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SMALL ZEROS OF ADDITIVE FORMS IN MANY VARIABLES[1]

BY

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ABSTRACT. It is shown that if s is large as a function of k and of $\varepsilon > 0$, then the diophantine equation $a_1x_1^k + \cdots + a_sx_s^k = b_1y_1^k + \cdots + b_sy_s^k$ with positive coefficients $a_1, \ldots, a_s, b_1, \ldots, b_s$ has a nontrivial solution in nonnegative integers $x_1, \ldots, x_s, y_1, \ldots, y_s$ not exceeding $m^{(1/k)+\varepsilon}$, where m is the maximum of the coefficients.

1. Introduction. A fairly direct application of the circle method shows that an equation

$$a_1x_1^k + \cdots + a_sx_s^k = 0 \tag{1.1}$$

where the coefficients $a_1, \ldots, a_s$ are not all of the same sign has a nontrivial solution in nonnegative integers $x_1, \ldots, s_s$, provided only that $s \geqslant c_1(k)$. (See, e.g., Davenport and Lewis [3], or Davenport [2].) As for the *size* of these solutions, it was shown by Pitman [6] that if the coefficients are as above, and each nonzero, and if $s \geqslant c_2(k)$ where $c_2(k)$ is explicitly given, then for given $\varepsilon > 0$ there is a nontrivial solution in nonnegative integers with

$$|a_1x_1^k| + \cdots + |a_sx_s^k| < c_3(k, \varepsilon)|a_1 \cdots a_s|^{k+\varepsilon}.$$

(Actually Pitman does not require the solutions to be nonnegative, hence for odd k allows the coefficients to be of arbitrary signs. But the result quoted is an immediate outcome of her method.) In particular, for $s \geqslant c_2(k)$ there is a solution with

$$\max(x_1, \ldots, x_s) < c_4(k)m^{c_5(k)} \tag{1.2}$$

where $m = \max(|a_1|, \ldots, |a_s|)$.

Under suitable conditions, and if s is very large, the estimate (1.2) may be considerably improved. Birch [1] combined Pitman's results with ideas contained in Linnik's elementary solution [4], [5] of Waring's problem to show that if k is *odd* and if $s \geqslant c_6(k, \varepsilon)$, then (1.1) has a nontrivial solution in

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integers $x_1, \ldots, x_s$, which may be of arbitrary sign, and which have

$$\max(|x_1|, \ldots, |x_s|) < c_7(k, \varepsilon) m^{(1/k)+\varepsilon}. \tag{1.3}$$

This estimate is probably not the best possible. If the right-hand side of (1.3) could be improved to $c_8(k, \varepsilon) m^{\varepsilon}$, it would have the important consequence that a form of odd degree k with real coefficients in enough variables can be made arbitrarily small for suitable (not all zero) integer values of the variables (see the remark in Birch [1]). For certain other applications in diophantine approximation, it is desirable to have a version where k may be even as well as odd, and where each variable is of a prescribed sign.

THEOREM. *Suppose $s \geqslant c_9(k, \varepsilon)$, and suppose $a_1, \ldots, a_s$, $b_1, \ldots, b_s$ are positive integers. Then the equation*

$$a_1 x_1^k + \cdots + a_s x_s^k = b_1 y_1^k + \cdots + b_s y_s^k \tag{1.4}$$

has a nontrivial solution in nonnegative integers $x_1, \ldots, x_s, y_1, \ldots, y_s$ having

$$\max(x_1, \ldots, x_s, y_1, \ldots, y_s) \leqslant m^{(1/k)+\varepsilon},$$

where

$$m = \max(a_1, \ldots, a_s, b_1, \ldots, b_s).$$

This is stronger than Birch's result even if k is odd. The estimate is essentially best possible, for every nontrivial solution of

$$a(x_1^k + \cdots + x_s^k) = b(y_1^k + \cdots + y_s^k)$$

with coprime positive a, b has $x_1^k + \cdots + x_s^k \geqslant b$ and $y_1^k + \cdots + y_s^k \geqslant a$, whence

$$\max(x_1, \ldots, x_s, y_1, \ldots, y_s) \geqslant c_{10}(k, s) m^{1/k}$$

where $m = \max(a, b)$. But it is conceivable that the Theorem holds with $c_9(k, \varepsilon)$ replaced by some $c_9'(k)$, and the conclusion replaced by $\max(x_1, \ldots, x_s, y_1, \ldots, y_s) \leqslant c_{11}(k, \varepsilon) m^{(1/k)+\varepsilon}$. The constant $c_9(k, \varepsilon)$ obtainable by our method is computable but very large.

Our proof is similar to Birch's in that we reduce the problem to that of finding solutions of

$$a_1 x_1^k + \cdots + a_s x_s^k - (b_1 y_1^k + \cdots + b_s y_s^k) = z$$

with very small z. But we shall employ the circle method instead of elementary estimates à la Linnik.

Our Theorem is applied by Schlickewei [7] to obtain a result about small values of indefinite diagonal forms with real coefficients.

2. An inductive argument.

PROPOSITION 1. *Let $\lambda \geqslant 1/k$, $\varepsilon > 0$ and $s \geqslant c_{12}(k, \lambda, \varepsilon)$. Let $a_1, \ldots, a_s$, $b_1, \ldots, b_s$ be as in the Theorem. Then* (1.4) *has a nontrivial solution in* *nonnegative integers* $x_1, \dots, x_s, y_1, \dots, y_s$ *with*

$$\max(x_1, \dots, x_s, y_1, \dots, y_s) \leqslant m^{\lambda+\varepsilon}. \tag{2.1}$$

The case $\lambda = 1/k$ is the Theorem. Moreover, since the truth of the proposition for a particular value of λ implies its truth for every $\lambda' > \lambda$, the proposition is in fact equivalent to the Theorem.

It will suffice to prove the proposition when m is large, say $m \geqslant c_{13}(k, \lambda, \varepsilon)$. For if $m < c_{13}$ and if s is large, then the a_i will assume the same value a at least m times, and the b_i will assume the same value b at least m times, so that a occurs at least b times and b occurs at least a times, and from this one can construct a solution of the equation consisting of zeros and ones only. Proposition 1 is true for *some* values of λ: By Pitman's estimate (1.2) it is true for $\lambda > c_5(k)$. Because of the $\varepsilon > 0$ in the formulation of the proposition, the set of numbers λ (this set depends only on k) for which the proposition holds is closed. Thus to prove Proposition 1 (and hence the Theorem), it will suffice to prove the following[2]

"Inductive Assertion." *If* $\lambda > 1/k$ *and if the proposition is true for* λ, *then it is true for some* $\lambda' < \lambda$.

In what follows, λ will be a fixed number $> 1/k$ for which the proposition holds. Pick μ so small that

$$1/k + 6c_5(k)\mu + 20\mu < \lambda \quad \text{and} \quad 22k\mu < 1, \tag{2.2}$$

and put

$$\lambda' = \max\left(\lambda\left(1 - \tfrac{1}{2}\mu\right) + \mu/2k,\ 1/k + 6c_5(k)\mu + 20\mu\right), \tag{2.3}$$

so that indeed $\lambda' < \lambda$. We proceed to prove the proposition for λ'.

Write

$$\delta = \min(\varepsilon/8\lambda', \varepsilon/4)$$

and divide the interval $0 \leqslant x \leqslant 1$ into a finite number of subintervals I of length not exceeding δ. If s is large, one of these intervals I will be such that many of the coefficients $a_1, \dots, a_s$ are of the type $a_i = m^{\alpha_i}$ with $\alpha_i \in I$. We may therefore suppose without loss of generality that $a_i/a_j \leqslant m^{\delta}$ $(1 \leqslant i, j \leqslant s)$. Similarly we may suppose that $b_i/b_j \leqslant m^{\delta}$ $(1 \leqslant i, j \leqslant s)$. Put $a = m^{\delta}\max(a_1, \dots, a_s)$ and $b = m^{\delta}\max(b_1, \dots, b_s)$. Let p_i, q_i, respectively, be the largest integers with

$$a_i p_i^k \leqslant a \quad \text{and} \quad b_i q_i^k \leqslant b \qquad (i = 1, \dots, s).$$

Now $a/a_i \geqslant m^{\delta}$, and if m is large (which we may suppose), then $p_i \geqslant 2^{-1/k}(a/a_i)^{1/k}$, so that $a_i p_i^k \geqslant \frac{1}{2}a$. Similarly, $b_i q_i^k \geqslant \frac{1}{2}b$.

With $a_i' = a_i p_i^k$, $b_i' = b_i q_i^k$ and $x_i = p_i x_i'$, $y_i = q_i y_i'$ $(i = 1, \dots, s)$, (1.4)

[2]A reader who finds our nonconstructive argument distasteful should be able to replace it by a constructive one.

becomes

$$a_1'x_1'^k + \cdots + a_s'x_s'^k = b_1'y_1'^k + \cdots + b_s'y_s'^k. \tag{2.4}$$

If Proposition 1 holds for λ' and for the particular equation (2.4), then we have a nontrivial nonnegative solution with

$$\max(x_1', \ldots, x_s', y_1', \ldots, y_s') \leqslant (\max(a, b))^{\lambda' + (\varepsilon/4)}$$
$$\leqslant m^{(1+\delta)(\lambda'+(\varepsilon/4))} \leqslant m^{\lambda'+(\varepsilon/2)}.$$

But clearly $a_i \geqslant am^{-2\delta}$, so that $p_i \leqslant p_i^k \leqslant m^{2\delta} \leqslant m^{\varepsilon/2}$, whence $x_i \leqslant m^{\lambda'+\varepsilon}$ $(i = 1, \ldots, s)$, and similarly, $y_i \leqslant m^{\lambda'+\varepsilon}$, as desired.

(2.4) was special since $\frac{1}{2}a \leqslant a_i' \leqslant a$ and $\frac{1}{2}b \leqslant b_i' \leqslant b$. Thus we have the

REDUCTION. *In proving Proposition* 1 *for* λ' *we may suppose that*

$$\tfrac{1}{2}a \leqslant a_i \leqslant a \quad \textit{and} \quad \tfrac{1}{2}b \leqslant b_i \leqslant b \qquad (i = 1, \ldots, s)$$

for certain a, b.

3. Two cases. In what follows, h will be the integer

$$h = c_{12}(k, \lambda, \varepsilon)$$

occurring in Proposition 1, and s will be assumed to be much larger than h. Write

$$\nu = \mu/2k. \tag{3.1}$$

We distinguish two cases.

A. *There is a subset of* h *elements among* $a_1, \ldots, a_s$, *say* $a_1, \ldots, a_h$, *and there is a subset of* h *elements among* $b_1, \ldots, b_s$, *say* $b_1, \ldots, b_h$, *and there are natural integers*

$$p_1, \ldots, p_h, q_1, \ldots, q_h \leqslant m^\nu, \tag{3.2}$$

such that

$$d = \text{g.c.d.}\,(a_1p_1, \ldots, a_hp_h, b_1q_1, \ldots, b_hq_h) \geqslant m^\mu.$$

In this case put $x_i = p_ix_i'$, $y_i = q_iy_i'$ $(i = 1, \ldots, h)$ and $x_{h+1} = y_{h+1} = \cdots = x_s = y_s = 0$. After division by d, (1.4) becomes

$$a_1'x_1'^k + \cdots + a_h'x_h'^k = b_1'y_1'^k + \cdots + b_h'y_h'^k, \tag{3.3}$$

where $a_i' = a_ip_i^k/d$ and $b_i' = b_iq_i^k/d$ $(i = 1, \ldots, h)$. Because of the truth of the proposition for λ, and by our choice of h, (3.3) has a nontrivial nonnegative solution with

$$\max(x_1', \ldots, x_h', y_1', \ldots, y_h') \leqslant (m^{1+k\nu-\mu})^{\lambda+\varepsilon},$$

so that

$$\max(x_1, \ldots, x_s, y_1, \ldots, y_s) \leqslant m^{(1+k\nu-\mu)(\lambda+\varepsilon)+\nu} \leqslant m^{\lambda'+\varepsilon}.$$

We are thus reduced to case

B. *For any h elements, say $a_1, \ldots, a_h$, among $a_1, \ldots, a_s$, and for any h elements, say $b_1, \ldots, b_h$, among $b_1, \ldots, b_s$, and given* (3.2), *we have*

$$\text{g.c.d.}\,(a_1p_1, \ldots, a_hp_h, b_1q_1, \ldots, b_hq_h) < m^{\mu}. \tag{3.4}$$

Condition B depends on h, m, μ, ν, and if ν is given by (3.1), it is a condition B(k, h, m, μ).

PROPOSITION 2. *Let $h \geq 1$, $k \geq 1$, and*

$$0 < \mu < 1/22k. \tag{3.5}$$

Let $0 < a, b \leq m$ and let $a_1, \ldots, a_s, b_1, \ldots, b_s$ be integers with

$$\tfrac{1}{2}a \leq a_i \leq a, \quad \tfrac{1}{2}b \leq b_i \leq b \qquad (i = 1, \ldots, s)$$

with property B(k, h, m, μ). *Then if $s \geq c_{14}(k, h, \mu)$, the equation*

$$a_1x_1^k + \cdots + a_sx_s^k - \left(b_1y_1^k + \cdots + b_sy_s^k\right) = z \tag{3.6}$$

has a solution in nonnegative integers $x_1, \ldots, x_s, y_1, \ldots, y_s, z$ with

$$\max(x_1, \ldots, x_s, y_1, \ldots, y_s) \leq m^{(1/k)+20\mu}, \qquad z \leq m^{6\mu}.$$

This proposition implies the Inductive Assertion, as we now proceed to show. For let λ, μ, λ', ν be as above, in particular, with (2.2) (whence (3.5)), (2.3), (3.1). We may suppose that we are in the case B $=$ B(k, h, m, μ) with $h = c_{12}(k, \lambda, \varepsilon)$ where $\varepsilon > 0$. Suppose that

$$s = c_2(k)c_{14}(k, h, \mu) = nu,$$

say. After a change of notation, (1.4) becomes

$$\sum_{i=1}^{n} \left(a_{i1}x_{i1}^k + \cdots + a_{iu}x_{iu}^k - b_{i1}y_{i1}^k - \cdots - b_{iu}y_{iu}^k\right) = 0. \tag{3.7}$$

For each i, $1 \leq i \leq n$, the coefficients $a_{i1}, \ldots, a_{iu}, b_{i1}, \ldots, b_{iu}$ satisfy the conditions of Proposition 2. Hence there are nonnegative $x'_{i1}, \ldots, x'_{iu}$, $y'_{i1}, \ldots, y'_{iu}$, not all zero, having

$$a_{i1}x_{i1}'^k + \cdots + a_{iu}x_{iu}'^k - b_{i1}y_{i1}'^k - \cdots - b_{iu}y_{iu}'^k = z_i \tag{3.8}$$

with $\max(x'_{i1}, \ldots, x'_{iu}, y'_{i1}, \ldots, y'_{iu}) \leq m^{(1/k)+20\mu}$ and $0 \leq z_i \leq m^{6\mu}$. No. Hold it! Keep $0 \leq z_i \leq m^{6\mu}$ for $i = 1, \ldots, n-1$, but ask for $-m^{6\mu} \leq z_n \leq 0$. This is not asking for too much, in view of the symmetry in the $+$ and $-$ terms in (3.8). If some $z_i = 0$, we get a small solution of (3.7) straightaway. If $z_1, \ldots, z_n$ are each nonzero, then Pitman's estimate (1.2) gives nonnegative $w_1, \ldots, w_n$, not all zero, with

$$z_1w_1^k + \cdots + z_nw_n^k = 0$$

having $\max(w_1, \ldots, w_n) \leq c_4(k)m^{6\mu c_5(k)}$. Putting $x_{ij} = w_ix'_{ij}$, $y_{ij} = w_iy'_{ij}$ $(1 \leq i \leq n, 1 \leq j \leq u)$ we obtain a nontrivial solution of (3.7) with

$$\max(x_{ij}, y_{ij}) \leq c_4(k)m^{(1/k)+20\mu+6c_5(k)\mu} \leq m^{\lambda'}$$

if m is large. Thus Proposition 1 is true for λ'.

4. Weyl's inequality. Write $e(x) = e^{2\pi i x}$.

LEMMA 1. *Suppose that*

$$|\alpha - u/q| < 1/q^2 \quad \textit{where } q > 0, (u, q) = 1.$$

Then for $\eta > 0$,

$$\left|\sum_{x=1}^{N} e(\alpha x^k)\right| \leqslant c_{15}(k, \eta) N^{1+\eta}\left(N^{-1/K} + q^{-1/K} + (q/N^k)^{1/K}\right)$$

where $K = 2^{k-1}$.

PROOF. This is the well-known "Weyl Inequality." See, e.g., [2, Lemma 1].

COROLLARY. *Suppose that* $N \geqslant c_{16}(k, \eta)$, $C \geqslant N^{1-(1/K)+\eta}$ *and*

$$\left|\sum_{x=1}^{N} e(\alpha x^k)\right| \geqslant C.$$

Then there is a natural

$$q \leqslant (N/C)^K N^\eta \quad \textit{with } \|\alpha q\| \leqslant (N/C)^K N^{\eta - k},$$

where $\| \cdots \|$ *denotes the distance to the nearest integer.*

PROOF. We have $N^{k-\eta}(C/N)^K \geqslant N^{k-\eta-1+K\eta} \geqslant 1$. According to Dirichlet we may pick coprime q, u with

$$0 < q \leqslant N^{k-\eta}(C/N)^K$$

and

$$|\alpha q - u| = \|\alpha q\| \leqslant (N/C)^K N^{\eta-k}.$$

Now

$$N^{1+(\eta/2K)}\left(N^{-1/K} + (q/N^k)^{1/K}\right) \leqslant N^{1-(1/K)+(\eta/2K)} + CN^{-\eta/2K}$$

is of smaller order of magnitude than C if N is large. Thus by Lemma 1 (with $\eta/2K$ in place of η) we obtain that $N^{1+(\eta/2K)}q^{-1/K} \geqslant c_{17}(k, \eta)C$, whence that $q \leqslant (N/C)^K N^\eta$ if N is large.

5. Application of the Circle Method. Note that it will suffice to prove Proposition 2 for large m, say for $m \geqslant c_{18}(k, h, \mu)$. Put

$$A = [b^{1/k} m^{20\mu}], \quad B = [a^{1/k} m^{20\mu}], \quad H = [m^{6\mu}], \tag{5.1}$$

where $[\cdots]$ denotes the integer part. If m is sufficiently large, then

$$A \geqslant 2^{-1/k} b^{1/k} m^{20\mu}, \qquad B \geqslant 2^{-1/k} a^{1/k} m^{20\mu}. \tag{5.2}$$

Write Z for the number of solutions of (3.6) in integers $x_1, \ldots, x_s, y_1, \ldots, y_s,$

z subject to

$$1 \leqslant x_1, \dots, x_s \leqslant A, \qquad 1 \leqslant y_1, \dots, y_s \leqslant B, \qquad 1 \leqslant z \leqslant H. \tag{5.3}$$

We will show that under the assumptions made in the proposition, Z is positive; in fact we will show that Z is at least of the order of magnitude of $HA^sB^sa^{-1}b^{-1}m^{-20k\mu}$.

Recall the definition (3.1) of ν and pick $\eta > 0$ with

$$\eta \leqslant 1/2K, \qquad \eta(1 + 20\mu) < \tfrac{1}{2}\nu, \tag{5.4}$$

and pick s so large that

$$s > (6K/\nu) + h. \tag{5.5}$$

All of the parameters h, μ, ν, η, s will be fixed from now on. We shall employ the 0-notation or $\ll$ notation with the understanding that the implicit constants may depend on k, h, μ, ν, η, s, but they will be independent of $a_1, \dots, a_s, b_1, \dots, b_s, a, b, m$. We are going to show that the hypotheses of Proposition 2 imply

$$Z \gg HA^sB^sa^{-1}b^{-1}m^{-20k\mu}. \tag{5.6}$$

The number Z will be estimated by the Circle Method. Note that this method has already been used implicitly, via Pitman's estimate (1.2). We have

$$Z = \int_0^1 f(\alpha)\, d\alpha \tag{5.7}$$

where

$$f(\alpha) = \sum_{z=1}^{H} \sum_{x_1=1}^{A} \cdots \sum_{x_s=1}^{A} \sum_{y_1=1}^{B} \cdots \sum_{y_s=1}^{B} e\big(\alpha\big(a_1x_1^k + \cdots + a_sx_s^k - b_1y_1^k - \cdots - b_sy_s^k - z\big)\big). \tag{5.8}$$

We define the *major arcs* to be the intervals modulo 1 of the type

$$\mathfrak{M}_{qu}\colon |\alpha - u/q| < a^{-1}b^{-1}m^{-16k\mu}$$

where $q < m^{\mu}$ and $(q, u) = 1$.

LEMMA 2. *Suppose that* $|f(\alpha)| \geqslant HA^sB^sm^{-3}$. *Then* α *lies in a major arc.*

PROOF. The inequality of the hypothesis implies that

$$|S_1(\alpha)| \cdots |S_s(\alpha)|\, |T_1(\alpha)| \cdots |T_s(\alpha)| \geqslant A^sB^sm^{-3}, \tag{5.9}$$

where

$$S_i(\alpha) = \sum_{x=1}^{A} e(\alpha a_i x^k), \quad T_i(\alpha) = \sum_{y=1}^{B} e(\alpha b_i y^k). \tag{5.10}$$

If, say, $|S_1(\alpha)| \geqslant \cdots \geqslant |S_s(\alpha)|$, then the left-hand side of (5.9) is

$$\leqslant |S_h(\alpha)|^{s-h+1} A^{h-1} B^s,$$

so that (5.9) yields

$$|S_i(\alpha)| \geqslant |S_h(\alpha)| \geqslant A m^{-3/(s-h+1)} = C, \quad \text{say} \qquad (i = 1, \ldots, h).$$

Observe that

$$m^{3/(s-h+1)} \leqslant A^{1/6\mu(s-h+1)} \leqslant A^{(1/2K)} \leqslant A^{(1/K)-\eta}$$

by (5.2), (5.4), (5.5). We may apply the Corollary to Lemma 1 to each of the sums $S_1(\alpha), \ldots, S_h(\alpha)$ to obtain natural numbers $p_1, \ldots, p_h$ with

$$p_i \leqslant m^{3K/(s-h+1)} A^{\eta}, \quad \|\alpha a_i p_i\| \leqslant m^{3K/(s-h+1)} A^{\eta - k} \qquad (i = 1, \ldots, h).$$

Using (5.4) and (5.5) again we get

$$p_i \leqslant m^{\nu} \qquad (i = 1, \ldots, h), \tag{5.11}$$

$$\|\alpha a_i p_i\| \leqslant m^{\nu} A^{-k} \qquad (i = 1, \ldots, h). \tag{5.12}$$

Similarly, after a possible reordering of $b_1, \ldots, b_s$, there are natural $q_1, \ldots, q_h$ having

$$q_j \leqslant m^{\nu} \qquad (j = 1, \ldots, h), \tag{5.13}$$

$$\|\alpha b_j p_j\| \leqslant m^{\nu} B^{-k} \qquad (j = 1, \ldots, h). \tag{5.14}$$

There are integers $u_1, \ldots, u_h, v_1, \ldots, v_h$ with

$$\|\alpha a_i p_i\| = |\alpha a_i p_i - u_i| \qquad (i = 1, \ldots, h)$$

and

$$\|\alpha b_j q_j\| = |\alpha b_j q_j - v_j| \qquad (j = 1, \ldots, h).$$

Subtracting $a_i p_i$ times (5.14) from $b_j q_j$ times (5.12) and observing (5.11), (5.13) and (5.2), we obtain

$$\begin{aligned} |u_i b_j q_j - v_j a_i p_i| &\leqslant b m^{\nu} m^{\nu} A^{-k} + a m^{\nu} m^{\nu} B^{-k} \\ &\leqslant m^{2\nu - 18\mu k} + m^{2\nu - 18\mu k} < 1 \end{aligned}$$

if m is sufficiently large. Thus the $2h$ nonzero vectors $(a_i p_i, u_i)$ $(i = 1, \ldots, h)$ and $(b_j q_j, v_j)$ $(j = 1, \ldots, h)$ are proportional to each other. They are integer multiples of some vector (q, u) where $q > 0$ and q, u are coprime. Since q is a common divisor of $a_1 p_1, \ldots, a_h p_h, b_1 q_1, \ldots, b_h q_h$, condition (3.4) of case B yields $q < m^{\mu}$. If, say, the vector $(a_i p_i, u_i)$ is l_i times (q, u), then $l_i \geqslant \frac{1}{2} a q^{-1}$, whence

$$\begin{aligned} |\alpha q - u| = l_i^{-1} |\alpha a_i p_i - u_i| &\leqslant 2a^{-1} q \|\alpha a_i p_i\| \leqslant 2a^{-1} q m^{\nu} A^{-k} \\ &\leqslant 2qa^{-1} b^{-1} m^{\nu - 18k\mu} < qa^{-1} b^{-1} m^{-16k\mu} \end{aligned}$$

if m is large. Thus α lies in $\mathfrak{M}_{qu}$.

6. The major arcs. Since the major arcs do not overlap, and from Lemma 2, we obtain

$$Z = \sum_{q < m^\mu} \sum_{\substack{u=1 \\ (u,q)=1}}^{q} \int_{\mathfrak{M}_{qu}} f(\alpha)\, d\alpha + O(HA^sB^sm^{-3}). \tag{6.1}$$

LEMMA 3. *For $\alpha = u/q + \beta \in \mathfrak{M}_{qu}$ we have*

$$S_i(\alpha) = q^{-1}\hat{S}_i(u/q)I_i(\beta) + O(m^{5k\mu}) \qquad (i = 1, \ldots, s) \tag{6.2}$$

where

$$\hat{S}_i(u/q) = \sum_{x=1}^{q} e\left(\frac{a_i u}{q} x^k\right) \quad \text{and} \quad I_i(\beta) = \int_0^A e(a_i\beta\xi^k)\, d\xi.$$

PROOF. Write $x = qy + z$. Then

$$S_i(\alpha) = \sum_{z=1}^{q} e\left(\frac{a_i u}{q} z^k\right) \sum_y e\big(a_i\beta(qy + z)^k\big), \tag{6.3}$$

where the sum over y is over the integers in $1 \leqslant qy + z \leqslant A$. There will be a certain error if we replace the sum over y by the integral of $e(a_i\beta(q\zeta + z)^k)$ with respect to ζ, with the range of integration given by $0 \leqslant q\zeta + z \leqslant A$. The function

$$g(\zeta) = e\big(a_i\beta(q\zeta + z)^k\big)$$

has

$$|g'(\zeta)| \leqslant qa_i|\beta|A^{k-1}, \qquad |g(\zeta)| \leqslant 1$$

in this range, and this range is an interval of length A/q. Therefore

$$\left|\sum_y e\big(a_i\beta(qy + z)^k\big) - \int e\big(a_i\beta(q\beta + z)^k\big)\, d\zeta\right|$$
$$\leqslant (A/q)\big(qa_i|\beta|A^{k-1}\big) + 3 \leqslant A^ka|\beta| + 3$$
$$< A^kaa^{-1}b^{-1}m^{-16k\mu} + 3 < m^{4k\mu} + 3.$$

Taking the sum over z in (6.3) we get

$$S_i(\alpha) = \sum_{z=1}^{q} e\left(\frac{a_i u}{q} z^k\right) \int e\big(a_i\beta(q\zeta + z)^k\big)\, d\zeta + O(m^{5k\mu}).$$

The change of variables $\xi = q\zeta + z$ yields the desired result.

In analogy to Lemma 3 we obtain

$$T_i(\alpha) = q^{-1}\hat{T}_i(u/q)J_i(\beta) + O(m^{5k\mu}) \qquad (i = 1, \ldots, s) \tag{6.4}$$

where $\hat{T}_i$, J_i are defined in the obvious way.

LEMMA 4. *If $\mathfrak{M}$ is the totality of the major arcs, then*

$$\int_{\mathfrak{M}} f(\alpha)\, d\alpha = A^s B^s a^{-1} b^{-1} m^{-20k\mu} \mathfrak{S}(m^\mu, H) \mathfrak{J}(m^{4k\mu}) + O(HA^sB^sa^{-1}b^{-1}m^{-22k\mu}),$$

where the "singular series"

$$\mathfrak{S}(m^\mu, H) = \sum_{z=1}^{H} \sum_{q < m^\mu} \sum_{\substack{u=1\\(q,u)=1}}^{q} q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \hat{S}_s\left(\frac{u}{q}\right) \hat{T}_1\left(\frac{u}{q}\right) \cdots \hat{T}_s\left(\frac{u}{q}\right) e\left(-\frac{u}{q} z\right),$$

and the "singular integral"

$$\mathfrak{J}(m^{4k\mu}) = \int_{|\beta| < m^{4k\mu}} \prod_{i=1}^{s} \left(\int_0^1 e(\rho_i \xi_i^k \beta)\, d\xi_i \right) \prod_{j=1}^{s} \left(\int_0^1 e(-\sigma_i \zeta_i^k \beta)\, d\zeta \right) d\beta,$$

for certain constants $\rho_1, \ldots, \rho_s, \sigma_1, \ldots, \sigma_s$ in the interval $[\frac{1}{4}, 1]$.

PROOF. The integral in question is

$$\sum_{z=1}^{H} \sum_{q < m^\mu} \sum_{\substack{u=1\\(q,u)=1}}^{q} \int_{\mathfrak{M}_{qu}} S_1(\alpha) \ldots S_s(\alpha) T_1(\alpha) \ldots T_s(\alpha) e(-z\alpha)\, d\alpha. \tag{6.5}$$

If $\alpha = u/q + \beta$ lies in $\mathfrak{M}_{qu}$, then Lemma 3 and the trivial estimates $|I_i(\beta)| \leqslant A$, $|J_i(\beta)| \leqslant B$ yield

$$S_1(\alpha) \cdots S_s(\alpha) T_1(\alpha) \cdots T_s(\alpha) = q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \hat{S}_s\left(\frac{u}{q}\right) \hat{T}_1\left(\frac{u}{q}\right) \cdots \hat{T}_s\left(\frac{u}{q}\right) I_1(\beta) \cdots I_s(\beta) J_1(\beta) \cdots J_s(\beta) + O\left(A^sB^s \max(m^{5k\mu}A^{-1}, m^{5k\mu}B^{-1})\right).$$

The error term here is $O(A^sB^sm^{-15k\mu})$, and since $\mathfrak{M}_{qu}$ is of length $2a^{-1}b^{-1}m^{-16k\mu}$, the integral over $\mathfrak{M}_{qu}$ in (6.5) is

$$q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \hat{T}_s\left(\frac{u}{q}\right) e\left(-\frac{u}{q} z\right) \int_{|\beta| < a^{-1}b^{-1}m^{-16k\mu}} I_1(\beta) \cdots J_s(\beta) e(-\beta z)\, d\beta + O(A^sB^sa^{-1}b^{-1}m^{-30k\mu}). \tag{6.6}$$

In the integral in (6.6) we replace $e(-\beta z)$ by 1. The error is

$$\ll A^sB^sz(a^{-1}b^{-1}m^{-16k\mu})^2 \ll A^sB^sHa^{-2}b^{-2}m^{-32k\mu} \ll A^sB^sa^{-1}b^{-1}m^{-25k\mu}.$$

Thus the integral over $\mathfrak{M}_{qu}$ in (6.5) is

$$q^{-2s}\hat{S}_1\left(\frac{u}{q}\right)\cdots\hat{T}_s\left(\frac{u}{q}\right)\int_{|\beta|<a^{-1}b^{-1}m^{-16k\mu}} I_1(\beta)\cdots J_s(\beta)\,d\beta + O(A^sB^sa^{-1}b^{-1}m^{-25k\mu}). \quad (6.7)$$

To evaluate the integral in (6.7), put $\xi_i = A\xi_i'$ $(i = 1, \ldots, s)$, $\zeta_i = B\zeta_i'$ $(i = 1, \ldots, s)$ and $\beta = a^{-1}b^{-1}m^{-20k\mu}\beta'$. Then

$$a_i\beta\xi_i^k = (a_iA^k/abm^{20k\mu})\beta'\xi_i'^k = \rho_i\beta'\xi_i'^k \qquad (i = 1, \ldots, s),$$

say, where by (5.2), $\frac{1}{4} \leqslant \rho_i \leqslant 1$. Similarly, $-b_i\beta\zeta_i^k = -\sigma_i\beta'\zeta_i'^k$. The integral in (6.7) becomes

$$A^sB^sa^{-1}b^{-1}m^{-20k\mu}\mathfrak{J}(m^{4k\mu})$$

and the integral over $\mathfrak{M}_{qu}$ in (6.5) turns out to be

$$A^sB^sa^{-1}b^{-1}m^{-20k\mu}q^{-2s}\hat{S}_1(u/q)\cdots\hat{T}_s(u/q)e(-uz/q)\mathfrak{J}(m^{4k\mu}) + O(A^sB^sa^{-1}b^{-1}m^{-25k\mu}).$$

Taking the sum over z, q, u in (6.5) we obtain Lemma 4.

7. The singular integral. We have

$$\int_0^1 e(\rho_i\xi_i^k\beta)\,d\xi_i = k^{-1}\rho_i^{-1/k}\int_0^{\rho_i}\varphi_i^{-1+(1/k)}e(\varphi_i\beta)\,d\varphi_i$$

$$= k^{-1}(\rho_i\beta)^{-1/k}\int_0^{\rho_i\beta}\varphi_i^{-1+(1/k)}e(\varphi_i)\,d\varphi_i. \quad (7.1)$$

The last integral is bounded as a function of the upper limit of integration so that the integral on the left is $\ll \beta^{-1/k}$. It follows that as a function of m,

$$\mathfrak{J}(m^{4k\mu}) = \mathfrak{J}(\infty) + o(1), \quad (7.2)$$

where $\mathfrak{J}(\infty)$ is as $\mathfrak{J}(m^{4k\mu})$, but with the integral over β extended over the real line. Using the middle expression in (7.1) we get

$$\mathfrak{J}(\infty) = k^{-2s}(\rho_1\cdots\sigma_s)^{-1/k} \cdot\int_{-\infty}^{\infty}d\beta\int_0^{\rho_1}d\varphi_1\cdots\int_0^{\sigma_s}d\psi_s(\varphi_1\cdots\varphi_s\psi_1\cdots\psi_s)^{-1+(1/k)} \cdot e((\varphi_1+\cdots+\varphi_s-\psi_1-\cdots-\psi_s)\beta)$$

$$= k^{-2s}(\rho_1\cdots\sigma_s)^{-1/k} \cdot\lim_{\omega\to\infty}\int_0^{\rho_1}d\varphi_1\cdots\int_0^{\sigma_s}d\psi_s(\varphi_1\cdots\varphi_s\psi_1\cdots\psi_s)^{-1+(1/k)} \cdot\frac{\sin 2\pi\omega(\varphi_1+\cdots+\varphi_s-\psi_1-\cdots-\psi_s)}{\pi(\varphi_1+\cdots+\varphi_s-\psi_1-\cdots-\psi_s)},$$

as in [2, p. 27]. Continuing as in [2] we get

$$\mathfrak{J}(\infty) = k^{-2s}(\rho_1 \cdots \sigma_s)^{-1/k} \lim_{\omega\to\infty} \int_{-s}^{s} \Omega(\omega) \frac{\sin 2\pi\omega u}{\pi u}\, d\omega, \tag{7.3}$$

where

$$\Omega(\omega) = \underset{u<\varphi_1+\cdots+\varphi_s-\psi_1-\cdots-\psi_{s-1}<u+\sigma_s}{\int_0^{\rho_1} d\varphi_1 \cdots \int_0^{\rho_s} d\varphi_s \int_0^{\sigma_1} d\psi_1 \cdots \int_0^{\sigma_{s-1}} d\psi_{s-1}}$$
$$\cdot (\varphi_1 \cdots \varphi_s\psi_1 \cdots \psi_{s-1}(\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_{s-1} - u))^{-1+(1/k)}.$$

The limit in (7.3) equals

$$\Omega(0) \geqslant \underset{0<\varphi_1+\cdots+\varphi_s-\psi_1-\cdots-\psi_{s-1}<1/4}{\int_0^{1/4} d\varphi_1 \cdots \int_0^{1/4} d\varphi_s \int_0^{1/4} d\psi_1 \cdots \int_0^{1/4} d\psi_{s-1}}$$
$$\cdot (\varphi_1 \cdots \varphi_s\psi_1 \cdots \psi_{s-1}(\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_{s-1}))^{-1+(1/k)}$$
$$\gg 1.$$

Combining our estimates we find that for m sufficiently large,

$$\mathfrak{J}(m^{4k\mu}) \gg 1. \tag{7.4}$$

8. The singular series.

$$\mathfrak{S}(m^\mu, H) = \sum_{z=1}^{H} \sum_{q<m^\mu} \sum_{\substack{u=1\\(u,q)=1}}^{q} \sum_{x_1=1}^{q} \cdots \sum_{y_s=1}^{q} q^{-2s} e\Big(\frac{u}{q}(a_1x_1^k + \cdots - b_sy_s^k - z)\Big).$$

The summands with $q = 1$ give the contribution H.

When $q > 1$,

$$\sum_{z=1}^{H} e(-uz/q) \ll q,$$

so that the summands with fixed $q > 1$ contribute $\ll q^2$. Since Σq^2 over $q < m^\mu$ is $\ll m^{3\mu}$, we obtain

$$\mathfrak{S}(m^\mu, H) = H + O(m^{3\mu}) \gg H. \tag{8.1}$$

9. Conclusion. Combining (6.1), Lemma 4, (7.4) and (8.1) we get

$$Z \gg HA^sB^sa^{-1}b^{-1}m^{-20k\mu} + O(HA^sB^sm^{-3} + HA^sB^sa^{-1}b^{-1}m^{-22k\mu}).$$

Since $m^3 \geqslant abm \geqslant abm^{22k\mu}$ by (3.5), the error term here is smaller than the main term, and (5.6) follows.

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