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HIGHER ORDER MASSEY PRODUCTS AND LINKS

BY

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ABSTRACT. In this paper we generalize Steenrod's functional cup products, calling the generalizations functional triple products, and relate them with Massey 4-products. We then study certain links using this machinery and a new relation that is satisfied by 4-products under conditions on X which permit applications to links. Finally, many examples illustrating the connection between Massey higher products, both ordinary and matrix, and links are presented. This material constituted a portion of the author's doctoral dissertation. The author would like to thank his thesis advisor Professor W. S. Massey for his encouragement and guidance.

1. Introduction. We first present generalizations of Steenrod's functional cup products (see [S]), calling them functional triple products, and relate them with Massey 4-products. Then a new relation on 4-products is given under conditions on X which permit applications to links. It seems reasonable to conjecture that this relation holds under less restrictive hypotheses.

We consider the cohomology of the complement of certain links of four spheres in S^n in which there is a generator naturally associated with each sphere. Using the new relation and the machinery developed before, we see that there are essentially two integers which determine all twenty-four 4-products which contain the four distinct generators. This generalizes the result in [MW2] that for certain links of three spheres is S^n there is essentially one integer which determines in the complementary space of the link all six triple products which contain three distinct generators. We also show that these 4-products are invariant under certain homotopies of the embedding maps. We then treat the 4-products in the complement of certain links of three circles in S^3 and of two circles in S^3.

In the final section which is much more geometrical than the preceding sections of the paper, we give several examples of how Massey higher products, both ordinary and matrix, can be used to detect linking. To avoid complications due to questions of orientation we do our computations with $\mathbf{Z}_2$ coefficients and for convenience we use homology theory and intersections of cycles.

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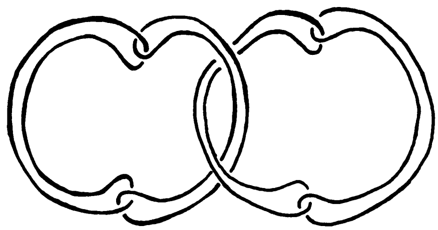
The first example presented there is the link of four circles in S^3.



We can see that any subcollection of three circles is not linked, yet the whole collection is linked. The Alexander polynomial for this link is zero, so it cannot be used to detect this behavior. We will show that the linking is detected by a nontrivial 4-product in the complement of the link.

In another example in the last section, a matrix triple product is used to detect the linking of a circle with two spectacles, a spectacle being a space homeomorphic to two circles joined by a line (○-○). And at the end, several examples are given indicating the use of matrix higher products to detect the linking of larger collections of circles and spectacles.

Advice to the Reader. At a first reading of this paper, we would suggest that the following not be omitted: Definitions 2.3a–2.3c, Propositions 3.3b and 3.4b, Theorem 4.2, Proposition 5.6 and Corollary 5.8, and finally the examples in §8.

2. Definitions. Since this paper is mainly concerned with the 4-products in the complement of certain links, we will present only the machinery that we will use for that purpose. However, our definitions could be generalized in two different ways: they could be given for higher order products and they could be given for matrices of elements, but we will not concern ourselves here with these extensions.

We first introduce generalizations of Steenrod's functional cup products, which we call functional triple products and investigate their relationship with Massey 4-products. In order to define these products a knowledge of the Steenrod functional cup products and of the Massey triple product is necessary. The literature contains different definitions of these products due to different sign conventions, so we will include their definitions.

Let X be a topological space and R a commutative ring with identity. In this paper $H^*(X; R)$ will denote the singular cohomology ring and $\mathcal{C}^*(X; R)$ the singular cochain complex. (For that matter we could use in the following any cochain complex which has an associative product.) If $a \in H^p(X; R)$ or $\mathcal{C}^p(X; R)$, we will write $\bar{a} = (-1)^p a$. If $f\colon Y \to X$ is a continuous map between topological spaces, then $f^*\colon H^*(X; R) \to H^*(Y; R)$ and $f^\#\colon \mathcal{C}^*(X; R) \to \mathcal{C}^*(Y; R)$ will denote the induced homomorphisms.

DEFINITION 2.1. Let u, v and w be homogeneous elements from $H^*(X; R)$ of degrees p, q and r respectively. Assume $uv = 0$, $vw = 0$ and $p + q + r - 1 = s$. Let u', v' and w' be representative cocycles for u, v and w respectively. Then we may select cochains a^{12} and a^{23} so that $\delta(a^{12}) = \overline{u'}v'$ and $\delta(a^{23}) = \overline{v'}w'$. Then the cochain

$$z' = \overline{u'}\,a^{23} + \overline{a^{12}}\,w'$$

is actually a cocycle of degree s. We define the Massey triple product $\langle u, v, w\rangle$ to be the collection of all cohomology classes $[z'] \in H^s(X; R)$ that we can obtain by the above procedure.

The indeterminacy of the triple product is

$$u \cdot H^{q+r-1}(X) + H^{p+q-1}(X) \cdot w$$

and this definition of the triple product differs by the sign (-1) from the definition in [MJ] in the case of 1×1 matrices and it differs by the sign $(-1)^{q+1}$ from Massey's original definition in [UM].

DEFINITION 2.2a. Let X and Y be topological spaces and $f\colon Y \to X$ be a continuous map. Let u and v be elements of $H^*(X; R)$ of degrees p and q respectively. Assume that $uv = 0$ and $f^*(u) = 0$. Let u' and v' be representative cocycles for u and v and since $uv = 0$ and $f^*(u) = 0$, we may select cochains a^{12} and a^{f1} such that $\delta(a^{12}) = \overline{u'}v'$ and $\delta(a^{f1}) = f^{\#}(u')$. Then the cochain

$$z' = f^{\#}(a^{12}) + \overline{a^{f1}}\, f^{\#}(v')$$

can be shown to be a cocycle of degree $p + q - 1$. We define the Steenrod left functional cup product, denoted by $P_{f,1}(u, v)$ or $L_f(u, v)$, to be the collection of all cohomology classes $[z'] \in H^{p+q-1}(Y; R)$ that we can obtain by the above procedure.

The indeterminacy of this product is

$$f^*\big(H^{p+q-1}(X)\big) + H^{p-1}(Y) \cdot f^*(v)$$

and this definition differs by the sign $(-1)^p$ from the definition given in [UM].

DEFINITION 2.2b. In the same setting as in Definition 2.2a, we now assume that $uv = 0$ and $f^*(v) = 0$. We select representative cocycles u' and v' and cochains a^{12} and a^{2f} such that $\delta(a^{12}) = \overline{u'}v'$ and $\delta(a^{2f}) = f^{\#}(\overline{v'})$. Then

$$z' = f^{\#}\left(\overline{a^{12}}\right) + f^{\#}\left(\overline{u'}\right)a^{2f}$$

is actually a cocycle of degree $p + q - 1$. The collection of all cohomology classes $[z'] \in H^{p+q-1}(Y; R)$ that we can obtain in the above manner is called the Steenrod right functional cup product and is denoted by $P_{f,2}(u, v)$ or $R_f(u, v)$.

The indeterminacy of this product is

$$f^*\big(H^{p+q-1}(X)\big) + f^*(u)\cdot H^{q-1}(Y)$$

and this definition differs by the sign $(-1)^{q+1}$ from the definition in **[UM]**.

We need the above definitions to introduce the three functional triple products which generalize Steenrod's functional cup products. (See **[S]**.)

Let $f\colon Y \to X$ be a continuous map between topological spaces, u, v and w be elements in $H^*(X; R)$ of degrees p, q and r respectively and assume $\langle u, v, w\rangle$ is defined in $H^{p+q+r-1}(X; R)$. We consider three cases:

Case I. $f^*(u) = (0)$; $P_{f,1}(u, v)$ and $\langle u, v, w\rangle$ vanish simultaneously.

Case II. $f^*(v) = (0)$; $P_{f,2}(u, v)$, $\langle u, v, w\rangle$ and $P_{f,1}(v, w)$ vanish simultaneously.

Case III. $f^*(w) = (0)$; $P_{f,2}(v, w)$ and $\langle u, v, w\rangle$ vanish simultaneously.

We recall that when we say, for example, that $P_{f,1}(u, v)$ and $\langle u, v, w\rangle$ vanish simultaneously, we mean that we may make our cocycle and cochain selections along the way so that the cocycle representatives of $P_{f,1}(u, v)$ and of $\langle u, v, w\rangle$:

$$f^{\#}(a^{12}) + \overline{a^{f1}}\, f^{\#}(v') \quad \text{and} \quad \overline{a^{12}}\, w' + \overline{u'}\, a^{23}$$

are simultaneously coboundaries. (See **[MW1**, pp. 147–148] and **[O]** for a discussion of vanishing simultaneously.) This is a stronger condition than both simply vanishing.

Definition 2.3a. Assume Case I holds. Let u', v' and w' be representative cocycles for u, v and w and select cochains a^{12}, a^{23} and a^{f1} such that

$$\delta(a^{12}) = \overline{u'}\, v', \quad \delta(a^{23}) = \overline{v'}\, w', \quad \delta(a^{f1}) = f^{\#}(u').$$

Since $P_{f,1}(u, v)$ and $\langle u, v, w\rangle$ vanish simultaneously, we may select cochains a^{f12} and a^{13} so that

$$\delta(a^{f12}) = f^{\#}(a^{12}) + \overline{a^{f1}}\, f^{\#}(v'), \qquad \delta(a^{13}) = \overline{a^{12}}\, w' + \overline{u'}\, a^{23}.$$

Then the cochain $z' = f^{\#}(a^{13}) + \overline{a^{f1}} f^{\#}(a^{23}) + \overline{a^{f12}} f^{\#}(w')$ is actually a cocycle of degree $p + q + r - 2$. We define $P_{f,1}(u, v, w)$ to be the collection of all cohomology classes $[z'] \in H^{p+q+r-2}(Y; R)$ that we can obtain in this manner.

Definition 2.3b. Assume Case II holds. Let u', v', w', a^{12} and a^{23} be as in Definition 2.3a. Select a cochain a^{2f} so that $\delta(a^{2f}) = f^{\#}(\overline{v'})$. Since $P_{f,2}(u, v)$, $\langle u, v, w\rangle$, and $P_{f,1}(v, w)$ vanish simultaneously, we may select cochains a^{13}, a^{12f} and a^{f23} such that

$$\delta(a^{13}) = \overline{a^{12}}\, w' + \overline{u'}\, a^{23},$$

$$\delta(a^{12f}) = f^{\#}\big(\overline{a^{12}}\big) + f^{\#}\big(\overline{u'}\big)a^{2f},$$

$$\delta(a^{f23}) = f^{\#}(a^{23}) - a^{2f} f^{\#}(w').$$

Then $z' = f^\#(a^{13}) - a^{12f}f^\#(w') - f^\#(u')a^{f23}$ is actually a cocycle of degree $p + q + r - 2$. We define $P_{f,2}(u, v, w)$ to be the collection of all cohomology classes $[z'] \in H^{p+q+r-2}(Y; R)$ that we can obtain in the above manner.

DEFINITION 2.3c. Assume Case III holds. Let u', v', w', a^{12} and a^{23} be as in Definition 2.3a. Select a cochain a^{3f} such that $\delta(a^{3f}) = f^\#(\overline{w'})$. Since $P_{f,2}(v, w)$ and $\langle u, v, w\rangle$ vanish simultaneously, we may select cochains a^{23f} and a^{13} so that

$$\delta(a^{23f}) = f^\#\left(\overline{a^{23}}\right) + f^\#\left(\overline{v'}\right)a^{3f}, \qquad \delta(a^{13}) = \overline{a^{12}}\,w' + \overline{u'}\,a^{23}.$$

Then the cochain $z' = f^\#(\overline{a^{13}}) + f^\#(\overline{u'})a^{23f} + f^\#(\overline{a^{12}})a^{3f}$ is actually a cocycle of degree $p + q + r - 2$. We define $P_{f,3}(u, v, w)$ to be the collection of all cohomology classes $[z'] \in H^{p+q+r-2}(Y; R)$ that we can obtain in this way.

J. Peter May has a general setting in which matrix higher products can be considered. (See Example 2, p. 535 in **[MJ]**.) If we use his notation we may consider the following:

Case 1. $(M, U, N) = (\mathcal{C}^*(Y), \mathcal{C}^*(X), \mathcal{C}^*(X))$,

Case 2. $(M, U, N) = (\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(Y))$

where we consider $\mathcal{C}^*(Y)$ as a two-sided $\mathcal{C}^*(X)$-module via the homomorphism $f^\#\colon \mathcal{C}^*(X) \to \mathcal{C}^*(Y)$. One can easily see that $P_{f,1}(u, v, w) = -\langle (1'), u, v, w\rangle$ in Case 1, and $P_{f,3}(u, v, w) = -\langle u, v, w, (1')\rangle$ in Case 2, where $1' \in H^0(Y)$ is the unit of the ring and the negative sign arises because of different conventions. There does not seem to be a natural setting for $P_{f,2}(u, v, w)$.

In light of the above, an element in the indeterminacy of $P_{f,1}(u, v, w)$ can be written as an element in the matrix triple product of Case 1,

$$\left\langle (1', H^{p-1}(Y)), \begin{pmatrix} u & H^{p+q-1}(X) \\ 0 & v \end{pmatrix}, \begin{pmatrix} H^{q+r-1}(X) \\ w \end{pmatrix} \right\rangle$$

and an element in the indeterminacy of $P_{f,3}(u, v, w)$ can be written as an element in the matrix triple product of Case 2,

$$\left\langle (u, H^{p+q-1}(X)), \begin{pmatrix} v & H^{q+r-1}(X) \\ 0 & w \end{pmatrix}, \begin{pmatrix} H^{r-1}(Y) \\ 1' \end{pmatrix} \right\rangle.$$

A nice description of the indeterminacy of $P_{f,2}(u, v, w)$ has not been found.

The last definition we present is of the Massey 4-product.

DEFINITION 2.4. Let u, v, w and x be elements of $H^*(X; R)$ of degrees p, q, r and s respectively. Let $p + q + r + s - 2 = t$, $uv = 0$, $vw = 0$ and $wx = 0$; and assume that $\langle u, v, w\rangle = 0$ and $\langle v, w, x\rangle = 0$ and that these triple products vanish simultaneously. (See **[MW1**, pp. 147–148].) Select representative cocycles u', v', w' and x' for u, v, w and x and also cochains a^{12}, a^{23} and a^{34} so that $\delta(a^{12}) = \overline{u'}v'$, $\delta(a^{23}) = \overline{v'}w'$ and $\delta(a^{34}) = \overline{w'}x'$. Since the triple products vanish simultaneously, we may select cochains a^{13} and a^{24} so that

$$\delta(a^{13}) = \overline{a^{12}}\, w' + \overline{u'}\, a^{23}, \qquad \delta(a^{24}) = \overline{a^{23}}\, x' + \overline{v'}\, a^{34}.$$

Then the cochain $z' = \overline{a^{13}}\, x' + \overline{a^{12}} a^{34} + \overline{u'} a^{24}$ is actually a cocycle of degree t. We define the Massey 4-product $\langle u, v, w, x\rangle$ to be the collection of all cohomology classes $[z'] \in H^t(X;\ R)$ that we can obtain by the above procedure.

A result due to Kraines (see **[MJ**, p. 543]) states that an element in the indeterminacy of the 4-product is an element of a matrix triple product which we can write as

$$\left\langle \left(u, H^{p+q-1}(X)\right), \begin{pmatrix} v & H^{q+r-1}(X) \\ 0 & w \end{pmatrix}, \begin{pmatrix} H^{r+s-1}(X) \\ x \end{pmatrix} \right\rangle.$$

This definition of the 4-product differs by the sign (-1) from the definition in **[MJ]** and by the sign $(-1)^{p+r+1}$ from the definition in **[MW1]**.

REMARK. We have generalized Steenrod's functional cup products and defined above three functional triple products, $P_{f,i}(u, v, w)$, $1 \leq i \leq 3$. Proceeding analogously we could define four functional 4-products and so forth for functional higher products, but we will not make use of them in this paper. Additionally, the above definitions could be given for matrices of elements.

3. Properties. In this section we will present properties of the Massey triple product and 4-product and of the Steenrod functional cup products and functional triple products. There will usually be a property stated for a lower order product followed by a similar property for the next higher order product in order to give an indication of how the generalization to even higher order products might proceed. Yet because most of the proofs are straightforward, only a few proofs will be given.

PROPOSITION 3.1. *The Massey triple product and 4-product and the Steenrod functional cup products and functional triple products are natural.*

PROPOSITION 3.2a. *Let u and v be homogeneous elements in $H^*(X;\ R)$. Assume $uv = 0$, $f^*(u) = 0$, and $f^*(v) = 0$. Then $P_{f,1}(u, v)$ and $P_{f,2}(u, v)$ are both defined and*

$$\overline{P_{f,1}(u, v)} = P_{f,2}(u, v).$$

PROPOSITION 3.2b. *Let u, v and w be homogeneous elements in $H^*(X;\ R)$. Assume that $\langle u, v, w\rangle$ is defined, $f^*(u) = 0$, $f^*(w) = 0$, and that $P_{f,1}(u, v)$, $\langle u, v, w\rangle$ and $P_{f,2}(v, w)$ vanish simultaneously. Then $P_{f,1}(u, v, w)$ and*

$P_{f,3}(u, v, w)$ are defined and

$$\overline{P_{f,1}(u, v, w)} = P_{f,3}(u, v, w)$$

modulo the sum of the indeterminacies.

The next propositions relate f^* of a Massey product with Steenrod functional products of lower orders.

PROPOSITION 3.3a. *Let u, v and w be homogeneous elements in $H^*(X; R)$. Assume that $uv = 0$, $vw = 0$ and $f^*(v) = 0$. Then $\langle u, v, w\rangle$ is defined and*

$$f^*(\langle u, v, w\rangle) = P_{f,2}(u, v)f^*(w) + f^*(\bar{u})P_{f,1}(v, w)$$

modulo $f^*(u \cdot H^*(X) + H^*(X) \cdot w) + f^*(u) \cdot H^*(Y) \cdot f^*(w)$.

PROPOSITION 3.3b. *Let u, v, w and x be homogeneous elements in $H^*(X; R)$. Assume $f^*(v) = 0$, $f^*(w) = 0$ and $\langle u, v, w\rangle$, $P_{f,1}(v, w)$, $P_{f,2}(v, w)$ and $\langle v, w, x\rangle$ vanish simultaneously. Then $\langle u, v, w, x\rangle$ is defined and*

$$f^*(\langle u, v, w, x\rangle) = P_{f,3}(u, v, w)f^*(x) + P_{f,2}(u, v)P_{f,1}(w, x) + f^*(\bar{u})P_{f,1}(v, w, x)$$

modulo the sum of the indeterminacies.

PROOF. Choose representative cocycles u', v', w' and x' for u, v, w and x respectively, and choose cochains a^{12}, a^{23}, a^{34}, a^{f2} and a^{f3} such that

$$\delta(a^{12}) = \overline{u'}\, v', \quad \delta(a^{34}) = \overline{w'}\, x', \quad \delta(a^{f3}) = f^{\#}(w'),$$
$$\delta(a^{23}) = \overline{v'}\, w', \quad \delta(a^{f2}) = f^{\#}(v').$$

Since $\langle u, v, w\rangle$, $P_{f,1}(v, w)$, $P_{f,2}(v, w)$ and $\langle v, w, x\rangle$ vanish simultaneously we may select cochains a^{13}, a^{f23}, a^{23f} and a^{24} such that

$$\delta(a^{13}) = \overline{a^{12}}\, w' + \overline{u'}\, a^{23}, \qquad \delta(a^{24}) = \overline{a^{23}}\, x' + \overline{v'}\, a^{34},$$
$$\delta(a^{f23}) = f^{\#}(a^{23}) + \overline{a^{f2}}\, f^{\#}(w'), \qquad \delta(a^{23f}) = f^{\#}\left(\overline{a^{23}}\right) - f^{\#}\left(\overline{v'}\right)\overline{a^{f3}}.$$

Then $P_{f,3}(u, v, w)f^*(x) + P_{f,2}(u, v)P_{f,1}(w, x) + f^*(\bar{u})P_{f,1}(v, w, x)$ is represented by

$$\begin{aligned}
&\left[f^{\#}\left(\overline{a^{13}}\right) + f^{\#}\left(\overline{u'}\right)a^{23f} - f^{\#}\left(\overline{a^{12}}\right)\overline{a^{f3}}\right]f^{\#}(x') \\
&\quad + \left[f^{\#}\left(\overline{a^{12}}\right) - f^{\#}\left(\overline{u'}\right)\overline{a^{f2}}\right]\cdot\left[f^{\#}(a^{34}) + \overline{a^{f3}}\, f^{\#}(x')\right] \\
&\quad + f^{\#}\left(\overline{u'}\right)\left[f^{\#}(a^{24}) + \overline{a^{f2}}\, f^{\#}(a^{34}) + \overline{a^{f23}}\, f^{\#}(x')\right] \\
&= f^{\#}\left(\overline{a^{13}}\, x' + \overline{a^{12}}\, a^{34} + \overline{u'}\, a^{24}\right) \\
&\quad + f^{\#}\left(\overline{u'}\right)\left[a^{23f} + \overline{a^{f23}} - \overline{a^{f2}a^{f3}}\right]f^{\#}(x')
\end{aligned}$$

where the first term is a representative of $f^*(\langle u, v, w, x\rangle)$ and the second term belongs to $f^*(u)H^{q+r-2}(Y)f^*(x)$ which is in the indeterminacy.

The last propositions give conditions when Massey products are in the kernel of f^*.

PROPOSITION 3.4a. *Let u, v, and w be homogeneous elements in $H^*(X; R)$. Assume that $uv = 0$, $vw = 0$, $f^*(u) = 0$ and $f^*(w) = 0$. Then the coset $\langle u, v, w\rangle$ is defined and*

$$f^*(\langle u, v, w\rangle) = 0.$$

PROPOSITION 3.4b. *Let u, v, w and x be homogeneous elements in $H^*(X; R)$. Assume that $f^*(u) = 0, f^*(x) = 0$ and $P_{f,1}(u, v)$, $\langle u, v, w\rangle$ and $\langle v, w, x\rangle$ vanish simultaneously. Then $\langle u, v, w, x\rangle$ is defined and $f^*(\langle u, v, w, x\rangle) = 0$.*

In the statement of this proposition we could replace the hypothesis that $P_{f,1}(u, v)$, $\langle u, v, w\rangle$ and $\langle v, w, x\rangle$ vanish simultaneously by the hypothesis $P_{f,2}(w, x)$, $\langle u, v, w\rangle$ and $\langle v, w, x\rangle$ vanish simultaneously, and we would obtain the same conclusion.

PROOF. Choose representative cocycles u', v', w' and x' for u, v, w and x and choose cochains a^{12}, a^{23}, a^{34} as in the proof of Proposition 3.3b. Also select cochains a^{f1} and a^{f4} so that $\delta(a^{f1}) = f^{\#}(u')$ and $\delta(a^{f4}) = f^{\#}(x')$. Since $P_{f,1}(u, v)$, $\langle u, v, w\rangle$ and $\langle v, w, x\rangle$ vanish simultaneously, we may select cochains a^{f12}, a^{13} and a^{24} so that

$$\delta(a^{13}) = \overline{a^{12}}\,w' + \overline{u'}\,a^{23}, \quad \delta(a^{24}) = \overline{a^{23}}\,x' + \overline{v'}\,a^{34},$$
$$\delta(a^{f12}) = f^{\#}(a^{12}) + \overline{a^{f1}}\,f^{\#}(v').$$

Then $\langle u, v, w, x\rangle$ is represented by

$$z' = \overline{a^{13}}\,x' + \overline{a^{12}}\,a^{34} + \overline{u'}\,a^{24}$$

and if we let

$$z'' = f^{\#}(a^{13})a^{f4} - \overline{a^{f12}}\,f^{\#}(a^{34}) + \overline{a^{f12}}\,f^{\#}(w')a^{f4}$$
$$- \overline{a^{f1}}\,f^{\#}(a^{24}) + \overline{a^{f1}}\,f^{\#}(a^{23})a^{f4}$$

we can show that $\delta(z'') = f^{\#}(z')$ and we are finished.

4. New relation on 4-products. In this section we will introduce a new relation on 4-products. We will need the notion of a strictly defined 4-product. (See [**MJ**, p. 538].)

DEFINITION 4.1. A 4-product $\langle w_1, w_2, w_3, w_4\rangle$ is said to be strictly defined if $\langle w_1, w_2, w_3\rangle$ and $\langle w_2, w_3, w_4\rangle$ are defined and contain only the zero element.

In the theorem we need to assume that the 4-products are strictly defined because in our proof we need the vanishing of certain triple products after specially selected cochains have been used to obtain a cocycle representative of them. This cocycle representative need not be a coboundary in general, if the 4-products were not strictly defined.

THEOREM 4.2. *Let X be a space such that $C^*(X; R)$ possesses an anticommutative cup product. Then if the given 4-products are strictly defined, in $H^*(X; R)$ we have*

$$0 \in \langle w_1, w_2, w_3, w_4 \rangle + (-1)^{(r+1)(s+1)} \langle w_1, w_2, w_4, w_3 \rangle + (-1)^{pq+pr+qr} \left[\langle w_3, w_2, w_1, w_4 \rangle + (-1)^{(p+1)(s+1)} \langle w_3, w_2, w_4, w_1 \rangle \right]$$

where the degrees of w_1, w_2, w_3 and w_4 are p, q, r and s respectively.

It seems reasonable to conjecture that the relation in this theorem should hold true for an arbitrary coefficient ring in general, without the assumption of the anticommutativity of the cochains, but using their homotopy commutativity.

Note that the assumptions on X are satisfied if: (1). X is a manifold and we use anti-commutative deRham cochains with coefficients in $\mathbf{R}$: (2) X is a finite complex, since we can embed X in a Euclidean space as a deformation retract of a manifold; and (3) X is a simply connected, pointed space, because Quillen in [Q] obtains the existence of a cohomology theory with rational coefficients where the cochains are anticommutative.

PROOF OF THEOREM 4.2. Since the 4-products are defined, we may select cocycles w_i' representing the cohomology classes w_i, $1 \leqslant i \leqslant 4$, and cochains a_{12}, a_{23}, a_{34}, a_{14} and a_{24} such that

$$\delta(a_{12}) = \overline{w_1'}\, w_2', \quad \delta(a_{23}) = \overline{w_2'}\, w_3', \quad \delta(a_{34}) = \overline{w_3'}\, w_4',$$
$$\delta(a_{14}) = \overline{w_1'}\, w_4', \quad \delta(a_{24}) = \overline{w_2'}\, w_4'.$$

Since the cochains are anticommutative,

$$\delta\left((-1)^{p(q+1)+q} a_{12}\right) = \overline{w_2'}\, w_1', \qquad \delta\left((-1)^{q(r+1)+r} a_{23}\right) = \overline{w_3'}\, w_2',$$
$$\delta\left((-1)^{r(s+1)+s} a_{34}\right) = \overline{w_4'}\, w_3', \qquad \delta\left((-1)^{p(s+1)+s} a_{14}\right) = \overline{w_4'}\, w_1'.$$

Since all the 4-products are strictly defined, we may select cochains a_{ijk} such that $\delta(a_{ijk}) = \overline{w_i'}\, a_{jk} + \overline{a_{ij}}\, w_k'$, where $(i, j, k) = (1, 2, 3)$, $(2, 3, 4)$ and $(1, 2, 4)$. Also cochains a_{243}, a_{214}, a_{324} and a_{241} can be found such that

$$\delta(a_{243}) = \overline{w_2'}\,(-1)^{r(s+1)+s} a_{34} + \overline{a_{24}}\, w_3',$$
$$\delta(a_{214}) = \overline{w_2'}\, a_{14} + (-1)^{p(q+1)+q}\, \overline{a_{12}}\, w_4',$$
$$\delta(a_{324}) = \overline{w_3'}\, a_{24} + (-1)^{q(r+1)+r}\, \overline{a_{23}}\, w_4',$$
$$\delta(a_{241}) = \overline{w_2'}\,(-1)^{p(s+1)+s} a_{14} + \overline{a_{24}}\, w_1'.$$

We have by the anticommutativity of the cochains

$$\left((-1)^{pq+pr+qr-1}a_{123}\right) = \overline{w_3'}(-1)^{p(q+1)+q}a_{12} + (-1)^{q(r+1)+r}\overline{a_{23}}\,w_1',$$

where the cocycle on the right hand side represents $\langle w_3, w_2, w_1\rangle$. If we let $z_1' = a_{234} + (-1)^{(r+1)(s+1)}a_{243} + (-1)^{(q+1)(r+1)}a_{324}$ and $z_3' = a_{214} + (-1)^{(p+1)(q+1)}a_{124} + (-1)^{(p+1)(s+1)}a_{241}$, we note that $\delta(z_1') = \delta(z_3') = 0$.

The sum of the 4-products we are interested in can be represented by the cochain

$$\begin{aligned}
&\overline{w_1'}(a_{234} - z_1') + \overline{a_{12}}\,a_{34} + \overline{a_{123}}\,w_4' + (-1)^{(r+1)(s+1)}\\
&\quad\cdot\left(\overline{w_1'}\,a_{243} + \overline{a_{12}}(-1)^{r(s+1)+s}a_{34} + \overline{a_{124}}\,w_3'\right)\\
&\quad + (-1)^{pq+pr+qr}\Big[\overline{w_3'}(a_{214} - z_3') + (-1)^{q(r+1)+r}\overline{a_{23}}\,a_{14}\\
&\qquad\qquad\qquad\qquad + (-1)^{pq+pr+qr-1}\overline{a_{123}}\,w_4'\Big]\\
&\quad + (-1)^{pq+pr+qr+(p+1)(s+1)}\\
&\quad\cdot\left(\overline{w_3'}\,a_{241} + (-1)^{q(r+1)+r}\overline{a_{23}}(-1)^{p(s+1)+s}a_{14} + \overline{a_{324}}\,w_1'\right).
\end{aligned}$$

This simplifies to become

$$\begin{aligned}
&\overline{w_1'}\left[(a_{234} - z_1') + (-1)^{(r+1)(s+1)}a_{243} + (-1)^{(q+1)(r+1)}a_{324}\right]\\
&\qquad + (-1)^{pq+pr+qr}\overline{w_3'}\Big[(a_{214} - z_3') + (-1)^{(p+1)(q+1)}a_{124}\\
&\qquad\qquad\qquad + (-1)^{(p+1)(s+1)}a_{241}\Big]
\end{aligned}$$

which is zero, using the definitions of z_1' and z_3'. Thus the relation is established.

Remark. In the next section we will apply this theorem to study the 4-products in the complement of certain links. Let us assume for example that four spheres S_1, S_2, S_3 and S_4 are embedded in S^n and the linking number of any two of them is zero. We will usually assume that the embeddings are piecewise linear, but that assumption is not necessary here. If $X = S^n - \bigcup_{i=1}^4 S_i$, then X is an open manifold with anticommutative deRham cochains with real coefficients. If the 4-products are defined in $H^*(X; \mathbf{R})$ they will be strictly defined since all cup products vanish because the linking number of any two of the spheres is assumed to be zero. Thus Theorem 4.2 gives us a relation in $H^*(X; \mathbf{R})$. We will see in §5 that this relation will yield a relation in $H^*(X; \mathbf{Z})$.

5. Links of four spheres in S^n. We will apply Theorem 4.2 to study the 4-products in the complement of a link of four spheres in S^n. Unless otherwise stated, cohomology will be with integer coefficients.

Let S_i, $1 \leqslant i \leqslant 4$, be disjoint oriented spheres embedded piecewise linearly in S^n, where $\dim S_i = p_i$, $1 \leq p_i \leq n-2$, let the linking number of any two of the spheres be zero, and let

$$\sum_{i=1}^{4} p_i = 3n - 5. \tag{5.1}$$

Let $X = S^n - \bigcup_{i=1}^4 S_i$ and let $w_i \in H^{q_i}(X)$, $q_i = n - p_i - 1$, $1 \leq i \leq 4$, be the cohomology class which is the Alexander dual of the fundamental class of the sphere S_i appropriately oriented. By the Alexander Duality Theorem $H^{n-1}(X)$ is a free abelian group of rank three, while in dimensions between 0 and $n-1$ the cohomology of X is freely generated by the classes w_i, $1 \leq i \leq 4$. Since the linking number of the spheres is zero, the cup products $w_i \cup w_j$ are all 0. Therefore all possible triple products $\langle w_i, w_j, w_k\rangle$ are defined without indeterminacy. We will also assume that these triple products vanish. Because the indeterminacy is zero, the 4-products $\langle w_i, w_j, w_k, w_l\rangle$ are strictly defined for any permutation (i, j, k, l) of the integers $(1, 2, 3, 4)$. It follows from equation (5.1) that $\Sigma_{i=1}^4 q_i = n+1$, and thus $\langle w_i, w_j, w_k, w_l\rangle \in H^{n-1}(X)$.

The indeterminacy of $\langle w_i, w_j, w_k, w_l\rangle$ can be written according to Proposition 2.4 in [MJ] as

$$\bigcup_{(x_1,x_2,x_3)} (\langle x_1, w_k, w_l\rangle + \langle w_i, x_2, w_l\rangle + \langle w_i, w_j, x_3\rangle) \tag{5.2}$$

where $x_1 \in H^{q_i+q_j-1}(X)$, $x_2 \in H^{q_j+q_k-1}(X)$ and $x_3 \in H^{q_k+q_l-1}(X)$. Thus the indeterminacy is zero.

There are 24 possible 4-products with all four classes appearing corresponding to the 24 different permutations of the integers $(1, 2, 3, 4)$. According to Theorems 8 and 10 in [K], these 4-products are subject to the following two relations:

$$\langle w_i, w_j, w_k, w_l\rangle = (-1)^h \langle w_l, w_k, w_j, w_i\rangle \tag{5.3}$$

where $h = q_i(q_j + q_k + q_l) + q_j(q_k + q_l) + q_k q_l + q_i + q_j + q_k + q_l + 1$, and

$$(-1)^\alpha \langle w_i, w_j, w_k, w_l\rangle + (-1)^\beta \langle w_j, w_k, w_l, w_i\rangle \\ + (-1)^\gamma \langle w_k, w_l, w_i, w_j\rangle + (-1)^\delta \langle w_l, w_i, w_j, w_k\rangle = 0 \tag{5.4}$$

where $\alpha = q_i + q_k + 1$, $\beta = q_i(q_j + q_k + q_l) + q_j + q_l$, $\gamma = (q_i + q_j)(q_k + q_l) + q_i + q_k + 1$, and $\delta = (q_i + q_j + q_k)q_l + q_j + q_l$.

Because of the remark after the proof of Theorem 4.2, if we considered cohomology with real coefficients we would also have

$$\langle w_i, w_j, w_k, w_l\rangle + (-1)^{(q_k+1)(q_l+1)} \langle w_i, w_j, w_l, w_k\rangle + (-1)^{q_iq_j + q_iq_k + q_jq_k} \\ \cdot \left[\langle w_k, w_j, w_i, w_l\rangle + (-1)^{(q_i+1)(q_l+1)} \langle w_k, w_j, w_l, w_i\rangle\right] = 0. \tag{5.5}$$

But since $H^*(X;\ \mathbf{Z})$ is torsion-free, the map in the universal coefficient theorem from $H^*(X;\ \mathbf{Z})$ to $H^*(X;\ \mathbf{R})$ is a monomorphism. Since the indeterminacy of the 4-products vanishes, our relation pulls back to a relation in $H^*(X;\ \mathbf{Z})$. So (5.5) is a third relation on the 4-products. Although these three relations will be used shortly when the subscripts on the cohomology classes are all distinct, they also hold with repetitions of subscripts.

In constrast to the result in **[MW2]** concerning the six triple products of different classes in the complement of certain links of three spheres in S^n being determined up to sign by one integer, we will see that in our case the 24 quadruple products are essentially determined by two integers. In order to state the precise results we will require as in **[MW2]** the manifold with boundary which is naturally associated with X.

Since each sphere S_i is embedded piecewise linearly in S^n, we may take an open regular neighborhood U_i of S_i in S^n such that the closures of the U_i are pairwise disjoint and we will denote the boundary of U_i by B_i. Let $U = \bigcup_{i=1}^4 U_i$, $B = \bigcup_{i=1}^4 B_i$ and $M = S^n - U$. Then M is an orientable manifold with boundary B which is a deformation retract of X and whose cohomology is naturally isomorphic to the cohomology of X. We will identify the cohomology rings and investigate the 4-products in $H^*(M)$.

We assume that S^n is oriented and that M has the induced orientation. We orient the components B_i of B by singling out a generator $\mu_i \in H^{n-1}(B_i)$ which corresponds to a generator in $H^{n-1}(B)$ which we denote by the same symbol and which satisfies $\delta(\mu_i) = \mu$, where μ is the orientation class in $H^n(M, B)$.

In the exact sequence

$$H^{n-1}(M) \xrightarrow{g^*} H^{n-1}(B) \xrightarrow{\delta} H^n(M, B) \to 0$$

we have free abelian groups of ranks 3, 4 and 1 respectively. Exactness causes g^* to be a monomorphism and thus any element in $H^{n-1}(M)$ is completely determined by describing its image in $H^{n-1}(B)$ under g^*. We will use the natural basis for $H^{n-1}(B)$ selected above: $\{\mu_1, \mu_2, \mu_3, \mu_4\}$.

PROPOSITION 5.6. *If* (i, j, k, l) *is any permutation of the integers* $(1, 2, 3, 4)$, *then there is an integer* m_{ijkl} *such that*

$$g^*(\langle w_i, w_j, w_k, w_l\rangle) = m_{ijkl}(\mu_i - \mu_l)$$

where μ_t *are the generators of* $H^{n-1}(B)$ *determined above*, $1 \leqslant t \leqslant 4$.

PROOF. Since $\langle w_i, w_j, w_k, w_l\rangle \in H^{n-1}(M)$ we know we can write $g^*(\langle w_i, w_j, w_k, w_l\rangle) = a_1\mu_1 + a_2\mu_2 + a_3\mu_3 + a_4\mu_4$, where $a_t \in \mathbf{Z}$, $1 \leqslant t \leqslant 4$. Since $\delta(\mu_t) = \mu$, $1 \leqslant t \leqslant 4$, and $\delta \circ g^* = 0$, we must have $a_1 + a_2 + a_3 + a_4 = 0$. We must only show that $a_j = 0$ and $a_k = 0$.

Since 4-products are natural, this will follow if we show that

$g_j^*(\langle w_i, w_j, w_k, w_l\rangle) = 0$ and $g_k^*(\langle w_i, w_j, w_k, w_l\rangle) = 0$, where $g_t\colon B_t \to M$ is the inclusion. But applying the result in [**MW2**, p. 182],

$$g_j^*(w_i) = g_j^*(w_l) = 0, \qquad g_k^*(w_i) = g_k^*(w_l) = 0 \tag{5.7}$$

and Proposition 3.4b above, we obtain the desired conclusion.

COROLLARY 5.8. *The* 24 *integers* m_{ijkl} *may be grouped into three collections of eight integers apiece. All the integers in a collection are equal in absolute value. Knowledge of two of the m's, each from a different collection, determines all the other m's. To be precise we have*:

$$m_{ijkl} = (-1)^{t_1} m_{lkji},$$
$$m_{ijkl} = (-1)^{t_2} m_{jkli},$$
$$m_{ijkl} = (-1)^{t_3} m_{ijlk} + (-1)^{t_4} m_{kjli}$$

where $t_1 = q_i(q_j + q_k + q_l) + q_j(q_k + q_l) + q_k q_l + q_i + q_j + q_k + q_l$, $t_2 = q_i(q_j + q_k + q_l) + q_i + q_j + q_k + q_l + 1$, $t_3 = (q_k + 1)(q_l + 1) + 1$, $t_4 = (q_i + 1)(q_l + 1) + 1$, *and* (i, j, k, l) *is any permutation of* $(1, 2, 3, 4)$.

The identities in the corollary follow from relations (5.3), (5.4) and (5.5). We list the three groups of integers mentioned in the corollary:

$$\begin{Bmatrix} m_{1234} & m_{4123} & m_{3412} & m_{2341} \\ m_{4321} & m_{1432} & m_{2143} & m_{3214} \end{Bmatrix}, \quad \begin{Bmatrix} m_{3241} & m_{1324} & m_{4132} & m_{2413} \\ m_{1423} & m_{3142} & m_{2314} & m_{4231} \end{Bmatrix},$$
$$\begin{Bmatrix} m_{1243} & m_{3124} & m_{4312} & m_{2431} \\ m_{3421} & m_{1342} & m_{2134} & m_{4213} \end{Bmatrix}.$$

If we had the values of two of the m's each from a different group above, then all the remaining m's are completely determined by the identities in the corollary.

This corollary suggests some connection with Milnor's work on almost trivial links in which we obtains two homotopy invariants for an almost trivial link with four components. (See [**MI2**, p. 189 following].) And this also lends support to the conjecture of long standing that the higher products and Milnor's $\bar{\mu}$-invariants are related since the relations among the m's agree with his symmetry formulas (27), (21), and (22) in [**MI1**] in the case of $\bar{\mu}$ of four different indices. R. Porter has obtained some recent results relating the $\bar{\mu}$-invariants and generalized Massey products. (See [**P**].)

It would be nice if after assigning arbitrary integer values to two of the m's from different collections, we had an example of a link that realized the 4-products that would be completely determined. Unfortunately it is not easy to construct examples and the computation of these invariants is difficult.

We now obtain a result analogous to the result in **[MW2]** which relates certain functional cup products with triple products. We will relate certain functional triple products with quadruple products. In our situation we let

$$X_i = S^n - \bigcup_{\substack{j=1 \\ j \neq i}}^{4} S_j$$

be the complement in S^n of all spheres except S_i, and we will consider the embeddings $f_i\colon S_i \to X_i$, $1 \leqslant i \leqslant 4$.

We will abuse notation and let w_j, w_k and w_l denote the cohomology classes in $H^*(X_i)$ which are the Alexander duals of the fundamental homology classes of the spheres S_j, S_k and S_l respectively. The inclusion $X \to X_i$ will send classes denoted by the same symbol onto themselves, so no confusion should result. Since the linking number of any two spheres is zero, we have $w_j \cup w_k = w_k \cup w_l = 0$. Also since $\langle w_i, w_j, w_k \rangle$ vanishes by assumption, and according to **[MW2**, p. 184] this triple product determines $P_{f_i,1}(w_j, w_k)$, we see that $P_{f_i,1}(w_j, w_k)$ vanishes. Using (5.7) and the naturality of the 4-product, we can show that $f_i^*(w_j) = f_i^*(w_k) = f_i^*(w_l) = 0$. This implies that the indeterminacy of $P_{f_i,1}(w_j, w_k)$ vanishes; the indeterminacy of $\langle w_j, w_k, w_l \rangle$ is zero because cup products are trivial. Thus $P_{f_i,1}(w_j, w_k, w_l)$ is strictly defined.

Proposition 2.4 in **[MJ]** says that the indeterminacy of $P_{f_i,1}(w_j, w_k, w_l)$ can be written as

$$\bigcup_{(x_1,x_2,x_3)} \left(\langle x_1, w_k, w_l \rangle + P_{f_j,1}(x_2, w_k) + P_{f_j,1}(w_j, x_3)\right)$$

where $x_1 \in H^{q_j-1}(S_i)$, $x_2 \in H^{q_j+q_l-1}(X_i)$, $x_3 \in H^{q_k+q_l-1}(X_i)$ and the first triple product is defined in Peter May's setting, Case 1, as mentioned after Definition 2.3c. The triple product $\langle x_1, w_k, w_l \rangle$ does not enter into consideration for dimension reasons unless $q_j - 1 = 0$ when it becomes, using Proposition 2.7 in **[MJ]**, a multiple of the functional product $P_{f_i,1}(w_k, w_l)$ which is zero. The other functional products also vanish and we see that $P_{f_i,1}(w_j, w_k, w_l)$ is defined in $H^{p_i}(S_i)$ without indeterminacy. Note that $q_j + q_k + q_l - 2 = p_i$ by equation (5.1).

To be precise about signs when we describe $P_{f_i,1}(w_j, w_k, w_l)$ in terms of 4-products, we must use the manifold M discussed above and be careful in indicating how the generators w_i, w_j, w_k and w_l in $H^*(M)$ are selected.

Instead of looking at $P_{f_i,1}(w_j, w_k, w_l)$ in $H^{p_i}(S_i)$, we can use the manifold M and look at $P_{g_i,1}(w_j, w_k, w_l)$ in $H^{p_i}(B_i)$ for the following reason. We have a commutative diagram with all homomorphism except f_i^* induced by inclusions:

$$\begin{array}{ccc} H^*(S_i) & \xleftarrow{f_i^*} & H^*(X_i) \\ \approx\uparrow\rho_i^* & & \approx\downarrow\sigma_i^* \\ H^*(\overline{U}_i) & & H^*(M \cup U_i) \\ \text{mono}\downarrow\phi_i^* & & \text{mono}\downarrow\tau_i^* \\ H^*(B_i) & \xleftarrow{g_i^*} & H^*(M) \end{array}$$

where ρ_i^* and τ_i^* are induced by deformation retraction maps and are isomorphisms, ϕ_i^* is a monomorphism onto a direct summand of $H^*(B_i)$ in dimensions less than $n - 1$ by a Meyer-Vietoris argument (see **[MW3**, p. 282]), and τ_i^* is a monomorphism because of naturality and Alexander duality. Thus if $P_{f_i,1}(w_j, w_k, w_l)$ is a multiple of a generator of $H^{p_i}(S_i)$, $P_{g_i,1}(w_j, w_k, w_l)$ is the same multiple of the corresponding generator of $H^{p_i}(B_i)$. We have abused notation and denoted the images of w_j, w_k, w_l in $H^*(M)$ under $\tau_i^* \circ \sigma_i^*$ by the same symbols, but no confusion should result.

We select the generator $w_i \in H^{q_i}(M)$ in the following way. (w_j, w_k and w_l are selected in a similar manner.) Since S_i is oriented we have a distinguished generator $t_i \in H^{p_i}(S_i)$ which determines a generator $u_i \in H^{p_i}(\overline{U}_i)$ because S_i is a deformation retract of $\overline{U}_i$. Alexander duality shows that $H^{q_i}(S^n - U_i)$ is an infinite cyclic group and a Meyer-Vietoris argument gives that $H^*(B_i)$ is generated in degrees $< n - 1$ by Image ϕ_i^* and Image ψ_i^*, where ϕ_i: $B_i \to \overline{U}_i$ and ψ_i: $B_i \to S^n - U_i$ are the inclusions. Poincaré duality gives us that $\phi_i^*(u_i) \cup \psi_i^*(v_i) = \mu_i$, where μ_i is our generator of $H^{n-1}(B_i)$ and $v_i \in H^{q_i}(S^n - U_i)$ is a generator chosen to satisfy this equation. We select w_i to be $\eta_i^*(v_i)$ where η_i: $M \to S^n - U_i$ is the inclusion. Note that $\psi_i^*(v_i) = g_i^*(w_i)$ because the following diagram induced by inclusion maps is commutative:

$$\begin{tikzcd} H^*(S^n - U_i) \arrow[rr, "\eta_i^*"] \arrow[dr, "\psi_i^*"'] & & H^*(M) \arrow[dl, "g_i^*"] \\ & H^*(B_i) & \end{tikzcd}$$

Now with the above conventions about the selection of w_i, w_j, w_k and w_l in $H^*(M)$ and using w_j, w_k and w_l to denote the corresponding classes in $H^*(X_i)$, we have the following proposition.

PROPOSITION 5.9. *The functional triple product* $P_{f_i,1}(w_j, w_k, w_l)$ *is equal to* $(-1)^{q_i(p_i+1)} m_{ijkl} t_i$, *where* t_i *is the generator in* $H^{p_i}(S_i)$ *mentioned above and* m_{ijkl} *is the integer of Proposition* 5.6 *determined by the* 4*-product* $\langle w_i, w_j, w_k, w_l \rangle$. *This holds for any permutation* (i, j, k, l) *of integers* (1, 2, 3, 4).

PROOF. By the comments preceding the proposition, it suffices to prove that $P_{g_i,1}(w_j, w_k, w_l) = (-1)^{q_i(p_i+1)} m_{ijkl}\phi_i(u_i)$ where $\phi_i(u_i)$ is our generator of $H^{p_i}(B_i)$ and $w_j, w_k, w_l \in H^*(M)$.

Since $g_i^*(w_j) = g_i^*(w_k) = 0$ by (5.7), we can apply Proposition 3.3b and obtain

$$g_i^*(\langle w_i, w_j, w_k, w_l\rangle) = P_{g_i,3}(w_i, w_j, w_k)\, g_i^*(w_l) + P_{g_i,2}(w_i, w_j) P_{g_i,1}(w_k, w_l) + g_i^*(\overline{w_i}) P_{g_i,1}(w_j, w_k, w_l).$$

But $g_i^*(w_l) = 0$ and $P_{g_i,1}(w_k, w_l)$ is determined by $\langle w_i, w_k, w_l\rangle$ which we have assumed to vanish. So we obtain

$$g_i^*(\langle w_i, w_j, w_k, w_l\rangle) = g_i^*(\overline{w_i}) P_{g_i,1}(w_j, w_k, w_l).$$

But $g_i^*(\langle w_i, w_j, w_k, w_l\rangle) = m_{ijkl}\mu_i$ and $g_i^*(\overline{w_i}) = \psi_i^*(\overline{v_i})$. So

$$m_{ijkl}\mu_i = \psi_i^*(\overline{v_i}) \cup P_{g_i,1}(w_j, w_k, w_l).$$

Using the fact that $\phi_i^*(u_i) \cup \psi_i^*(v_i) = \mu_i$, we obtain $P_{g_i,1}(w_j, w_k, w_l) = (-1)^{q_i(p_i+1)} m_{ijkl}\phi_i(u_i)$ as desired.

COROLLARY 5.10. *The 4-products $\langle w_i, w_j, w_k, w_l\rangle$ are invariant under homotopies of the embedding maps in the complement of the other spheres, where (i, j, k, l) is any permutation of the integers* (1, 2, 3, 4).

PROOF. If we replaced the embedding f_i by an embedding $\tilde{f}_i$ in X_i homotopic to it, we would have $P_{f_i,1}(w_j, w_k, w_l) = P_{\tilde{f}_i,1}(w_j, w_k, w_l)$ and the integer m_{ijkl} would not change. Similarly we would have $P_{f_i,1}(w_l, w_j, w_k) = P_{\tilde{f}_i,1}(w_l, w_j, w_k)$ and the integer m_{iljk} would not change. But these two integers suffice to determine all 4-products by Corollary 5.8 and we would have all 4-products invariant.

In [**MI2**, p. 177] Milnor defines homotopic links of circles in S^3 and our corollary implies that the 4-products with four different generators in the complementary space of such links would be invariant under homotopies of links.

It should be noted that the assumption in this section that the linking number of any two of the spheres is zero could be relaxed somewhat. If we used as coefficients the integers modulo the greatest common divisor of all the linking numbers, most of the considerations of this section would apply.

6. Links of three circles in S^3. We will now consider certain links of circles in S^3 with three components S_1, S_2 and S_3. All cohomology will be with integer coefficients.

Let $X = S^3 - \bigcup_{i=1}^3 S_i$ and let $w_i \in H^1(X)$ be the cohomology class which is the Alexander dual of the fundamental class of the circle S_i appropriately oriented, $1 \leqslant i \leqslant 3$. The linking number of any two of the circles we assume to be zero and the circles will be embedded piecewise linearly. By Theorem 3.1 in **[MW2]** we know that the triple products $\langle w_i, w_j, w_k\rangle \in H^2(X)$ are determined by a single integer, where (i, j, k) is any permutation of (1, 2, 3). We assume that this integer is zero.

Proposition 6.1 in **[MW2]** and naturality of triple products gives $\langle w_i, w_i, w_j\rangle = \langle w_i, w_j, w_i\rangle = \langle w_j, w_i, w_i\rangle = 0$, where one cohomology class is repeated, and the Jacobi identity gives $\langle w_i, w_i, w_i\rangle = 0$, $1 \leqslant i \neq j \leqslant 3$. Thus all triple products vanish and their indeterminacy is zero. We shall investigate the 4-products which are strictly defined in $H^2(X)$.

The indeterminacy of the 4-products can be described by using (5.2) above, and since all triple products vanish, the indeterminacy is zero.

There are thirty-six 4-products with all three cohomology classes w_1, w_2 and w_3 present. We shall see that the six of these of the form $\langle w_i, w_j, w_k, w_i\rangle$, where (i, j, k) is any permutation of (1, 2, 3), must be zero. The remaining thirty 4-products are completely determined by the relations (5.3), (5.4) and (5.5) and knowledge of the six 4-products of the form $\langle w_i, w_i, w_j, w_k\rangle$, where (i, j, k) is any permutation of (1, 2, 3).

As above in §5 before Proposition 5.6 we must introduce a manifold M with boundary B which is a deformation retract of X. We identify $H^*(X)$ and $H^*(M)$ and study the 4-products in $H^*(M)$.

We assume S^3 is oriented and that M has the induced orientation. We orient the three components B_i of B by selecting a generator $\mu_i \in H^2(B_i)$ which corresponds to a generator in $H^2(B)$ which we denote by the same symbol and which satisfies $\delta(\mu_i) = \mu$, where μ is the orientation class in $H^3(M, B)$.

In the exact sequence

$$H^2(M) \xrightarrow{g^*} H^2(B) \xrightarrow{\delta} H^3(M, B) \to 0$$

we have free abelian groups of ranks 2, 3 and 1 respectively and g^* is again a monomorphism.

Proposition 6.1. *If* (i, j, k) *is any permutation of* (1, 2, 3), *we have in* $H^2(M)$, $\langle w_i, w_j, w_k, w_i\rangle = 0$.

Proof. Since g^* is a monomorphism, we need only show that $g^*(\langle w_i, w_j, w_k, w_i\rangle) = 0$. But $g^*(\langle w_i, w_j, w_k, w_i\rangle) = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$, where $a_t \in \mathbf{Z}$, $1 \leqslant t \leqslant 3$. We will show $a_1 = a_2 = a_3 = 0$.

If we let $g_t\colon B_t \to M$ be the inclusion map, $t = j, k$, then (5.7) would give us $g_j^*(w_i) = g_k^*(w_i) = 0$ and we can apply Proposition 3.4b to obtain

$$g_j^*(\langle w_i, w_j, w_k, w_i\rangle) = g_k^*(\langle w_i, w_j, w_k, w_i\rangle) = 0.$$

Thus by naturality we have $a_j = a_k = 0$. But $\delta \circ g^* = 0$ implies $a_i = 0$, as desired.

We now show that the remaining thirty 4-products are determined by relations (5.3), (5.4) and (5.5) and the six 4-products $\langle w_i, w_i, w_j, w_k\rangle$ where (i, j, k) runs over the permutations of (1, 2, 3).

Applying relation (5.5) we have

$$\langle w_i, w_i, w_j, w_k\rangle + \langle w_i, w_i, w_k, w_j\rangle - \langle w_j, w_i, w_i, w_k\rangle - \langle w_j, w_i, w_k, w_i\rangle = 0$$

and

$$\langle w_j, w_i, w_k, w_i\rangle + \langle w_j, w_i, w_i, w_k\rangle - \langle w_k, w_i, w_j, w_i\rangle - \langle w_k, w_i, w_i, w_j\rangle = 0.$$

Applying relation (5.4) we have

$$\langle w_i, w_i, w_j, w_k\rangle + \langle w_i, w_j, w_k, w_i\rangle + \langle w_j, w_k, w_i, w_i\rangle + \langle w_k, w_i, w_i, w_j\rangle = 0.$$

Adding the above relations and using Proposition 6.1 we obtain

$$2\langle w_i, w_i, w_j, w_k\rangle + \langle w_i, w_i, w_k, w_j\rangle + \langle w_j, w_k, w_i, w_i\rangle - \langle w_k, w_i, w_j, w_i\rangle = 0.$$

But (5.3) gives $\langle w_i, w_i, w_k, w_j\rangle + \langle w_j, w_k, w_i, w_i\rangle = 0$ and we obtain

$$\langle w_k, w_i, w_j, w_i\rangle = 2\langle w_i, w_i, w_j, w_k\rangle. \tag{6.2}$$

Now if we apply (5.5) we obtain

$$\langle w_j, w_i, w_i, w_k\rangle + \langle w_j, w_i, w_k, w_i\rangle - \langle w_i, w_i, w_j, w_k\rangle - \langle w_i, w_i, w_k, w_j\rangle = 0.$$

By using (6.2) this becomes

$$\langle w_j, w_i, w_i, w_k\rangle = \langle w_i, w_i, w_j, w_k\rangle - \langle w_i, w_i, w_k, w_j\rangle. \tag{6.3}$$

We summarize the above in the following proposition.

PROPOSITION 6.4. *In $H^*(X)$ all thirty-six 4-products which contain all three cohomology classes are determined by the six 4-products $\langle w_i, w_i, w_j, w_k\rangle$, where (i, j, k) runs over the permutations of* (1, 2, 3), *and the following relations*:

$$\langle w_i, w_j, w_k, w_i\rangle = 0,$$
$$\langle w_k, w_i, w_j, w_i\rangle = 2\langle w_i, w_i, w_j, w_k\rangle,$$
$$\langle w_j, w_i, w_i, w_k\rangle = \langle w_i, w_i, w_j, w_k\rangle - \langle w_i, w_i, w_k, w_j\rangle$$

and the relation $\langle w_\alpha, w_\beta, w_\gamma, w_\delta\rangle = -\langle w_\delta, w_\gamma, w_\beta, w_\alpha\rangle$, where (i, j, k) is any permutation of (1, 2, 3).

The situation in which we have 4-products with only two classes present will be treated at the end of §7. And (5.3) yields $\langle w_i, w_i, w_i, w_i\rangle = 0$, $1 < i < 3$.

If we considered the case of four circles linked in S^3 under the assumptions in §5, the discussion above would apply to the 4-products which contained only three classes. Thus if we knew the twenty-four 4-products $\langle w_i, w_i, w_j, w_k\rangle$, where now (i, j, k) runs over all possible triples of distinct integers taken from $\{1, 2, 3, 4\}$, and we used the identities and relation in Proposition 6.4, all of the one hundred and forty-four possible 4-products which contain three different classes would be determined.

By looking at $X_l = S^3 - (S_i \cup S_j \cup S_k)$, $l \notin \{i, j, k\}$, and the natural inclusion $X \to X_l$, we could show that the 4-product $\langle w_i, w_i, w_j, w_k \rangle$ in $H^2(X)$ is independent of how circle S_l is embedded in the complement of the other three circles, so long as the linking number of S_l with each of the circles is zero and all triple products vanish.

7. Links of two circles in S^3. We will now consider certain links of circles in S^3 with two components S_1 and S_2, where the linking number of S_1 and S_2 is zero. All cohomology will be with integer coefficients.

Let S_1 and S_2 be embedded in S^3 piecewise linearly, let $X = S^3 - (S_1 \cup S_2)$, and let $w_1, w_2 \in H^1(X)$ be the Alexander duals of the fundamental classes of the circles S_1 and S_2 appropriately oriented. Then all triple products of the form $\langle w_i, w_i, w_j \rangle$, $\langle w_i, w_j, w_i \rangle$ and $\langle w_j, w_i, w_i \rangle$, $1 \leqslant i \neq j \leqslant 2$, are zero by Proposition 6.1 in **[MW2]**. The Jacobi identity for triple products gives that $\langle w_1, w_1, w_1 \rangle = \langle w_2, w_2, w_2 \rangle = 0$. Thus for any 2-component link in S^3 where the linking number of the two circles is zero, all triple products vanish without indeterminacy since all cup products are zero. We shall investigate the 4-products which are strictly defined.

Using (5.2) above we see that the indeterminacy of the 4-products can be described in terms of triple products and therefore vanishes.

Sixteen 4-products can be formed using w_1 and w_2 and we will show that twelve of these are forced to be zero. The four remaining 4-products are completely determined by relations (5.3) and (5.5) and knowledge of either $\langle w_1, w_1, w_2, w_2 \rangle$ or $\langle w_2, w_2, w_1, w_1 \rangle$.

Using relation (5.3) we obtain that the following 4-products are zero.

$$\begin{aligned} \langle w_1, w_1, w_1, w_1 \rangle &= 0, \qquad \langle w_2, w_2, w_2, w_2 \rangle = 0, \\ \langle w_1, w_2, w_2, w_1 \rangle &= 0, \qquad \langle w_2, w_1, w_1, w_2 \rangle = 0. \end{aligned} \tag{7.1}$$

As before Proposition 5.6 we consider a manifold M with boundary $B = B_1 \cup B_2$ which is a deformation retract of X. We identify $H^*(X)$ and $H^*(M)$ and orient the components B_i of B by selecting a generator $\mu_i \in H^2(B_i)$ which corresponds to a generator in $H^2(B)$ which we denote by the same symbol and which satisfies $\delta(\mu_i) = \mu$, where μ is the orientation class in $H^3(M, B)$.

In the exact sequence

$$H^2(M) \xrightarrow{g^*} H^2(B) \xrightarrow{\delta} H^3(M, B) \to 0$$

we have free abelian groups of ranks 1, 2 and 1 respectively, and g^* is a monomorphism. If g_1: $B_1 \to M$ is the inclusion map, (5.7) would give us $g_1^*(w_2) = 0$. Proposition 3.4b would yield

$$g_1^*(\langle w_2, w_1, w_2, w_2 \rangle) = 0.$$

But if $g^*(\langle w_2, w_1, w_2, w_2\rangle) = a_1\mu_1 + a_2\mu_2$, the above would imply $a_1 = 0$ and exactness would then give $a_2 = 0$. Therefore

$$\langle w_2, w_1, w_2, w_2\rangle = 0.$$

Similarly,

$$\langle w_1, w_2, w_1, w_1\rangle = 0.$$

And using (5.3),

$$\begin{aligned}\langle w_2, w_2, w_1, w_2\rangle &= 0,\\ \langle w_1, w_1, w_2, w_1\rangle &= 0.\end{aligned} \tag{7.2}$$

Applying relation (5.5) we obtain

$$\langle w_2, w_1, w_1, w_1\rangle + \langle w_2, w_1, w_1, w_1\rangle - \langle w_1, w_1, w_2, w_1\rangle - \langle w_1, w_1, w_1, w_2\rangle = 0.$$

But by (7.2) and (5.3) we obtain $3\langle w_2, w_1, w_1, w_1\rangle = 0$. Therefore

$$\langle w_2, w_1, w_1, w_1\rangle = 0.$$

Similarly,

$$\langle w_1, w_2, w_2, w_2\rangle = 0.$$

And by (5.3),

$$\begin{aligned}\langle w_1, w_1, w_1, w_2\rangle &= 0,\\ \langle w_2, w_2, w_2, w_1\rangle &= 0.\end{aligned} \tag{7.3}$$

We have shown that twelve of the sixteen 4-products are zero. Now (5.5) can be applied to obtain

$$\langle w_1, w_2, w_2, w_1\rangle + \langle w_1, w_2, w_1, w_2\rangle - \langle w_2, w_2, w_1, w_1\rangle - \langle w_2, w_2, w_1, w_1\rangle = 0.$$

But (7.1) gives $\langle w_1, w_2, w_2, w_1\rangle = 0$ and we have

$$\langle w_1, w_2, w_1, w_2\rangle = 2\langle w_2, w_2, w_1, w_1\rangle,$$

and by (5.3)

$$\langle w_2, w_1, w_2, w_1\rangle = 2\langle w_1, w_1, w_2, w_2\rangle. \tag{7.4}$$

We summarize the above in the following proposition.

PROPOSITION 7.5. *In $H^*(X)$ there are only four possible nontrivial 4-products which contain two cohomology classes and these are determined by a knowledge of either class of the form $\langle w_i, w_i, w_j, w_j\rangle$ and the following relations*:

$$\langle w_j, w_i, w_j, w_i\rangle = 2\langle w_i, w_i, w_j, w_j\rangle$$

and the relation

$$\langle w_\alpha, w_\beta, w_\gamma, w_\delta \rangle = -\langle w_\delta, w_\gamma, w_\beta, w_\alpha \rangle,$$

where $(i, j) = (1, 2)$ *or* $(2, 1)$.

If we considered the case of three circles linked in S^3 as in §6, the discussion above would apply to the 4-products containing only two classes. Thus all forty-two of these 4-products would be completely determined by the identity and relation in Proposition 7.5 and knowledge of say the three 4-products $\langle w_1, w_1, w_2, w_2 \rangle$, $\langle w_1, w_1, w_3, w_3 \rangle$ and $\langle w_2, w_2, w_3, w_3 \rangle$.

If we considered $X_k = S^3 - (S_i \cup S_j)$, $k \notin \{i, j\}$, and the natural inclusion $X \to X_k$, we could show that the 4-product $\langle w_i, w_i, w_j, w_j \rangle$ in $H^2(X)$ is completely independent of how S_k is embedded in the complement of the other two circles, so long as the linking number of S_k with each of the circles is zero and the triple product $\langle w_i, w_j, w_k \rangle$ vanishes.

In §§6 and 7, we have treated 4-products where cohomology classes corresponding to the circles of the link have been repeated, but we have not been able to obtain a result similar to the one in Corollary 5.10. It seems reasonable to conjecture that these 4-products with repetition among the classes are not homotopy invariants of the link.

8. Examples. In this final section we will present examples where higher order products, both ordinary and matrix, are used to detect linking in S^3. The first example will present four circles embedded in S^3 in a manner similar to that of the Borromean rings; every subcollection of three circles is not linked, yet the whole collection of four circles is linked. This linking will be detected by a nontrivial 4-product. The second example will show the use of a matrix triple product to detect the linking of circles in S^3. The third example will again use a matrix triple product, but it will detect a different kind of linking. If we call a space homeomorphic to two circles joined by a line (⚯) a *spectacle* (intentionally singular), the matrix triple product will be used to detect the linking of a circle with two spectacles. At the end an indication is given via examples of how matrix higher products can be used to detect the linking of larger collections of circles and spectacles.

Because it is more convenient, the necessary computations in all the examples of this section will be performed as in **[MW2]** using homology theory and intersections of cycles instead of cohomology theory and cup products. To make this shift we must use duality theorems for manifolds.

In all the examples A will denote the union of the circles or the union of the spectacles and circles embedded in S^3. We will let $X = S^3 - A$ and construct a manifold $M = S^3 - U$ with boundary B, where U is the union of sufficiently small regular neighborhoods of the circles and spectacles and B is the union of the boundaries of these neighborhoods. M is a deformation retract of X and we will identify the cohomology groups of X and M.

By the Lefschetz duality theorem,

$$H^q(M) \approx H_{3-q}(M, B)$$

and the cup product pairing on cohomology

$$H^p(M) \otimes H^q(M) \xrightarrow{\cup} H^{p+q}(M)$$

is equivalent to the intersection theoretic pairing on homology

$$H_{3-p}(M, B) \otimes H_{3-q}(M, B) \xrightarrow{\circ} H_{3-(p+q)}(M, B).$$

By excision we have $H_i(M, B) \approx H_i(S^3, \bar{U})$ and since each component of A is a deformation retract of the corresponding component of $\bar{U}$, we have

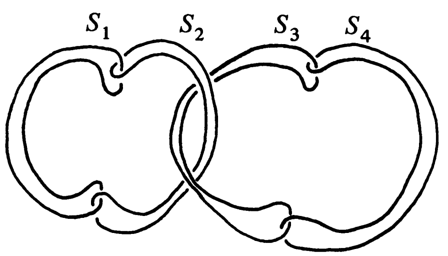
$$H_i(S^3, \bar{U}) \approx H_i(S^3, A).$$

Instead of computing higher products in $H^*(X)$ we will compute their analogue in $H_*(S^3, A)$ using intersection theory. To avoid complications with orientations and signs we will use $\mathbf{Z}_2$ coefficients.

Let $u_i(u_i') \in H^1(X)$ be the Alexander dual of the fundamental homology class of the circle $S_i(S_i')$ appropriately oriented. According to the Alexander duality theorem, $H^1(X)$ is a free abelian group on generators in one-to-one correspondence with the circles, including those that make up the spectacles, that are contained in A, and $H^2(X)$ is a free abelian group whose rank is one less than the number of components of A. Under the above isomorphisms $H^1(X) \approx H_2(S^3, A)$ and $H^2(X) \approx H_1(S^3, A)$. We may select a basis over $\mathbf{Z}_2$ for $H_1(S^3, A)$ by fixing one component of A and choosing a path from this component to each of the other components of A. These paths are mod 2 relative cycles and their relative homology classes in $H_1(S^3, A)$ are a basis.

In all the examples of this section we will prescribe the positions of the circles and spectacles embedded in S^3 by means of a regular projection onto a plane. (See Figure (8.1) and [**CF**, pp. 6-7].) Under the above isomorphisms, the dual of $u_i(u_i') \in H^1(X)$ is a relative homology class $w_i(w_i') \in H_2(S^3, A)$ which is represented by a singular disc $D_i(D_i')$ with boundary $S_i(S_i')$. We will consider the disc $D_i(D_i')$ as a cone on the circle $S_i(S_i')$ with vertex the point at infinity, and look at this cone as an infinite half-cylinder with base the circle $S_i(S_i')$ and generating rays perpendicular to the plane of our diagram. All cones we will consider will have the point at infinity as vertex.

First example. Consider the following link of four circles in S^3:



(8.1)

We can see that any subcollection of three circles is not linked, yet the whole collection is linked. The Alexander polynomial for this link is zero, so it cannot be used to detect this behavior. We will show that this linking is detected by a nontrivial 4-product $\langle u_1, u_2, u_3, u_4 \rangle$ in the complement of the link.

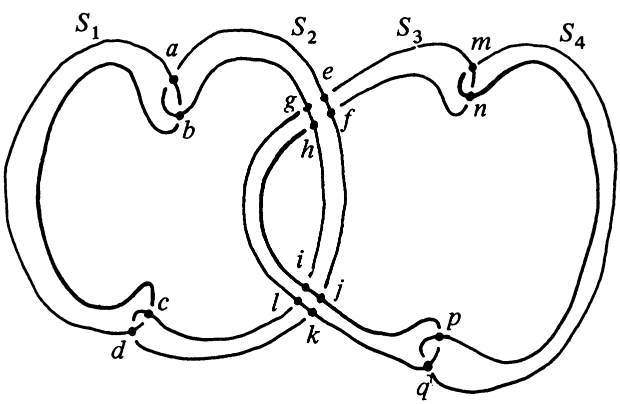
Since the linking number of any two circles is zero, we see that all cup products $u_i \cup u_j$ vanish, so the triple products $\langle u_1, u_2, u_3 \rangle$ and $\langle u_2, u_3, u_4 \rangle$ will be defined without indeterminacy. Since any collection of three circles is splittable, these triple products are zero; thus the 4-product $\langle u_1, u_2, u_3, u_4 \rangle$ is strictly defined in $H^2(X)$.

Using (5.2) above we see that the indeterminacy of $\langle u_1, u_2, u_3, u_4 \rangle$ is determined by triple products. We can use Proposition 2.7 in **[MJ]** to split up the triple products into triple products of the classes u_i, $1 \leqslant i \leqslant 4$. If the classes in the triple product are all distinct, it will vanish by our comment above, and if a class is repeated in the triple product, Proposition 6.1 in **[MW2]** gives that it vanishes. Thus the indeterminacy of $\langle u_1, u_2, u_3, u_4 \rangle$ is zero.

We will now compute the analogue of $\langle u_1, u_2, u_3, u_4 \rangle$ in $H_*(S^3, A)$ using the machinery and notation introduced above. The dual of $u_i \in H^1(X)$ is $w_i \in H_2(S^3, A)$ which is represented by the mod 2 relative cycle $D_i = C(S_i)$, the cone on S_i, $1 \leqslant i \leqslant 4$. These cycles are in general position and their intersections are

$$D_1 \circ D_2 = C(\{a, b, c, d\}),$$
$$D_2 \circ D_3 = C(\{e, f, g, h, i, j, k, l\}),$$
$$D_3 \circ D_4 = C(\{m, n, p, q\})$$

where the points $a, b, \ldots, q$ are as in figure (8.2) and $C(Y)$ denotes the cone on the set Y.



(8.2)

We note that

$$D_1 \circ D_2 = \partial(a_{12}), \quad D_2 \circ D_3 = \partial(a_{23}), \quad D_3 \circ D_4 = \partial(a_{34})$$

where a_{12}, a_{23} and a_{34} are mod 2 relative chains given by

$$a_{12} = C(A[a, d] \cup A[b, g, c]),$$
$$a_{23} = C(A[e, f] \cup A[g, h] \cup A[i, j] \cup A[k, l]),$$
$$a_{34} = C(A[m, l, q] \cup A[n, p])$$

where $A[x, y]$ denotes the shortest arc on the circle joining x with y and $A[x, w, y]$ denotes the arc on the circle from x through w to y. Next we have

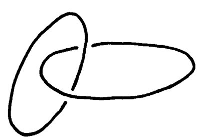
$$D_1 \circ a_{23} + a_{12} \circ D_3 = C(\{g, h, i, l\}) = \partial(a_{13}),$$
$$D_2 \circ a_{34} + a_{23} \circ D_4 = C(\{e, g, l, k\}) = \partial(a_{24})$$

where $a_{13} = C(A[g, h] \cup A[i, j, l])$ and $a_{24} = C(A[e, b, g] \cup A[l, k])$ are mod 2 relative chains. We see that

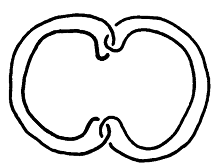
$$D_1 \circ a_{24} = C(\{a, b\}), \quad a_{12} \circ a_{34} = C(\{g, l\}), \quad a_{13} \circ D_4 = C(\{p, q\}).$$

The sum $D_1 \circ a_{24} + a_{12} \circ a_{34} + a_{13} \circ D_4$ is a representative of the dual of the 4-product $\langle u_1, u_2, u_3, u_4 \rangle$ and is seen to be nontrivial since the mod 2 relative cycles $C(\{a, b\})$, $C(\{g, l\})$ and $C(\{p, q\})$ are paths joining different components of A and could be selected as generators of $H_1(S^3, A)$.

REMARK. We can inductively construct examples of collections of n disjoint circles, $n \geqslant 2$, embedded in S^3 with the property that any subcollection of $n - 1$ circles is not linked, yet the whole collection of n circles is linked. We start with



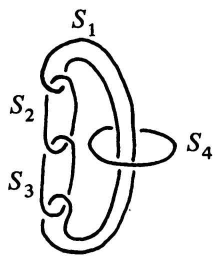
for $n = 2$. Then we replace one of the circles by two unlinked circles



for $n = 3$. To be more precise, we place the two unlinked circles in a small tubular neighborhood of the circle removed. In the given example for $n = 3$ we obtain the Borromean rings. We proceed in this manner and obtain a collection of n circles with the desired property, for $n = 2, 3, \ldots$. Computations of nontrivial higher products can be made as in the cases $n = 3, 4$.

SECOND EXAMPLE. In this example we will show how a matrix triple product can be used to detect the linking of circles in S^3. In order for nontrivial matrix products to be defined where ordinary products are not, we must have a nontrivial cup product structure in the cohomology of the complementary space. In terms of the linking of circles in S^3, some circles in our collection must link. The existence of nontrivial cup products also means that the indeterminacy will not vanish and care must be taken with it. Thus examples showing the use of matrix products to detect linking will not be similar to those for ordinary products.

Consider four circles embedded in S^3 as follows:



(8.3)

We note that these circles are linked in S^3 and we will show that, ignoring signs, the matrix triple product

$$\left\langle (u_1, u_2, u_3), \begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix}, (u_4) \right\rangle$$

is nontrivial in $H^2(X)$.

The indeterminacy of

$$\left\langle (u_1, u_2, u_3), \begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix}, (u_4) \right\rangle$$

is

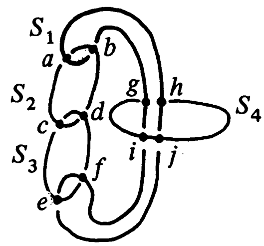
$$(u_1, u_2, u_3) \cdot \begin{bmatrix} H^1(X) \\ H^1(X) \\ H^1(X) \end{bmatrix} + H^1(X) \cdot u_4.$$

But since the linking number of S_4 with each of the other circles is zero, the second term is zero. Thus the indeterminacy is nontrivial and consists of the subgroup of $H^2(X)$ generated by the elements decomposable in terms of the u_i, $1 < i < 3$. $H^2(X)$ has rank 3 by Alexander duality, and we will see in our computations that the subgroup of elements decomposable in terms of the u_i, $1 < i < 3$, has rank 2. We will show that this matrix triple product does not belong to the indeterminacy.

The dual of $u_i \in H^1(X)$ is $w_i \in H_2(S^3, A)$ which is represented by the mod 2 relative cycle $D_i = C(S_i)$, the cone on S_i, and since these cycles are in general position we have the following intersections:

$$D_1 \circ D_2 = C(\{a, b\}), \qquad D_2 \circ D_3 = C(\{c, d\}), \quad D_3 \circ D_1 = C(\{e, f\}),$$
$$D_1 \circ D_4 = C(\{g, h, i, j\}), \quad D_2 \circ D_4 = 0, \qquad D_3 \circ D_4 = 0.$$

where the points $a, b, \ldots, j$ are as in figure (8.4):



(8.4)

Again letting $A[x, y]$ denote the shortest arc on the circle joining x with y and $A[x, w, y]$ denote the arc on the circle from x through w to y, we have

$$(D_1 \circ D_2 + D_2 \circ D_3 + D_3 \circ D_1) = \partial(a_{12}),$$

$$\begin{bmatrix} D_2 \\ D_3 \\ D_1 \end{bmatrix} \circ (D_4) = \partial(a_{23})$$

where a_{12} and a_{23} are the matrices of mod 2 relative chains,

$$a_{12} = \big(C(A[b, c]) + C(A[d, e]) + C(A[a, h, f])\big),$$

$$a_{23} = \begin{bmatrix} 0 \\ 0 \\ C(A[i, j]) + C(A[g, a, h]) \end{bmatrix}.$$

We can see that $(D_1, D_2, D_3) \circ a_{23} = 0$ and $a_{12} \circ (D_4) = C(\{h, j\})$. The sum $(D_1, D_2, D_3) \circ a_{23} + a_{12} \circ (D_4) = C(\{h, j\})$ is a representative of the relative homology class in $H_1(S^3, A)$ which is dual to the matrix triple product

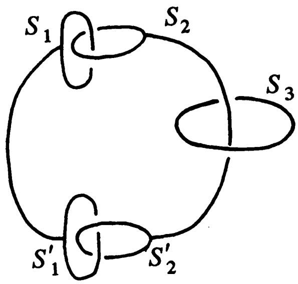
$$\left\langle (u_1, u_2, u_3), \begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix}, (u_4) \right\rangle \text{ in } H^2(X).$$

In the computations above we saw that $w_1 \circ w_2 + w_2 \circ w_3 + w_3 \circ w_1 = 0$ and we note that $w_4 \circ w_i = 0$, $1 \leqslant i \leqslant 3$. Thus the rank of the subgroup of $H_1(S^3, A)$ generated by elements decomposable in terms of w_1, w_2 and w_3 is seen to be two. We could select $w_1 \circ w_2$ and $w_1 \circ w_3$, represented by paths from S_1 to S_2 and from S_1 to S_3 respectively, to generate this subgroup. But then $C(\{h,j\})$ is a path from S_1 to S_4 and represents a third generator of $H_1(S^3, A)$ and is not in the indeterminacy. Therefore the matrix triple product

$$\left\langle (u_1, u_2, u_3), \begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix}, (u_4) \right\rangle$$

is nontrivial.

THIRD EXAMPLE. In this example we will show how the matrix triple product can be used to detect the linking of a circle and two spectacles embedded in S^3. We will consider the following:



(8.5)

The linking will be detected by the computation of the matrix triple product

$$\left\langle (u_1, u'_1), \begin{pmatrix} u_2 \\ u'_2 \end{pmatrix}, (u_3) \right\rangle$$

in $H^2(X)$. In the figure S_i and S'_i are two circles which are part of one spectacle, $i = 1, 2$.

The indeterminacy of this matrix triple product is

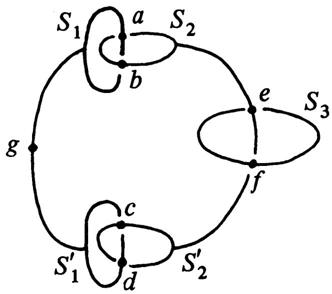
$$(u_1, u'_1)\begin{bmatrix} H^1(X) \\ H^1(X) \end{bmatrix} + H^1(X) \cdot u_3.$$

But since S_3 has linking number zero with the other circles, we only have the first term to consider. The indeterminacy is therefore nontrivial and is the subgroup generated by those elements which are decomposable with one factor being u_1 or u'_1. Alexander duality shows that $H^2(X)$ is free abelian of rank 2 and we will see in the computations that the indeterminacy has rank 1. The matrix triple product will not belong to this indeterminacy.

The dual of $u_i(u'_i) \in H^1(X)$ is $w_i(w'_i) \in H_2(S^3, A)$ which is represented by the mod 2 relative cycle $D_i = C(S_i)$ $(D'_i = C(S'_i))$. These cycles are in general position and their intersections are:

$$D_1 \circ D_2 = C(\{a, b\}), \quad D'_1 \circ D'_2 = C(\{c, d\}), \quad D_2 \circ D_3 = 0, \quad D'_2 \circ D_3 = 0$$

where the points $a, b, \ldots, g$ are as in figure (8.6):



(8.6)

We have

$$(D_1 \circ D_2 + D_1' \circ D_2') = \partial(a_{12}), \qquad \begin{pmatrix} D_2 \\ D_2' \end{pmatrix} \circ (D_3) = \partial(a_{23})$$

where a_{12} and a_{23} are the following matrices of mod 2 relative chains:

$$a_{12} = \big(C(A[a, g, d]) + C(A[b, e, c])\big), \quad a_{23} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

letting $A[x, w, y]$ denote the shortest path on a spectacle from x through w to y. We then have $(D_1, D_1') \circ a_{23} = 0$, $a_{12} \circ (D_3) = C(\{e, f\})$, and the sum $(D_1, D_1') \circ a_{23} + a_{12} \circ (D_3) = C(\{e, f\})$ represents the relative homology class in $H_1(S^3, A)$ which is dual to

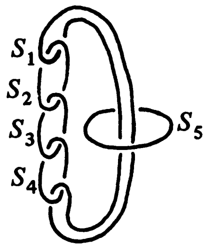
$$\left\langle (u_1, u_1'), \begin{pmatrix} u_2 \\ u_2' \end{pmatrix}, (u_3) \right\rangle.$$

In the computations we saw that $w_1 \circ w_2 + w_1' \circ w_2' = 0$ and that the intersection of w_3 with the other 2-dimensional relative homology classes was zero. Thus the subgroup of $H_1(S^3, A)$ generated by those elements which are decomposable with one factor being w_1 or w_1' has rank 1 and $w_1 \circ w_2$, represented by a path from S_2 on the second spectacle to S_1 on the first spectacle, could be selected as a generator. But then $C(\{e, f\})$ is a path from the second spectacle to the circle and represents a second generator of $H_1(S^3, A)$ which is therefore not in the indeterminacy. Thus

$$\left\langle (u_1, u_1'), \begin{pmatrix} u_2 \\ u_2' \end{pmatrix}, (u_3) \right\rangle$$

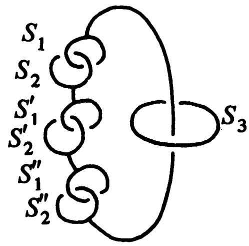
is nontrivial.

Further examples. In the following examples the indicated matrix higher product, taken modulo two, is nontrivial in $H^2(X)$. We thus see that matrix higher products play a striking role in detecting linking.



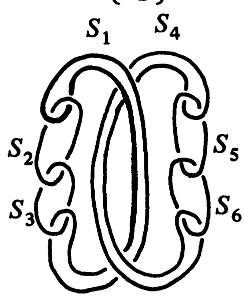
(8.7)

$$\left\langle (u_1, u_2, u_3, u_4), \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_1 \end{bmatrix}, (u_5) \right\rangle$$



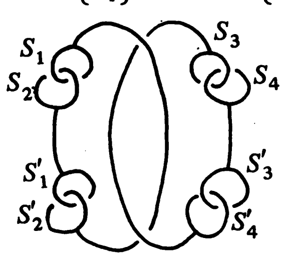
(8.8)

$$\left\langle (u_1, u_1', u_1''), \begin{bmatrix} u_2 \\ u_2' \\ u_2'' \end{bmatrix}, (u_3) \right\rangle$$



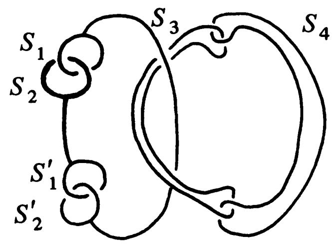
(8.9)

$$\left\langle (u_1 u_2 u_3), \begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix}, (u_4 u_5 u_6), \begin{bmatrix} u_5 \\ u_6 \\ u_4 \end{bmatrix} \right\rangle$$



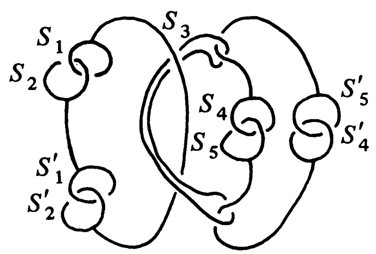
(8.10)

$$\left\langle (u_1u_1'), \begin{pmatrix} u_2 \\ u_2' \end{pmatrix}, (u_3u_3'), \begin{pmatrix} u_4 \\ u_4' \end{pmatrix} \right\rangle$$



(8.11)

$$\left\langle (u_1u_1'), \begin{pmatrix} u_2 \\ u_2' \end{pmatrix}, (u_3), (u_4) \right\rangle$$



(8.12)

$$\left\langle (u_1u_1'), \begin{pmatrix} u_2 \\ u_2' \end{pmatrix}, (u_3), (u_4u_4'), \begin{pmatrix} u_5 \\ u_5' \end{pmatrix} \right\rangle.$$

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