. We shall describe them briefly here and refer the reader to Kolchin [7, Chapter I, 12] for details.

If $y_1, \ldots, y_n$ are differential indeterminates over $\mathfrak{U}$, the power series ring $\mathfrak{U}[[(\theta y_i)_{\theta \in \Theta, 1 \leqslant i \leqslant n}]]$ is a local differential ring. A typical element can be written as an infinite sum $\sum_{j=0}^{\infty} f_j$, where $f_j \in \mathfrak{U}\{y_1, \ldots, y_n\}$ and is homogeneous of degree j. If $\delta \in \Delta$, then $\delta f = \sum_{j=0}^{\infty} \delta f_j$ is again in the power series ring since δf_j is homogeneous of degree j. The unique maximal ideal consisting of all power series $\sum_{j=1}^{\infty} f_j$ is a differential ideal. We denote this ring of power series by $\mathfrak{U}\{\{y_1, \ldots, y_n\}\}$ and call its elements *differential power series* in $y_1, \ldots, y_n$.

If t is a constant transcendental over $\mathfrak{U}$ the power series ring $\mathfrak{U}[[t]]$ is a local differential ring. If $s = \sum_{j=0}^{\infty} a_j t^j$ and $\delta \in \Delta$ we define δs to be the power series $\sum_{j=0}^{\infty} \delta a_j t^j$. The unique maximal ideal $\mathfrak{m} = t\mathfrak{U}[[t]]$ is clearly a differential ideal.

Let $z_1, \ldots, z_r$ be differential indeterminates and let $f_1, \ldots, f_n$ be the maximal ideal of $\mathfrak{U}\{\{z_1, \ldots, z_r\}\}$. The substitution of $f_1, \ldots, f_n$ for $y_1, \ldots, y_n$ defines a homomorphism of local differential algebras over $\mathfrak{U}$ from $\mathfrak{U}\{\{y_1, \ldots, y_n\}\}$ to $\mathfrak{U}\{\{z_1, \ldots, z_r\}\}$. Also, if $s_1, \ldots, s_n$ are in $t\mathfrak{U}[[t]]$ the substitution of $s_1, \ldots, s_n$ for $y_1, \ldots, y_n$ defines a homomorphism of local differential algebras over $\mathfrak{U}$ from $\mathfrak{U}\{\{y_1, \ldots, y_n\}\}$ into $\mathfrak{U}[[t]]$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and $\mathbf{z} = (z_1, \ldots, z_n)$ be n-tuples of differential indeterminates over $\mathfrak{U}$.

An *n-dimensional formal differential group* is an n-tuple $\mathbf{f} = (f_1, \ldots, f_n)$ of differential power series in the maximal ideal of $\mathfrak{U}\{\{\mathbf{x}, \mathbf{y}\}\}$ that satisfies the following conditions:

(1) $\mathbf{f}(\mathbf{x}, \mathbf{0}) = \mathbf{x}$ and $\mathbf{f}(\mathbf{0}, \mathbf{y}) = \mathbf{y}$,

(2) $\mathbf{f}(\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{f}(\mathbf{x}, \mathbf{f}(\mathbf{y}, \mathbf{z}))$

(Ritt [**12**, p. 708]).

The first condition implies that for $i = 1, \ldots, n$ $f_i \equiv x_i + y_i + a_i(\mathbf{x}, \mathbf{y})$ mod deg 3, where a_i is a homogeneous differential polynomial of degree 2 and, moreover, for fixed $\mathbf{x}$ (resp. fixed $\mathbf{y}$) is a homogeneous linear differential polynomial in $\mathbf{y}$ (resp. $\mathbf{x}$). We write

$$\mathbf{f} \equiv \mathbf{x} + \mathbf{y} + \mathbf{a}(\mathbf{x}, \mathbf{y}) \quad \text{mod deg } 3.$$

For those of us who feel uneasy about groups without elements, we observe that $\mathbf{f}$ gives us a bona fide group $C(\mathbf{f})$, which was used to great effect by Ritt in his final paper [**14**]. The elements of $C(\mathbf{f})$ are n-tuples of power series in $\mathfrak{m} = t\mathfrak{U}[[t]]$, where t is a transcendental constant. The law of composition in

$C(\mathbf{f})$ is given by the formal group $\mathbf{f}$. We might, following Cartier (see Lubin's review of Lazard [**10**]) call the elements of $C(\mathbf{f})$ "curves in f." Power series in a transcendental constant were used in DAG, Chapter III, to define "1-parameter subgroups" of differential algebraic matric groups. Although we make no further use of the group $C(\mathbf{f})$, it is perhaps mildly interesting to note that $C(\mathbf{f})$ can be given a structure of prodifferential algebraic group.

We shall assume until further notice that $\mathcal{U}$ is an ordinary differential field.

Let $\mathbf{b}$ be a $\mathcal{K}$-bilinear differential polynomial map from $\mathbf{G}_a^n \times \mathbf{G}_a^n$ to $\mathbf{G}_a^n$. Then for fixed $\mathbf{x}$ (resp. $\mathbf{y}$) the ith coordinate function b_i is a homogeneous linear differential polynomial in $\mathbf{y}$ (resp. $\mathbf{x}$). Therefore, b_i is homogeneous quadratic and every term in b_i involves some derivative of an x_j and of a y_k. Thus, $b_i = \sum_{j,\alpha,k,\beta} c^i_{j\alpha k\beta} x_j^{(\alpha)} y_k^{(\beta)}$, where $c^i_{j\alpha k\beta} \in \mathcal{U}$ and equals 0 for all but finitely many (j, α, k, β). We define a product operation, written $(\mathbf{u}, \mathbf{v}) \to [\mathbf{u}, \mathbf{v}]$, on $\mathbf{G}_a^n$ by the formula $[\mathbf{u}, \mathbf{v}] = \mathbf{b}(\mathbf{u}, \mathbf{v})$.

Given a family $\mathbf{c} = (c^i_{j\alpha k\beta})_{1 \leqslant i \leqslant n, j, \alpha, k, \beta \in \mathbf{N}}$ of elements of $\mathcal{U}$, with all but finitely many of the components equal to 0, we associate to it a unique $\mathcal{K}$-bilinear differential polynomial map $\mathbf{b}$ from $\mathbf{G}_a^n \times \mathbf{G}_a^n \to \mathbf{G}_a^n$ by defining the ith coordinate function b_i of $\mathbf{b}$ to be $\mathrm{b}_i = \sum_{j,\alpha,k,\beta} c^i_{j\alpha k\beta} x_j^{(\alpha)} y_k^{(\beta)}$. We call $\mathbf{b}(\mathbf{x}, \mathbf{y})$ the *product defined by the family* $\mathbf{c}$.

We now describe the conditions that the family $\mathbf{c}$ must satisfy in order that the product defined by it be a Lie product. These identities, which resemble the usual structure conditions, are somewhat complicated by the presence of derivatives of the differential indeterminates. Let $\mathbf{b}$ be the product defined by $\mathbf{c}$ and let λ be a natural number. Let $b_{i\lambda}$ be the λth derivative of b_i; in particular, $b_i = b_{i0}$. Let $c^i_{j\alpha k\beta}$ be the coefficient of $x_j^{(\alpha)} y_k^{(\beta)}$ in b_i. For given (i, λ) there are only finitely many nonzero $c^{i\lambda}_{j\alpha k\beta}$ and $c^{i\lambda}_{j\alpha k\beta}$ is a homogeneous linear differential polynomial in the $c^{i0}_{j\alpha k\beta}$. It is tedious but straightforward to show that the product $\mathbf{b}$ is skew-symmetric and satisfies the identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ if and only if $\mathbf{c}$ satisfies the following *structure conditions*:

(1) $c^P_{JK} + c^P_{KJ} = 0$,

(2) $\sum_S (c^P_{IS} \cdot c^S_{JK} + c^P_{JS} \cdot c^P_{KI} + c^P_{KS} \cdot c^S_{IJ}) = 0$,

where $I = (i, \lambda)$, $J = (j, \alpha)$, $K = (k, \beta)$, $P = (p, 0)$, $S = (s, \sigma)$. If $\mathbf{c}$ satisfies identities 1 and 2 its components are called *structure elements*.

Let $\mathbf{f}$ be a formal differential group and write $\mathbf{f} \equiv \mathbf{x} + \mathbf{y} + \mathbf{a}(\mathbf{x}, \mathbf{y}) \bmod \deg 3$. As we observed earlier, $a_i(\mathbf{x}, \mathbf{y})$ is a $\mathcal{K}$-bilinear differential polynomial. Let $\mathbf{b}(\mathbf{x}, \mathbf{y}) = \mathbf{a}(\mathbf{x}, \mathbf{y}) - \mathbf{a}(\mathbf{y}, \mathbf{x})$ be the antisymmetrization of $\mathbf{a}$. Then $\mathbf{b}$ is clearly a skew-symmetric $\mathcal{K}$-bilinear differential polynomial map from $\mathbf{G}_a^n \times \mathbf{G}_a^n$ to $\mathbf{G}_a^n$. Ritt proves [**12**, pp. 710, 722] that if we write $b_i = \sum_{j\alpha k\beta} c^i_{j\alpha k\beta} x_j^{(\alpha)} y_k^{(\beta)}$ then the associativity of $\mathbf{f}$ implies that the coefficients of the b_i, $1 \leqslant i \leqslant n$, are structure elements. Therefore, the product defined by $(c^i_{j\alpha k\beta})$ is a Lie product on $\mathbf{G}^n_a$. We put on the $\mathcal{K}$-vector space $\mathbf{G}^n_a$ a structure of δ-Lie algebra by defining $[\mathbf{u}, \mathbf{v}] = \mathbf{b}(\mathbf{u}, \mathbf{v})$. We call this δ-Lie algebra structure on $\mathbf{G}^n_a$ the *Lie algebra of the formal differential group* $\mathbf{f}$ and denote it by $\mathfrak{L}(\mathbf{f})$. We call the $c^i_{j\alpha k\beta}$ the *structure elements of f.*

Conversely, if g is a δ-Lie algebra such that $\mathrm{g}^+ = \mathbf{G}^n_a$ then the Lie product is an everywhere defined differential rational map from $\mathbf{G}^n_a \times \mathbf{G}^n_a$ to $\mathbf{G}^n_a$ and hence is given by a $\mathcal{K}$-bilinear differential polynomial map $\mathbf{b}$. $b_i(\mathbf{x}, \mathbf{y}) = \sum_{j\alpha k\beta} c^i_{j\alpha k\beta} x_j^{(\alpha)} y_k^{(\beta)}$. Since $\mathbf{b}$ is a Lie product on $\mathbf{G}^n_a$, it follows from the above remarks that its coefficients are structure elements. The following theorem, which parallels Lie's theorem, and is the subject of Ritt's beautiful and difficult paper [**12**], states that the $c^i_{j\alpha k\beta}$ are the structure elements of a formal differential group.

THEOREM 2 (RITT [**12**, p. 722]). *Let there be given a family* $\mathbf{c} = (c^i_{j\alpha k\beta})_{1 \leqslant i \leqslant n, j\alpha, k, \beta \in \mathbf{N}}$ *of structure elements. There is an n-dimensional formal differential group* $\mathbf{f}$ *whose structure elements are the* $c^i_{j\alpha k\beta}$.

We paraphrase the theorem in the following corollary.

COROLLARY. *Let* g *be a δ-Lie algebra whose additive group is* $\mathbf{G}^n_a$. *There is an n-dimensional formal differential group whose Lie algebra is* g.

We must discuss the degree of uniqueness of the formal differential group whose Lie algebra is g.

We define a natural equivalence of formal differential groups. Let $\mathfrak{m}$ denote the maximal ideal of $\mathfrak{U}\{\{y_1, \ldots, y_n\}\}$. Let G_n be the set of transformations φ of $\underbrace{\mathfrak{m} \times \cdots \times \mathfrak{m}}_{n}$ such that $\varphi \equiv \lambda$ mod deg 2 and λ is a δ-automorphism of $\mathbf{G}^n_a$. Thus, $\varphi = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_i \in \mathfrak{m}$ and $\varphi_i \equiv L_i \bmod m^2$, with L_i a homogeneous linear differential polynomial and $\lambda = (L_1, \ldots, L_n)$ is invertible. We see easily, by the well-known method of "successive approximation", that there is a transformation $\psi \in G_n$ such that $\psi \equiv \lambda^{-1}$ mod deg 2 and φ and ψ are inverse to one another. Thus, G_n is a group relative to composition. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be n-dimensional formal differential groups. Then $\mathbf{f}_1$ is *equivalent* to $\mathbf{f}_2$ if there is a transformation φ in G_n such that $\mathbf{f}_2 = \varphi\mathbf{f}_1(\varphi^{-1}(\mathbf{x}), \varphi^{-1}(\mathbf{y}))$.[1] We write $\mathbf{f}_2 = \varphi\mathbf{f}_1\varphi^{-1}$. Clearly, equivalence of formal groups is an equivalence relation.

THEOREM 3 (RITT [**12**, p. 719, italicized remark]). *Let* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *be n-dimensional formal differential groups. If* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *have the same structure elements, then they are equivalent.*

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[1] If two differential groups with coefficients in F are equivalent, then they are clearly equivalent over a finitely generated Picard-Vessiot extension of F.

COROLLARY. *Let* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *be n-dimensional formal differential groups. If* $\mathfrak{L}(\mathbf{f}_1) = \mathfrak{L}(\mathbf{f}_2)$ *then* $\mathbf{f}_1$ *is equivalent to* $\mathbf{f}_2$.

We prove a stronger statement.

THEOREM 4. *Let* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *be n-dimensional formal differential groups. Then* $\mathfrak{L}(\mathbf{f}_1)$ *is* δ*-isomorphic to* $\mathfrak{L}(\mathbf{f}_2)$ *if and only if* $\mathbf{f}_1$ *is equivalent to* $\mathbf{f}_2$.

PROOF. Let $\mathbf{f}_j \equiv \mathbf{x} + \mathbf{y} + \mathbf{a}_j$ mod deg 3, and let $\mathbf{b}_j = \mathbf{a}_j(\mathbf{x}, \mathbf{y}) - \mathbf{a}_j(\mathbf{y}, \mathbf{x})$, $j = 1, 2$.

Let $\varphi \in G_n$ be such that $\mathbf{f}_2 = \varphi \mathbf{f}_1 \varphi^{-1}$. $\varphi = \lambda + \mu$, $\varphi^{-1} = \lambda^{-1} + \nu$, where λ is a δ-automorphism of $\mathbf{G}_a^n$ and the components of μ and ν are in $\mathfrak{m}^2$. We shall show that λ is an isomorphism of δ-Lie algebras. Since λ is a δ-automorphism of $\mathbf{G}_a^n$ we need only show that $\mathbf{b}_2(\mathbf{x}, \mathbf{y}) = \lambda \mathbf{b}_1(\lambda^{-1}(x), \lambda^{-1}(y))$. Write $\mu \equiv \mathbf{q}$ mod deg 3 and $\nu \equiv \mathbf{r}$ mod deg 3. We compute $\mathbf{f}_2 = \varphi \mathbf{f}_1 \varphi^{-1}$ through terms of degree 2.

$$\begin{aligned}\varphi \mathbf{f}_1 \varphi^{-1} &\equiv \varphi \mathbf{f}_1\big(\lambda^{-1}(\mathbf{x}) + \mathbf{r}(\mathbf{x}), \lambda^{-1}(\mathbf{y}) + \mathbf{r}(\mathbf{y})\big) \\ &\equiv \varphi\big(\lambda^{-1}(\mathbf{x}) + \mathbf{r}(\mathbf{x}) + \lambda^{-1}(\mathbf{y}) + \mathbf{r}(\mathbf{y}) + \mathbf{a}_1(\lambda^{-1}(\mathbf{x}), \lambda^{-1}(\mathbf{y}))\big) \\ &\equiv \lambda\big(\lambda^{-1}(\mathbf{x}) + \lambda^{-1}(\mathbf{y}) + \mathbf{r}(\mathbf{x}) + \mathbf{r}(\mathbf{y}) + \mathbf{a}_1(\lambda^{-1}(\mathbf{x}), \lambda^{-1}(\mathbf{y}))\big) \\ &\quad + \mathbf{q}\big(\lambda^{-1}(\mathbf{x}) + \lambda^{-1}(\mathbf{y})\big) \\ &\equiv \mathbf{x} + \mathbf{y} + \lambda\big(\mathbf{r}(\mathbf{x}) + \mathbf{r}(\mathbf{y})\big) + \lambda \mathbf{a}_1\big(\lambda^{-1}(\mathbf{x}), \lambda^{-1}(y)\big) \\ &\quad + \mathbf{q}\big(\lambda^{-1}(x) + \lambda^{-1}(y)\big).\end{aligned}$$

Therefore,

$$\mathbf{a}_2 = \lambda \mathbf{a}_1\big(\lambda^{-1}(\mathbf{x}), \lambda^{-1}(\mathbf{y})\big) + \lambda\big(\mathbf{r}(\mathbf{x}) + \mathbf{r}(\mathbf{y})\big) + \mathbf{q}\big(\lambda^{-1}(\mathbf{x}) + \lambda^{-1}(\mathbf{y})\big).$$

Thus,

$$\mathbf{b}_2 = \lambda \mathbf{b}_1\big(\lambda^{-1}(\mathbf{x}), \lambda^{-1}(\mathbf{y})\big),$$

whence λ is an isomorphism of δ-Lie algebras.

Conversely, let $\lambda \colon \mathfrak{L}(\mathbf{f}_1) \to \mathfrak{L}(\mathbf{f}_2)$ be a δ-isomorphism. Then λ is a δ-automorphism of $\mathbf{G}_a^n$. The Lie product on $\mathfrak{L}(\mathbf{f}_j)$ is given by $\mathbf{b}_j(\mathbf{x}, \mathbf{y})$. Since λ is a Lie algebra isomorphism, $\mathbf{b}_2(\mathbf{x}, \mathbf{y}) = \lambda \mathbf{b}_1(\lambda^{-1}(\mathbf{x}), \lambda^{-1}(\mathbf{y}))$. Since λ is clearly in G_n, $\lambda \mathbf{f}_1 \lambda^{-1}$ is an n-dimensional formal differential group equivalent to f_1. Since $\lambda \mathbf{f}_1 \lambda^{-1}$ and $\mathbf{f}_2$ have the same structure elements they are equivalent by Theorem 3. Therefore, $\mathbf{f}_1$ is equivalent to $\mathbf{f}_2$.

CHAPTER IV. δ-LIE ALGEBRA STRUCTURES ON $\mathbf{G}_a$ AND ON $\mathbf{G}_a \times \mathbf{G}_a$

1. The δ-Lie algebra structures on $\mathbf{G}_a$. Corollary 2 of Proposition 4 states that if g is a Δ-Lie algebra whose additive group is a vector group of dimension 1, then either g is abelian or $D[\mathrm{g}, \mathrm{g}] = \mathrm{g}$. Thus, a Δ-Lie algebra structure on $\mathbf{G}_a$ is either abelian or extremely nonabelian. These extremes are exemplified in the ordinary case by $\mathfrak{g}_a$, where the Lie product is 0, and by the substitution Lie algebra $\mathfrak{g}_s$, where the Lie product is $xy' - y'x$. If the cardinality of Δ is 1, we can show that there are precisely two isomorphism classes of δ-Lie algebras with additive group $\mathbf{G}_a$–the isomorphism class of abelian Lie algebras, represented by $\mathfrak{g}_a$, and the isomorphism class of Lie algebras equal to their derived algebras, represented by $\mathfrak{g}_s$.

In the Ritt theory of formal differential groups, the 1-dimensional additive group is of course the differential power series $x + y$. The 1-dimensional substitution group is the differential power series $x + y + \sum_{k=1}^{\infty}(y^{(k)}x^k/k!)$. When we antisymmetrize the degree 2 terms we see that the Lie algebra of the additive group is $\mathfrak{g}_a$ and the Lie algebra of the substitution group is $\mathfrak{g}_s$.

THEOREM 5 (RITT [**11**, p. 757]). *A 1-dimensional formal differential group ("associative differential operation of the first rank") either is equivalent to the additive group or else is equivalent to the substitution group.*

COROLLARY. *A δ-Lie algebra $\mathfrak{g}$ whose additive group is $\mathbf{G}_a$ either is isomorphic to $\mathfrak{g}_a$ or else is isomorphic to $\mathfrak{g}_s$.*

Just as there is no counterpart in 1-parameter Lie groups of the noncommutative substitution group, there is no counterpart of the 1-parameter nonabelian Lie algebra $\mathfrak{g}_s$. $\mathfrak{g}_s$ can be realized as a Lie algebra of derivations. Indeed, in the Lie algebra $\mathcal{U} \cdot \delta$ over $\mathcal{K}$ of derivation operators on $\mathcal{U}$, $[u\delta, v\delta] = u\delta \circ v\delta - v\delta \circ u\delta = (uv' - u'v)\delta$. However, $\mathfrak{g}_s$ cannot be realized as the Lie algebra of invariant differential derivations on a linear differential algebraic group, i.e., $\mathfrak{g}_s$ is not integrable in $GL_{\mathcal{U}}(n)$ for any n.[2] In fact, this 1-parameter Lie algebra, which is equal to its derived algebra and has trivial center, also has trivial automorphism group, and thus provides a 1-dimensional counterexample to the analog in the category of δ-Lie algebras of Ado's theorem.

THEOREM 6. *The substitution Lie algebra $\mathfrak{g}_s$ is not linear.*

PROOF. We show that the only δ-automorphism of $\mathfrak{g}_s$ is the identity automorphism. Suppose σ is a δ-automorphism of $\mathfrak{g}_s$. Then σ is a δ-automorphism of its additive group $\mathbf{G}_a$. Therefore, there is an element $a \in \mathbf{G}_m$ such that $\sigma(u) = au$ for all $u \in \mathcal{U}$ (since $\mathrm{Hom}_\delta(\mathbf{G}_a, \mathbf{G}_a) = \mathcal{U}[\delta]$, the ring of linear differential operators). Since σ is a Lie algebra homomorphism, $\sigma[u, v] = [\sigma(u), \sigma(v)]$. Therefore, $a(uv' - u'v) = a^2(uv' - u'v)$ for all $u, v \in \mathbf{G}_a$. Thus, $a = 1$, and σ is the identity auomorphism. Proposition 18 now implies that $\mathfrak{g}_s$ is not linear.

[2]Thus, if G is a linear δ-group whose Lie algebra $\mathfrak{g}$ has additive group a vector group of dimension 1, then G is abelian.

2. The isomorphism classes of δ-Lie algebra structures on $\mathbf{G}_a \times \mathbf{G}_a$. In [13] Ritt determines up to equivalence all 2-dimensional formal differential groups. It follows from his theorems, restated here in Chapter III, that to do this it suffices to compute all possible families of structure elements, which is what he does. Thus, Ritt computes the δ-Lie algebra structures on $\mathbf{G}_a \times \mathbf{G}_a$ up to isomorphism.

In contrast to the classical case of 2-dimensional Lie algebras, where there are only two isomorphism classes, δ-Lie algebra structures on the plane abound. It is a tribute to Ritt's formidable computational ability that he was able to determine all of them. There are infinitely many isomorphism classes, divided into thirteen types, three *finite types* and ten *substitutional types*. We list the isomorphism classes according to type by giving in each case the Lie product of a representative (which we call a basic representative). In this list, for all $i \geqslant 2$ we will denote $\delta^i z$ by $z^{(i)}$. We will continue our practice of writing z for $z^{(0)}$ and z' for $z^{(1)}$.

First Finite Type

$$[\mathbf{x}, \mathbf{y}] = (0, 0);$$

Second Finite Type

$$[\mathbf{x}, \mathbf{y}] = \Big(\sum_{0 \leqslant i \leqslant g} a_i (x_1 y_2^{(i)} - y_1 x_2^{(i)}), 0 \Big), \qquad (a_0, \ldots, a_g) \neq (0, \ldots, 0);$$

Third Finite Type

$$[\mathbf{x}, \mathbf{y}] = \Big(\sum_{0 \leqslant i < j \leqslant g} a_{ij} (x_2^{(i)} y_2^{(j)} - y_2^{(i)} x_2^{(j)}), 0 \Big), \quad \text{not all } a_{ij} = 0;$$

First Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = (0, x_2 y_2' - y_2 x_2');$$

Second Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = (x_1 y_1' - y_1 x_1', x_2 y_2' - y_2 x_2');$$

Third Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = (c(x_1 y_2' - y_1 x_2') + x_2 y_1' - y_2 x_1', x_2 y_2' - y_2 x_2'), \qquad c \in \mathcal{K};$$

Fourth Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = (x_2 y_1' - y_2 x_1' + a(x_2 y_2' - y_2 x_2'), x_2 y_2' - y_2 x_2'), \qquad a \in \mathcal{U}, a \neq 0;$$

Fifth Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = \big(x_2 y_1' - y_2 x_1' + x_2' y_1 - y_2' x_1 + a(x_2 y_2^{(2)} - y_2 x_2^{(2)}), x_2 y_2' - y_2 x_2'\big), \qquad a \in \mathcal{U}, a \neq 0;$$

Sixth Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = \big(x_2y_1' - y_2x_1' + 2(x_2'y_1 - y_2'x_1) + a\big(x_2y_2^{(3)} - y_2x_2^{(3)}\big), x_2y_2' - y_2x_2'\big), \quad a \in \mathcal{U}, a \neq 0;$$

Seventh Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = \big(x_2y_1' - y_2x_1' + x_2'y_1 + y_2'x_1 + a\big(x_2y_2^{(2)} - y_2x_2^{(2)}\big) + x_2'y_2^{(2)} - y_2'x_2^{(2)}, x_2y_2' - y_2x_2'\big), \qquad a \in \mathcal{U};$$

Eighth Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = \big(x_2y_1' - y_2x_1' + 2(x_2'y_1 - y_2'x_1) + a\big(x_2y_2^{(3)} - y_2x_2^{(3)}\big) + x_2'y_2^{(3)} - y_2'x_2^{(3)}, x_2y_2' - y_2x_2'\big), \qquad a \in \mathcal{U};$$

Ninth Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = \big(x_2y_1' - y_2x_1' + 5(x_2'y_1 - y_2'x_1) + x_2^{(2)}y_2^{(5)} - y_2^{(2)}x_2^{(5)}, x_2y_2' - y_2x_2'\big);$$

Tenth Substitutional Type

$$[\mathbf{x}, \mathbf{y}] = \big(x_2y_1' - y_2x_1' + 7(x_2'y_1 - y_2'x_1) + (9/2)\big(x_2^{(5)}y_2^{(4)} - x_2^{(4)}y_2^{(5)}\big) + x_2^{(3)}y_2^{(6)} - y_2^{(3)}x_2^{(6)}, x_2y_2' - y_2x_2'\big).$$

Ritt discusses the distinctness of the equivalence classes of the formal differential groups whose Lie algebras are represented in the above list, in the last paragraph of **[13]**.[3] An isomorphism class cannot be of two different types. So, if two δ-Lie algebras are isomorphic they must be of the same type. Distinct $(a_0, \ldots, a_g)$ give rise to basic representatives of distinct isomorphism classes of second finite type. The only isomorphisms holding among the representatives of third finite type are diagonal transformations $\lambda = (L_1, L_2)$, where $L_1(\mathbf{x}) = a_1x_1$ and $L_2(\mathbf{x}) = a_2x_2$. Distinct constants c give rise to distinct isomorphism classes of third substitutional type. So, we have a 1-constant-parameter family of isomorphism classes of third substitutional type. Two isomorphism classes of fourth or fifth or sixth substitutional type are equal if and only if the ratio of their parameters a is in $\mathcal{K}$. Distinct $a \in \mathcal{U}$ give rise to distinct isomorphism classes of seventh and eighth substitutional type. There is only one isomorphism class of first, second, ninth and tenth substitutional type, respectively.

3. Solvability of δ-Lie algebra structures on $\mathbf{G}_a \times \mathbf{G}_a$. In the 1-dimensional case, the Lie algebra $\mathfrak{g}_a$ of finite type is abelian, hence solvable, whereas the Lie algebra $\mathfrak{g}_s$ of substitutional type is clearly not solvable since it is equal to its derived algebra. This linking of solvability with finiteness carries over to

[3]Ritt actually displays the differential power series in twelve of the thirteen cases. The series all involve exponentials and logarithms of series of substitutional type.

the δ-Lie algebra structures on the plane. We shall first show that the δ-Lie algebras of finite type are all solvable, which parallels the case of the classical 2-dimensional Lie algebras.

A δ-Lie algebra of first finite type is abelian, hence solvable. Suppose $\mathfrak{g}$ is a basic representative of third finite type. The center of $\mathfrak{g}$ has additive group $\mathbf{G}_a \times W$, where W is the set of zeros in $\mathbf{G}_a$ of a homogeneous linear differential polynomial. The derived algebra of $\mathfrak{g}$ has additive group $\mathbf{G}_a \times 0$, hence is central.[4] Therefore, $\mathfrak{g}$ is nilpotent (hence solvable) of nil class 2. In particular, $\mathfrak{g}$ is a nonabelian central extension of $\mathfrak{g}_a$ by $\mathfrak{g}_a$. So, every δ-Lie algebra of third finite type is a central extension of $\mathfrak{g}_a$ by $\mathfrak{g}_a$.

We shall now show that a δ-Lie algebra $\mathfrak{g}$ of second finite type is solvable, and in fact, is a split extension of $\mathfrak{g}_a$ by $\mathfrak{g}_a$. It suffices to consider basic representatives. The classical 2-dimensional nonabelian Lie algebra whose Lie product is given by the formula $[\mathbf{x}, \mathbf{y}] = (x_1y_2 - y_1x_2, 0)$ is, of course, of second finite type. If $\mathfrak{g}$ is any basic representative of second finite type, $\mathfrak{g}$ is not nilpotent. However, the additive group of the derived algebra is $\mathbf{G}_a \times 0$. The derived algebra is readily seen to be an abelian δ-ideal of $\mathfrak{g}$ isomorphic to $\mathfrak{g}_a$. Thus, $\mathfrak{g}$ is solvable of solv class 2. Now, for fixed $u_2 \in \mathfrak{g}_a$, the map D_{u_2} from $\mathfrak{g}_a$ to $\mathfrak{g}_a$ defined by the formula $D_{u_2}u_1 = u_1(-\sum_{i=0}^{g} a_i u_2^{(i)})$ is a derivation of the $\mathcal{K}$-Lie algebra $\mathfrak{g}_a$ (and, in fact, since it is merely scalar multiplication, it is even $\mathfrak{U}$-linear). It is easily seen that the map that sends $(u_1, u_2) \mapsto D_{u_2}u_1$ defines an action of the δ-Lie algebra $\mathfrak{g}_a$ on itself. Evidently, the Lie algebra whose Lie product is given by the formula $[\mathbf{x}, \mathbf{y}] = (\sum_{i=0}^{g} a_i(x_1y_2^{(i)} - y_1x_2^{(i)}), 0)$ is the split extension of $\mathfrak{g}_a$ by $\mathfrak{g}_a$ relative to this action. So, the δ-Lie algebras of second finite type are all split extensions of $\mathfrak{g}_a$ by $\mathfrak{g}_a$.

THEOREM 7. *Let* $\mathfrak{g}$ *be a δ-Lie algebra whose additive group is* $\mathbf{G}_a \times \mathbf{G}_a$. *Then* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{g}$ *is of finite type.*

PROOF. We must show that none of the δ-Lie algebras of substitutional type is solvable. We first observe that $\mathfrak{g}_s$ is not solvable since $[\mathfrak{g}_s, \mathfrak{g}_s] = \mathfrak{g}_s$.

Let $\mathfrak{g}$ be a δ-Lie algebra of first substitutional type. $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_a \times \mathfrak{g}_s$. The derived algebra of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_s$ and the center to $\mathfrak{g}_a$. $\mathfrak{g}$ is not solvable since it contains a δ-subalgebra isomorphic to $\mathfrak{g}_s$. If $\mathfrak{g}$ has second substitutional type, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_s \times \mathfrak{g}_s$, which is clearly not solvable. The derived algebra is equal to $\mathfrak{g}$ and the center is trivial.

A δ-Lie algebra of third substitutional type is isomorphic to a δ-Lie algebra $\mathfrak{g}$ in which the Lie product is given by the formula

$$[\mathbf{x}, \mathbf{y}] = (c(x_1y_2' - y_1x_2') + x_2y_1' - y_2x_1', x_2y_2' - y_2x_2'), \qquad c \in \mathcal{K}.$$

[4]Note that $D[\mathfrak{g}, \mathfrak{g}]$ has as additive group a vector group as expected, but that the center does not.

We claim g is not solvable, and in fact, g is a split extension of g_s by g_a. It is easy to see that the δ-subgroup $\mathbf{G}_a \times 0$ of the additive group of g is the additive group of a δ-ideal isomorphic to g_a, and $0 \times \mathbf{G}_a$ is the additive group of a δ-subalgebra isomorphic to g_s. In particular, g is not solvable. The δ-Lie algebra g_s acts on g_a. For fixed u_2 in g_s, the map D_{u_2} from g_a into g_a defined by the formula $D_{u_2}u_1 = u_2u_1' - cu_1u_2'$, $c \in \mathcal{K}$, is a derivation of g_a and the map that sends $(u_1, u_2) \mapsto D_{u_2}u_1$ defines an action of g_s on g_a.

A δ-Lie algebra of fourth substitutional type is isomorphic to a Lie algebra g whose Lie product is given by the formula $[x, y] = (x_2y_1' - y_2x_1' + a(x_2y_2' - y_2x_2'), x_2y_2' - y_2x_2'))$. We show that g is not solvable by showing that the derived algebra is equal to g. Let $\mathbf{u} = (0, 1)$ and let $(v_1, v_2) \in$ g. We can solve simultaneously the differential equations

$$y_1' + ay_2' = v_1, \qquad y_2' = v_2.$$

Therefore, ad u is surjective, whence [g, g] = g.

The same technique shows that if g is a basic representative of an isomorphism class of substitutional type > 4, then [g, g] = g, whence g is not solvable. In each case, we let $\mathbf{u} = (0, 1)$. We then solve the following systems of linear differential equations:

Fifth and Seventh Substitutional Types

$$y_1' + ay_2'' = v_1, \qquad y_2' = v_2;$$

Sixth and Eighth Substitutional Types

$$y_1' + ay_2^{(3)} = v_1, \qquad y_2' = v_2;$$

Ninth and Tenth Substitutional Types

$$y_1' = v_1, \qquad y_2' = v_2.$$

4. Linearity of δ-Lie algebra structures on $\mathbf{G}_a \times \mathbf{G}_a$. The dichotomy between finite and substitutional types carries over to the question of the linearity of δ-Lie algebra structures on the plane. The δ-Lie algebras of substitutional type give us an infinity of δ-Lie algebras with no faithful representation as matric algebras.

THEOREM 8. *Let* g *be a δ-Lie algebra whose additive group is $\mathbf{G}_a \times \mathbf{G}_a$. Then* g *is linear if and only if* g *is of finite type.*[5]

PROOF. $g_a \times g_a$ is clearly linear. Suppose g is a basic representative of second finite type. We define an isomorphism α of δ-Lie algebras from g into

[5] It follows from Theorems 7 and 8 that a linear δ-group whose Lie algebra g has as additive group a vector group of dimension 2 is solvable.

the Lie algebra of 2×2 upper triangular matrices by the formula

$$\alpha(u_1, u_2) = \begin{bmatrix} -a_0u_2 & u_1 \\ 0 & \sum_{i=1}^{g} a_iu_2^{(i)} \end{bmatrix}, \quad \text{if } a_0 \neq 0,$$

and

$$= \begin{bmatrix} u_2 & u_1 \\ 0 & \sum_{i=1}^{g} a_iu_2^{(i)} + u_2 \end{bmatrix}, \quad \text{if } a_0 = 0.$$

If g is a basic representative of third finite type we define an isomorphism α of δ-Lie algebras from g onto a Lie algebra of upper triangular nilpotent matrices as follows: The entries of $\alpha(\mathbf{u})$ are 0 except for those in the first row and last column. If not all a_{0j} equal 0, the first row is the n-tuple $(0, a_{01}u_2, \ldots, a_{0g}u_2, \ldots, a_{g-1,g}u_2^{(g-1)}, u_1)$, and the last column is the n-tuple $(u_1, u_2', \ldots, u_2^{(g)}, u_2'', \ldots, u_2^{(g)}, \ldots, u_2^{(g)}, 0)$. If $a_{0j} = 0$, $1 \leqslant j \leqslant g$, the first row is the n-tuple $(0, u_2, a_{12}u_2', \ldots, a_{1g}u_2', \ldots, a_{g-1,g}u_2^{(g-1)}, u_1)$ and the last column is the n-tuple $(u_1, 0, u_2'', \ldots, u_2^{(g)}, u_2'', \ldots, u_2^{(g)}, \ldots, u_2^{(g)}, 0)$. For example, if the Lie product is given by the formula $(x_2y_2' - y_2x_2', 0)$, then

$$\alpha(\mathbf{u}) = \begin{bmatrix} 0 & u_2 & u_1 \\ 0 & 0 & u_2' \\ 0 & 0 & 0 \end{bmatrix}.$$

If g is a basic representative of first, second, or third substitutional type then g is not linear since g contains a δ-subalgebra isomorphic to g_s.

If g is a basic representative of substitutional type $\geqslant 4$, then it is easy to see that the δ-subgroup $\mathbf{G}_a \times 0$ of g^+ is the additive group of an abelian ideal a of g. Now, (v_1, v_2) centralizes a if and only if for every $u \in \mathfrak{U}[(u, 0), (v_1, v_2)] = (0, 0)$. In each case, (v_1, v_2) must satisfy an equation of the form $u'v_2 + bv_2' = 0$ for all $u \in \mathfrak{U}$. Therefore, $v_2 = 0$. Thus, a is its own centralizer in g. Moreover g/a has additive group a vector group of dimension 1 (Chapter 1, §2). Since g = [g, g], g/a cannot be abelian. Therefore, g/a is isomorphic to g_s (corollary of Theorem 5). Suppose g is linear. We may suppose that $g \subset gl_{\mathfrak{U}}(n)$ for some n. The Zariski closure A(a) of the abelian ideal a of g is an abelian ideal of the Zariski closure A(g) of g. Since every finite-dimensional Lie algebra over $\mathfrak{U}$ is linear by Ado's theorem, there is a homomorphism α: $A(g) \to gl_{\mathfrak{U}}(r)$, for some r, with kernel A(a). The restriction $\alpha|g$ is a homomorphism of δ-Lie algebras from g into $gl_{\mathfrak{U}}(r)$ with kernel $A(a) \cap g$. Since $A(a) \cap g$ is clearly abelian and contains a it is contained in the centralizer in g of a. Thus, $A(a) \cap g = a$. But, α(g) is a matric δ-Lie algebra isomorphic to g_s, which contradicts Theorem 6.

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