TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 248, Number 1, February 1979

UNIQUELY ARCWISE CONNECTED PLANE CONTINUA HAVE THE FIXED-POINT PROPERTY

BY

CHARLES L. HAGOPIAN[1]

ABSTRACT. This paper contains a solution to a fixed-point problem of G. S. Young [**17**, p. 884] and R. H. Bing [**4**, Question 4, p. 124]. Let M be an arcwise connected plane continuum that does not contain a simple closed curve. We prove that every continuous function of M into M has a fixed point.

1. Introduction. A space S has the *fixed-point property* if for each continuous function f of S into S, there exists a point x of S such that $f(x) = x$. It is known that every arcwise connected plane continuum that does not separate the plane has the fixed-point property [**6**], [**7**].[2] In this paper we consider another class of arcwise connected plane continua.

A continuum M is *uniquely arcwise connected* if for each pair p, q of points of M, there exists exactly one arc in M with endpoints p and q. Note that a continuum is uniquely arcwise connected if and only if it is arcwise connected and does not contain a simple closed curve. The sin $1/x$ circle (Warsaw circle) is the simplest example of a uniquely arcwise connected plane continuum that separates the plane.

In [**4**, p. 123], Bing gave a dog-chases-rabbit argument that shows the sin $1/x$ circle has the fixed-point property. Recently L. Mohler [**12**] used the Markov-Kakutani theorem (measure theory) to prove that every homeomorphism of a uniquely arcwise connected continuum into itself has a fixed point.

We prove that every uniquely arcwise connected plane continuum has the fixed-point property. An example of Young [**17**, p. 884] shows that this theorem cannot be extended to all uniquely arcwise connected continua in Euclidean 3-space.

Received by the editors October 4, 1977.

AMS (MOS) subject classifications (1970). Primary 54F20, 54F50, 54H25; Secondary 54C05, 55C20, 57A05.

Key words and phrases. Fixed-point property, uniquely arcwise connected continua, plane continua, indecomposable continua, Warsaw circle.

[1] The author presented this result at the Louisiana State University Topology Conference on March 11, 1977.

[2] It is not known whether every nonseparating plane continuum has the fixed-point property [**1**, p. 336], [**4**, Question 3, p. 122].

0002-9947/79/0000-0053/$06.00

Our proof involves a continuous image of a ray defined by K. Borsuk [5] and a nested sequence of polygonal disks constructed by K. Sieklucki [15]. In [5], Borsuk established the fixed-point property for every hereditarily unicoherent arcwise connected continuum. Sieklucki [15] and H. Bell [2] proved that every nonseparating plane continuum with a hereditarily decomposable boundary has the fixed-point property.

2. Preliminaries. A *continuum* is a nondegenerate compact connected metric space.

Throughout this paper R^2 is the Cartesian plane with metric ρ. We denote the boundary, closure, and interior of a given set K by Bd K, Cl K, and Int K respectively. The union of a collection $\mathcal{K}$ of sets is denoted by St $\mathcal{K}$.

For each real number ζ, let $I(\zeta)$ be the interval $\{(x, y) \in R^2\colon 0 \leqslant x \leqslant 1$ and $y = \zeta\}$.

DEFINITION. Suppose A is an arc, H is a continuum, and $A \cup H \subset R^2$. Then H *straddles* A if for each homeomorphism h of $\mathrm{St}\{I(\zeta)\colon -1 \leqslant \zeta \leqslant 1\}$ into R^2 with $h[I(0)] = A$, there exists a positive real number η such that $H \cap h[I(\zeta)] \neq \varnothing$ when $|\zeta| < \eta$.

Henceforth M is a uniquely arcwise connected continuum in R^2.

Notation. If u and v are distinct points of M, then the arc, the half-open arc, and the arc segment (open arc) in M with endpoints u, v are denoted by $M[u, v]$, $M[u, v)$, and $M(u, v)$ respectively.

Let P be the image in M of the ray $[1, +\infty)$ under a one-to-one continuous function ψ. For each positive integer n, let $a_n = \psi(n)$. The function ψ determines a linear ordering $\prec$ of P with a_1 as the first point. In this section, if u and v are points of P, the notation $M[u, v]$, $M[u, v)$, and $M(u, v)$ will be used only when $u \prec v$.

Notation. For each point u of P, let $P(u)$ denote $\{v \in P\colon u = v \text{ or } u \prec v\}$.

Let $L = \bigcap_{n=1}^{\infty} \mathrm{Cl}\, P(a_n)$. In this section we assume L is not degenerate. Hence L is a continuum.

LEMMA 1. *Suppose u and v are distinct points of M that belong to a complementary domain D of L. Then* Cl D *contains* $M[u, v]$.

PROOF. Assume there is a point w of $M(u, v)$ in $R^2 \setminus \mathrm{Cl}\, D$. Let J be an arc in D that is irreducible between $M[u, w]$ and $M[w, v]$. Every arc in M that intersects D and $R^2 \setminus \mathrm{Cl}\, D$ is straddled by Bd D. Hence Bd D intersects each complementary domain of $J \cup M[u, v]$. Since Bd $D \subset L$ and M does not contain a simple closed curve, for each positive integer n, there is a point z_n of $P(a_n)$ in J. Let z be a limit point of $\{z_n\}_{n=1}^{\infty}$. It follows that $z \in J \cap L$, and this contradicts the fact that $J \subset R^2 \setminus L$. Hence $M[u, v] \subset \mathrm{Cl}\, D$. □

LEMMA 2. *There exist a complementary domain D of L and a positive integer n such that* Cl D *contains* $P(a_n)$.

PROOF. Assume the contrary. Then by Lemma 1,

(1) for each complementary domain D of L, there exists a positive integer n such that $D \cap P(a_n) = \varnothing$.

Let $\{Y_n\}_{n=1}^{\infty}$ be the elements of a countable base for M that intersect L. For each n, let $Z_n = \{z \in L \setminus \{a_1\}: Y_n \cap M[a_1, z] = \varnothing\}$.

Note that $L \setminus \{a_1\} = \bigcup_{n=1}^{\infty} Z_n$. To see this let z be a point of $L \setminus \{a_1\}$. If $L \subset M[a_1, z]$, then L is an arc and the closure of the complementary domain of L (being R^2) contains P, contrary to our assumption. Thus $L \not\subset M[a_1, z]$. Hence there exists an integer n such that $M[a_1, z] \subset M \setminus Y_n$. It follows that $z \in Z_n$. Therefore $L \setminus \{a_1\} = \bigcup_{n=1}^{\infty} Z_n$.

By the Baire category theorem, there exists an integer α such that Cl Z_α contains a nonempty open subset U of L. Let Y and Z be disjoint disks in R^2 such that

(2) $L \cap \operatorname{Int} Y \neq \varnothing$ and $L \cap \operatorname{Int} Z \neq \varnothing$ and

(3) $M \cap Y \subset Y_\alpha$ and $L \cap Z \subset U$.

Notation. For each component C of $P \setminus Y$ that misses a_1, let Dom C denote the complementary domain of $C \cup Y$ that misses a_1.

Let $\mathfrak{M}$ be the collection of all components C of $P \setminus Y$ such that $a_1 \notin C$ and $Z \cap \operatorname{Dom} C \neq \varnothing$. Note that each element of $\mathfrak{M}$ is an arc segment with both endpoints in Bd Y.

For each element C of $\mathfrak{M}$,

(4) $U \cap \operatorname{Dom} C = \varnothing$ and

(5) $C \cap \operatorname{Int} Z \neq \varnothing$.

Statement (4) is true; for otherwise, since $U \subset \operatorname{Cl} Z_\alpha$, $M \setminus Y_\alpha$ contains an arc A that runs from a_1 to $U \cap \operatorname{Dom} C$, and $A \cup P$ contains a simple closed curve, violating the unique arcwise connectivity of M. Since $Z \cap \operatorname{Dom} C \neq \varnothing$ and $L \cap \operatorname{Int} Z$ is a nonempty subset of U, (5) follows immediately from (4).

Since Y and Z are disjoint, it follows from (2) that $\mathfrak{M}$ is infinite. By (5), for each positive integer n,

(6) $P(a_n)$ contains all but finitely many elements of $\mathfrak{M}$.

For each element C of $\mathfrak{M}$,

(7) Dom C contains at most finitely many elements of $\mathfrak{M}$ and

(8) only finitely many elements of $\mathfrak{M}$ separate a_1 from C in $R^2 \setminus Y$.

To verify (7) assume there exists an infinite subcollection $\mathfrak{N}$ of $\mathfrak{M}$ such that St $\mathfrak{N} \subset \operatorname{Dom} C$. By (5), each element of $\mathfrak{N}$ intersects Z. Hence by (6), there is a point t of L in $Z \cap \operatorname{Cl} \operatorname{St} \mathfrak{N}$. By (4), $t \notin \operatorname{Dom} C$. Therefore $t \in C$.

Let J be an arc segment in Dom C with endpoints in $C \setminus \{t\}$ such that the bounded complementary domain K of $C \cup J$ has the following properties. The point t belongs to Cl K and U contains $K \cap L$.

Since infinitely many elements of $\mathfrak{N}$ intersect K, it follows from (1) and (6) that no complementary domain of L contains $K \cap \operatorname{St} \mathfrak{N}$. Therefore $K \cap L \neq \varnothing$. But since $K \subset \operatorname{Dom} C$, this contradicts (4). Hence (7) is true.

To establish (8) assume the contrary. By (7), there exists a sequence $\{C_n\}_{n=1}^{\infty}$ of distinct elements of $\mathfrak{M}$ such that for each n, Dom $C_n \subset$ Dom C_{n+1}. By (1), (4), (5), (6), and Lemma 1, for some n, L separates C_1 from C_n in Z. Let A be an arc segment in Z such that Cl A is an arc irreducible between C_1 and C_n. Since $C_1 \subset$ Dom C_n, $A \subset$ Dom C_n. But since $A \cap L \neq \varnothing$, this contradicts (4). Hence (8) is true.

Let $\mathcal{O} = \{C \in \mathfrak{M}$: no element of $\mathfrak{M}$ separates a_1 from C in $R^2 \setminus Y\}$. Since $\mathfrak{M}$ is infinite, it follows from (7) and (8) that $\mathcal{O}$ is infinite.

Next we define a pair p, q of points of $P \cap$ Int Y. We consider two cases.

Case 1.1. Suppose $L \cap P \cap$ Int $Y \neq \varnothing$. Define p to be a point of $L \cap P \cap$ Int Y, and let q be a point of $(P(p) \cap$ Int $Y) \setminus \{p\}$.

Case 1.2. Suppose $L \cap P \cap$ Int $Y = \varnothing$. Applying (1), we define p and q to be points of $P \cap$ Int Y such that $P(q)$ misses the p-component of $R^2 \setminus L$.

An element $M(u, v)$ of $\mathcal{O}$ in $P(q)$, a point z of $M(u, v) \cap$ Int Z, and an arc segment I in (Int Z) $\setminus$ Cl Dom $M(u, v)$ exist such that

(9) (Cl I) $\cap$ $(M[a_1, v] \cup$ Bd $Z) = \{z\}$,

(10) $L \cap$ Cl $I \neq \varnothing$, and

(11) either $z \in L$ or $P \cap$ Cl $I = \{z\}$.

To verify this consider two cases.

Case 2.1. Suppose $L \cap P(q) \cap$ Int $Z \neq \varnothing$. Let z be a point of $L \cap P(q) \cap$ Int Z. Let $M(u, v)$ be the element of $\mathfrak{M}$ that contains z. It follows from (4) that $M(u, v) \in \mathcal{O}$. Let I be an arc segment in (Int Z) $\setminus$ Cl Dom $M(u, v)$ that satisfies (9). Since $z \in L$, (10) and (11) hold.

Case 2.2. Suppose $L \cap P(q) \cap$ Int $Z = \varnothing$. Note that $(L \setminus P) \cap$ Int $Z \neq \varnothing$. To see this assume otherwise. Since $L \cap$ Int $Z \neq \varnothing$ and $L \cap P(q) \cap$ Int $Z = \varnothing$, there exists a point a of $M[a_1, q]$ in $L \cap$ Int Z. Let V be an open disk in Z such that $a \in V$ and $V \setminus M[a_1, q]$ has exactly two components. Since $a \in L$, infinitely many elements of $\mathfrak{M}$ intersect $V \setminus M[a_1, q]$. But since both components of $V \setminus M[a_1, q]$ are in $R^2 \setminus L$, this contradicts (1) and (6).

Let t be a point of $(L \setminus P) \cap$ Int Z. Let W be an open disk in Z that contains t and misses $M[a_1, q]$. By (1), an element $M(w, x)$ of $\mathfrak{M}$ and a point r of $W \cap M(w, x)$ exist such that $W \cap P(x)$ misses the r-component G of $R^2 \setminus L$.

Let T be an arc in W that runs from r to t. Define s to be the first point of T that belongs to L. Let I be the arc segment in T that precedes s with the property that Cl I is irreducible between s and $M[q, x]$. Define z to be the endpoint of I opposite s. Let $M(u, v)$ be the element of $\mathfrak{M}$ that contains z. It follows from (4) that $s \notin$ Cl Dom $M(u, v)$. Since $z \in G$, $P \cap$ Cl $I = \{z\}$. Hence by (4), $M(u, v) \in \mathcal{O}$. Clearly (9), (10), and (11) hold.

Let J be a polygonal arc segment in

$$Z \setminus \left(M[a_1, v] \cup \text{Dom } M(u, v) \cup \text{Cl } I\right)$$

with endpoints b and b_0 in $M(u, v)$ such that $z \in M(b, b_0) \subset Z$.

Define K_0 to be the component of $R^2 \setminus (J \cup M[b, b_0])$ that contains I. Note that $K_0 \subset Z$.

The following statements and definitions $(12_n)-(21_n)$ will be used inductively.

By (7), (8), and (10), for $n = 1$, there exists an element $M(w_n, x_n)$ of $\mathcal{O}$ in $P(v)$ such that

(12_n) $K_{n-1} \cap M(w_n, x_n) \neq \emptyset$ and

(13_n) no element of $\mathcal{O}$ in $M(b_{n-1}, w_n)$ intersects K_{n-1}.

For $n = 1$,

(14_n) let $H_n = M(b_{n-1}, x_n) \cap \mathrm{Cl}\, K_{n-1}$,

(15_n) let I_n be the component of $I \setminus M(p, x_n)$ whose closure contains z,

(16_n) let J_n be the arc in Cl J that is irreducible between b and $M(w_n, x_n)$, and

(17_n) let b_n be the endpoint of J_n that belongs to $M(w_n, x_n)$.

Let J' be the arc in Cl J that is irreducible between b_0 and $M(w_1, x_1)$. Since $Y \cup \mathrm{Dom}\, M(u, v)$ misses $M(w_1, x_1) \cup \mathrm{Cl}\, K_0$, there exists an arc segment I' in $Y \cup \mathrm{Dom}\, M(u, v)$ from p to z such that Cl I_1 and Cl I' abut on $M[b, b_0]$ from opposite sides with respect to the simple closed curve Γ in $J_1 \cup J' \cup M(b, b_0) \cup M(w_1, x_1)$ [**13**, Theorem 32, p. 181]. Since $\Gamma \cap (I_1 \cup I' \cup \{p\}) = \emptyset$, $J_1 \cup J' \cup M(u, x_1)$ separates p from I_1 in R^2. Since $J_1 \cap J' = \emptyset$, either $J_1 \cup M(u, x_1)$ or $J' \cup M(u, x_1)$ separates p from I_1 in R^2 [**13**, Theorem 20, p. 173]. For convenience we assume that $J_1 \cup M(u, x_1)$ separates p from I_1 in R^2.

For $n = 1$,

(18_n) let K_n be the complementary domain of $J_n \cup M(b, b_n)$ that contains I_n.

Clearly, for $n = 1$,

(19_n) $p \notin K_n$.

By (10) and (11), for $n = 1$,

(20_n) $L \cap (K_n \cup \{z\}) \neq \emptyset$.

Next we show that for $n = 1$,

(21_n) $K_n \cap Y \cap P(b_n) = \emptyset$.

Let A be an arc in J_1 such that $A \cap M(p, x_1) = \{b\}$. Since $M(u, v)$ and $M(w_1, x_1)$ belong to $\mathcal{O}$, $M(u, v)$ and $M(w_1, x_1)$ are not separated in $R^2 \setminus Y$ by an element of $\mathfrak{M}$. Hence there exists a polygonal arc segment B in $R^2 \setminus (Y \cup M(p, x_1))$ such that $A \cup \{b_1\} \subset \mathrm{Cl}\, B$. Let I_1' be an arc in $(\mathrm{Cl}\, I_1) \setminus B$ that contains z. Let K be the complementary domain of $B \cup M[b, b_1]$ that intersects I_1'.

Note that

(22) $K_1 \cap Y \subset K$.

To see this let y be a point of $K_1 \cap Y$. We must show that $y \in K$. Let F be a polygonal arc in K_1 from y to I_1' such that $B \cap F$ is finite and F crosses B at each point of $B \cap F$. Clearly $y \in K$ if $B \cap F = \varnothing$, so we assume $B \cap F \neq \varnothing$. Since $I_1' \cup Y \cup M(z, v)$ misses $B \cup J_1$, one complementary domain of $B \cup J_1$ contains $I_1' \cup \{y\}$. Since $F \cap J_1 = \varnothing$, it follows that F crosses B an even number of times. Therefore F crosses $B \cup M[b, b_1]$ an even number of times. Thus $y \in K$. Hence (22) is true.

It follows from (22) that (21_1) can be established by proving

(23) $K \cap Y \cap P(b_1) = \varnothing$.

To verify (23) first note that since $p \notin K_1$ and $M[p, b) \cap (J_1 \cup M(b, b_1)) = \varnothing$, I_1' and $M[p, b]$ abut on $A \cup M[b, v]$ from opposite sides with respect to a simple closed curve in $J_1 \cup M(b, b_1)$. Hence I_1' and $M[p, b]$ abut on $A \cup M[b, v]$ from opposite sides with respect to $B \cup M[b, b_1]$ [**13**, Theorem 32, p. 181]. Since $M[p, b) \cap (B \cup M[b, b_1]) = \varnothing$, it follows that $p \notin K$.

Let Q be an arc from p to q in Int Y. Either $p \in L$ (Case 1.1) or $L \cap P \cap \text{Int}\, Y = \varnothing$ and $P(q)$ misses the p-component of $R^2 \setminus L$ (Case 1.2). Hence the p-component of $Q \setminus \text{Cl}\, K$ contains a point r of L. Let G be an open set in $Y \setminus \text{Cl}\, K$ that contains r.

Now suppose that (23) is false. Since $r \in L$, there exists an arc $M[s, t]$ in $P(b_1)$ such that $s \in K \cap Y$ and $t \in G$.

Let $M(s', t')$ be an arc segment in $M[s, t] \setminus Y$ such that $s' \in K \cap \text{Bd}\, Y$ and $t' \in (\text{Bd}\, Y) \setminus K$. A component of $(B \cup M[b, b_1]) \setminus \text{Int}\, Y$ separates s' from t' in $R^2 \setminus \text{Int}\, Y$ [**13**, Theorem 27, p. 177]. Since $B \cap M(b, b_1) = \varnothing$ and $M(s', t') \cap (Y \cup M(b, b_1)) = \varnothing$, it follows that $B \cup M[b, v] \cup M[w_1, b_1]$ separates s' from t' in $R^2 \setminus \text{Int}\, Y$. Hence $\{v, w_1\}$ separates s' from t' in Bd Y. Thus $M(s', t')$ is an element of $\mathfrak{M}$ that separates $M(u, v)$ from $M(w_1, x_1)$ in $R^2 \setminus Y$ [**13**, Theorem 30, p. 158], and this contradicts the fact that $M(u, v)$ and $M(w_1, x_1)$ belong to $\mathcal{O}$. Hence (23) is true. Consequently (21_1) is true.

Proceeding inductively, for each integer $n > 1$, we define $M(w_n, x_n)$, H_n, I_n, J_n, b_n, and K_n satisfying (12_n)–(21_n). For $n > 1$, (12_n) and (13_n) follow from (7), (8), and (20_{n-1}). To verify (19_n) for $n > 1$, note that by (19_{n-1}), there exists an arc A in J_n such that $M[p, b]$ and Cl I_n abut on $A \cup M[b, v]$ from opposite sides with respect to a simple closed curve in $J_n \cup M(b, b_n)$ [**13**, Theorem 32, p. 181]. The arguments given for (20_n) and (21_n) when $n = 1$ hold when $n > 1$.

Since $Y \cap Z = \varnothing$, for each positive integer m,

(24) there exists an integer n such that $a_m \notin P(x_n)$.

Let H be the limit superior of $\{H_n\}_{n=1}^{\infty}$. By (24), $H \subset L$.

Since $K_0 \cap Y = \varnothing$, it follows from (21_n) that

(25) $Y \cap \bigcup_{n=1}^{\infty} H_n = \varnothing$.

Since $\{J_n\}_{n=1}^{\infty}$ is a nested sequence of arcs, $\{b_n\}_{n=1}^{\infty}$ converges to a point c of $H \cap J_1$. For each positive integer n, let B_n be the polygonal arc in J_n with endpoints c and b_n.

Since $J_1 \cap M[a_1, u] = \varnothing$, it follows from ($13_n$) and ($16_n$) that $(J_1 \setminus B_1) \cap \bigcup_{n=1}^{\infty} H_n = \varnothing$. For each positive integer n, every component of H_{n+1} intersects B_n. Hence H is connected.

For each component C of $P(u) \setminus Y$,

(26) $H \cap \text{Int}(Y \cup \text{Dom } C) = \varnothing$.

To see this let x be the last point of Cl C with respect to the ordering of P. By (13_n) and (16_n), $c \notin M(b, x)$. Let i be a positive integer such that $B_i \cap M(b, x) = \varnothing$. Since $c \in U$, it follows from (4) that $B_i \cap \text{Dom } C = \varnothing$. Since P does not contain a simple closed curve, $C \cap \bigcup_{n=i+1}^{\infty} H_n = \varnothing$. Since B_i intersects each component of $\bigcup_{n=i+1}^{\infty} H_n$, it follows from (25) that $\bigcup_{n=i+1}^{\infty} H_n$ misses $Y \cup \text{Dom } C$ [**13**, Theorem 28, p. 156]. Hence (26) is established.

Next we prove that Knaster's chainable indecomposable continuum with one endpoint [**11**, Example 1, p. 204] is a continuous image of H. We use a result [**8**, Theorem 1] that was derived from an argument of D. P. Bellamy [**3**]. According to Theorem 1 of [**8**], H can be mapped continuously onto Knaster's continuum if there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of nonempty open sets in H such that $(\text{Cl } G_1) \cap \text{Cl } G_2 = \varnothing$ and for each n,

(27_n) $G_{2n+1} \cup G_{2n+2} \subset G_{2n-1}$ and

(28_n) there exists a separation $E_n \cup F_n$ of $M \setminus G_{2n}$ such that $G_{2n+1} \subset E_n$ and $G_{2n+2} \subset F_n$.

To establish the existence of $\{G_n\}_{n=1}^{\infty}$, order B_1 so that b_1 is its first point. Let c_1 be the first point of $B_1 \cap L$ with respect to the ordering of B_1. By (1) and (24), $c_1 \neq c$.

If $b_1 \neq c_1$, define C_1 to be the arc in B_1 from b_1 to c_1. If $b_1 = c_1$, let $C_1 = \{c_1\}$.

Note that

(29) $c_1 \in H$.

To see this consider two cases.

Case 3.1. Suppose $c_1 \in P(z)$. Let m be an integer such that $J_m \cap M[z, c_1] = \varnothing$. For each integer $n \geqslant m$, $M[z, c_1] \subset \text{Cl } K_n$. It follows from (7) that $c_1 \in H$.

Case 3.2. Suppose $c_1 \notin P(z)$. For some positive integer n, $C_1 \cap P(a_n) = \varnothing$; for otherwise, by (1), $L \cap (C_1 \setminus \{c_1\}) \neq \varnothing$, and this contradicts the definition of c_1. Let d be the last point of $C_1 \cap P$ that precedes c_1 with respect to the ordering of B_1. Since $C_1 \cap M[a_1, z] = \varnothing$, $d \in P(z)$.

Let Δ be the arc in C_1 from d to c_1. Let $M(w, x)$ be the d-component of $P \setminus Y$. Since $c_1 \in U$ and $\Delta \cap P = \{d\}$, it follows from (4) that $M(w, x) \in \mathcal{O}$.

Let A be an arc in $J_1 \setminus \Delta$ such that $A \cap M(p, x) = \{b\}$. Observe that

(30) A and Δ abut on $M[u, x]$ from the same side.

To verify (30) first note that, by (4), $\Delta \cap \text{Dom } M(w, x) = \varnothing$. Hence there exists a polygonal arc segment B in $R^2 \setminus (Y \cup M(p, x))$ such that $A \cup \Delta \subset \text{Cl } B$. Let F be an arc in $R^2 \setminus M(p, x)$ from p to x such that $B \cap F$ is finite and F crosses B at each point of $B \cap F$.

Suppose (30) is false. Then F crosses B an odd number of times. It follows that p and x are separated in R^2 by $B \cup M[b, d]$. By the argument for (23), there is a component of $P(z) \setminus Y$ that separates $M(u, v)$ from $M(w, x)$ in $R^2 \setminus Y$, and this contradicts the fact that $M(u, v)$ and $M(w, x)$ belong to $\mathcal{O}$. Hence (30) is true.

Let i be a positive integer such that $J_i \cap M(b, x) = \varnothing$. Let E be an arc in $\mathrm{Cl}\, I_i$ such that $E \cap M[u, x] = \{z\}$. Since E and A abut on $M[u, x]$ from the same side, it follows from (30) that E and Δ abut on $M[b, x]$ from the same side. Since $\Delta \cap P = \{d\}$, for each integer $j \geqslant i$, $c_1 \in K_j$.

Let V be a disk in K_i such that $c_1 \in \mathrm{Int}\, V$. Since $c_1 \in L$, $V \cap P(b_i) \neq \varnothing$. For each integer $j > i$, since $c_1 \in K_j$, if $V \cap M(b_i, b_j) = \varnothing$, then $V \subset K_j$. Hence for some $j > i$, H_j contains the first point of $V \cap P(b_i)$ with respect to the ordering of P (recall (14_n)). It follows that $c_1 \in H$. Thus (29) is established.

Let D_1 and D_2 be open disks in R^2 such that $B_1 \subset D_1$, $\mathrm{Cl}\, I_1 \subset D_2$, and $(\mathrm{Cl}\, D_1) \cap \mathrm{Cl}\, D_2 = \varnothing$. Let $i_1 = 1$.

Let j_1 be an integer greater than 1 such that

(31) $C_1 \cap P(w_{j_1}) = \varnothing$ and $J_{j_1} \cap M[z, b_1] = \varnothing$.

By (7), (8), (10), and (11), there exists an integer $i_2 > j_1$ such that $D_2 \cap K_1 \cap M(w_{i_2}, x_{i_2}) \neq \varnothing$.

Let Λ be an arc segment in

$$\big((\mathrm{Bd}\, Y) \cap \mathrm{Cl}\, \mathrm{Dom}\, M(w_{i_2}, x_{i_2})\big) \setminus M[p, x_{i_2}]$$

that has x_{i_2} as an endpoint. Let Λ_1 be a polygonal arc segment in $\mathrm{Dom}\, M(w_{i_2}, x_{i_2}) \setminus M[b, b_i]$ from a point e of Λ to $D_2 \cap K_1$ such that $B_1 \cap \Lambda_1$ is finite and Λ_1 crosses B_1 at each point of $B_1 \cap \Lambda_1$.

By (21_1), $e \notin K_1$.

Let Π_1 be the arc in B_1 from c_1 to c. Since Λ_1 misses $M[b, b_1] \cup (J_1 \setminus \Pi_1)$, it follows that Λ_1 crosses Π_1 an odd number of times.

By (26), there exists a simple closed curve Σ_1 in

$$Y \cup \Lambda_1 \cup D_2 \cup \mathrm{Dom}\, M(u, v)$$

such that $\Lambda_1 \subset \Sigma_1$, $H \cap \Sigma_1 \subset D_2$, and $\Pi_1 \cap \Sigma_1 \subset \Lambda_1$. Since Π_1 crosses Σ_1 an odd number of times, Σ_1 separates c from c_1 in R^2.

Define Ω_1 to be the c-component of $R^2 \setminus \Sigma_1$. Note that $B_{i_2} \subset \Omega_1 \subset R^2 \setminus C_1$.

Let $E_1 = \Omega_1 \cap (H \setminus D_2)$ and $F_1 = H \setminus (D_2 \cup E_1)$.

For $i = 1$ and 2, let $G_i = D_i \cap H$. Note that $E_1 \cup F_1$ is a separation of $H \setminus G_2$.

Let c_2 be the first point of $L \cap B_{i_2}$ with respect to the ordering of B_1. By (1) and (24), $c_2 \neq c$.

If $b_{i_2} \neq c_2$, define C_2 to be the arc in B_1 from b_{i_2} to c_2. If $b_{i_2} = c_2$, let $C_2 = \{c_2\}$. By the argument for (29), $c_2 \in H$.

Let D_3 and D_4 be open disks such that $B_{i_2} \subset D_3 \subset D_1 \cap \Omega_1$ and $C_1 \subset D_4 \subset D_1 \setminus \Omega_1$. Note that $D_3 \cap H \subset E_1$ and $D_4 \cap H \subset F_1$.

It follows from (31) and the arguments for Cases 3.1 and 3.2 that $c_1 \in \text{Cl } K_i$ for each integer $i \geqslant i_2$.

Proceeding inductively, we let n be an integer greater than 1. We assume that for each integer m $(1 < m \leqslant n)$, an integer i_m, a point c_m of $H \setminus \{c\}$, a subset C_m of B_1, and disjoint open disks D_{2m-1}, D_{2m} have been defined such that

(32_m) c_m is the first point of $L \cap B_{i_m}$ with respect to the ordering of B_1,

(33_m) C_m is a minimal connected set containing $\{b_{i_m}, c_m\}$,

(34_m) $B_{i_m} \subset D_{2m-1}$,

(35_m) $C_{m-1} \subset D_{2m}$, and

(36_m) $c_{m-1} \in \text{Cl } K_i$ for each integer $i \geqslant i_m$.

For each integer i $(2 < i \leqslant 2n)$, let $G_i = D_i \cap H$. We assume that for each positive integer i less than n, (27_i) and (28_i) are satisfied.

Let j_n be an integer greater than i_n such that $C_n \cap P(w_{j_n}) = \varnothing$ and $J_{j_n} \cap M[z, b_{i_n}] = \varnothing$. Define i_{n+1} to be an integer greater than j_n such that

$$D_{2n} \cap K_{i_n} \cap M(w_{i_{n+1}}, x_{i_{n+1}}) \neq \varnothing.$$

Let Π_n be the arc in B_1 from c_n to c. Define Λ_n to be a polygonal arc segment in Dom $M(w_{i_{n+1}}, x_{i_{n+1}})$ from (Bd Y) $\setminus H$ to $D_{2n} \cap K_{i_n}$ that crosses Π_n an odd number of times.

Let Σ_n be a simple closed curve in

$$Y \cup \Lambda_n \cup D_{2n} \cup \text{Dom } M(w_{i_{n-1}}, x_{i_{n-1}})$$

such that $\Lambda_n \subset \Sigma_n$, $H \cap \Sigma_n \subset D_{2n}$, and $\Pi_n \cap \Sigma_n \subset \Lambda_n$. Since Π_n crosses Σ_n an odd number of times, Σ_n separates c from c_n in R^2.

Define Ω_n to be the c-component of $R^2 \setminus \Sigma_n$. Let $E_n = \Omega_n \cap (H \setminus D_{2n})$ and $F_n = H \setminus (D_{2n} \cup E_n)$.

To complete the inductive step define c_{n+1}, C_{n+1}, D_{2n+1}, D_{2n+2} satisfying (32_{n+1})$-$(36_{n+1}), (27_n), and (28_n) when $G_{2n+1} = D_{2n+1} \cap H$ and $G_{2n+2} = D_{2n+2} \cap H$. It follows from the existence of $\{G_n\}_{n=1}^{\infty}$ that Knaster's continuum is a continuous image of H [**8**, Theorem 1].

Since Knaster's continuum is indecomposable, H contains an indecomposable continuum Φ [**11**, Theorem 4, p. 208]. According to a theorem of J. Krasinkiewicz [**10**, Theorem 3.1], Φ has a composant Ψ with the property that

(37) no arc segment in $R^2 \setminus \Psi$ has an endpoint in Ψ.

Note that

(38) $\Psi \cap P(u) = \varnothing$.

To see this assume the contrary. Let t be a point of $\Psi \cap P(u)$. Since $\Psi \subset H \subset M \setminus \text{Int } Y$, it follows from (37) that $t \notin Y$. Let C be the t-component of $P(u) \setminus Y$. By (37), $\Psi \cap \text{Dom } C \neq \varnothing$, and this contradicts (26). Hence (38) is true.

Since M is arcwise connected, there exists an arc A in M that intersects both Ψ and $M \setminus \Psi$. By (37), there exist points r and s of $A \cap \Psi$ such that $M(r, s) \not\subset \Psi$. Let B be a continuum in Ψ that contains $\{r, s\}$. Note that $A \cup B$ separates R^2 [13, Theorem 22, p. 175]. By (37), each component of $R^2 \setminus (A \cup B)$ intersects Ψ. Since $\Psi \subset L$, it follows from (38) that $A \cup P$ contains a simple closed curve, and this violates the unique arcwise connectivity of M. This contradiction completes the proof of Lemma 2. □

LEMMA 3. *For each positive integer n, $P(a_n)$ intersects $M \setminus L$.*

PROOF. Suppose $P(a_n) \subset L$ for some positive integer n. Assume without loss of generality that $P \subset L$. It follows from the proof of Lemma 2 (with Cases 1.2, 2.2, and 3.2 deleted) that this assumption involves a contradiction. □

We assume without loss of generality that D (in Lemma 2) is the unbounded complementary domain of L and $P \subset \mathrm{Cl}\, D$. We also assume without loss of generality that $a_1 \in D$ (Lemma 3).

Let L' be a subcontinuum of L. Define D' to be the unbounded complementary domain of L'. Note that $D \subset D'$.

Let X be the nonseparating plane continuum $R^2 \setminus D'$. Since M is arcwise connected, there is an arc segment S in $M \cap D'$ with an endpoint s in X. Since M does not contain a simple closed curve, we can assume without loss of generality that $P \cap \mathrm{Cl}\, S = \varnothing$.

Sieklucki [15, Lemma 5.5, p. 267] proved that X has the following properties. There exists a sequence $\{Q_n\}_{n=1}^{\infty}$ of disks in R^2 such that $X = \bigcap_{n=1}^{\infty} Q_n$ and for each n,

(i) $Q_{n+1} \subset \mathrm{Int}\, Q_n$,

(ii) the boundary B_n of Q_n is a polygonal simple closed curve with consecutive vertices $b_n^1, b_n^2, \ldots, b_n^{\mu_n}, b_n^{\mu_n+1} = b_n^1$, and

(iii) for $j = 1, 2, \ldots, \mu_n$, the interval in B_n from b_n^j to b_n^{j+1} has diameter less than 2^{-n}.

For every b_n^j ($n = 1, 2, \ldots$ and $j = 1, 2, \ldots, \mu_n$) there exists a vertex $b_{n+1}^{\nu(j)}$ such that

(iv) the interval N_n^j in R^2 from b_n^j to $b_{n+1}^{\nu(j)}$ has diameter less than 2^{-n},

(v) $N_n^j \setminus \{b_n^j, b_{n+1}^{\nu(j)}\} \subset (\mathrm{Int}\, Q_n) \setminus Q_{n+1}$, and

(vi) $N_n^j \cap N_n^k = \varnothing$ for each integer $k \neq j$ $(1 \leqslant k \leqslant \mu_n)$.

Let $N = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\mu_n} N_n^j$. Note that each component of N is a half-open arc in $R^2 \setminus X$ with an endpoint in X.

Let m be a given positive integer. Define N_m to be the union of all components of N that intersect B_m. Let O_m be a subset of N_m that is maximal with respect to the property that each component of O_m is a component of N_m and each pair of components of O_m with a common endpoint is separated in $Q_m \setminus X$ by another pair of components of O_m.

Let $c_m^1, c_m^2, \ldots, c_m^{\xi_m}, c_m^{\xi_m+1} = c_m^1$ denote the consecutive vertices of B_m that belong to O_m. Since X is not degenerate, we can assume without loss of generality that $\xi_m > 3$. Assume without loss of generality that $B_m \cap S \neq \varnothing$.

Let n be an integer greater than m. Define E to be the closure of a component of $Q_n \setminus (O_m \cup X)$. We call E an *(m, n)-link* on X. The polygonal arc $B_n \cap E$ is called the *bottom* of E. The two components of $E \cap O_m$ are called the *sides* of E. Note that the sides of E are half-open arcs in $Q_n \setminus X$ with distinct endpoints in X. The diameter of the union of the sides of E is less than 2^{3-m}.

For $j = 1, 2, \ldots, \xi_m$, let E_j be the (m, n)-link whose sides are contained in the components of O_m that intersect $\{c_m^j, c_m^{j+1}\}$.

Suppose there exist two (m, n)-links E and F that have a common side such that $Q_n \cap S \subset E \cup F$ and Cl S misses the closure of each uncommon side of E and F. Change the indexing of the (m, n)-links (if necessary) so that $E = E_1$, $F = E_{\xi_m}$, and each pair of consecutive links has a common side.

Define F_1 to be the closure of the component of $(E_1 \cup E_{\xi_m}) \setminus (S \cup X)$ that contains a side of E_2. Let $F_j = E_j$ for $1 < j < \xi_m$. Define F_{ξ_m} to be the closure of the component of $(E_1 \cup E_{\xi_m}) \setminus (S \cup X)$ that contains a side of $E_{\xi_m - 1}$. We call $\mathfrak{F} = \{F_j\colon 1 \leqslant j \leqslant \xi_m\}$ an *m-chain* on (X, S). We call F_1 and F_{ξ_m} the *end links* of $\mathfrak{F}$. Each F_j $(1 < j < \xi_m)$ is called an *interior link* of $\mathfrak{F}$. Let T be the arc in Cl S that is irreducible between s and B_n. The half-open arc $T \setminus \{s\}$ is called the common side of F_1 and F_{ξ_m}.

Since S is an arc segment, for each positive integer m, there exists an m-chain on (X, S).

LEMMA 4. *For each positive integer i, there exists an m-chain $\mathfrak{F}$ on (X, S) such that $m > i$ and no pair of consecutive links of $\mathfrak{F}$ contains* Bd X *in its union.*

PROOF. Let m and m' be integers such that $0 < m \leqslant m'$. Suppose $\mathfrak{F}$ is an m-chain on (X, S) and U is the union of a pair of consecutive links of an m'-chain on (X, S). Then the union of some pair of consecutive links of $\mathfrak{F}$ contains $U \cap$ Bd X. Hence it is sufficient to show that there exists an m-chain $\mathfrak{F}$ on (X, S) such that no pair of consecutive links of $\mathfrak{F}$ contains Bd X in its union.

Assume that for each positive integer m, every m-chain on (X, S) has a pair of consecutive links whose union contains Bd X. Then for each m, there exist a positive number ε_m, a pair of consecutive links E_m, F_m of an m-chain on (X, S), and an arc segment A_m in $B_m \cup O_m \cup S$ such that $\{\varepsilon_m\}_{m=1}^{\infty}$ converges to zero, Bd $X \subset E_m \cup F_m$, A_m has diameter less than ε_m and contains the uncommon sides of E_m and F_m, and $A_m \cup X$ separates (Int E_m) $\setminus X$ from $R^2 \setminus Q_m$ in R^2.

For each positive integer m, let x_m and y_m be the endpoints of A_m. Note that for each m, $\{x_m, y_m\} \subset \mathrm{Bd}\, X$. For each m, let W_m be the complementary domain of $A_m \cup X$ whose closure contains $E_m \cup F_m$. Let y be a limit point of $\{y_m\}_{m=1}^{\infty}$.

The continuum Bd X is nonaposyndetic at y with respect to each point of $(\mathrm{Bd}\, X) \setminus \{y\}$ [**9**]. For assume otherwise. Then a continuum Y, an open disk G, and a point z of $(\mathrm{Bd}\, X) \setminus \mathrm{Cl}\, G$ exist such that

$$y \in G \cap \mathrm{Bd}\, X \subset Y \subset (\mathrm{Bd}\, X) \setminus \{z\}.$$

Let Z be an open disk such that $z \in Z \subset R^2 \setminus (G \cup Y)$.

Let i be an integer such that $B_i \cap Z \neq \varnothing$. Define m to be an integer greater than i such that $A_m \subset G$. Let p be a point of $Z \cap (Q_i \setminus Q_m)$. Let q be a point of $W_m \cap Z$.

There exists a polygonal arc I in $Q_i \setminus X$ from p to q such that $A_m \cap I$ is finite and I crosses A_m at each point of $A_m \cap I$. Since A_m separates p from q in $Q_i \setminus X$, I crosses A_m an odd number of times. It follows that $I \cup Z$ contains a simple closed curve that separates x_m from y_m in R^2. Since $\{x_m, y_m\} \subset Y \subset R^2 \setminus (I \cup Z)$, this violates the connectivity of Y. Hence Bd X is nonaposyndetic at y with respect to each point of $(\mathrm{Bd}\, X) \setminus \{y\}$.

According to a theorem of H. E. Schlais [**14**, Theorem 9], [**8**, Theorem 4], Bd X contains an indecomposable continuum Φ. Let Ψ be a composant of Φ with the property that no arc segment in $R^2 \setminus \Psi$ has an endpoint in Ψ [**10**, Theorem 3.1].

Note that $\Psi \cap P = \varnothing$. To see this assume there is a point u of P in Ψ. By Lemma 3, there is a point v of $P(u)$ in D. Let J be an arc in D that is irreducible between $M[a_1, u]$ and $M[u, v]$. Since $M[a_1, v) \cap P(v) = \varnothing$ and each complementary domain of $J \cup M[a_1, v]$ intersects Ψ, for each positive integer n, $J \cap P(a_n) \neq \varnothing$. Thus $J \cap L \neq \varnothing$, and this contradicts the definition of J. Hence $\Psi \cap P = \varnothing$.

From the last paragraph in the proof of Lemma 2, we see that the existence of Ψ implies that M contains a simple closed curve. This contradiction completes the proof of Lemma 4. □

LEMMA 5. *Suppose F is an element of an m-chain $\mathfrak{F}$, u is a point of $(P \cap \mathrm{Int}\, F) \setminus X$, v is a point of $P \setminus F$, and* St $\mathfrak{F}$ *contains $M[u, v]$. Then $M[u, v]$ intersects the closure of a side of F.*

PROOF. Assume $M[u, v]$ misses the closure of each side of F. By Lemma 3, there is a point y of $P(v)$ in D. Let x be a point of $X \cap M[u, v] \cap \mathrm{Bd}\, F$ such that every arc in $M[u, x]$ from $P \setminus X$ to x intersects Int F, and every arc in $M[x, y]$ from x to $P \setminus X$ intersects $P \setminus F$. Let J be an arc in D that is irreducible between $M[a_1, x]$ and $M[x, y]$.

The continuum $X \cap \mathrm{Bd}\, F$ straddles every arc in P that contains x and has both endpoints in D. Consequently each complementary domain of $J \cup M[a_1, y]$ intersects $X \cap \mathrm{Bd}\, F$. Since $M[a_1, y) \cap P(y) = \varnothing$, it follows that

$J \cap L \neq \emptyset$, and this contradicts the definition of J. Hence $M[u, v]$ intersects the closure of a side of F. □

In the remaining part of this section we assume $L = L'$. Hence $D = D'$ and L is the boundary of the nonseparating plane continuum X.

Let $\mathcal{F} = \{F_j: 1 \leqslant j \leqslant \xi_m\}$ be an m-chain on (X, S). Let $\mathcal{G}$ be the set of all elements F_j of $\mathcal{F}$ such that for each point u of P, $P(u)$ intersects $(\text{Int } F_j) \setminus X$. It follows from Lemma 3 that $\mathcal{G}$ is not empty. Let F_i and F_k be the first and last links, respectively, of $\mathcal{F}$ that belong to $\mathcal{G}$.

Lemma 6. *Suppose B is an arc segment in $F_j \setminus X$ $(i < j < k)$ that has an endpoint in X. Then there exists a point u of P such that each arc in $P(u)$ that intersects both* $(\text{Int } F_i) \setminus X$ *and* $(\text{Int } F_k) \setminus X$ *also intersects B.*

Proof. Let $\mathcal{H}$ be an m-chain on (X, S) such that B intersects the bottom of a link of $\mathcal{H}$. Define u to be a point of P such that $P(u) \subset (\text{St } \mathcal{H}) \setminus ((\text{Cl } B) \setminus B)$.

Suppose there is an arc $M[v, w]$ in $P(u) \setminus B$ that intersects both $(\text{Int } F_i) \setminus X$ and $(\text{Int } F_k) \setminus X$. Assume without loss of generality that $v \in (\text{Int } F_i) \setminus X$ and $w \in (\text{Int } F_k) \setminus X$.

Let V be the v-component of $(\text{St } \mathcal{H}) \setminus (B \cup S \cup X)$. Note that V misses F_k and contains $(\text{St } \mathcal{H}) \cap ((\text{Int } F_i) \setminus X)$. Since $M[v, w] \cap \text{Cl}(B \cup S) = \emptyset$, the continuum $X \cap \text{Bd } V$ straddles each subarc of $M[v, w]$ that has one endpoint in V and the other endpoint in $M \setminus \text{Cl } V$.

Since $F_i \in \mathcal{G}$, there is a point z of $P(w)$ in $(\text{Int } F_i) \setminus X$. Let J be an arc in V that is irreducible between $M[v, w]$ and $M[w, z]$. Each complementary domain of $J \cup M[v, z]$ intersects $X \cap \text{Bd } V$. Since $M[v, z) \cap P(z) = \emptyset$, it follows that $J \cap L \neq \emptyset$, and this contradicts the definition of J. Hence each arc in $P(u)$ that intersects $(\text{Int } F_i) \setminus X$ and $(\text{Int } F_k) \setminus X$ intersects B. □

It follows from Lemma 6 that $\mathcal{G}$ is the subchain $\{F_j: i \leqslant j \leqslant k\}$ of $\mathcal{F}$.

Definition. Suppose K is an arcwise connected subset of M that is contained in $(\text{St } \mathcal{G}) \setminus \text{Bd}(X \cup \text{St } \mathcal{F})$ and intersects $\text{Int } F_i$ and $\text{Int } F_k$. Then K is a *trace* of $\mathcal{G}$ if for each arc A in K, there exists a function g of A into $\mathcal{G}$ with the following properties:

(1) For each point a of A, $a \in g(a)$.

(2) If a and b are points of A and $g(a) \neq g(b)$, then $M[a, b]$ intersects a side of $g(a)$ and the interior of each link of $\mathcal{G}$ between $g(a)$ and $g(b)$ (with respect to the index ordering of $\mathcal{G}$).

Definition. An arcwise connected set K *agrees* with $\mathcal{G}$ if K is a trace of $\mathcal{G}$, $\mathcal{G} \setminus \{F_i\}$, $\mathcal{G} \setminus \{F_k\}$, or $\mathcal{G} \setminus \{F_i, F_k\}$.

Lemma 7. *There exists a point u of $P \cap$* $\text{Int } F_i$ *such that $P(u)$ is a trace of $\mathcal{G}$.*

Proof. Let $W = \{x \in X: x$ is an endpoint of a side of a link of $\mathcal{G}\}$. Define u to be a point of $P \cap ((\text{Int } F_i) \setminus X)$ such that $P(u)$ is contained in St $\mathcal{F}$ and misses $W \cup \mathrm{Bd}(X \cup \mathrm{St}\,\mathcal{F})$ and $\bigcup\{(\mathrm{Int}\, F_j) \setminus X: 1 \leqslant j \leqslant i$ or $k < j \leqslant \xi_m\}$.

Using Lemma 5, we define a function g^* of $P(u)$ onto $\mathcal{G}$ such that

(i) $v \in g^*(v)$ for each point v of $P(u)$ and

(ii) if v and w are points of $P(u)$ and $g^*(v) \neq g^*(w)$, then $M[v, w]$ intersects a side of $g^*(v)$ and the interior of each link of $\mathcal{G}$ between $g^*(v)$ and $g^*(w)$.

By considering the restriction of g^* to each arc in $P(u)$, we see that $P(u)$ is a trace of $\mathcal{G}$. □

3. Principal result.

THEOREM. *If M is a uniquely arcwise connected plane continuum, then M has the fixed-point property.*

PROOF. Assume there exists a continuous function f of M into M that moves each point of M. Let ε be a positive number such that

(1) $\rho(z, f(z)) > \varepsilon$ for each point z of M.

According to Borsuk [5], there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points of M such that for each n,

(2) $\rho(a_n, a_{n+1}) = \varepsilon/3$ [5, p. 19, (4_n)],

(3) if $z \in M(a_n, a_{n+1})$, then $\rho(a_n, z) < \varepsilon/3$ [5, p. 19, (5_n)],

(4) $M[a_1, a_n] \cap M[a_n, a_{n+1}] = \{a_n\}$ for $n > 1$ [5, p. 19, (11)], and

(5) $\{a_n, a_{n+1}\} \subset M[a_1, f(a_n))$ [5, p. 19, (7_n), (13)].

For each positive integer n, let ψ_n be a homeomorphism of the half-open real line interval $[n, n+1)$ onto $M[a_n, a_{n+1})$. For each point x of $[1, +\infty)$, let $\psi(x) = \psi_n(x)$ if $n \leqslant x < n + 1$.

Let $P = \bigcup_{n=2}^{\infty} M[a_1, a_n)$. It follows from (4) that ψ is a one-to-one continuous function of $[1, +\infty)$ onto P.

The function ψ determines a linear ordering $\prec$ of P with a_1 as the first point. As in §2, for each point u of P, we let $P(u)$ denote $\{v \in P: u = v$ or $u \prec v\}$.

For each point u of P,

(6) $u \in M[a_1, f(u)]$.

To see this assume $u \notin M[a_1, f(u)]$. Suppose $u \in M[a_n, a_{n+1})$. Since M does not contain a simple closed curve, $a_{n+1} \notin M[a_1, f(u)]$. By (1), (2), and (3), $a_{n+1} \notin f[M[u, a_{n+1}]]$. Thus $M[a_1, f(u)] \cup f[M[u, a_{n+1}]]$ misses a_{n+1} and contains an arc that runs from a_1 to $f(a_{n+1})$, and this contradicts (5). Hence (6) holds.

By (2), $P \not\subset M[a_1, f(a_1)]$. Since M does not contain a simple closed curve, there exists a point a of P such that $P(a) \cap M[a_1, f(a_1)] = \{a\}$.

Next we prove that

(7) $a \in f[M[a_1, a)]$.

Statement (7) is obviously true if $f(a_1) = a$, so we assume $f(a_1) \neq a$. Suppose $a \in M[a_n, a_{n+1})$. By (5), $a_2 \in M[a_1, f(a_1))$. Hence $M[a_1, a_2] \subset$

$M[a_1, f(a_1))$ and $n > 1$. Note that $a_{n+1} \notin M[a_1, f(a_1)]$ and $f(a_1) \notin M[a_1, a_{n+1}]$. By (5), $a_{n+1} \in M[a_1, f(a_n))$. Since M does not contain a simple closed curve, it follows that $M[a, f(a_n)] \cap M[a, f(a_1)] = \{a\}$.

Suppose (7) is false. Then $a \notin f[M[a_1, a_n]]$. Consequently

$$M[a, f(a_1)] \cup M[a, f(a_n)] \cup f[M[a_1, a_n]]$$

contains a simple closed curve, and this violates the unique arcwise connectivity of M. Hence (7) is true.

For each point c of $P(a)$,

(8) $c \in f[M[a_1, c)]$.

To establish (8) assume the contrary. Let y be the point of $P(a)$ that is the greatest lower bound of $\{z \in P(a)\colon z \notin f[M[a_1, z)]\}$ relative to $\prec$. By (7) and the continuity of f, there exists a point x of $M[a_1, y)$ such that $f(x) = y$. Assume without loss of generality that $y \notin f[M(x, y)]$.

Suppose $x \in M[a_i, a_{i+1})$ and $y \in M[a_n, a_{n+1})$. By (1), (2), and (3), $n > i + 1$. Since M does not contain a simple closed curve, $M[y, a_{n+1}] \cap f[M[x, a_{i+1}]] = \{y\}$. By (5), $a_{n+1} \in M[a_1, f(a_n))$. Therefore $M[y, f(a_n)] \cap M[y, f(a_{i+1})] = \{y\}$. Since $y \notin f[M(x, y)]$, it follows that

$$M[y, f(a_n)] \cup M[y, f(a_{i+1})] \cup f[M[a_{i+1}, a_n]]$$

contains a simple closed curve, and this violates the unique arcwise connectivity of M. Hence (8) is true.

For each integer $i > 1$,

(9) there exists a positive integer n such that $P(a_n) \cap f[M[a_1, a_i]] = \varnothing$.

To see this assume there exists an integer $i > 1$ such that for each positive integer n, $P(a_n) \cap f[M[a_1, a_i]] \neq \varnothing$. Since M does not contain a simple closed curve, there exists a positive integer j such that $P(a_j) \subset f[M[a_1, a_i]]$. Since $f[M[a_1, a_i]]$ is a dendrite, Cl $P(a_j)$ is a dendrite [**11**, Theorem 4, p. 301], and this contradicts (2) (see Theorem 5 of [**11**, p. 302]). Therefore (9) is true.

Let $L = \bigcap_{n=1}^{\infty} \mathrm{Cl}\, P(a_n)$. It follows from (2) that L is not degenerate. Hence L is a continuum.

According to Lemma 2, there exist a complementary domain D of L and a positive integer α such that $P(a_\alpha) \subset \mathrm{Cl}\, D$. Assume without loss of generality that $P \subset \mathrm{Cl}\, D$, $a_1 \in D$ (Lemma 3), and D is the unbounded complementary domain of L.

Let X be the continuum $R^2 \setminus D$. Since M is arcwise connected, there is an arc segment S in $M \cap D$ with an endpoint in X. Since M does not contain a simple closed curve, we can assume without loss of generality that $P \cap \mathrm{Cl}\, S = \varnothing$.

Using Sieklucki's nested sequence of polygonal disks (described in §2 above), define a sequence $\{\mathcal{F}_m\}_{m=1}^{\infty}$ with the property that for each m, $\mathcal{F}_m$ is an m-chain on (X, S) refined by $\mathcal{F}_{m+1}$.

For each positive integer m, let $\mathcal{G}_m$ be the set of all elements F of $\mathcal{F}_m$ such that for each point u of P, $P(u)$ intersects $(\mathrm{Int}\, F) \setminus X$. By Lemma 6, for each m, $\mathcal{G}_m$ is a subchain of $\mathcal{F}_m$. Note that if m and n are integers and $0 < m < n$, then $\mathcal{G}_n$ refines $\mathcal{G}_m$ and each end link of $\mathcal{G}_m$ contains an end link of $\mathcal{G}_n$.

For each positive integer m, let $G_1^m, G_2^m, \ldots, G_{\lambda_m}^m$ be the consecutive links of $\mathcal{G}_m$.

By Lemma 7, for each positive integer m, there exists a point u_m of $P \cap \mathrm{Int}\, G_1^m$ such that $P(u_m)$ is a trace of $\mathcal{G}_m$. Hence for each m, $L \subset \mathrm{St}\, \mathcal{G}_m$. Since M does not contain a simple closed curve, it follows from the proof of Lemma 6 that for each m,

(10) no arc segment in $(M \setminus X) \cap \mathrm{St}\{G_i^m\colon 1 < i < \lambda_m\}$ has an endpoint in L.

For each positive integer m, there exists an arc B_m in $P(u_m)$ such that $f[B_m] \subset (\mathrm{St}\, \mathcal{G}_m) \setminus X$ and $f[B_m]$ is a trace of $\mathcal{G}_m$. To see this let v be a point of $P(u_m) \cap \mathrm{Int}\, G_{\lambda_m}^m$. By (8), (9), and (10), there exist points w and x of $P(v)$ such that $f(w) \in P(w) \cap \mathrm{Int}\, G_1^m$, $f(x) \in P(x) \cap \mathrm{Int}\, G_{\lambda_m}^m$, and the arc $M[f(w), f(x)]$ is between L and $M[u_m, v]$ in $\mathrm{St}\, \mathcal{G}_m$. By (6), $M[u_m, v] \cap f[M[w, x]] = \varnothing$. Since M does not contain a simple closed curve, it follows from (10) that there exists a subarc B_m of $M[w, x]$ such that $f[B_m] \subset (\mathrm{St}\, \mathcal{G}_m) \setminus X$ and $f[B_m]$ is a trace of $\mathcal{G}_m$.

Note that X has the following property:

Reduction Property. The continuum X does not separate R^2 and there exist an arc segment S in $M \setminus X$ with an endpoint in X, a sequence $\{A_m\}_{m=1}^{\infty}$ of arcs in P converging to Bd X, and a sequence $\{\mathcal{G}_m\}_{m=1}^{\infty}$ of chains such that for each m,

(i) $\mathcal{G}_m$ is a subchain of an m-chain on (X, S),

(ii) $\mathcal{G}_{m+1}$ refines $\mathcal{G}_m$,

(iii) each end link of $\mathcal{G}_m$ contains an end link of $\mathcal{G}_{m+1}$,

(iv) A_m agrees with $\mathcal{G}_m$, and

(v) either $f[A_m] \subset (\mathrm{St}\, \mathcal{G}_m) \setminus X$ or there exists a subarc B_m of A_m such that $f[B_m] \subset (\mathrm{St}\, \mathcal{G}_m) \setminus X$ and $f[B_m]$ agrees with $\mathcal{G}_m$.

Next we prove that X contains a continuum that is irreducible with respect to the Reduction Property. Assume $\{X_n\}_{n=1}^{\infty}$ is a nested sequence of nonseparating plane continua in X. For each n, assume there exist an arc segment S_n in $M \setminus X_n$ that has an endpoint in X_n, a sequence $\{A_m^n\}_{m=1}^{\infty}$ of arcs in P converging to Bd X_n, and a sequence $\{\mathcal{G}_m^n\}_{m=1}^{\infty}$ of chains with the following property. For each m, $\mathcal{G}_m^n$ is a subchain of an m-chain on (X_n, S_n), $\mathcal{G}_{m+1}^n$ refines $\mathcal{G}_m^n$, each end link of $\mathcal{G}_m^n$ contains an end link of $\mathcal{G}_{m+1}^n$, A_m^n agrees with $\mathcal{G}_m^n$, and either $f[A_m^n] \subset (\mathrm{St}\, \mathcal{G}_m^n) \setminus X_n$ or there exists a subarc B_m^n of A_m^n such that $f[B_m^n] \subset (\mathrm{St}\, \mathcal{G}_m^n) \setminus X_n$ and $f[B_m^n]$ agrees with $\mathcal{G}_m^n$. Let $X_0 = \bigcap_{n=1}^{\infty} X_n$. According to the Brouwer reduction theorem [**16**, p. 17], it is sufficient to prove that X_0 is a continuum with the Reduction Property.

Since f is continuous, for each positive integer n, either $f[\text{Bd } X_n] \subset \text{Bd } X_n$ or $\text{Bd } X_n \subset f[\text{Bd } X_n]$. Since $\{\text{Bd } X_n\}_{n=1}^{\infty}$ converges to $\text{Bd } X_0$, it follows that $f[\text{Bd } X_0] \subset \text{Bd } X_0$ or $\text{Bd } X_0 \subset f[\text{Bd } X_0]$. Since f moves each point of M, $\text{Bd } X_0$ is not degenerate. Hence X_0 is a continuum.

Since $R^2 \setminus X_0 = \bigcup_{n=1}^{\infty} R^2 \setminus X_n$ and for each n, $R^2 \setminus X_n$ is connected, it follows that

(11) $R^2 \setminus X_0$ is connected.

Since M is arcwise connected, there is an arc segment S_0 in $M \setminus X_0$ with an endpoint in X_0. Assume without loss of generality that $P \cap \text{Cl } S_0 = \varnothing$.

Define a sequence $\{\mathcal{F}_m^0\}_{m=1}^{\infty}$ with the property that for each m, $\mathcal{F}_m^0$ is an m-chain on (X_0, S_0) refined by $\mathcal{F}_{m+1}^0$.

There exists a sequence $\{\mathcal{G}_m^0\}_{m=1}^{\infty}$ of chains such that for each m,

(12) $\mathcal{G}_m^0$ is a subchain of $\mathcal{F}_m^0$,

(13) $\mathcal{G}_{m+1}^0$ refines $\mathcal{G}_m^0$,

(14) each end link of $\mathcal{G}_m^0$ contains an end link of $\mathcal{G}_{m+1}^0$, and

(15) there exist integers i_m and j_m ($j_m > m$) such that

(i) $\text{St } \mathcal{G}_{i_m}^{j_m} \subset (\text{St } \mathcal{G}_m^0) \setminus \text{Bd}(X_0 \cup \text{St } \mathcal{F}_m^0)$,

(ii) the interior of each interior link of $\mathcal{G}_m^0$ contains the sides of two consecutive links of $\mathcal{G}_{i_m}^{j_m}$,

(iii) no endpoint of a side of a link of $\mathcal{G}_m^0$ belongs to $A_{i_m}^{j_m} \cup f[A_{i_m}^{j_m}]$ (recall (6)), and

(iv) the Hausdorff distance [**11**, p. 47] from $A_{i_m}^{j_m}$ to $\text{Bd } X_{j_m}$ is less than m^{-1}.

For each positive integer m, let $A_m^0 = A_{i_m}^{j_m}$. Since $\{\text{Bd } X_n\}_{n=1}^{\infty}$ converges to $\text{Bd } X_0$, it follows from (15(iv)) that

(16) $\{A_m^0\}_{m=1}^{\infty}$ converges to $\text{Bd } X_0$.

By (15(i)–(iii)) and Lemma 5, for each positive integer m,

(17) A_m^0 agrees with $\mathcal{G}_m^0$, and either

(18) $f[A_m^0] \subset (\text{St } \mathcal{G}_m^0) \setminus X_0$, or

(19) there exists a subarc B_m^0 of A_m^0 ($B_m^0 = B_{i_m}^{j_m}$) such that $f[B_m^0] \subset (\text{St } \mathcal{G}_m^0) \setminus X_0$ and $f[B_m^0]$ agrees with $\mathcal{G}_m^0$.

It follows from (11)–(14) and (16)–(19) that X_0 has the Reduction Property. Hence there exists a subcontinuum of X that is irreducible with respect to the Reduction Property.

For convenience we assume that

(20) no proper subcontinuum of X_0 has the Reduction Property.

According to Lemma 4, there exists a positive integer β such that $\varepsilon > 2^{1-\beta}$ and no pair of consecutive links of $\mathcal{G}_\beta^0$ contains $\text{Bd } X_0$ in its union.

By (6), $S_0 \cap f[P(u)] = \varnothing$ for some point u of P. Assume without loss of generality that for each integer $m \geqslant \beta$,

(21) $S_0 \cap f[A_m^0] = \varnothing$.

Let $G_1, G_2, \ldots, G_\gamma$ be the consecutive links of $\mathcal{G}_\beta^0$. For $i = 1, 2, \ldots, \gamma - 1$, let $V_i = \bigcup_{j=1}^{i} G_j$, and let W_i be the common side of G_i and G_{i+1}.

Let $\mathcal{W} = \{W_i: 2 \leqslant i \leqslant \gamma - 2\}$. Note that each element of $\mathcal{W}$ has diameter less than ε.

For $m = \beta, \beta + 1, \ldots$ and $i = 2, 3, \ldots, \gamma - 2$, let $W_i^m = A_m^0 \cap W_i$.

DEFINITION. A point x of W_i^m is *sent back* by f if $f(x) \in V_i$; otherwise, x is *sent forward* by f.

DEFINITION. The arc A_m^0 has the *switch property* if a component of $A_m^0 \setminus \mathrm{St}\, \mathcal{W}$ has endpoints in $\mathrm{St}\, \mathcal{W}$ that are sent in opposite directions by f.

Statement (18) is true for only finitely many integers $m \geqslant \beta$. To see this assume the contrary. Suppose without loss of generality that (18) is true for each integer $m \geqslant \beta$.

For each integer $m \geqslant \beta$, if f sends two points of $\bigcup_{i=2}^{\gamma-2} W_i^m$ in opposite directions, then, by (1), (18), and (21), A_m^0 has the switch property.

Suppose that for infinitely many integers $m \geqslant \beta$, two points of $\bigcup_{i=2}^{\gamma-2} W_i^m$ are sent in opposite directions by f. Then infinitely many elements of $\{A_m^0\}_{m=\beta}^{\infty}$ have the switch property. Assume without loss of generality that there exists a component G of $(\mathrm{St}\, \mathcal{G}_\beta^0) \setminus (S_0 \cup X_0 \cup \mathrm{St}\, \mathcal{W})$ such that for each integer $m \geqslant \beta$, A_m^0 has the switch property on a component T_m of $A_m^0 \setminus \mathrm{St}\, \mathcal{W}$ that is contained in $\mathrm{Cl}\, G$.

For each integer $m \geqslant \beta$, we have three cases.

Case 1. Suppose $f[\mathrm{Cl}\, T_m] \subset G$. Then $\mathrm{Cl}\, G$ is a link of $\mathcal{G}_m^0$ and T_m has an endpoint in each side of $\mathrm{Cl}\, G$.

Case 2. Suppose $f[\mathrm{Cl}\, T_m]$ intersects two components of $(\mathrm{St}\, \mathcal{G}_\beta^0) \setminus (G \cup S_0 \cup X_0)$. Then $\mathrm{Cl}\, G$ is a link of $\mathcal{G}_\beta^0$ and there exists an arc A in $\mathrm{Cl}\, T_m$ such that $f[A] \subset \mathrm{Cl}\, G$ and $f[A]$ intersects each side of $\mathrm{Cl}\, G$.

Case 3. Suppose $f[\mathrm{Cl}\, T_m]$ intersects only one component of $(\mathrm{St}\, \mathcal{G}_\beta^0) \setminus (G \cup S_0 \cup X_0)$. Then there exist an element W_i of $\mathcal{W}$ in $\mathrm{Cl}\, G$ and an arc A in $\mathrm{Cl}\, T_m$ with an endpoint in W_i such that $f[A] \subset \mathrm{Cl}\, G$ and $W_i \cap f[A] \neq \varnothing$.

Since one of these three cases holds for infinitely many elements of $\{T_m\}_{m=\beta}^{\infty}$, there is a continuum Y in $X_0 \cap \mathrm{Cl}\, G$ with the following properties. A sequence $\{H_m\}_{m=1}^{\infty}$ of arcs in P converging to Y and a sequence $\{\mathcal{H}_m\}_{m=1}^{\infty}$ of chains exist such that for each positive integer m,

(22) $\mathrm{St}\, \mathcal{H}_m \subset \mathrm{Cl}\, G$,

(23) $\mathcal{H}_m$ is a subchain of $\mathcal{G}_{m+\beta}^0$,

(24) $\mathcal{H}_{m+1}$ refines $\mathcal{H}_m$,

(25) each end link of $\mathcal{H}_m$ contains an end link of $\mathcal{H}_{m+1}$,

(26) H_m agrees with $\mathcal{H}_m$, and

(27) either $f[H_m] \subset (\mathrm{St}\, \mathcal{H}_m) \setminus X_0$ or there exists a subarc I_m of H_m such that $f[I_m] \subset (\mathrm{St}\, \mathcal{H}_m) \setminus X_0$ and $f[I_m]$ agrees with $\mathcal{H}_m$.

Let U be the complementary domain of Y that contains D. Let X' be the nonseparating plane continuum $R^2 \setminus U$. Note that $Y = \mathrm{Bd}\, X'$.

Since M is arcwise connected, there exists an arc segment S' in $M \setminus X'$ with an endpoint in X'. Since M does not contain a simple closed curve, we can assume without loss of generality that $P \cap \mathrm{Cl}\, S' = \emptyset$.

Define a sequence $\{\mathcal{J}_m\}_{m=1}^{\infty}$ with the property that for each m, $\mathcal{J}_m$ is an m-chain on (X', S') refined by $\mathcal{J}_{m+1}$.

There exists a sequence $\{\mathcal{K}_m\}_{m=1}^{\infty}$ of chains such that for each m,

(28) $\mathcal{K}_m$ is a subchain of $\mathcal{J}_m$,

(29) $\mathcal{K}_{m+1}$ refines $\mathcal{K}_m$,

(30) each end link of $\mathcal{K}_m$ contains an end link of $\mathcal{K}_{m+1}$, and

(31) there exists an integer i_m such that

(i) $\mathrm{St}\, \mathcal{K}_{i_m} \subset (\mathrm{St}\, \mathcal{K}_m) \setminus \mathrm{Bd}(X' \cup \mathrm{St}\, \mathcal{J}_m)$,

(ii) the interior of each interior link of $\mathcal{K}_m$ contains the sides of two consecutive links of $\mathcal{K}_{i_m}$, and

(iii) no endpoint of a side of a link of $\mathcal{K}_m$ belongs to $H_{i_m} \cup f[H_{i_m}]$.

It follows from (26)–(31) that X' has the Reduction Property. Since Cl G is either a link or the union of two consecutive links of $\mathcal{G}^0_\beta$, Bd $X_0 \not\subset$ Bd X'. But X' and X_0 are nonseparating plane continua and Bd $X' \subset$ Bd X_0. Consequently X' is a proper subcontinuum of X_0, and this contradicts (20). Hence for all but finitely many integers $m \geqslant \beta$, f sends each point of $\bigcup_{i=2}^{\gamma-2} W_i^m$ in the same direction.

Assume without loss of generality that for each integer $m \geqslant \beta$, every point of $\bigcup_{i=2}^{\gamma-2} W_i^m$ is sent back by f.

The set $\{m: f[A^0_m \cap (G_1 \cup G_2)] \not\subset G_1 \cup G_2\}$ is finite; for otherwise, Case 3 (with Cl $G = G_1 \cup G_2$ and $W_i = W_2$) holds for infinitely many elements of $\{A^0_m\}_{m=\beta}^{\infty}$, and we have shown that this is impossible. Hence we can assume without loss of generality that for each integer $m \geqslant \beta$,

(32) $f[A^0_m \cap (G_1 \cup G_2)] \subset G_1 \cup G_2$.

For each positive integer m, let $\mathcal{K}_m$ be the chain consisting of all links of $\mathcal{G}^0_{m+\beta}$ that intersect $\mathrm{Int}(G_1 \cup G_2)$, and let H_m be an arc in $A^0_{m+\beta} \cap (G_1 \cup G_2)$ that intersects W_2 and agrees with $\mathcal{K}_m$.

The sequence $\{H_m\}_{m=1}^{\infty}$ converges to a continuum in $X_0 \cap (G_1 \cup G_2)$. For each positive integer m, $\mathrm{St}\, \mathcal{K}_m \subset G_1 \cup G_2$ and conditions (23)–(26) are satisfied. By (18) and (32), for each positive integer m, $f[H_m] \subset (\mathrm{St}\, \mathcal{K}_m) \setminus X_0$. According to the argument following (27), a proper subcontinuum of X_0 has the Reduction Property, and this contradicts (20). Hence (18) is true for at most finitely many integers.

Assume without loss of generality that (19) holds for each integer $m \geqslant \beta$.

By the preceding argument, for infinitely many integers $m \geqslant \beta$, A^0_m does not have the switch property on a component of $A^0_m \setminus \mathrm{St}\, \mathcal{W}$ that is contained in B^0_m. Hence we can assume without loss of generality that for each integer $m \geqslant \beta$, every point of $B^0_m \cap \bigcup_{i=2}^{\gamma-2} W_i^m$ is sent forward by f. It follows from a similar argument that for infinitely many integers $m \geqslant \beta$,

(33) for each point x of $B_m^0 \setminus V_i$ $(2 \leqslant i \leqslant \gamma - 2)$, $f(x) \notin V_i$.

We assume without loss of generality that (33) holds for each integer $m \geqslant \beta$.

Since for each positive integer m, $B_{m+\beta}^0$ has the properties given in (19) and (33), there exist a sequence $\{H_m\}_{m=1}^{\infty}$ of arcs in P converging to a continuum in $X_0 \cap (G_1 \cup G_2)$ and a sequence $\{\mathcal{H}_m\}_{m=1}^{\infty}$ of chains with the following properties. For each positive integer m, St $\mathcal{H}_m \subset G_1 \cup G_2$, conditions (23)–(26) are satisfied, $H_m \cap W_2 \neq \varnothing$, and there exists a subarc I_m of H_m such that $f[I_m] \subset (\text{St } \mathcal{H}_m) \setminus X_0$ and $f[I_m]$ agrees with $\mathcal{H}_m$.

By the argument following (27), a proper subcontinuum of X_0 has the Reduction Property, and this contradicts (20). Hence every continuous function of M into M has a fixed point. □

References

1. W. L. Ayres, *Some generalizations of the Scherrer fixed-point theorem*, Fund. Math. **16** (1930), 332–336.
2. H. Bell, *On fixed point properties of plane continua*, Trans. Amer. Math. Soc. **128** (1967), 539–548.
3. D. P. Bellamy, *Composants of Hausdorff indecomposable continua; a mapping approach*, Pacific J. Math. **47** (1973), 303–309.
4. R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132.
5. K. Borsuk, *A theorem on fixed points*, Bull. Acad. Polon. Sci. **2** (1954), 17–20.
6. C. L. Hagopian, *A fixed point theorem for plane continua*, Bull. Amer. Math. Soc. **77** (1971), 351–354.
7. ______, *Another fixed point theorem for plane continua*, Proc. Amer. Math. Soc. **31** (1972), 627–628.
8. ______, *λ connectivity and mappings onto a chainable indecomposable continuum*, Proc. Amer. Math. Soc. **45** (1974), 132–136.
9. F. B. Jones, *Concerning non-aposyndetic continua*, Amer. J. Math. **70** (1948), 403–413.
10. J. Krasinkiewicz, *Concerning the accessibility of composants of indecomposable plane continua*, Bull. Acad. Polon. Sci. **21** (1973), 621–628.
11. K. Kuratowski, *Topology*, Vol. 2, 3rd ed., Monografie Mat., Tom 21, PWN, Warsaw, 1961; English transl., Academic Press, New York; PWN, Warsaw, 1968.
12. L. Mohler, *The fixed-point property for homeomorphisms of* 1-*arcwise connected continua*, Proc. Amer. Math. Soc. **52** (1975), 451–456.
13. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R.I., 1962.
14. H. E. Schlais, *Non-aposyndesis and non-hereditary decomposability*, Pacific J. Math. **45** (1973), 643–652.
15. K. Sieklucki, *On a class of plane acyclic continua with the fixed point property*, Fund. Math. **63** (1968), 257–278.
16. G. T. Whyburn, *Analytic topology*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R.I., 1963.
17. G. S. Young, *Fixed-point theorems for arcwise connected continua*, Proc. Amer. Math. Soc. **11** (1960), 880–884.

Department of Mathematics, California State University, Sacramento, California 95819