THE RIEMANN PROBLEM IN GAS DYNAMICS[1]

BY

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ABSTRACT. We consider the Riemann problem (R.P.) for the 3×3 system of gas dynamics equations in a single space variable. We assume that the specific internal energy $e = e(v, s)$ (s = specific entropy, v = specific volume) satisfies the usual hypotheses, $p_v < 0, p_{vv} > 0, p_s > 0$ ($p = -e_v$ = pressure); we also assume some reasonable hypotheses about the asymptotic behavior of e. We call functions e satisfying these hypotheses *energy functions*.

THEOREM 1. *For any initial data* (U_l, U_r) $(U_l = (v_l, p_l, u_l), U_r = (v_r, p_r, u_r),$ u = *flow velocity), the R. P. has a solution.*

We introduce two conditions:

$$\frac{\partial}{\partial v} \, p(v, e) < \frac{p^2}{2e} \qquad (v, e > 0), \tag{I}$$

$$\frac{\partial}{\partial v} \, e(v, p) > -\frac{p}{2} \qquad (v, p > 0). \tag{II}$$

THEOREM 2. (I) *is necessary and sufficient for uniqueness of solutions of the R. P. Nonuniqueness persists under small perturbations of the initial data.*

(I) is implied by the known condition

$$\frac{\partial}{\partial v} \, e(v, p) > 0 \qquad (v, p > 0), \tag{$*$}$$

which holds for all usual gases. (I) implies (II). We construct energy functions e that violate (II), that satisfy (II) but violate (I), and that satisfy (I) but violate ($*$).

In all solutions considered, the shocks satisfy the entropy condition and the Lax shock conditions.

1. Introduction. In this paper we investigate existence and uniqueness of solutions of the Riemann problem (R. P.), with arbitrary initial data, for the equations of compressible fluid flow in one space variable. The results depend on the energy function $e = e(\tau, S)$ (τ = specific volume, S = specific entropy). In addition to the assumptions normally made (see (1.2)–(1.7)), we always assume some physically reasonable limiting behaviors of the energy

and pressure functions near the boundaries of the domains on which they are defined (see (1.8)–(1.10)); ideal gases, for example, always satisfy these assumptions. We prove (§3) that the R. P. always has a solution. However, solutions need not always be unique, even if they satisfy the entropy condition and the Lax shock conditions. Our analysis is based on the behavior of the Hugoniot curve H (see (1.20)ff.) in the (τ, p)-plane (p = pressure). It is sometimes assumed (see [1]) that H is *monotone* in the sense that along H, one of the variables τ, p is a decreasing function of the other. This holds for polytropic gases, but need not hold for ideal gases. (See *Ideal gases*, §6.) It turns out that whether uniqueness fails, and if so how drastically, depends on how much H deviates from this monotonicity. We introduce three conditions on e: WEAK, MEDIUM and STRONG (see Definition 1.4, Proposition 1.5), which we relate to the behavior of H (see Lemmas 4.1, 4.2, 4.10, 4.11, 4.12 and Conventions 4.6, 4.7). WEAK and STRONG are already known ([2], [4]). The following implications hold:

polytropic gas $\Rightarrow$ ideal gas $\Rightarrow$ STRONG $\Rightarrow$ MEDIUM $\Rightarrow$ WEAK,   (*)

where all implications are proper. In §4 we prove that whenever MEDIUM holds, solutions of the R. P. are unique. However, whenever MEDIUM is violated, it is possible to construct initial data (U_l, U_r) for which the R. P. has three distinct solutions; we call this particular situation *type* I *nonuniqueness*. We show that whenever WEAK fails, one cannot only obtain type I nonuniqueness, but also get nonuniqueness a different way, by a different construction; this is called *type* II *nonuniqueness*. The initial data that exhibit type II nonuniqueness also exhibit type I, and we then obtain five distinct solutions of the R. P. In §5 we show that for both types of nonuniqueness the data can always be chosen so that the nonuniqueness is stable under perturbation of the initial data. In §6, we show by constructing various examples of energy functions, that all the possible situations abstractly considered in §§1–5 do indeed occur, e.g., that the implications (*) are proper, and that it is possible to have five solutions of one R. P. For all solutions considered, the shocks always satisfy the entropy condition (see (1.18)) and the Lax shock conditions (see (1.19)).

The equations of compressible flow in one space variable (in Eulerian coordinates) are

$$\rho_t + (\rho u)_x = 0,$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0,$$

$$\left[\rho\left(\tfrac{1}{2}u^2 + e\right)\right]_t + \left[\rho u\left(\tfrac{1}{2}u^2 + e\right) + up\right]_x = 0, \qquad (1.1)$$

where $t \geqslant 0$ is time, x is position, ρ is density, p is pressure, e is specific internal energy, and u is flow velocity. We also let S = specific entropy,

T = temperature, and τ = specific volume ($= \rho^{-1}$). Here e is assumed to be a C^3 function $e(\tau, S)$ of τ and S, $p = -e_\tau$, and $T = e_S$. The letter g will always denote the function $-e_\tau$, i.e., $p = g(\tau, S)$. The function e is assumed to satisfy the following

Standard assumptions.

$$e > 0, \tag{1.2}$$

$$p > 0 \quad \text{(i.e., } g > 0\text{)}, \tag{1.3}$$

$$T > 0 \quad \text{(i.e., } e_S > 0\text{)}, \tag{1.4}$$

$$g_\tau < 0, \tag{1.5}$$

$$g_{\tau\tau} > 0, \tag{1.6}$$

$$g_S > 0. \tag{1.7}$$

In addition, we shall always assume that e satisfies the following *Limit assumptions.*

$$\lim_{S \to -\infty} g(\tau, S) = 0, \qquad \lim_{S \to +\infty} g(\tau, S) = +\infty, \tag{1.8}$$

$$\lim_{S \to -\infty} e(\tau, S) = 0, \qquad \lim_{S \to +\infty} e(\tau, S) = +\infty, \tag{1.9}$$

$$\lim_{\tau \to 0^+} e(\tau, p) = 0. \tag{1.10}$$

Here the expression "$e(\tau, p)$" requires definition. Suppose $\tau, p > 0$. Then by (1.8), there is an S such that $g(\tau, S) = p$, and by (1.7) S is unique; we denote $S \equiv S(\tau, p)$. Then we define

$$e(\tau, p) \equiv e(\tau, S(\tau, p)) \tag{1.11}$$

(whenever p is understood to be a pressure rather than an entropy; cf. [1, §64]). We also define $g_\tau(\tau, p) \equiv g_\tau(\tau, S(\tau, p))$, etc.

REMARK. If we stipulate that $S > 0$ and replace (1.8) and (1.9) by

$$\lim_{S \to 0^+} g(\tau, S) = 0, \qquad \lim_{S \to +\infty} g(\tau, S) = +\infty \tag{1.8$'$}$$

and

$$\lim_{S \to 0^+} e(\tau, S) = 0, \qquad \lim_{S \to +\infty} e(\tau, S) = +\infty, \tag{1.9$'$}$$

then it is easy to see that all results in this paper still hold, with the exception of that of §6.3; in this case an ideal gas (i.e., a function e satisfying (1.2)–(1.7), (1.8)$'$, (1.9)$'$, (1.10) and (6.75)) cannot exist.

Analogously, by (1.9) and (1.4) we can use ρ, e (i.e., τ, e) as independent variables so that S and p are functions of (ρ, e). If we let $U = (\rho, u, e)$, then the equations (1.1) can be written as

$$U_t + A(U)U_x = 0, \tag{1.12}$$

for a 3×3 matrix $A(U)$ with eigenvalues $u - c$, u and $u + c$, where c (the sound speed) is defined by

$$c^2 = f_\rho \qquad (1.13)$$

where $f(\rho, S) \equiv g(\rho^{-1}, S)$.

The Riemann Problem (R. P.) (see [1], [4], [5]) for (1.1) or (1.12) is the initial-value problem for (1.1) with data

$$U(x, 0) = \begin{cases} U_l, & \text{if } x < 0, \\ U_r, & \text{if } x > 0, \end{cases} \qquad (1.14)$$

where $U_l = (\rho_l, u_l, e_l)$, $U_r = (\rho_r, u_r, e_r)$ are arbitrary constant vectors.

By a solution of (1.1), (1.12) we mean a function $U(x/t)$ consisting of at most four constant states, U_l, U_1, U_3, U_r. Here U_l (on the left) is connected to U_1 (on the right) by either a 1-shock or a 1-rarefaction wave, U_1 (left) is connected to U_3 (right) by a contact discontinuity, and U_3 (left) is connected to U_r (right) by a 3-shock or a 3-rarefaction wave. (See [1] or [3] for the definition of these terms.)

From now on, we shall use the notation "$U_1 \xrightarrow{\text{1-shock}} U_2$" to express the fact that U_1 (on the left) is, or can be, connected to U_2 (on the right) by a 1-shock. We define similarly the expressions "$U_1 \xrightarrow{\text{1-wave}} U_2$", "$U_1 \xrightarrow{\text{contact}} U_2$", "$U_1 \xrightarrow{\text{3-shock}} U_2$" and "$U_1 \xrightarrow{\text{3-wave}} U_2$", where "1-wave" (resp. "3-wave") is just an abbreviation for "1-rarefaction wave" (resp. "3-rarefaction wave"). We also abbreviate the sentence "$U_1 \xrightarrow{\text{1-shock}} U_2$ or $U_1 \xrightarrow{\text{1-wave}} U_2$" by "$U_1 \xrightarrow{1} U_2$", etc. Thus for a solution of the R. P. with initial data (U_l, U_r) we must have U_1 and U_3 such that

$$U_l \xrightarrow{1} U_1 \xrightarrow{\text{contact}} U_3 \xrightarrow{3} U_r.$$

Finally, since (ρ, e) and (τ, p) determine each other, we can, and shall, describe states $U_1, U_2, \ldots$ by triples (τ_1, p_1, u_1), (τ_2, p_2, u_2), $\ldots$.

By definition, $U_1 \xrightarrow{\text{contact}} U_2$ iff $u_1 = u_2$ and $p_1 = p_2$. For shocks, we have the following well-known result (see [1]):

PROPOSITION 1.1. U_0 (left) can be connected to U (right) by a shock with shock speed s iff the following hold.

Jump conditions.

$$\rho v = \rho_0 v_0 \equiv m \neq 0, \qquad (1.15)$$

$$(\rho u)v + p = (\rho_0 u_0)v_0 + p_0, \qquad (1.16)$$

$$\rho\left(\tfrac{1}{2}u^2 + e\right)v + pu = \rho_0\left(\tfrac{1}{2}u_0^2 + e_0\right)v_0 + p_0 u_0, \qquad (1.17)$$

where $v \equiv u - s$, $v_0 \equiv u_0 - s$;

Entropy condition.

$$mS_0 \leqslant mS. \qquad (1.18)$$

Moreover, the shock is a 1-shock (resp. 3-shock) iff $s < u, u_0$ (resp. $s > u, u_0$).

In this paper, shocks will also be required to satisfy the
Lax shock conditions.

$$U_0 \xrightarrow{\text{1-shock}} U \Rightarrow v_0^2 > c_0^2 \text{ and } v^2 < c^2;$$

$$U \xrightarrow{\text{3-shock}} U_0 \Rightarrow v_0^2 > c_0^2 \text{ and } v^2 < c^2. \qquad (1.19)$$

(See (1.13).)

The *Hugoniot function* with center (τ_0, p_0) is defined by

$$H(\tau, p) \equiv e(\tau, p) - e(\tau_0, p_0) + (\tau - \tau_0)\frac{p + p_0}{2} \qquad (\tau, p > 0). \quad (1.20)$$

We define the *Hugoniot curve H* with center (τ_0, p_0) by

$$H \equiv \{(\tau, p) | H(\tau, p) = 0\}.$$

Let U, U_0 be constant states, and let H have center (τ_0, p_0). For $a > 0$, let $\mathcal{H}_a$ be the upper branch of the hyperbola, $(\tau_0 - \tau)(p - p_0) = a$. Let $\mathcal{H}'_a$ be the lower branch. (See Figure 1.) The importance of the Hugoniot curve lies in the following two facts.

PROPOSITION 1.2.

$$U_0 \xrightarrow{\text{1-shock}} U \quad \textit{iff } u < u_0 \textit{ and } (\tau, p) \in \mathcal{H}_{(u_0 - u)^2} \cap H.$$

PROPOSITION 1.3.

$$U \xrightarrow{\text{3-shock}} U_0 \quad \textit{iff } u > u_0 \textit{ and } (\tau, p) \in \mathcal{H}_{(u - u_0)^2} \cap H.$$

These will be proved in §2.

DEFINITION 1.4. $e = e(\tau, S)$ is an *energy function* iff it satisfies the Standard assumptions (1.2)–(1.7) and the Limit assumptions (1.8)–(1.10).
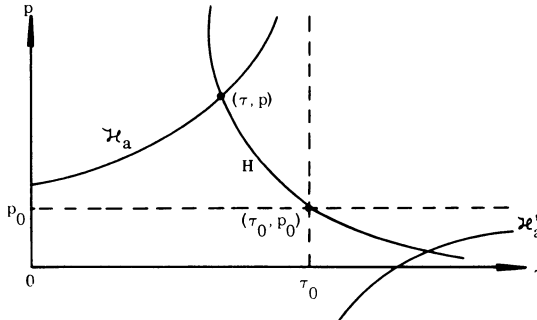

FIGURE 1. The hyperbola $H_a \cup H'_a$ and a Hugoniot curve H.

We introduce the following conditions on an energy function e (see (1.11)):

$$\text{STRONG} \quad \frac{\partial}{\partial \tau} e(\tau, p) \geqslant 0 \qquad (\tau, p > 0),$$

$$\text{MEDIUM} \quad \left(\frac{\partial p}{\partial \tau} \right)_e \leqslant \frac{p^2}{2e} \qquad (\tau, e > 0),$$

$$\text{WEAK} \quad \frac{\partial}{\partial \tau} e(\tau, p) \geqslant -\frac{p}{2} \quad (\tau, p > 0).$$

PROPOSITION 1.5. *The conditions* STRONG, MEDIUM *and* WEAK *have the following equivalent formulations*:

$$\text{STRONG} \quad \frac{\partial}{\partial p} e(\tau, p) \geqslant \frac{p}{-g_\tau(\tau, p)} \qquad (\tau, p > 0),$$

$$\text{MEDIUM} \quad \frac{\partial}{\partial p} e(\tau, p) \geqslant \frac{2e(\tau, p)}{-g_\tau(\tau, p) \cdot (2e(\tau, p)/p) + p} \qquad (\tau, p > 0),$$

$$\text{WEAK} \quad \frac{\partial}{\partial p} e(\tau, p) \geqslant \frac{p}{-2g_\tau(\tau, p)} \qquad (\tau, p > 0).$$

PROOF. We have

$$\frac{\partial}{\partial p} e(\tau, p) = \frac{\partial}{\partial p} e(\tau, S(\tau, p)) = e_S \frac{\partial}{\partial p} S(\tau, p) = \frac{e_S}{g_S}. \qquad (1.21)$$

Here we have abbreviated $e_S(\tau, S(\tau, p))$ $(= e_S(\tau, p))$ by e_S; we shall do this often, e.g., $g_\tau = g_\tau(\tau, p)$. In particular

$$\partial e(\tau, p)/\partial p > 0. \qquad (1.22)$$

Moreover

$$\frac{\partial}{\partial \tau} e(\tau, p) = \frac{\partial}{\partial \tau} e(\tau, S(\tau, p)) = e_\tau + e_S \frac{\partial}{\partial \tau} S(\tau, p).$$

But $p = g(\tau, S(\tau, p))$, so taking $\partial/\partial\tau$ of both sides yields $0 = g_\tau + g_S \partial S(\tau, p)/\partial\tau$; hence

$$\frac{\partial}{\partial \tau} e(\tau, p) = e_\tau + e_S \left(-\frac{g_\tau}{g_S} \right),$$
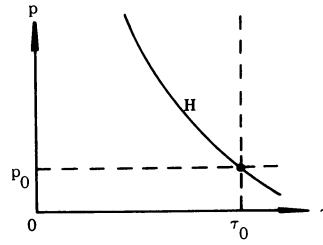
so

$$\frac{\partial}{\partial \tau} e(\tau, p) = -p + (-g_\tau) \frac{\partial}{\partial p} e(\tau, p), \qquad (1.23)$$

by (1.21). Solving (1.23) for $-g_\tau$, and substituting in the inequalities of Proposition 1.5, we obtain the result. $\square$

<u>polytropic</u>

H is monotone

existence and uniqueness

<u>MEDIUM or ideal</u>

τ is a function of p along H

$\forall a > 0$, $\mathcal{H}_a \cap H$ consists of one point

(H may not be monotone)

existence and uniqueness
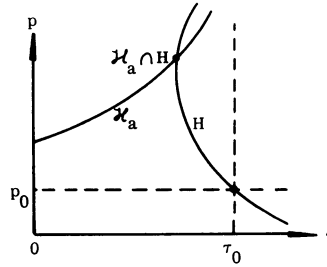
<u>WEAK and not-MEDIUM</u>

τ is a function of p along H

(*) $\exists a > 0 \ni \mathcal{H}_a \cap H$ contains more than one point

existence

(*) type I nonuniqueness

<u>not-WEAK</u>

(*) $\exists p > 0$, $\exists \tau_1, \tau_2 \in (0, \tau_0) \ni$
$\quad (\tau_1, p), (\tau_2, p) \in H$

existence

(*) both type I and type II nonuniqueness

FIGURE 2.   Hugoniot curves in different situations.

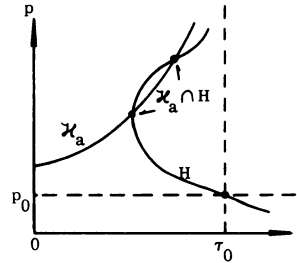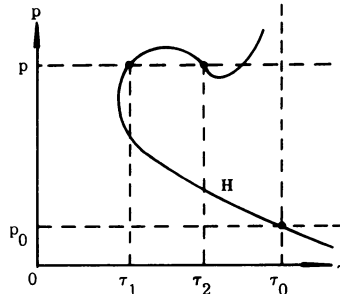LEMMA 1.6. *Let ϕ be defined on $(0, +\infty)$ with $\phi > 0$, $\phi' < 0$ and $\phi'' > 0$. Then for every $\tau > 0$,*

$$\phi(\tau)^2 < -2\phi'(\tau) \int_\tau^{+\infty} \phi.$$

PROOF. Straightforward.

PROPOSITION 1.7. STRONG $\Rightarrow$ MEDIUM $\Rightarrow$ WEAK.

PROOF. Apply Lemma 1.6 to the function $g(\cdot, S)$, note that

$$\int_\tau^{+\infty} g(v, S) \, dv < e(\tau, S),$$

and use Proposition 1.5.

We conclude with a rough summary, in table form, of what will be proved in the remainder of this paper regarding the connection between: (1), the conditions MEDIUM and WEAK; (2), the behavior of the Hugoniot curve H; and (3), existence and uniqueness of solutions of the R. P. The properties marked with a (∗) are proved to hold for *some* choice of (τ_0, p_0) as center of H, or of (U_l, U_r) as initial data; the others hold in general.

2. Properties of the Hugoniot curve. This section establishes some properties of H that hold for all energy functions e, under no additional assumptions. We first summarize some known results due to Weyl; these are proved in [1, §65].

PROPOSITION 2.1 (WEYL). *Let H have center (τ_0, p_0) and let $(\tau_1, p_1) \in H \setminus \{(\tau_0, p_0)\}$. Let $\mathfrak{R}$ be the line through (τ_0, p_0) and (τ_1, p_1), parametrized by*

$$p(s) = p_0 + as, \qquad \tau(s) = \tau_0 + bs,$$

where $a = p_1 - p_0$, $b = \tau_1 - \tau_0$. Then

$$\frac{d}{ds} S(\tau(s), p(s))|_{s=0} > 0; \qquad \frac{d}{ds} S(\tau(s), p(s))|_{s=1} < 0.$$

COROLLARY 2.2. *Let H have center (τ_0, p_0) and let $(\tau_1, p_1) \in H \setminus \{(\tau_0, p_0)\}$. Let L be the line through (τ_0, p_0) and (τ_1, p_1). Then L cannot be tangent to H at (τ_1, p_1); i.e., if the C^1 functions $\tau(t), p(t)$, t in a neighborhood of t_1, satisfy*

$$(\tau(t_1), p(t_1)) = (\tau_1, p_1), \qquad (\tau(t), p(t)) \in H \quad \text{for all } t,$$

then there cannot exist $\gamma \neq 0$ such that

$$(\tau'(t_1), p'(t_1)) = \gamma(\tau_1 - \tau_0, p_1 - p_0).$$

COROLLARY 2.3. *$dS \neq 0$ along H except at (τ_0, p_0); i.e., if $\tau_1, p_1, t_1, \tau(t), p(t)$ are as in Corollary 2.2, and $(\tau'(t_1), p'(t_1)) \neq 0$, then*

$$\frac{d}{dt} S(\tau(t), p(t))|_{t=t_1} \neq 0.$$

COROLLARY 2.4. *Whenever $\tau_1 < \tau_0$ and $(\tau_1, p_1) \in H$,*

$$- g_\tau(\tau_0, S_0) < \frac{p_1 - p_0}{\tau_0 - \tau_1} < - g_\tau(\tau_1, S_1);$$

thus if $U_0 \xrightarrow{\text{1-shock}} U_1$ or $U_1 \xrightarrow{\text{3-shock}} U_0$, then $v_0^2 > c_0^2$ and $v_1^2 < c_1^2$. (Cf. (1.19) and Propositions 1.1–1.3.)

The proofs that follow are long and tedious; hence we preface the formal proofs with an informal introduction. We define two subsets, Q and Q', of the (τ, p)-plane by

$$Q = \{(\tau, p) \mid 0 < \tau < \tau_0 \text{ and } p > p_0\},$$

$$Q' = \{(\tau, p) \mid \tau > \tau_0 \text{ and } 0 < p < p_0\}, \qquad (2.1)$$

where (τ_0, p_0) is the center of the Hugoniot curve H. We prove that grad $H \neq 0$ everywhere on H, and that $H \subset Q \cup Q' \cup \{(\tau_0, p_0)\}$.

DEFINITION 2.5. For all $\tau, p > 0$, define

$$\theta = \theta(\tau, p) = \frac{p - p_0}{\tau_0 - \tau} \quad \text{if } \tau \neq \tau_0;$$

$$\theta_0 = \theta(\tau_0, p_0) = - \frac{dp}{d\tau}\Big|_{(\tau_0, p_0)} \quad \text{along } H.$$

Then we find that the function θ is C^1 on H; we plan to use θ as parameter along H. We shall be concerned only with the "compression branch" $H \cap Q^{\text{cl}}$ of H. For all θ, let L_θ be the line through (τ_0, p_0) with slope $-\theta$. We want to prove that along H, (τ, p) is a C^1 function of θ; i.e., as L_θ sweeps through Q in the clockwise direction, starting at θ_0, there is always exactly one $(\tau(\theta), p(\theta)) \in H \cap L_\theta$. (See Figure 3.) This is true locally by Corollary 2.2 and the fact that grad $H \neq 0$ on H, since then $d\theta \neq 0$ along H. In particular, H is represented by $(\tau(\theta), p(\theta))$ uniquely in a neighborhood of (τ_0, p_0). Here $(\tau(\theta), p(\theta))$ is defined for θ in some neighborhood of θ_0. We try to define $(\tau(\theta), p(\theta))$ on the largest possible interval $[\theta_0, \theta_1)$. Any two such functions $(\tau_1(\theta), p_1(\theta))$, $(\tau_2(\theta), p_2(\theta))$ on $[\theta_0, \theta_1)$ must be the same, since otherwise there would be a first point $(\bar{\tau}, \bar{p})$ along H at which the two curves begin to separate, which contradicts grad $H \neq 0$ at $(\bar{\tau}, \bar{p})$. (See Figure 4.) Hence there is a maximal interval $[\theta_0, \theta_{\text{max}})$ on which such a function $(\tau(\theta), p(\theta))$ can be defined. Then we prove $\theta_{\text{max}} = +\infty$ by showing that if not, then we can obtain an accumulation point in Q of $(\tau(\theta), p(\theta))$ as $\theta \uparrow \theta_{\text{max}}$, and using this, extend $(\tau(\theta), p(\theta))$ to some $[\theta_0, \theta_1)$, $\theta_1 > \theta_{\text{max}}$. Thus we have a C^1 branch $H' \subset H$ along which (τ, p) is a function of $\theta \in [\theta_0, +\infty)$, and we also prove that along H', $p \to +\infty$ as $\theta \to +\infty$; moreover, for every $a > 0$, $\mathcal{H}_a \cap H' \neq \varnothing$. (Cf. Figure 1.) Finally we show $H' = H \cap Q^{\text{cl}}$ as follows: we pick any $(\bar{\tau}, \bar{p}) \in H \cap Q$; we start with a representation $(\tau_1(\theta), p_1(\theta))$ (θ near $\bar{\theta} \equiv \theta(\bar{\tau}, \bar{p})$) of H near $(\bar{\tau}, \bar{p})$; we extend (τ_1, p_1) to a maximal left interval of existence

$(\theta_{\min}, \bar{\theta}]$; we then prove that this curve $J = \{(\tau_1(\theta), p_1(\theta)) | \theta \in (\theta_{\min}, \bar{\theta}]\}$ must intersect H', so $J \subset H'$; hence $(\bar{\tau}, \bar{p}) \in H'$.
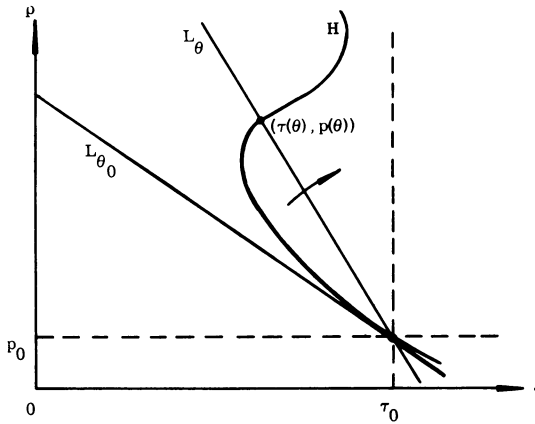
FIGURE 3. (τ, p) is a function of θ along H.
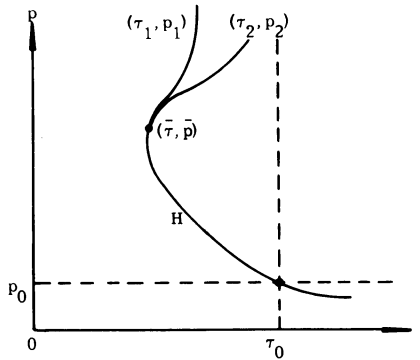
FIGURE 4.   The images of (τ_1, p_1) and (τ_2, p_2) begin to separate at $(\bar{\tau}, \bar{p})$.

We hope the above will aid the reader in understanding and ignoring various tedious details of the proofs that follow.

LEMMA 2.6. (a) $H_\tau(\tau, p) > 0$ for $0 < p \leqslant p_0$; (b) $H_p(\tau, p) > 0$ for $\tau \geqslant \tau_0$; (c) grad $H \neq 0$ everywhere on H.

PROOF. Use (1.23) and Corollary 2.4.

COROLLARY 2.7. $H \subset Q \cup Q' \cup \{(\tau_0, p_0)\}$.

Notation. For $x \in (-\infty, +\infty)$, $\varepsilon > 0$, let

$$N_\varepsilon(x) = (x - \varepsilon, x + \varepsilon). \tag{2.2}$$

We know [1, §65] that there exist δ, $\sigma > 0$ and a C^2 function G on $N_\delta(\tau_0)$ such that

$$G(\tau_0) = p_0, \quad G'(\tau_0) = g_\tau(\tau_0, S_0), \quad G''(\tau_0) = g_{\tau\tau}(\tau_0, S_0), \qquad (2.3)$$

and for all (τ, p),

$$(\tau, p) \in H \cap (N_\delta(\tau_0) \times N_\sigma(p_0)) \quad \text{iff } p = G(\tau).$$

Then Definition 2.5 implies

$$\theta_0 \equiv \theta(\tau_0, p_0) = -G'(\tau_0) > 0, \qquad (2.4)$$

by (2.3) and (1.5). Corollary 2.7 then yields

$$\theta(\tau, p) > 0 \quad \text{for every } (\tau, p) \in H. \qquad (2.5)$$

LEMMA 2.8. *Define*

$$\theta(\tau) = \theta(\tau, G(\tau)) \quad \text{for } \tau \in N_\delta(\tau_0).$$

Then θ is C^1 and $\theta'(\tau_0) < 0$.

PROOF. Straightforward.

LEMMA 2.9. $d\theta \neq 0$ *along H.*

PROOF. $d\theta \neq 0$ at (τ_0, p_0) by Lemma 2.8; at $(\tau, p) \neq (\tau_0, p_0)$, use Corollary 2.2. $\square$

DEFINITION 2.10. *A θ-parametrization (θ-p) on an interval I is a C^1 vector function (τ^0, p^0) defined on I such that*

(a) $(\tau^0(\theta), p^0(\theta)) \in H \ (\theta \in I)$;

(b) $\theta(\tau^0(\theta), p^0(\theta)) = \theta \ (\theta \in I)$;

(c) *if $\theta \in I$ and $\theta > \theta_0$, then $(\tau^0(\theta), p^0(\theta)) \in Q$.*

(See Definition 2.5 for θ_0.) *A θ-p is centered (c-θ-p) if $\theta_0 \in I$ implies $(\tau^0(\theta_0), p^0(\theta_0)) = (\tau_0, p_0)$.*

REMARK. The superscript "0" of (τ^0, p^0) refers to the subscript in (τ_0, p_0), the center of H. Later, this superscript will be varied to distinguish functions arising from Hugoniot curves with different centers.

LEMMA 2.11. *Let $(\bar{\tau}, \bar{p}) \in H \cap Q^{\text{cl}}$, $\bar{\theta} = \theta(\bar{\tau}, \bar{p})$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$, a θ-p (τ^*, p^*) on $(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$, and an open rectangle R about $(\bar{\tau}, \bar{p})$, such that $(\tau^*(\bar{\theta}), p^*(\bar{\theta})) = (\bar{\tau}, \bar{p})$, and $\forall(\tau, p) \in H \cap R$, $\theta(\tau, p) \in (\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$ and $(\tau, p) = (\tau^*(\theta(\tau, p)), p^*(\theta(\tau, p)))$.*

PROOF. By Lemma 2.6(c), grad $H \neq 0$ at $(\bar{\tau}, \bar{p})$.

Case 1. $H_p(\bar{\tau}, \bar{p}) \neq 0$ or $(\bar{\tau}, \bar{p}) = (\tau_0, p_0)$. Then $H_p(\bar{\tau}, \bar{p}) \neq 0$, so by the implicit function theorem, $\exists \delta$, $\sigma > 0$ and a C^1 function G on $N_\delta(\bar{\tau})$ such that for all (τ, p),

$$(\tau, p) \in H \cap (N_\delta(\bar{\tau}) \times N_\sigma(\bar{p})) \quad \text{iff } p = G(\tau). \qquad (2.6)$$

Define

$$\theta^*(\tau) = \theta\big(\tau, G(\tau)\big) \qquad \big(\tau \in N_\delta(\bar{\tau})\big). \tag{2.7}$$

Then by Lemma 2.9 if $\bar{\tau} < \tau_0$, or by Lemma 2.8 if $\bar{\tau} = \tau_0$, θ^* is C^1 on a neighborhood of $\bar{\tau}$ and $(\theta^*)'(\bar{\tau}) \neq 0$. Thus there exist $\delta_1 \in (0, \delta]$ and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\theta^* \text{ is a } C^1 \text{ diffeomorphism of } N_{\delta_1}(\bar{\tau}) \text{ onto } \big(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2\big). \tag{2.8}$$

Let

$$\tau^* = (\theta^*)^{-1}; \tag{2.9}$$

then τ^* is a C^1 diffeomorphism of $(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$ onto $N_{\delta_1}(\bar{\tau})$. Let $p^* = G \circ \tau^*$; then p^* is C^1 on $(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$. Since $\bar{p} = G(\bar{\tau})$, $\bar{\theta} = \theta^*(\bar{\tau})$, so $(\tau^*(\bar{\theta}), p^*(\bar{\theta})) = (\bar{\tau}, \bar{p})$. Clearly for $\theta \in (\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$ we have $(\tau^*(\theta), p^*(\theta)) = (\tau^*(\theta), G(\tau^*(\theta))) \in H$ and $\theta(\tau^*(\theta), p^*(\theta)) = \theta^*(\tau^*(\theta)) = \theta$. We claim that $\theta > \theta_0$ implies $(\tau^*(\theta), p^*(\theta)) \in Q$. If $(\bar{\tau}, \bar{p}) = (\tau_0, p_0)$, then θ^* is decreasing on $N_{\delta_1}(\bar{\tau})$ by Lemma 2.8 and the fact that it is a diffeomorphism; hence τ^* is decreasing on $(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$ and we are done by Lemma 2.7. If $(\bar{\tau}, \bar{p}) \neq (\tau_0, p_0)$, then $(\bar{\tau}, \bar{p}) \in Q$. Thus we can make the above δ and σ (for (2.8)) sufficiently small that $N_\delta(\bar{\tau}) \times N_\sigma(\bar{p}) \subset Q$, and the claim is proved. Hence (τ^*, p^*) is a θ-p on $(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2)$. Let $R = N_{\delta_1}(\bar{\tau}) \times N_\sigma(\bar{p})$. Let $(\tau, p) \in H \cap R$. Then $(\tau, p) \in N_\delta(\bar{\tau}) \times N_\sigma(\bar{p})$, so $p = G(\tau)$. Since $\tau \in N_{\delta_1}(\bar{\tau})$, we have

$$\theta(\tau, p) = \theta^*(\tau) \in \big(\bar{\theta} - \varepsilon_1, \bar{\theta} + \varepsilon_2\big)$$

and

$$\tau^*(\theta(\tau, p)) = \tau,$$

$$p^*(\theta(\tau, p)) = G\big(\tau^*(\theta(\tau, p))\big) = p.$$

Case 2. $H_r(\bar{\tau}, \bar{p}) \neq 0$ and $(\bar{\tau}, \bar{p}) \neq (\tau_0, p_0)$. The proof is similar to Case 1.

COROLLARY 2.12. *Let* (τ_1^0, p_1^0), (τ_2^0, p_2^0) *be θ-p's on an interval I; let $\bar{\theta} \in I$ with $(\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) = (\tau_2^0(\bar{\theta}), p_2^0(\bar{\theta})) \in Q^{cl}$. Then $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$ on some neighborhood (in I) of $\bar{\theta}$.*

An easy consequence of Corollary 2.12 is

LEMMA 2.13. *Let* (τ_1^0, p_1^0), (τ_2^0, p_2^0) *be c-θ-p's on $[\theta_0, \theta_1)$. Then $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$.*

PROPOSITION 2.14. *There exists a maximal right-open interval $[\theta_0, \theta_{max})$ on which a c-θ-p can be defined; i.e., there is a c-θ-p (τ^0, p^0) on $[\theta_0, \theta_{max})$, and there cannot be a c-θ-p on any larger interval $[\theta_0, \theta_1)$, $\theta_1 > \theta_{max}$. Here $\theta_{max} \in (\theta_0, +\infty]$.*

PROOF. Let $\Theta = \{\theta_1 > \theta_0 | \exists a\ c\text{-}\theta\text{-}p$ on $[\theta_0, \theta_1)\}$. Let $\theta_{max} = \sup \Theta$. With ε_2 as in Lemma 2.11 for $(\bar{\tau}, \bar{p}) = (\tau_0, p_0)$, $\theta_0 + \varepsilon_2 \in \Theta$; hence $\theta_{max} > \theta_0$. We define (τ^0, p^0) on $[\theta_0, \theta_{max})$ as follows:

> Let $\theta \in [\theta_0, \theta_{max})$. Choose some $\theta_1 \in (\theta, +\infty) \cap \Theta$. $\exists a$ $c\text{-}\theta\text{-}p$ (τ_1^0, p_1^0) on $[\theta_0, \theta_1)$. Let $(\tau^0(\theta), p^0(\theta)) = (\tau_1^0(\theta), p_1^0(\theta))$.

By Lemma 2.13, (τ^0, p^0) is well-defined by this process and is a $c\text{-}\theta\text{-}p$ on $[\theta_0, \theta_{max})$. $\square$
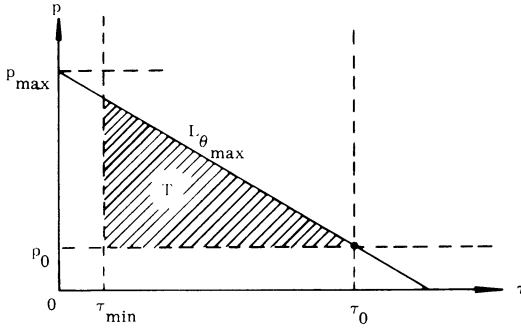

FIGURE 5. The region T.

PROPOSITION 2.15. $\theta_{max} = +\infty$.

PROOF. Suppose $\theta_{max} < +\infty$. Then $L_{\theta_{max}}$ has p-intercept $p_{max} = \tau_0\theta_{max} + p_0$. (See Figure 5.) Since $e(\tau_0, p_0) > 0$, and $\lim_{\tau \to 0^+} e(\tau, p_{max}) = 0$ by (1.10), $\exists \tau_{min} > 0$ such that $\forall \tau > 0$, $\tau \leqslant \tau_{min}$ implies $e(\tau, p_{max}) < e(\tau_0, p_0)$; we may assume $\tau_{min} \leqslant \tau_0$. Now suppose $0 < \tau \leqslant \tau_{min}$ and $0 < p \leqslant p_{max}$. Then from (1.20) and (1.22), we get

$$H(\tau, p) \leqslant e(\tau, p_{max}) - e(\tau_0, p_0) + (\tau - \tau_0)\frac{p + p_0}{2} < (\tau - \tau_0)\frac{p + p_0}{2} \leqslant 0;$$

thus

$$[\tau \leqslant \tau_{min} \text{ and } p \leqslant p_{max}] \Rightarrow (\tau, p) \notin H. \tag{2.10}$$

Let T be the open triangular region bounded by $L_{\theta_{max}}$ and the lines $\{p = p_0\}$ and $\{\tau = \tau_{min}\}$. (See Figure 5.) We assert that $(\tau^0(\theta), p^0(\theta)) \in T$ $\forall \theta \in (\theta_0, \theta_{max})$. For we know (Definition 2.10) that $(\tau^0(\theta), p^0(\theta)) \in H \cap Q$ and $\theta(\tau^0(\theta), p^0(\theta)) = \theta < \theta_{max}$. Hence $(\tau^0(\theta), p^0(\theta))$ lies below $L_{\theta_{max}}$, which then implies $p^0(\theta) \leqslant p_{max}$. From (2.10) we conclude $\tau^0(\theta) > \tau_{min}$, and the assertion is proved.

Now choose a sequence $\{\theta_n\}$ from $(\theta_0, +\infty)$ such that $\theta_n\uparrow\theta_{max}$. Then $(\tau^0(\theta_n), p^0(\theta_n)) \in T^{cl}$ $\forall n$, so there exist a subsequence of $\{\theta_n\}$, which we call $\{\theta_n\}$ again, and a point $(\bar{\tau}, \bar{p}) \in T^{cl}$ such that

$$(\tau^0(\theta_n), p^0(\theta_n)) \to (\bar{\tau}, \bar{p}). \tag{2.11}$$

Suppose $(\bar{\tau}, \bar{p}) = (\tau_0, p_0)$; let ε_1, ε_2, R be as in Lemma 2.11. Let N be so large that $(\tau^0(\theta_n), p^0(\theta_n)) \in R$ whenever $n \geqslant N$; then from Lemma 2.11 we conclude that $\forall n \geqslant N$, $\theta_n = \theta(\tau^0(\theta_n), p^0(\theta_n)) \in (\theta_0 - \varepsilon_1, \theta_0 + \varepsilon_2)$. But by Lemma 2.13 and (2.7)–(2.9) of the proof of Lemma 2.11, τ^0 is a C^1 diffeomorphism on $(\theta_0 - \varepsilon_1, \theta_0 + \varepsilon_2)$, hence (strictly) monotone. Then since $\{\theta_n\}$ is increasing and $(\tau^0(\theta_0), p^0(\theta_0)) = (\tau_0, p_0)$, we cannot have $\tau^0(\theta_n) \to \tau_0$; this contradicts (2.11). Therefore $(\bar{\tau}, \bar{p}) \neq (\tau_0, p_0)$. But since H is closed, $(\bar{\tau}, \bar{p}) \in H$; then Corollary 2.7 gives $(\bar{\tau}, \bar{p}) \in Q$. Since θ is continuous on Q,

$$\theta(\bar{\tau}, \bar{p}) = \lim_{n \to \infty} \theta(\tau^0(\theta_n), p^0(\theta_n)) = \lim_{n \to \infty} \theta_n = \theta_{\max};$$

hence Lemma 2.11 applies with $\bar{\theta} = \theta_{\max}$; let ε_1, ε_2, R, (τ^*, p^*) be as in Lemma 2.11 for $(\bar{\tau}, \bar{p})$, $\bar{\theta}$. Then $(\tau^0(\theta_N), p^0(\theta_N)) \in R \cap Q$ for some N, and we have

$$\theta_N = \theta(\tau^0(\theta_N), p^0(\theta_N)) \in (\theta_{\max} - \varepsilon_1, \theta_{\max} + \varepsilon_2),$$

$$(\tau^0(\theta_N), p^0(\theta_N)) = (\tau^*(\theta_N), p^*(\theta_N)).$$

Then (τ^0, p^0) and (τ^*, p^*) are c-θ-p's on a neighborhood of θ_N that agree at θ_N, so by Corollary 2.12, $(\tau^0, p^0) = (\tau^*, p^*)$ on some neighborhood of θ_N. Thus we may extend (τ^0, p^0) to $[\theta_0, \theta_{\max} + \varepsilon_2)$ by defining $(\tau^0, p^0) = (\tau^*, p^*)$ on $(\theta_N, \theta_{\max} + \varepsilon_2)$ and obtain a C^1 function, hence a c-θ-p, on $[\theta_0, \theta_{\max} + \varepsilon_2)$, which contradicts the maximality of $\theta_{\max}$. Hence $\theta_{\max} = +\infty$. $\square$

PROPOSITION 2.16. *Let $(\bar{\tau}, \bar{p}) \in H \cap Q$. Then $(\bar{\tau}, \bar{p}) = (\tau^0(\theta), p^0(\theta))$ for some $\theta > \theta_0$.*

PROOF. Routine, using preceding results and methods of proof.

Thus, combining Propositions 2.14–2.16, we see that the entire curve $H \cap Q^{\mathrm{cl}}$ is given as a function of θ by the function (τ^0, p^0) on $[\theta_0, +\infty)$.

LEMMA 2.17. *Along $H \cap Q^{\mathrm{cl}}$, S increases with increasing θ.*

PROOF. We know [1, §65] that near (τ_0, p_0), S increases along H with decreasing τ. Then by Lemma 2.8, S increases along $H \cap Q^{\mathrm{cl}}$ with increasing θ for θ near θ_0. Lemma 2.11 and its proof imply $((\tau^0)'(\theta), (p^0)'(\theta)) \neq 0$ for $\theta > \theta_0$; hence

$$\frac{d}{d\theta} S(\tau^0(\theta), p^0(\theta)) \neq 0 \quad \text{for } \theta > \theta_0,$$

by Corollary 2.3. Then we must have

$$\frac{d}{d\theta} S(\tau^0(\theta), p^0(\theta)) > 0 \quad \text{for } \theta > \theta_0;$$

hence $\theta \to S(\tau^0(\theta), p^0(\theta))$ is increasing on $[\theta_0, +\infty)$. $\square$

Let us remark that our assumptions (1.8) and (1.9) imply

$$\lim_{p \to 0^+} e(\tau, p) = 0 \quad \forall \tau > 0. \tag{2.12}$$

Using this, Lemma 2.6 and Corollary 2.7, it is easy to prove

LEMMA 2.18. $H \cap (Q')^{cl}$ is given by a decreasing C^1 function $\tau = V(p)$ from $(0, p_0]$ onto $[\tau_0, \tau_{max})$, where

$$\tau_{max} = \tau_0 + 2e(\tau_0, p_0)/p_0.$$

COROLLARY 2.19. Along $H \cap (Q')^{cl}$, S increases with increasing p.

PROOF. Similar to that of Lemma 2.17, using Corollary 2.3, the function G of Lemma 2.8, and Lemma 2.18.

We can now give the

PROOF OF PROPOSITIONS 1.2 AND 1.3. We prove Proposition 1.2; Proposition 1.3 is similar.

Suppose $U_0 \xrightarrow{\text{1-shock}} U$ with shock speed s. By Proposition 1.1, the conditions (1.15)–(1.18) hold. It is known [1, §§54, 55, 59, 64] that (1.15)–(1.17) imply

$$(\tau, p) \in (\mathcal{H}_{(u_0 - u)^2} \cup \mathcal{H}'_{(u_0 - u)^2}) \cap H,$$

where H, $\mathcal{H}$, $\mathcal{H}'$ have center (τ_0, p_0). Proposition 1.1 also gives v, $v_0 > 0$. Then (1.18) yields $S > S_0$ (where $S_0 = S(\tau_0, p_0)$, $S = S(\tau, p)$). From Corollary 2.19 we then conclude $(\tau, p) \notin (Q')^{cl}$; hence $(\tau, p) \in \mathcal{H}_{(u_0 - u)^2} \cap H$. In particular $(\tau, p) \in Q$, so $\tau < \tau_0$; then (1.15) yields $v < v_0$, so $u < u_0$.

Now suppose $u < u_0$ and $(\tau, p) \in \mathcal{H}_{(u_0 - u)^2} \cap H$. If s is to be the shock speed, then (1.15) is satisfied iff

$$s = (\rho u - \rho_0 u_0)/(\rho - \rho_0).$$

Using this value of s, one shows easily, by reversing the arguments of [1] cited above, that U_0, U and s also satisfy (1.16) and (1.17). Since $(\tau, p) \in \mathcal{H}_{(u_0 - u)^2}$, $\rho > \rho_0$ and $p > p_0$. Then

$$s = \frac{\rho u - \rho_0 u_0}{\rho - \rho_0} < \frac{\rho u - \rho_0 u}{\rho - \rho_0} = u.$$

So $s < u$, u_0, and $m > 0$. But $S > S_0$ by Lemma 2.17 (and Propositions 2.14–2.16, etc.), since $(\tau, p) \in H \cap Q$; thus (1.18) holds, and $U_0 \xrightarrow{\text{1-shock}} U$.

Now recall that in this paper all shocks are required to satisfy the Lax shock conditions (1.19). Thus for the purposes of this paper, the proof is not complete until (1.19) is proved. But (1.19) is a simple consequence of Corollary 2.4 and the fact that $\rho > \rho_0$ and $(\tau, p) \in H$. □

Observe that the above proof also establishes

PROPOSITION 2.20. Shocks always satisfy (1.19).

LEMMA 2.21. $\lim_{\theta \to +\infty} p^0(\theta) = +\infty$.

PROOF. Suppose not. Then we can find $M > 0$ and a sequence $\{\theta_n\}$ from $(\theta_0, +\infty)$ such that $\theta_n \to +\infty$ and $p^0(\theta_n) \leqslant M \ \forall n$. We have, for each n,

$$\theta_n = \theta\left(\tau^0(\theta_n), p^0(\theta_n)\right) = \frac{p^0(\theta_n) - p_0}{\tau_0 - \tau^0(\theta_n)} \leqslant \frac{M - p_0}{\tau_0 - \tau^0(\theta_n)},$$

since $(\tau^0(\theta_n), p^0(\theta_n)) \in Q$. (See Definition 2.10(c).) Since $\theta_n \to +\infty$, we must have $\tau^0(\theta_n) \to \tau_0$. There exist a subsequence of $\{\theta_n\}$, which we call $\{\theta_n\}$ again, and a number $\bar{p} \in [p_0, M]$ such that $p^0(\theta_n) \to \bar{p}$. Suppose $\bar{p} = p_0$. Let R be as in Lemma 2.11 for (τ_0, p_0), θ_0. Then since $(\tau^0(\theta_n), p^0(\theta_n)) \to (\tau_0, \bar{p}) = (\tau_0, p_0)$, $(\tau^0(\theta_n), p^0(\theta_n)) \in R$ for arbitrarily large θ_n, a contradiction. Hence $\bar{p} > p_0$. But since H is closed, it follows that $(\tau_0, \bar{p}) \in H$, which contradicts Corollary 2.7. $\square$

LEMMA 2.22. *For every* $a > 0$, $\mathcal{K}_a \cap H \neq \varnothing$. *In other words, the function* $a^0(\theta)$, $\theta \in [\theta_0, +\infty)$ *defined by*

$$a^0(\theta) = \sqrt{\left(\tau_0 - \tau^0(\theta)\right)\left(p^0(\theta) - p_0\right)}$$

assumes all nonnegative values.

PROOF. Suppose $\exists a > 0$ such that $\forall \theta \geqslant \theta_0$, $(\tau^0(\theta), p^0(\theta))$ lies to the right of $\mathcal{K}_a$, i.e.,

$$\tau^0(\theta) \geqslant \tau_0 - a/\left(p^0(\theta) - p_0\right) \qquad (\theta \geqslant \theta_0). \tag{2.13}$$

Then from (1.20) and (2.13),

$$e\left(\tau^0(\theta), p^0(\theta)\right) - e(\tau_0, p_0) = \left(\tau_0 - \tau^0(\theta)\right)\frac{p^0(\theta) + p_0}{2}$$

$$\leqslant \frac{a}{2}\, \frac{p^0(\theta) + p_0}{p^0(\theta) - p_0} \to \frac{a}{2} \quad \text{as } \theta \to +\infty,$$

by Lemma 2.21. Hence $e(\tau^0(\theta), p^0(\theta))$ is bounded above for $\theta \geqslant \theta_0$.

Choose $\theta_1 \geqslant \theta_0$ so large that $L_{\theta_1} \cap \mathcal{K}_a$ is a point $(\tau_1, p_1) \in Q$, i.e., $\tau_1 > 0$. Then $\forall \theta \geqslant \theta_1$, $(\tau^0(\theta), p^0(\theta))$ lies above L_{θ_1} and to the right of $\mathcal{K}_a$; thus

$$\tau^0(\theta) \geqslant \tau_1 \qquad (\theta \geqslant \theta_1). \tag{2.14}$$

Let $S^0(\theta) = S(\tau^0(\theta), p^0(\theta))$ $(\theta \geqslant \theta_0)$ and suppose $\exists M > 0$ such that $\forall \theta \geqslant \theta_0$, $S^0(\theta) \leqslant M$. By Lemma 2.17, $S^0(\theta) \geqslant S(\tau_0, p_0)$ $\forall \theta \geqslant \theta_0$. Thus $\forall \theta \geqslant \theta_1$, $(\tau^0(\theta), S^0(\theta)) \in [\tau_1, \tau_0] \times [S(\tau_0, p_0), M]$. But g is bounded above on $[\tau_1, \tau_0] \times [S(\tau_0, p_0), M]$, so $p^0(\theta) = g(\tau^0(\theta), S^0(\theta))$ is bounded above for $\theta \geqslant \theta_1$, which contradicts Lemma 2.21. Therefore $S^0(\theta)$ assumes arbitrarily large values for $\theta \geqslant \theta_0$.

Finally, observe that $\forall \theta \geqslant \theta_0$,

$$e\left(\tau^0(\theta), p^0(\theta)\right) = e\left(\tau^0(\theta), S^0(\theta)\right) \geqslant e\left(\tau_0, S^0(\theta)\right), \tag{2.15}$$

by (1.3). But the left member of (2.21) is bounded above, whereas by (1.9) the right member is not, yielding a contradiction. $\square$

3. Existence of solutions. In this section we prove that for any energy function e (see Definition 1.4), the R. P. with arbitrary initial data (U_l, U_r) always has a solution. In order to do this, it is sometimes necessary, as usual, to allow a "void" constant state $(\rho, e, p = 0)$, corresponding to a region in which there is no gas. (See [5], and Theorem 3.5, below.)

We first use the results of §2 to investigate shocks. Throughout this section we assume that we are given arbitrary fixed initial data (U_l, U_r). We define

$$S_1 = \left\{ U | U_l \xrightarrow{\text{1-shock}} U \right\}, \qquad S_3 = \left\{ U | U \xrightarrow{\text{3-shock}} U_r \right\}.$$

Recall $U_l = (\tau_l, p_l, u_l)$, $U_r = (\tau_r, p_r, u_r)$. Let H^l, Q^l, θ^l, θ_l, L_θ^l, (τ^l, p^l), $\mathcal{K}_a^l$ and a^l be the H, Q, θ, θ_0, L_θ, (τ^0, p^0), $\mathcal{K}_a$ and a^0 of §2 referred to the center (τ_l, p_l) instead of (τ_0, p_0); thus, for example, H^l is the Hugoniot curve with center (τ_l, p_l), $\theta^l(\tau, p) = (p - p_l)/(\tau_l - \tau)$ for $\tau \neq \tau_l$, and

$$a^l(\theta) = \sqrt{(\tau_l - \tau^l(\theta))(p^l(\theta) - p_l)} \qquad (\theta \in [\theta_l, +\infty)).$$

Define H^r, Q^r, θ^r, θ_r, L_θ^r, (τ^r, p^r), $\mathcal{K}_a^r$ and a^r similarly relative to (τ_r, p_r). Then we have

PROPOSITION 3.1. $S_1 = \{ U_1(\theta) | \theta > \theta_l \}$ where $U_1(\theta) = (\tau^l(\theta), p^l(\theta), u^l(\theta))$ $(\theta \geqslant \theta_l)$, and $u^l(\theta) = u_l - a^l(\theta)$ $(\theta \geqslant \theta_l)$. Thus (τ^l, p^l) is C^1 on $[\theta_l, +\infty)$ with $(\tau^l(\theta_l), p^l(\theta_l)) = (\tau_l, p_l)$, and

$$\left(\tau^l(\theta), p^l(\theta) \right) \in H^l \cap Q^l \cap L_\theta^l \quad \text{for } \theta > \theta_l. \tag{3.1}$$

Moreover, (p^l, u^l) is continuous and injective on $[\theta_l, +\infty)$, $(p^l(\theta_l), u^l(\theta_l)) = (p_l, u_l)$, $\lim_{\theta \to +\infty} p^l(\theta) = +\infty$, and $u^l(\theta)$ is not bounded below as $\theta \to +\infty$.

PROOF. Propositions 2.14 and 2.15 imply the existence of a c-θ-p (τ^l, p^l) on $[\theta_l, +\infty)$. Thus by Definition 2.10, (τ^l, p^l) is C^1, $(\tau^l(\theta_l), p^l(\theta_l)) = (\tau_l, p_l)$, and (3.1) holds. By Lemma 2.21, $\lim_{\theta \to +\infty} p^l(\theta) = +\infty$; by Lemma 2.22, $u^l(\theta)$ is not bounded below as $\theta \to +\infty$. Also u^l is clearly continuous; hence so is (p^l, u^l). It is easily verified that $S_1 = \{ U_1(\theta) | \theta > \theta_l \}$, and that (p^l, u^l) is injective. $\square$

Similarly we have

PROPOSITION 3.2. $S_3 = \{ U_3(\theta) | \theta > \theta_r \}$ where $U_3(\theta) = (\tau^r(\theta), p^r(\theta), u^r(\theta))$, $u^r(\theta) = u_r + a^r(\theta)$ $(\theta \geqslant \theta_r)$. Thus (τ^r, p^r) is C^1 on $[\theta_r, +\infty)$ with $(\tau^r(\theta_r), p^r(\theta_r)) = (\tau_r, p_r)$, and

$$\left(\tau^r(\theta), p^r(\theta) \right) \in H^r \cap Q^r \cap L_\theta^r \quad \text{for } \theta > \theta_r. \tag{3.2}$$

Moreover, (p^r, u^r) is continuous and injective on $[\theta_r, +\infty)$, $(p^r(\theta_r), u^r(\theta_r)) = (p_r, u_r)$, $\lim_{\theta \to +\infty} p^r(\theta) = +\infty$, and $u^r(\theta)$ is not bounded above as $\theta \to +\infty$.

We now study the rarefaction wave curves. Let

$$W_1 = \left\{ U \mid U_l \xrightarrow{\text{1-wave}} U \right\}, \qquad W_3 = \left\{ U \mid U \xrightarrow{\text{3-wave}} U_r \right\}.$$

PROPOSITION 3.3. *Along W_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p \in (0, p_l]$; u_1 is continuous and decreasing and maps $(0, p_l]$ onto $[u_l, u_{\max})$ for some $u_{\max} \in (u_l, +\infty]$.*

PROOF. Suppose v_1 and v_2 are independent 1-Riemann invariants. Then by [3], $U \in W_1$ iff

$$v_1(U) = v_1(U_l), \quad v_2(U) = v_2(U_l), \quad u - c \geqslant u_l - c_l. \tag{3.3}$$

Moreover, using equations equivalent to (1.1) with unknowns ρ, u, S, we see [3] that we may take

$$v_1(U) = S, \qquad v_2(U) = u + h(\rho, S), \tag{3.4}$$

where

$$h(\rho, S) = \int_{\rho_l}^{\rho} \bar{\rho}^{-1} \sqrt{f_\rho(\bar{\rho}, S)} \; d\bar{\rho},$$

so that

$$\partial h(\rho, S)/\partial \rho = c/\rho. \tag{3.5}$$

(Here $f(\rho, S) \equiv g(\rho^{-1}, S)$; see (1.13).) Then by (3.3) and (3.4), $U \in W_1$ iff

$$S = S_l, \tag{3.6}$$

$$u + h(\rho, S_l) = u_l + h(\rho_l, S_l), \tag{3.7}$$

$$u - c \geqslant u_l - c_l. \tag{3.8}$$

By (3.7) and (3.5), u is a decreasing function of ρ along W_1. Its domain is restricted by (3.8) as follows. Since $g_{\tau\tau}(\tau, S) = \rho^3[\rho f_{\rho\rho}(\rho, S) + 2f_\rho(\rho, S)]$, (1.7) implies

$$\rho f_{\rho\rho} + 2f_\rho > 0. \tag{3.9}$$

Then along W_1 we have

$$\frac{d}{d\rho}(u - c) = -\frac{\partial}{\partial \rho} h(\rho, S_l) - \frac{\partial}{\partial \rho} \sqrt{f_\rho(\rho, S_l)} = -\frac{1}{2\rho c}(2f_\rho + \rho f_{\rho\rho}) < 0.$$

So assuming (3.6) and (3.7), (3.8) is equivalent to $\rho \in (0, \rho_l]$; thus along W_1, u is a continuous, decreasing function of ρ, defined for $\rho \in (0, \rho_l]$. Since $p = g(\rho^{-1}, S_l)$ along W_1, (1.2), (1.3) and (1.5) imply that along W_1, p is a continuous, increasing function of ρ mapping $(0, \rho_l]$ onto $(0, p_l]$. It follows that along W_1, τ is a continuous, decreasing function $\tau_1(p)$ of $p \in (0, p_l]$, and u is a continuous, decreasing function $u_1(p)$ of $p \in (0, p_l]$, with range $[u_l, u_{\max})$ for some $u_{\max} \in (u_l, +\infty]$. Thus along W_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p \in (0, p_l]$ as desired. $\square$

Similarly we have

PROPOSITION 3.4. *Along W_3, U is a continuous function $U_3(p) = (\tau_3(p), p,$ $u_3(p))$ of $p \in (0, p_r]$; u_3 is continuous and increasing and maps $(0, p_r]$ onto $(u_{\min},$ $u_r]$ for some $u_{\min} \in [-\infty, u_r)$.*

THEOREM 3.5. *If $u_{\min} < u_{\max}$, then the R. P. (with initial data (U_l, U_r)) has a solution. If $u_{\min} \geqslant u_{\max}$, then the R. P. has no solution; however, if we allow ρ, p, $e = 0$, then the R. P. has a solution.*

PROOF. Let $C_1 = S_1 \cup W_1$, $C_3 = S_3 \cup W_3$. Thus

$$C_1 = \left\{ U \,|\, U_l \overset{1}{-} U \right\}, \qquad C_3 = \left\{ U \,|\, U \overset{3}{-} U_r \right\}.$$

Case 1. $u_{\max} > u_{\min}$. Let $\bar{C}_1$ (resp. $\bar{C}_3$) be the projection of C_1 (resp. C_3) onto the (p, u)-plane. Then from Propositions 3.1–3.4 it is easy to see that $\bar{C}_1$ and $\bar{C}_3$ must intersect, i.e., there is a $(\bar{p}, \bar{u}) \in \bar{C}_1 \cap \bar{C}_3$. Hence $\exists \tau_1$, $\tau_3 > 0 \ni (\tau_1, \bar{p}, \bar{u}) \in C_1$ and $(\tau_3, \bar{p}, \bar{u}) \in C_3$. Then the R. P. has the solution

$$U_l \overset{1}{-} (\tau_1, \bar{p}, \bar{u}) \xrightarrow{\text{contact}} (\tau_3, \bar{p}, \bar{u}) \overset{3}{-} U_r.$$

Case 2. $u_{\max} \leqslant u_{\min}$. If the problem had a solution, then there would exist states U_1 and U_3 such that

$$U_l \overset{1}{-} U_1 \xrightarrow{\text{contact}} U_3 \overset{3}{-} U_r.$$
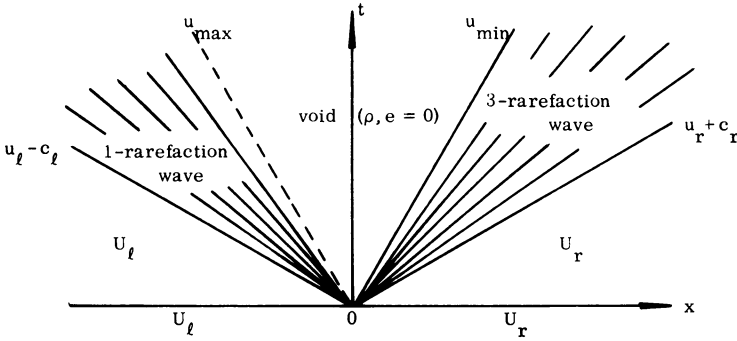

FIGURE 6.   The case of the vacuum.

These relations imply $U_1 \in C_1$, $u_1 = u_3$ and $U_3 \in C_3$; then, using Propositions 3.1–3.4, also $u_1 < u_{\max}$ and $u_3 > u_{\min}$, a contradiction. Hence there is no solution. But if we allow $\rho, p, e = 0$ (corresponding to a vacuum), then we can connect U_l and U_r each by a (complete) rarefaction wave to a constant "void" state (ρ, $e = 0$, u undefined). (See [5], and Figure 6.) To see that these two

waves do not overlap, we note that $u - c < u_{\max}$ on W_1, and $u + c > u_{\min}$ on W_3. Thus our "solution" is

$$U_l \xrightarrow{\text{1-wave}} \text{void} \xrightarrow{\text{3-wave}} U_r. \quad \Box$$

4. Uniqueness and nonuniqueness of solutions. In this section we show that the condition MEDIUM is necessary and sufficient for uniqueness of solutions of the R. P. (See Theorem 4.5 and Theorem 4.13.) We first prove the sufficiency. To obtain uniqueness, we replace Propositions 3.1 and 3.2 by Propositions 4.3 and 4.4, which assert that when MEDIUM holds, the curves S_1 and S_3 have a simpler form: along $\overline{S}_1$ (the projection of S_1 onto the (p, u)-plane), u is a decreasing function of p; along $\overline{S}_3$, u is an increasing function of p. Then the curves $\overline{C}_1$ and $\overline{C}_3$ can intersect in at most one point. (See Figure 8.)

LEMMA 4.1. *Suppose e satisfies* WEAK. *Then along H, $\tau = V(p)$, where V is a C^1 function defined on $(0, +\infty)$.*

PROOF. Straightforward.
When WEAK holds, we define

$$a_0(p) = \sqrt{(\tau_0 - V(p))(p - p_0)} \qquad (p > 0). \tag{4.1}$$

LEMMA 4.2. *Suppose e satisfies* MEDIUM. *Then*

$$\frac{d}{dp} a_0(p) > 0 \qquad (p > p_0).$$

PROOF. Let H^0 denote the Hugoniot curve with center (τ_0, p_0). Since WEAK holds, we have $\tau = V^0(p)$ along H^0 by Lemma 4.1; also,

$$H_\tau(\tau, p) > 0. \tag{4.2}$$

As in (4.1), we define a_0 by

$$a_0(p) = \sqrt{(\tau_0 - V^0(p))(p - p_0)} \qquad (p > 0).$$

Then

$$\frac{d}{dp} V^0(p) = -\frac{H_p(V^0(p), p)}{H_\tau(V^0(p), p)}.$$

We write $\tau = V^0(p)$; then, using (1.23), we obtain

$$\frac{d}{dp} V^0(p)$$

$$= \frac{1}{2a_0(p)} \left[(\tau_0 - \tau) + (p - p_0) \frac{(\partial/\partial p)e(\tau, p) - (\tau_0 - \tau)/2}{(-g_\tau)(\partial/\partial p)e(\tau, p) - (p - p_0)/2} \right].$$

But

$$(-g_\tau) \frac{\partial}{\partial p} e(\tau, p) - \frac{p - p_0}{2} > 0$$

by (4.2) and (1.23), so

$$\frac{d}{dp} a_0(p) > 0 \Leftrightarrow \frac{\partial}{\partial p} e(\tau, p) > \frac{(p - p_0)(\tau_0 - \tau)}{(-g_\tau)(\tau_0 - \tau) + (p - p_0)} . \quad (4.3)$$

We also have

$$\frac{d}{dp} a_0(p) < 0 \Leftrightarrow \frac{\partial}{\partial p} e(\tau, p) < \frac{(p - p_0)(\tau_0 - \tau)}{(-g_\tau)(\tau_0 - \tau) + (p - p_0)} , \quad (4.4)$$

which we shall use later.

Let $(\tau_1, p_1) \in H^0 \cap Q$ (see (2.1)); then $\tau_1 = V^0(p_1)$. We must prove

$$\frac{d}{dp} a_0(p)|_{p = p_1} > 0.$$

By (4.3), it suffices to prove

$$\frac{\partial}{\partial p} e(\tau_1, p_1) > \frac{(p_1 - p_0)(\tau_0 - \tau_1)}{(-g_\tau(\tau_1, p_1))(\tau_0 - \tau_1) + (p_1 - p_0)} . \quad (4.5)$$

Let H^1 be the Hugoniot curve with center (τ_1, p_1), and let $\tau = V^1(p)$ along H^1 as in Lemma 4.1. Then $(\tau_0, p_0) \in H^1$, so $\tau_0 = V^1(p_0)$. Since $\tau_0 > \tau_1$ and $p_0 < p_1$, Lemma 2.18 gives $\tau_0 = V^1(p_0) \in [\tau_1, \tau_{max}^1)$, where $\tau_{max}^1 = \tau_1 + [2e(\tau_1, p_1)/p_1]$. Hence

$$0 < \tau_0 - \tau_1 < 2e(\tau_1, p_1)/p_1. \quad (4.6)$$

Also,

$$0 < p_1 - p_0 < p_1. \quad (4.7)$$

Now observe that

$$\frac{(p_1 - p_0)(\tau_0 - \tau_1)}{(-g_\tau(\tau_1, p_1))(\tau_0 - \tau_1) + (p_1 - p_0)} = f(\tau_0 - \tau_1, p_1 - p_0), \quad (4.8)$$

where f is the function

$$f(x, y) = xy / (ax + by) \quad (x, y > 0),$$

where $a = -g_\tau(\tau_1, p_1) > 0$, $b = 1 > 0$. But f satisfies $\partial f / \partial x > 0$, $\partial f / \partial y > 0$; hence from (4.6) and (4.7) we get

$$f(\tau_0 - \tau_1, p_1 - p_0) < f\left(\frac{2e(\tau_1, p_1)}{p_1}, p_1 \right)$$

$$= \frac{2e(\tau_1, p_1)}{(-g_\tau(\tau_1, p_1))2e(\tau_1, p_1)/p_1 + p_1} \leq \frac{\partial}{\partial p} e(\tau_1, p_1),$$

by MEDIUM and Proposition 1.5. This, together with (4.8), proves (4.5). $\square$

PROPOSITION 4.3. *Suppose* e *satisfies* MEDIUM. *Then along* S_1, U *is a continuous function* $U_1(p) = (\tau_1(p), p, u_1(p))$ *of* $p > p_l$, *and* u_1 *is a continuous, decreasing function mapping* $(p_l, +\infty)$ *onto* $(-\infty, u_l)$.

PROOF. Let H^l be the Hugoniot curve with center (τ_l, p_l); let $\tau = V^l(p)$ along H^l as in Lemma 4.1, and define

$$a_l(p) = \sqrt{(\tau_l - V^l(p))(p - p_l)} \qquad (p > 0).$$

Define $\tau_1 = V^l$ and $u_1(p) = u_l - a_l(p) \, (p > p_l)$. $\square$
  Similarly we have

PROPOSITION 4.4. *Suppose* e *satisfies* MEDIUM. *Then along* S_3, U *is a continuous function* $U_3(p) = (\tau_3(p), p, u_3(p))$ *of* $p > p_r$, *and* u_3 *is a continuous, increasing function mapping* $(p_r, +\infty)$ *onto* $(u_r, +\infty)$.

THEOREM 4.5. *Suppose* e *satisfies* MEDIUM. *Then the R. P. always has a unique solution (in the sense of Theorem 3.5).*

PROOF. We are given arbitrary initial data (U_l, U_r). As in Theorem 3.5, let $C_1 = S_1 \cup W_1$, $C_3 = S_3 \cup W_3$. Combining Propositions 3.3 and 4.3, we have

> Along C_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p > 0$; u_1 is a continuous, decreasing function mapping $(0, +\infty)$ onto $(-\infty, u_{max})$, for some $u_{max} \in (u_l, +\infty]$. (4.9)

(See Figure 7.) Similarly, Propositions 3.4 and 4.4 give

> Along C_3, U is a continuous function $U_3(p) = (\tau_3(p), p, u_3(p))$ of $p > 0$; u_3 is a continuous, increasing function mapping $(0, +\infty)$ onto $(u_{min}, +\infty)$, for some $u_{min} \in [-\infty, u_r)$. (4.10)
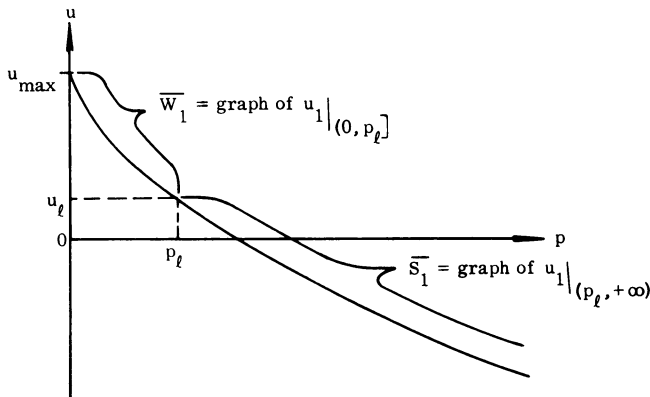

FIGURE 7. u is a decreasing function of p along $\overline{C}_1$.

Case 1. $u_{max} > u_{min}$. Let $\overline{C}_1$ (resp. $\overline{C}_3$) be the projection of C_1 (resp. C_3) onto the (p, u)-plane. Then $\overline{C}_1$ is the graph of u_1, and $\overline{C}_3$ is the graph of u_3. A solution of R. P. is uniquely determined by states $\overline{U}_1$, $\overline{U}_3$ such that

$$U_l \xrightarrow{\;\;1\;\;} \overline{U}_1 \xrightarrow{\;contact\;} \overline{U}_3 \xrightarrow{\;\;3\;\;} U_r.$$

For such states $\overline{U}_1$, $\overline{U}_3$ we must have $\overline{U}_1 \in C_1$, $\overline{U}_3 \in C_3$ and $(\overline{p}_1, \overline{u}_1) = (\overline{p}_3, \overline{u}_3)$. In this case, denote $\overline{p}_1$ and $\overline{p}_3$ by $\overline{p}$; then by (4.9) and (4.10), $\overline{p}$ uniquely determines $\overline{U}_1$ and $\overline{U}_3$, and $u_1(\overline{p}) = u_3(\overline{p})$. But u_1 is decreasing and u_3 is increasing, so $u_1 - u_3$ is decreasing; hence $u_1(\overline{p}) = u_3(\overline{p})$ can hold for at most one $\overline{p}$. (See Figure 8.) Thus there can be at most one solution of the R. P.
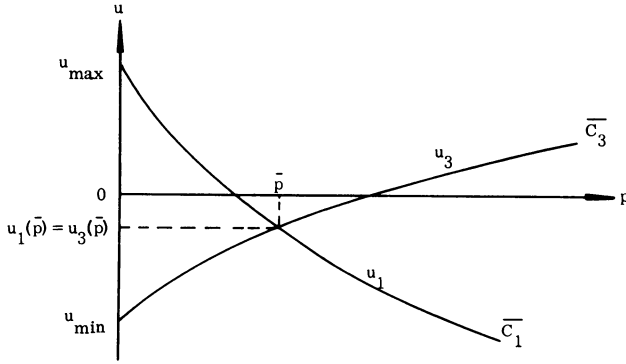

FIGURE 8. The strict monotonicity of $\overline{C}_1$ and $\overline{C}_3$ assures uniqueness.

Here it is necessary to stipulate that in the case $u_{max} > u_{min}$, we do not admit the void state (ρ, $e = 0$, u undefined). (See Theorem 3.5, Case 2.)

Case 2. $u_{max} \leqslant u_{min}$. Clearly there is only one way to connect U_l (left) to the void state on the right by a complete 1-rarefaction wave; similarly for

$$\text{void} \xrightarrow{\;3\text{-wave}\;} U_r. \quad \square$$

We turn now to nonuniqueness. We begin by showing two different ways in which a property of the Hugoniot curve allows one to construct initial data (U_l, U_r) for which the R. P. has more than one solution.

Convention 4.6. H will always denote the Hugoniot curve with center (τ_0, p_0). Similarly, we shall always denote

$Q = (0, \tau_0) \times (p_0, +\infty)$,

$\theta = (p - p_0)/(\tau_0 - \tau)$ $(\tau \neq \tau_0)$,

$\mathcal{H}_a$ = upper branch of hyperbola $(\tau_0 - \tau)(p - p_0) = a$,

$L_{\overline{\theta}}$ = line through (τ_0, p_0) with slope $-\overline{\theta}$ $(\overline{\theta} > 0)$.

(See, for example, Lemma 4.10, below.)

First, suppose we have two distinct points (τ_1, p_1), (τ_2, p_2) and $a > 0$ such

that
$$(\tau_i, p_i) \in \mathcal{H}_{a^2} \cap H \qquad (4.11)$$
for $i = 1, 2$. (See Figure 9.) Let u_l be arbitrary and let $U_l = (\tau_0,\ p_0,\ u_l)$, $U_r = (\tau_0,\ p_0,\ u_l - 2a)$. Then (4.11) and Propositions 1.2 and 1.3 yield the following two solutions of the R. P. with data $(U_l,\ U_r)$:

$$U_l \xrightarrow{\text{1-shock}} (\tau_1, p_1, u_l - a) \xrightarrow{\text{3-shock}} U_r, \qquad (4.12)$$

$$U_l \xrightarrow{\text{1-shock}} (\tau_2, p_2, u_l - a) \xrightarrow{\text{3-shock}} U_r. \qquad (4.13)$$
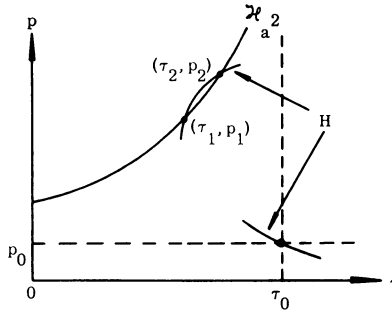

FIGURE 9.   The behavior of H that gives rise to type I nonuniqueness.

Clearly, if we have n distinct points $(\tau_1, p_1), \ldots, (\tau_n, p_n)$ that satisfy (4.11) for $i = 1, \ldots, n$, then we obtain n distinct solutions of the R. P. with data (U_l, U_r). Motivated by the above, we make

DEFINITION 4.7. We say that *type* I *nonuniqueness holds for* (U_l, U_r) if U_l and U_r have the same thermodynamic variables (τ_0, p_0) and there exist distinct (τ_1, p_1) and (τ_2, p_2) that satisfy (4.11), where $2a = u_l - u_r$; then the R. P. with data (U_l, U_r) has the two distinct solutions (4.12) and (4.13), which we call *type* I *solutions*. The (type I) nonuniqueness is *n-fold* if there exist n (type I) solutions.

Now suppose we have $\bar{p} > p_0$ and $0 < \tau_1 < \tau_2 < \tau_0$ such that
$$(\tau_1, \bar{p}), (\tau_2, \bar{p}) \in H. \qquad (4.14)$$

(See Figure 10.) Let a and b satisfy

$$(\tau_1, \bar{p}) \in \mathcal{H}_{b^2}, \qquad (\tau_2, \bar{p}) \in \mathcal{H}_{a^2} \qquad (4.15)$$

(e.g., $a = \sqrt{(\tau_0 - \tau_2)(\bar{p} - p_0)}$); then $b > a > 0$. Let u_l be arbitrary and define $U_l = (\tau_0,\ p_0,\ u_l)$, $U_r = (\tau_0,\ p_0,\ u_l - (a + b))$. Then by (4.14), (4.15), Propositions 1.2 and 1.3, and the meaning of contact discontinuity, the R. P. with data (U_l, U_r) has the following two solutions:

$$U_l \xrightarrow{\text{1-shock}} (\tau_1, \bar{p}, u_l - b) \xrightarrow{\text{contact}} (\tau_2, \bar{p}, u_l - b) \xrightarrow{\text{3-shock}} U_r, \qquad (4.16)$$

$$U_l \xrightarrow{\text{1-shock}} (\tau_2, \bar{p}, u_l - a) \xrightarrow{\text{contact}} (\tau_1, \bar{p}, u_l - a) \xrightarrow{\text{3-shock}} U_r. \qquad (4.17)$$
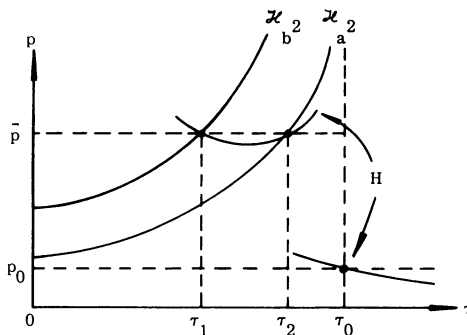
FIGURE 10. The behavior of H that gives rise to type II nonuniqueness.

DEFINITION 4.8. We say that *type* II *nonuniqueness holds for* (U_l, U_r) if U_l and U_r have the same thermodynamic variables (τ_0, p_0) and there exist $\bar{p} > p_0, \tau_1, \tau_2 \in (0, \tau_0)$ with $\tau_1 < \tau_2, a > 0$ and $b > 0$ such that $u_l - u_r = a + b$ and (4.14) and (4.15) hold; then the R. P. with data (U_l, U_r) has the two solutions (4.16) and (4.17), which we call *type* II *solutions*.

We now prove that MEDIUM is necessary for uniqueness. Some of the lemmas below are stronger than necessary; these strengthened statements will be used in §5 to obtain stability.

Convention 4.9. We shall henceforth often use a more informal notation, in which all derivatives are assumed to be taken along H. Here are two examples:

$$\frac{d\tau}{d\theta}\bigg|_{(\tau_1, \bar{p})} = \frac{d}{d\theta} \tau^0(\theta)\big|_{\theta = \theta_1},$$

where $\tau_1, \bar{p}$ are as above and $\theta_1 = \theta(\tau_1, \bar{p})$; $d\tau/dp = V'(p)$ when WEAK holds and V is as in Lemma 4.1.

LEMMA 4.10. *Suppose e satisfies* WEAK, *but violates* MEDIUM. *Then there exist* (τ_0, p_0), *and* $(\tau_1, p_1) \in H \cap Q$, *such that*

$$\frac{da}{dp}\bigg|_{(\tau_1, p_1)} < 0, \qquad \frac{dp}{d\theta}\bigg|_{(\tau_1, p_1)} > 0.$$

(*See Conventions* 4.6 *and* 4.9; *here, and often in what follows, we also denote* $a = \sqrt{(\tau_0 - \tau)(p - p_0)}$.)

PROOF. Since e violates MEDIUM, then by Proposition 1.5 there exists (τ_1, p_1) such that

$$\frac{\partial}{\partial p} e(\tau_1, p_1) < \frac{2e(\tau_1, p_1)}{-g_\tau(\tau_1, p_1) \cdot (2e(\tau_1, p_1)/p_1) + p_1}. \tag{4.18}$$

Let H^1 be the Hugoniot curve with center (τ_1, p_1). We obtain the first inequality by choosing $(\tau_0, p_0) \in H^1$ with $p_0 > 0$ sufficiently small, using Lemma 2.18 and (4.4). The second inequality is clear. $\square$

Now suppose (τ, p) satisfies $H(\tau, p) = 0$ and $H_\tau(\tau, p) \neq 0$. Then τ is a function of p along H, near (τ, p); hence $d\tau/dp$ exists and

$$\frac{d\theta}{dp} = \frac{(\tau_0 - \tau) + (p - p_0)d\tau/dp}{(\tau_0 - \tau)^2} \tag{4.19}$$

if $\tau \neq \tau_0$; by Lemma 2.9, $d\theta/dp \neq 0$; thus $dp/d\theta$ exists and is nonzero. In particular, if $(\tau, p) \in H$, $p > p_0$ and $H_\tau(\tau, p) \neq 0$, then by (4.19) the following hold:

$$\frac{dp}{d\theta} \neq 0, \tag{4.20}$$

$$\frac{dp}{d\theta} > 0 \Leftrightarrow \frac{d\tau}{dp} > \frac{\tau - \tau_0}{p - p_0}, \tag{4.21}$$

$$\frac{dp}{d\theta} < 0 \Leftrightarrow \frac{d\tau}{dp} < \frac{\tau - \tau_0}{p - p_0} \Rightarrow \frac{d\tau}{dp} < 0. \tag{4.22}$$

LEMMA 4.11. *Suppose e violates* WEAK. *Then there exist (τ_0, p_0), and $(\tau_2, \bar{p}) \in H \cap Q$, such that*

$$\left.\frac{d\tau}{dp}\right|_{(\tau_2, \bar{p})} < 0, \qquad \left.\frac{dp}{d\theta}\right|_{(\tau_2, \bar{p})} < 0.$$

PROOF. Since WEAK fails, $\exists(\tau_2, \bar{p})$ such that

$$\frac{\partial}{\partial p} e(\tau_2, \bar{p}) < \frac{\bar{p}}{-2g_\tau(\tau_2, \bar{p})}. \tag{4.23}$$

Let $\bar{H}$ be the Hugoniot curve with center $(\tau_2, \bar{p})$, and let $\tau = \bar{V}(p)$ along $\bar{H}$ for $p \in (0, \bar{p}]$, as in Lemma 2.18. By (4.23),

$$\left(-g_\tau(\tau_2, \bar{p})\right)\frac{\partial}{\partial p} e(\tau_2, \bar{p}) - \frac{\bar{p}}{2} < 0;$$

hence we can find $p_0 \in (0, \bar{p})$ such that

$$\left(-g_\tau(\tau_2, \bar{p})\right)\frac{\partial}{\partial p} e(\tau_2, \bar{p}) - \frac{\bar{p} - p_0}{2} < 0. \tag{4.24}$$

Write $\tau_0 = \bar{V}(p_0)$; then $(\tau_0, p_0) \in \bar{H}$, so $(\tau_2, \bar{p}) \in H$. But (4.19) holds for any $(\tau, p) \in H$ for which $H_\tau(\tau, p) \neq 0$ and $p > p_0$; in particular (4.19) holds for $(\tau, p) = (\tau_2, \bar{p})$, by (4.24) and (2.2). Then by (4.24) we see that

$$\frac{d\theta}{dp}\Big|_{(\tau_2,\bar{p})} < 0 \Leftrightarrow \left[\left(-g_\tau(\tau_2,\bar{p})\right)\frac{\partial}{\partial p}e(\tau_2,\bar{p}) - \frac{\bar{p}-p_0}{2}\right](\tau_0-\tau_2)$$

$$> (\bar{p}-p_0)\left[\frac{\partial}{\partial p}e(\tau_2,\bar{p}) - \frac{\tau_0-\tau_2}{2}\right]$$

$$\Leftrightarrow \frac{\partial}{\partial p}e(\tau_2,\bar{p})\left[\left(-g_\tau(\tau_2,\bar{p})\right)(\tau_0-\tau_2) - (\bar{p}-p_0)\right] > 0,$$

which is true by Corollary 2.4 and (1.22). Then also

$$dp/d\theta|_{(\tau_2,\bar{p})} < 0.$$

It follows from (4.22) that

$$d\tau/dp|_{(\tau_2,\bar{p})} < 0. \quad \square$$

Combining Lemmas 4.10 and 4.11 gives

LEMMA 4.12. *Suppose e violates* MEDIUM. *Then there exist* (τ_0, p_0), *and* $(\tau_1, p_1) \in H \cap Q$, *such that*

$$\frac{da}{d\theta}\Big|_{(\tau_1,p_1)} < 0, \qquad \frac{dp}{d\theta}\Big|_{(\tau_1,p_1)} \neq 0.$$

THEOREM 4.13. *Suppose e violates* MEDIUM. *Then uniqueness fails; there exist initial data* (U_l, U_r) *for which 3-fold type I nonuniqueness holds.*

PROOF. By Lemma 4.12, there exist (τ_0, p_0), and $(\tau_2, p_2) \in H \cap Q$, such that

$$da/d\theta|_{(\tau_2,p_2)} < 0.$$

Let $\theta_2 = \theta(\tau_2, p_2)$ (and, as always, $\theta_0 = \theta(\tau_0, p_0) = -dp/d\tau|_{(\tau_0,p_0)}$). Then

$$da^0(\theta)/d\theta|_{\theta=\theta_2} < 0.$$

Thus $a^0(\theta) > a^0(\theta_2)$ for $\theta < \theta_2$, θ near θ_2; $a^0(\theta) < a^0(\theta_2)$ for $\theta > \theta_2$, near θ_2. But $a^0(\theta_0) = 0$, so $\exists \theta_1 \in (\theta_0, \theta_2)$ such that $a^0(\theta_1) = a^0(\theta_2)$. Moreover, by Lemma 2.22, we can find arbitrarily large θ such that $a^0(\theta) > a^0(\theta_2)$, so $\exists \theta_3 \in (\theta_2, +\infty)$ such that $a^0(\theta_3) = a^0(\theta_2)$. Let $a = a^0(\theta_2)$, and let $(\tau_i, p_i) = (\tau^0(\theta_i), p^0(\theta_i))$ for $i = 1, 2, 3$. Then the (τ_i, p_i)'s are distinct and satisfy (4.11). Let $U_l = (\tau_0, p_0, u_l)$, $U_r = (\tau_0, p_0, u_l - 2a)$, where u_l is arbitrary. Then 3-fold type I nonuniqueness holds for (U_l, U_r). (See Figure 11.) $\square$

Thus we see from Theorem 4.5 and Theorem 4.13 that the condition MEDIUM is necessary and sufficient for uniqueness of solutions of the R. P.

We conclude this section by showing that when WEAK fails, type II nonuniqueness occurs.

LEMMA 4.14. *Suppose e violates* WEAK. *Then there exist a point* (τ_0, p_0), *an open interval* $N \subset (p_0, +\infty)$ *and three* C^1 *functions* V_1, V_2, V_3 *on* N *such that*

$$(V_i(p), p) \in H \qquad (p \in N, i = 1, 2, 3),$$

$$V_1(p) < V_2(p) < V_3(p) \qquad (p \in N), \qquad (4.25)$$

$$V_2'(p) < 0 \qquad (p \in N), \qquad (4.26)$$

$$H_\tau(V_i(p), p) \neq 0 \qquad (p \in N, i = 1, 2, 3). \qquad (4.27)$$

(*See Figure* 12.)

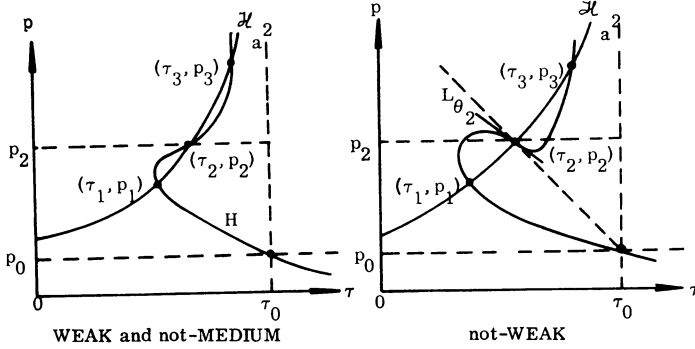PROOF. Use Lemma 4.11.

FIGURE 11. 3-fold type I nonuniqueness occurs whenever MEDIUM fails.
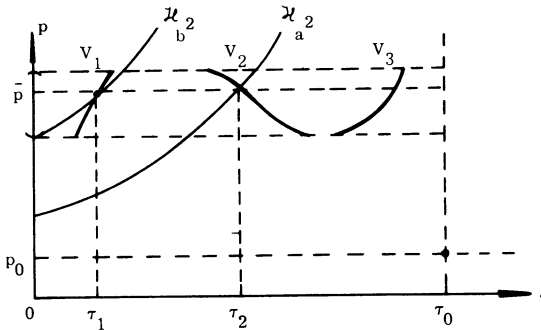

FIGURE 12. Type II nonuniqueness.

THEOREM 4.15. *Suppose e violates* WEAK. *Then there exist initial data* $(U_l,$ $U_r)$ *for which type* II *nonuniqueness holds.*

PROOF. Let (τ_0, p_0), N, V_1, V_2 and V_3 be as in Lemma 4.14. Choose $\bar{p} \in N$, and let $\tau_1 = V_1(\bar{p})$, $\tau_2 = V_2(\bar{p})$; then $(\tau_1, \bar{p})$, $(\tau_2, \bar{p}) \in H$. Also, $\bar{p} > p_0$, so by Corollary 2.7, τ_1, $\tau_2 < \tau_0$; by (4.26), $\tau_1 < \tau_2$. (See Figure 12.) It is now clear from Definition 4.8 and the remarks preceding it that data (U_l, U_r) can be found for which type II nonuniqueness holds. $\square$

Observe that, although the violation of WEAK implies the occurrence of both type I and type II nonuniqueness, the data (U_l, U_r) given by Theorems 4.13 and 4.15 need not be the same. In the next section we shall prove that (when WEAK fails) there always exists one set of data (U_l, U_r) for which type I and type II nonuniqueness hold simultaneously, and for which the resulting 5-fold nonuniqueness is stable.

5. Stability of the nonunique solutions. We begin by providing a meaning for the title of this section.

Suppose we have initial data (U_l^0, U_r^0) for which n-fold nonuniqueness (type I, type II, both, or neither) holds, i.e., the R. P. with data (U_l^0, U_r^0) has n distinct solutions,

$$U_l^0 \xrightarrow{\ 1\ } U_1^{i,0} \xrightarrow{\text{contact}} U_3^{i,0} \xrightarrow{\ 3\ } U_r^0 \qquad (1 \leqslant i \leqslant n). \qquad (5.1)$$

DEFINITION 5.1. We say *the n-fold nonuniqueness is stable* if there exists a neighborhood $\mathfrak{N}$ of (U_l^0, U_r^0) such that

(a) for any $(U_l, U_r) \in \mathfrak{N}$, the R. P. with data (U_l, U_r) has n solutions;

(b) these n solutions can be expressed,

$$U_l \xrightarrow{\ 1\ } U_1^i(U_l, U_r) \xrightarrow{\text{contact}} U_3^i(U_l, U_r) \xrightarrow{\ 3\ } U_r \qquad ((U_l, U_r) \in , i = 1, \ldots, n),$$

in such a way that

$$U_j^i(U_l, U_r) \to U_j^{i,0} \qquad (j = 1, 3; i = 1, \ldots, n)$$

as $(U_l, U_r) \to (U_l^0, U_r^0)$.

In this section we shall prove that for all types of n-fold nonuniqueness considered, the data can be chosen so that the n-fold nonuniqueness is stable. Thus when MEDIUM fails, there exist data that exhibit 3-fold type I nonuniqueness, and for which the 3-fold nonuniqueness is stable. (See Theorem 5.4.) When WEAK fails, we can find data that exhibit simultaneously 3-fold type I nonuniqueness and (2-fold) type II nonuniqueness, and for which the 5-fold nonuniqueness is stable. (See Theorem 5.6.)

We begin with type I nonuniqueness; the following two lemmas provide sufficient conditions for stability.

LEMMA 5.2. *Suppose* $\theta_0 < \theta_1 < \theta_2 < \theta_3, a > 0$ *and*

$$da/d\theta|_{\theta = \theta_i} \neq 0 \qquad (i = 1, 2, 3), \qquad (5.2)$$

$$dp/d\theta|_{\theta = \theta_i} \neq 0 \qquad (i = 1, 2, 3), \qquad (5.3)$$

$$a^0(\theta_i) = a \qquad (i = 1, 2, 3). \qquad (5.4)$$

Let u_0 *be arbitrary and let* $U_l^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - 2a)$. *Then the R. P. with data* (U_l^0, U_r^0) *has three type* I *solutions for which the 3-fold nonuniqueness is stable.*

PROOF. Let $(\tau_i, p_i) = (\tau^0(\theta_i), p^0(\theta_i))$ $(i = 1, 2, 3)$. Then by (5.4), the R. P. with data (U_l^0, U_r^0) has the three type I solutions,

$$U_l^0 \xrightarrow{\text{1-shock}} (\tau_i, p_i, u_0 - a) \xrightarrow{\text{contact}} (\tau_i, p_i, u_0 - a) \xrightarrow{\text{3-shock}} U_r^0$$

$$(i = 1, 2, 3), \quad (5.5)$$

which we have expressed in the form of (5.1). From (5.3), it is easy to see that $H_\tau(\tau_i, p_i) \neq 0$ $(i = 1, 2, 3)$. For any point (τ_c, p_c) as center, we define

$$H(\tau_c, p_c; \tau, p) = e(\tau, p) - e(\tau_c, p_c) + (\tau - \tau_c)\frac{p + p_c}{2} \qquad (\tau_c, p_c, \tau, p > 0).$$

$$(5.6)$$

Then $H_\tau(\tau_0, p_0; \tau_i, p_i) \neq 0$ $(i = 1, 2, 3)$, so by the implicit function theorem there exist neighborhoods N_i of (τ_0, p_0, p_i) and C^1 functions V_i on N_i such that

$$V_i(\tau_0, p_0; p_i) = \tau_i \qquad (i = 1, 2, 3), \tag{5.7}$$

$$H(\tau_c, p_c; V_i(\tau_c, p_c; p), p) = 0 \qquad ((\tau_c, p_c, p) \in N_i, i = 1, 2, 3), \tag{5.8}$$

and we may assume

$$N_i \cap N_j = \varnothing \qquad (1 \leqslant i < j \leqslant 3),$$

$$(\tau_c, p_c, p) \in N_i \Rightarrow p > p_c \qquad (i = 1, 2, 3). \tag{5.9}$$

For brevity, we shall usually write

$$V_i^c(p) = V_i(\tau_c, p_c; p) \qquad ((\tau_c, p_c, p) \in N_i). \tag{5.10}$$

We also define

$$\eta(\tau_c, p_c; \tau, p) = \sqrt{(\tau_c - \tau)(p - p_c)}, \tag{5.11}$$

and we abbreviate $\eta^c(\tau, p) = \eta(\tau_c, p_c; \tau, p)$. In similar fashion, we define

$$a_c(p) = \eta^c(V_i^c(p), p) \quad \text{when } (\tau_c, p_c, p) \in N_i. \tag{5.12}$$

By Lemma 2.9, (5.2), and the fact that $H_\tau(\tau_i, p_i) \neq 0$, we have

$$\left.\frac{da}{dp}\right|_{(\tau_i, p_i)} = \left.\frac{da}{d\theta}\right|_{(\tau_i, p_i)} \cdot \left.\frac{d\theta}{dp}\right|_{(\tau_i, p_i)} \neq 0 \qquad (i = 1, 2, 3),$$

which, in the notation (5.10)–(5.12), can be expressed,

$$a_0'(p_i) \neq 0 \qquad (i = 1, 2, 3). \tag{5.13}$$

Define

$$\Phi(\tau_l, p_l, u_l, \tau_r, p_r, p) = (\tau_l, p_l, u_l, \tau_r, p_r, u_l - (a_l(p) + a_r(p))),$$

for all $(\tau_l, p_l, u_l, \tau_r, p_r, p)$ for which $(\tau_l, p_l, p), (\tau_r, p_r, p) \in N_i$ for some i. (Here, for example, $a_l(p) = \eta^l(V_i^l(p), p)$ when $(\tau_l, p_l, p) \in N_i$; see (5.10)–(5.12).) Φ is

a C^1 function defined on an open set containing $\{(\tau_0, p_0, u_0, \tau_0, p_0, p_i) \mid i = 1, 2, 3\}$, and has Jacobian matrix of the form

$$
\begin{bmatrix}
1 & & & & & \\
& 1 & & & & \\
& & 1 & & & 0 \\
0 & & & 1 & & \\
& & & & 1 & \\
\hline
* & * & * & * & * & \dfrac{\partial}{\partial p}\left(u_l - (a_l(p) + a_r(p))\right)
\end{bmatrix};
$$

hence its Jacobian determinant is given by

$$
J_\Phi(\tau_l, p_l, u_l, \tau_r, p_r, p) = \frac{\partial}{\partial p}\left(u_l - (a_l(p) + a_r(p))\right) = -(a_l'(p) + a_r'(p)),
$$

whence

$$
J_\Phi(\tau_0, p_0, u_0, \tau_0, p_0, p_i) = -(a_0'(p_i) + a_0'(p_i)) = -2a_0'(p_i) \neq 0
$$

$$(i = 1, 2, 3).$$

Moreover,

$$
\Phi(\tau_0, p_0, u_0, \tau_0, p_0, p_i) = \left(\tau_0, p_0, u_0, \tau_0, p_0, u_0 - (a_0(p_i) + a_0(p_i))\right)
$$

$$
= (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - 2a) \qquad (i = 1, 2, 3), \quad (5.14)
$$

since $a_0(p_i) = \eta^0(\tau_i, p_i) = a^0(\theta_i) = a$ by (5.4) and (5.7). Hence there exist neighborhoods $\mathfrak{M}_i$ of $(\tau_0, p_0, u_0, \tau_0, p_0, p_i)$ and $\mathfrak{N}_i$ of $(\tau_0, p_0, u_0, \tau_0, p_0, u_0 - 2a)$ such that $\Phi|_{\mathfrak{M}_i}$ is a C^1 diffeomorphism of $\mathfrak{M}_i$ onto $\mathfrak{N}_i$; we may assume

$$
\mathfrak{M}_i \cap \mathfrak{M}_j = \varnothing \qquad (1 \leqslant i < j \leqslant 3). \quad (5.15)
$$

In addition, (5.9) implies

$$
(\tau_l, p_l, u_l, \tau_r, p_r, p) \in \mathfrak{M}_i \Rightarrow p > p_l, p_r \qquad (i = 1, 2, 3). \quad (5.16)
$$

Observe that $(U_l^0, U_r^0) = (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - 2a)$. Let $\mathfrak{N} = \mathfrak{N}_1 \cap \mathfrak{N}_2 \cap \mathfrak{N}_3$; $\mathfrak{N}$ is a neighborhood of (U_l^0, U_r^0). Let $(U_l, U_r) \in \mathfrak{N}$. Then $\forall i$, $(\tau_l, p_l, u_l, \tau_r, p_r, u_r) = (U_l, U_r) \in \mathfrak{N}_i$, so there is a $\bar{p}_i > 0$ with $(\tau_l, p_l, u_l, \tau_r, p_r, \bar{p}_i) \in \mathfrak{M}_i$ such that $\Phi(\tau_l, p_l, u_l, \tau_r, p_r, \bar{p}_i) = (\tau_l, p_l, u_l, \tau_r, p_r, u_r)$. Since also $\Phi(\tau_l, p_l, u_l, \tau_r, p_r, \bar{p}_i) = (\tau_l, p_l, u_l, \tau_r, p_r, u_l - (a_l(\bar{p}_i) + a_r(\bar{p}_i)))$, we have

$$
u_r = u_l - \left(a_l(\bar{p}_i) + a_r(\bar{p}_i)\right) \qquad (i = 1, 2, 3); \quad (5.17)
$$

moreover $\bar{p}_i > p_l, p_r$ by (5.16). Here $\bar{p}_i$ is a C^1 function of (U_l, U_r); in fact, it is the last coordinate function of the C^1 diffeomorphism $(\Phi|_{\mathfrak{M}_i})^{-1}|_{\mathfrak{N}}$. Note that

by (5.15), the $\overline{p_i}$'s are distinct. Thus from (5.8) and (5.17) we see that the R. P. with data (U_l, U_r) has the following three distinct solutions, for $i = 1, 2, 3$:

$$U_l \xrightarrow{\text{1-shock}} \left(V_i^l\left(\overline{p_i}\right), \overline{p_i}, u_l - a_l\left(\overline{p_i}\right) \right)$$

$$\xrightarrow{\text{contact}} \left(V_i^r\left(\overline{p_i}\right), \overline{p_i}, u_r + a_r\left(\overline{p_i}\right) \right) \xrightarrow{\text{3-shock}} U_r, \qquad (5.18)$$

which establishes condition (a) of Definition 5.1.

By (5.14) and the fact that $(\Phi|_{\mathfrak{N}_i})^{-1}$ is continuous on $\mathfrak{N}_i$, $\bar{p}_i \to p_i$ as $(U_l, U_r) \to (U_l^0, U_r^0)$. Letting

$$U_1^i(U_l, U_r) = \left(V_i^l\left(\overline{p_i}\right), \overline{p_i}, u_l - a_l\left(\overline{p_i}\right) \right),$$

$$U_3^i(U_l, U_r) = \left(V_i^r\left(\overline{p_i}\right), \overline{p_i}, u_r + a_r\left(\overline{p_i}\right) \right),$$

we see that the solutions (5.18) are expressible as

$$U_l \xrightarrow{1} U_1^i(U_l, U_r) \xrightarrow{\text{contact}} U_3^i(U_l, U_r) \xrightarrow{3} U_r \qquad (i = 1, 2, 3),$$

and $U_j^i(U_l, U_r) \to (\tau_i, p_i, u_0 - a)$ as $(U_l, U_r) \to (U_l^0, U_r^0)$ $(i = 1, 2, 3; j = 1, 3)$ (see (5.17)). Referring to (5.1) and (5.5), we see that condition (b) of Definition 5.1 holds. $\square$

We can also establish

LEMMA 5.3. *Let* $\theta_0 < \overline{\theta_1} < \bar{\theta} < \overline{\theta_2}$, *and suppose* $a^0(\overline{\theta_1}) > a^0(\bar{\theta}) > a^0(\overline{\theta_2})$, $(a^0)'(\bar{\theta}) \neq 0$ *and* $(p^0)'(\bar{\theta}) \neq 0$. *Then for any open interval* I *containing* $a^0(\bar{\theta})$, *there exist* $\theta_1, \theta_2, \theta_3$ *and* $a \in I$ *with* $\theta_0 < \theta_1 < \theta_2 < \theta_3$, *such that* $a^0(\theta_i) = a$, $(a^0)'(\theta_i) \neq 0$ *and* $(p^0)'(\theta_i) \neq 0$ $(i = 1, 2, 3)$.

As a corollary to the preceding two lemmas and Lemma 4.12, we have

THEOREM 5.4. *Suppose* e *violates* MEDIUM. *Then there exist data* (U_l^0, U_r^0) *for which 3-fold type I nonuniqueness holds and the 3-fold nonuniqueness is stable.*

PROOF. By Lemma 4.12, there exist a center (τ_0, p_0), and a point $(\bar{\tau}, \bar{p}) \in H \cap Q$ (see (2.1)), such that

$$\left.\frac{da}{d\theta}\right|_{(\bar{\tau}, \bar{p})} < 0, \qquad \left.\frac{dp}{d\theta}\right|_{(\bar{\tau}, \bar{p})} \neq 0.$$

Let $\bar{\theta} = \theta(\bar{\tau}, \bar{p})$; then $\bar{\theta} > \theta_0$, $(a^0)'(\bar{\theta}) < 0$ and $(p^0)'(\bar{\theta}) \neq 0$. Then there exist $\overline{\theta_1} \in (\theta_0, \bar{\theta})$ and $\overline{\theta_2} > \bar{\theta}$ such that $a^0(\overline{\theta_1}) > a^0(\bar{\theta}) > a^0(\overline{\theta_2})$. By Lemma 5.3, there exist $\theta_1, \theta_2, \theta_3$ and $a > 0$ with $\theta_0 < \theta_1 < \theta_2 < \theta_3$, such that $a^0(\theta_i) = a$, $(a^0)'(\theta_i) \neq 0$ and $(p^0)'(\theta_i) \neq 0$ $(i = 1, 2, 3)$. Let u_0 be arbitrary and let $U_l^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - 2a)$. By Lemma 5.2 the R. P. with data (U_l^0, U_r^0) has three type I solutions for which the 3-fold nonuniqueness is stable. $\square$

In preparation for Theorem 5.6, we prove the following lemma, which gives a sufficient condition for stability in the case of type II nonuniqueness.

LEMMA 5.5. *Suppose* $(\tau_1, \bar{p}), (\tau_2, \bar{p}) \in H \cap Q$ *with* $\tau_1 < \tau_2$ *and*

$$H_\tau(\tau_1, \bar{p}), H_\tau(\tau_2, \bar{p}) \neq 0, \tag{5.19}$$

$$\left.\frac{da}{dp}\right|_{(\tau_1, \bar{p})} + \left.\frac{da}{dp}\right|_{(\tau_2, \bar{p})} \neq 0. \tag{5.20}$$

Let u_0 *be arbitrary and let* $U_l^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - (a_0 + b_0))$, *where* $a_0 = \eta^0(\tau_2, \bar{p})$, $b_0 = \eta^0(\tau_1, \bar{p})$. *Then type* II *nonuniqueness holds for the data* (U_l^0, U_r^0), *and the (2-fold) nonuniqueness is stable.*

PROOF. First recall that the R. P. with data (U_l^0, U_r^0) has the two type II solutions,

$$\begin{array}{l} U_l^0 \xrightarrow{\text{1-shock}} (\tau_1, \bar{p}, u_l - b_0) \xrightarrow{\text{contact}} (\tau_2, \bar{p}, u_l - b_0) \xrightarrow{\text{3-shock}} U_r^0, \\ U_l^0 \xrightarrow{\text{1-shock}} (\tau_2, \bar{p}, u_l - a_0) \xrightarrow{\text{contact}} (\tau_1, \bar{p}, u_l - a_0) \xrightarrow{\text{3-shock}} U_r^0. \end{array} \tag{5.21}$$

By (5.19), $H_\tau(\tau_0, p_0; \tau_1, \bar{p})$, $H_\tau(\tau_0, p_0; \tau_2, \bar{p}) \neq 0$ (see (5.6)); hence we can find a neighborhood N of $(\tau_0, p_0, \bar{p})$ and C^1 functions V_1 and V_2 on N such that, abbreviating $V_i(\tau_c, p_c; p) = V_i^c(p)$, we have

$$V_i^0(\bar{p}) = \tau_i \qquad (i = 1, 2), \tag{5.22}$$

$$H(\tau_c, p_c; V_i^c(p), p) = 0 \qquad ((\tau_c, p_c, p) \in N, i = 1, 2), \tag{5.23}$$

$$(\tau_c, p_c, p) \in N \Rightarrow p > p_c, \tag{5.24}$$

$$(\tau_c, p_c, p), (\tau_d, p_d, q) \in N \Rightarrow V_1^c(p) < V_2^d(q). \tag{5.25}$$

Let $\mathcal{V} = \{(\tau_l, p_l, \tau_r, p_r, p, q) | (\tau_l, p_l, p), (\tau_l, p_l, q), (\tau_r, p_r, p), (\tau_r, p_r, q) \in N\}$; $\mathcal{V}$ is an open set containing $(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p})$. For $(\tau_l, p_l, \tau_r, p_r, p, q) \in \mathcal{V}$, we define

$$\begin{array}{ll} a_r = \eta^r(V_2^r(p), p), & a_l = \eta^l(V_2^l(q), q), \\ b_r = \eta^r(V_1^r(q), q), & b_l = \eta^l(V_1^l(p), p). \end{array}$$

(See Figure 13; $\mathcal{H}_a^c$ is the upper branch of the hyperbola $(\tau_c - \tau)(p - p_c) = a$.) Suppose u_l and u_r satisfy

$$u_r = u_l - (a_r + b_l); \tag{5.26}$$

then by (5.23) and (5.24), the R. P. with data $(U_l, U_r) = (\tau_l, p_l, u_l, \tau_r, p_r, u_r)$ has the solution,

$$U_l \xrightarrow{\text{1-shock}} (V_1^l(p), p, u_l - b_l) \xrightarrow{\text{contact}} (V_2^r(p), p, u_r + a_r) \xrightarrow{\text{3-shock}} U_r,$$
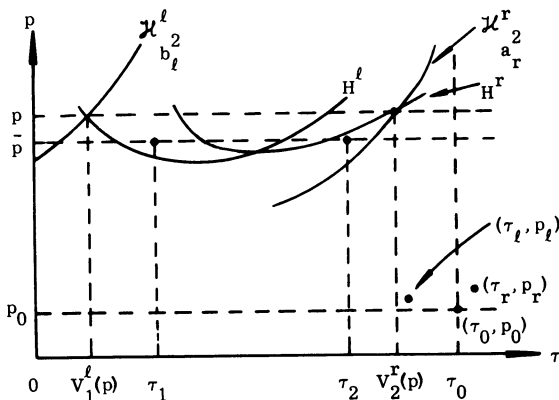
$$\tag{5.27}$$

FIGURE 13. One of the solutions.

since $u_l - b_l = u_r + a_r$ by (5.26). (See Figure 13.) If, on the other hand, u_l and u_r satisfy

$$u_r = u_l - (a_l + b_r), \tag{5.28}$$

then $u_l - a_l = u_r + b_r$, and the R. P. with data (U_l, U_r) has the solution,

$$U_l \xrightarrow{\text{1-shock}} (V_2^l(q), q, u_l - a_l) \xrightarrow{\text{contact}} (V_1^r(q), q, u_r + b_r) \xrightarrow{\text{3-shock}} U_r. \tag{5.29}$$

Thus if u_l and u_r satisfy (5.26) and (5.28) simultaneously, then the R. P. with data (U_l, U_r) has the two solutions (5.27) and (5.29), which are distinct by (5.25). But (5.26) and (5.28) hold iff

$$a_l + b_r = a_r + b_l. \tag{5.30}$$

Define $\Phi(\tau_l, p_l, \tau_r, p_r, p, q) = a_l + b_r - a_r - b_l$ $((\tau_l, p_l, \tau_r, p_r, p, q) \in \mathcal{V})$. (Note that, for example, a_l is *defined* as a number depending on the variables τ_l, p_l, q.) Then $\Phi(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p}) = 0$ (since, for example, $a_l(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p}) = \eta^0(V_2^0(\bar{p}), \bar{p}) = \eta^0(\tau_2, \bar{p}) = a_0$). Moreover,

$$\frac{\partial}{\partial q} \Phi(\tau_l, p_l, \tau_r, p_r, p, q) = \frac{d}{dq} \eta^l(V_2^l(q), q) + \frac{d}{dq} \eta^r(V_1^r(q), q);$$

then

$$\frac{\partial \Phi}{\partial q}(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p}) = \frac{da}{dp}\bigg|_{(\tau_2, \bar{p})} + \frac{da}{dp}\bigg|_{(\tau_1, \bar{p})} \neq 0,$$

by (5.6), (5.22), (5.23), Conventions 4.6 and 4.9, and (5.20). Thus by the implicit function theorem, there exist a neighborhood $\mathcal{W}$ of $(\tau_0, p_0, \tau_0, p_0, \bar{p})$ and a C^1 function $q = q(\tau_l, p_l, \tau_r, p_r, p)$ $((\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W})$ such that $q(\tau_0, p_0, \tau_0, p_0, \bar{p}) = \bar{p}$ and

$$\Phi(\tau_l, p_l, \tau_r, p_r, p, q(\tau_l, p_l, \tau_r, p_r, p)) = 0 \quad ((\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W}). \tag{5.31}$$

Henceforth we shall abbreviate $q(\tau_l, p_l, \tau_r, p_r, p) = q$. Thus q is a number depending continuously on the variables τ_l, p_l, τ_r, p_r and p. Using this q in the definitions of a_l and b_r above, we see that a_r, b_r, a_l and b_l likewise are numbers depending continuously on the variables τ_l, p_l, τ_r, p_r and p. Here q, a_r, b_r, a_l and b_l are defined whenever $(\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W}$. Thus (5.31) becomes

$$a_l + b_r = a_r + b_l \qquad ((\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W}). \qquad (5.32)$$

Now for $(\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W}$ and u_l arbitrary, we define

$$\Psi(\tau_l, p_l, u_l, \tau_r, p_r, p) = (\tau_l, p_l, u_l, \tau_r, p_r, u_l - (a_r + b_l)). \qquad (5.33)$$

Clearly Ψ has Jacobian determinant given by

$$J_\Psi(\tau_l, p_l, u_l, \tau_r, p_r, p) = \frac{\partial}{\partial p}(u_l - (a_r + b_l))$$

$$= -\left(\frac{d}{dp}\eta^r(V_2^r(p), p) + \frac{d}{dp}\eta^l(V_1^l(p), p)\right)$$

$$((\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W}).$$

Then clearly $-J_\Psi(\tau_0, p_0, u_0, \tau_0, p_0, \bar{p}) \neq 0$; moreover

$$\Psi(\tau_0, p_0, u_0, \tau_0, p_0, \bar{p}) = (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - (a_0 + b_0)) = (U_l^0, U_r^0).$$

Hence there exist neighborhoods $\mathfrak{M}$ of $(\tau_0, p_0, u_0, \tau_0, p_0, \bar{p})$ and $\mathfrak{N}$ of (U_l^0, U_r^0) such that

$$(\tau_l, p_l, u_l, \tau_r, p_r, p) \in \mathfrak{M} \Rightarrow (\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W},$$

and $\Psi|_{\mathfrak{M}}$ is a C^1 diffeomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$. Thus, for any data $(U_l, U_r) \in \mathfrak{N}$, there exists a number p such that

$$(\tau_l, p_l, \tau_r, p_r, p) \in \mathcal{W}, \qquad (5.34)$$

$$u_r = u_l - (a_r + b_l) \qquad (5.35)$$

(the latter by (5.33)); p depends continuously on (U_l, U_r), and $p = \bar{p}$ when $(U_l, U_r) = (U_l^0, U_r^0)$. Moreover, by (5.34) there is a number q, also depending continuously on $(U_l, U_r) \in \mathfrak{N}$, such that (5.30) holds; $q = \bar{p}$ when $(U_l, U_r) = (U_l^0, U_r^0)$. Therefore whenever $(U_l, U_r) \in \mathfrak{N}$, numbers p and q exist such that the R. P. with data (U_l, U_r) has the two distinct solutions (5.27) and (5.29); as $(U_l, U_r) \to (U_l^0, U_r^0)$, the constant states of the solutions (5.27) and (5.29) approach the respective constant states of the solutions (5.21), since p, $q \to \bar{p}$. $\square$

Lemmas 5.2, 5.3 and 5.5 give sufficient conditions for the occurrence of type I and type II nonuniqueness and their stability. Thus for our final result–that when WEAK fails we can get type I and type II nonuniqueness simultaneously, as well as their stability, for the same initial data–we must

construct data (U_l^0, U_r^0) that at once satisfy the hypotheses of Lemmas 5.2–5.3 and those of Lemma 5.5.

THEOREM 5.6. *Suppose e violates* WEAK. *Then there exist data* (U_l^0, U_r^0) *for which both 3-fold type* I *nonuniqueness and type* II *nonuniqueness hold, and the 5-fold nonuniqueness is stable.*

The proof, which uses Lemma 4.14, is long but rather routine, and hence is omitted.

6. Examples.

6.1. *Energy functions that distinguish the conditions* WEAK, MEDIUM *and* STRONG. All energy functions e considered in this subsection are assumed to satisfy

$$\lim_{\tau \to +\infty} e(\tau, S) = 0. \tag{6.1}$$

We shall focus throughout this section on the one-parameter family of functions $g(\cdot, S)$, $-\infty < S < +\infty$; each function is defined on $(0, +\infty)$ with τ as independent variable. To see what properties these functions have in general, let us fix $S \in (-\infty, +\infty)$ and define $\phi(\tau) = g(\tau, S)$ $(\tau > 0)$. We see immediately from (1.3), (1.5) and (1.6) that

$$\phi > 0, \quad \phi' < 0, \quad \phi'' > 0. \tag{6.2}$$

In addition, the following are easily verified:

$$\int_1^{+\infty} \phi < +\infty; \tag{6.3}$$

$$e(\tau, S) = \int_\tau^{+\infty} g(v, S)\, dv; \tag{6.4}$$

$$\lim_{\tau \to 0^+} \phi(\tau) = +\infty. \tag{6.5}$$

Note also that ϕ is C^2.

Motivated by the fact that the functions $g(\cdot, S)$ always satisfy (6.2), (6.3) and (6.5), we make

DEFINITION 6.1. Let $\mathscr{P}$ be the set of all C^2 functions f defined on $(0, +\infty)$ for which $f > 0, f' < 0, f'' > 0, \int_1^{+\infty} f < +\infty$ and $\lim_{\tau \to 0^+} f(\tau) = +\infty$.

"$\mathscr{P}$" signifies pressure. We have just proved

$$g(\cdot, S) \in \mathscr{P} \qquad (-\infty < S < +\infty), \tag{6.6}$$

for every energy function e. (Remember that e always satisfies (6.1) in this subsection.) In what follows, we shall construct and analyze one-parameter families $g(\cdot, S)$, $-\infty < S < +\infty$, and get our energy functions via (6.4).

LEMMA 6.2. *Suppose e satisfies* (1.2)–(1.9) *and* (6.1). *Suppose also that* $g(\cdot, S) \in \mathscr{P}$ *for all S. Then e satisfies* (1.10); *hence e is an energy function.*

PROOF. Fix $p > 0$. We first prove

$$\lim_{\tau \to 0^+} S(\tau, p) = -\infty. \tag{6.7}$$

(See (1.7)–(1.11) ff.) Let $S_0 \in (-\infty, +\infty)$. Since $g(\cdot, S_0) \in \mathscr{P}$, we may choose $\tau_0 > 0$ such that $g(\tau_0, S_0) = p$. Let $\tau \in (0, \tau_0)$ and suppose $S(\tau, p) \geqslant S_0$. Then $p = g(\tau, S(\tau, p)) > g(\tau_0, S(\tau, p)) \geqslant g(\tau_0, S_0) = p$, a contradiction. Hence $S(\tau, p) < S_0$. Thus (6.7) is proved.

Now let $\varepsilon > 0$. Choose $\tau_1 > 0$ so that $\tau_1 p \leqslant \frac{1}{2}\varepsilon$. Fix any S_1 and note that since $g(\cdot, S_1) \in \mathscr{P}$,

$$\int_{\tau_1}^{+\infty} g(\cdot, S_1) < +\infty.$$

By (1.7), $g(\cdot, S) < g(\cdot, S_1)$ whenever $S < S_1$. By (1.8), $g(\cdot, S) \to 0$ pointwise on $[\tau_1, +\infty)$ as $S \to -\infty$. Hence by the Lebesgue convergence theorem,

$$\lim_{S \to -\infty} \int_{\tau_1}^{+\infty} g(\cdot, S) = 0.$$

Thus $\exists M > 0$ such that

$$\int_{\tau_1}^{+\infty} g(\cdot, S) < \frac{1}{2}\varepsilon \quad \text{whenever } S < -M. \tag{6.8}$$

By (6.7), $\exists \delta \in (0, \tau_1)$ such that $S(\tau, p) < -M$ whenever $\tau \in (0, \delta)$. Let $\tau \in (0, \delta)$. By (1.5), $g(v, S(\tau, p)) \leqslant g(\tau, S(\tau, p)) = p$ whenever $v \geqslant \tau$. Then from (6.1) and (6.8) we get

$$e(\tau, p) = \int_{\tau}^{+\infty} g(v, S(\tau, p)) \, dv$$

$$= \int_{\tau}^{\tau_1} g(v, S(\tau, p)) \, dv + \int_{\tau_1}^{+\infty} g(v, S(\tau, p)) \, dv$$

$$< \int_{\tau}^{\tau_1} p \, dv + \frac{1}{2}\varepsilon$$

$$= (\tau_1 - \tau)p + \frac{1}{2}\varepsilon < \varepsilon.$$

Thus (1.10) is proved. □

PROPOSITION 6.3. *Let $\phi \in \mathscr{P}$. Let θ be a C^3 function mapping $(-\infty, +\infty)$ onto $(0, +\infty)$ such that $\theta' > 0$. Define*

$$e(\tau, S) = \theta(S) \int_{\tau}^{+\infty} \phi \qquad (\tau > 0, -\infty < S < +\infty).$$

Then e is an energy function satisfying MEDIUM. *ϕ can be chosen so that e violates* STRONG.

PROOF. Clearly e is C^3 in (τ, S) and $e > 0$. Define $g = -e_\tau$; then, using Lemma 6.2, it is easy to verify that e is an energy function.

Let $\tau, p > 0$. Let S satisfy $g(\tau, S) = p$; then

$$\theta(S)\phi(\tau) = p. \tag{6.9}$$

Moreover, $e(\tau, p) = e(\tau, S) = \theta(S)\int_\tau^{+\infty}\phi$; this, together with (6.9), gives

$$e(\tau, p) = \frac{p}{\phi(\tau)} \int_\tau^{+\infty} \phi \qquad (\tau, p > 0). \tag{6.10}$$

We also have

$$g_\tau(\tau, p) = g_\tau(\tau, S) = \theta(S)\phi'(\tau), \tag{6.11}$$

$$\frac{\partial}{\partial p} e(\tau, p) = \frac{1}{\phi(\tau)} \int_\tau^{+\infty} \phi. \tag{6.12}$$

Using Proposition 1.5, (6.9)–(6.12) and Lemma 1.6, we see that e satisfies MEDIUM.

Finally,

$$\text{STRONG} \Leftrightarrow \frac{\partial}{\partial p} e(\tau, p) \geqslant \frac{p}{-g_\tau} \Leftrightarrow \frac{1}{\phi(\tau)} \int_\tau^{+\infty} \phi \geqslant \frac{\theta(S)\phi(\tau)}{-\theta(S)\phi'(\tau)}$$

$$\Leftrightarrow \int_\tau^{+\infty} \phi \geqslant \phi(\tau)\left(\frac{\phi(\tau)}{-\phi'(\tau)} \right).$$

Referring to Figure 14, we see that $\phi(\tau) (\phi(\tau)/-\phi'(\tau))$ is the area of the rectangle $ABCD$, whereas ϕ can be chosen so as to make $\int_\tau^{+\infty}\phi$ arbitrarily close to the area of $\triangle ABD$. In particular

$$\int_\tau^{+\infty} \phi < \phi(\tau)\left(\frac{\phi(\tau)}{-\phi'(\tau)} \right)$$

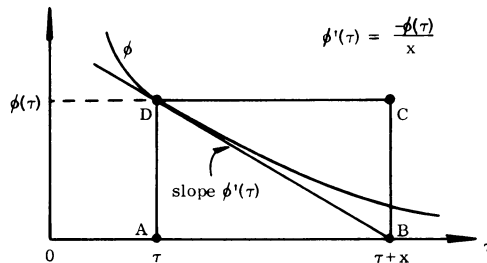for suitable choice of ϕ; STRONG fails for such ϕ. $\square$

FIGURE 14. ϕ can be chosen so that STRONG fails.

As a corollary we have

THEOREM 6.4. *There exists an energy function that satisfies* MEDIUM *but violates* STRONG.

To get an example of an energy function that satisfies WEAK but violates MEDIUM, we shall begin by establishing a criterion for WEAK in terms of the one-parameter family $g(\cdot, S)$, $-\infty < S < +\infty$, of elements of $\mathscr{P}$.

DEFINITION 6.5. Let $f, g \in \mathscr{P}$. We say $f \overset{W}{\sim} g$ iff $f \leqslant g$ and for any τ_1, τ_2, $p > 0$ such that $\tau_1 \leqslant \tau_2$ and $f(\tau_1) = g(\tau_2) = p$, we have

$$\int_{\tau_2}^{+\infty} g \geqslant \int_{\tau_1}^{+\infty} f - (\tau_2 - \tau_1)\tfrac{1}{2} p.$$

PROPOSITION 6.6. *e satisfies* WEAK *iff for all* $S_1, S_2 \in (-\infty, +\infty)$, $S_1 < S_2$ *implies* $g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2)$.

PROOF. Straightforward.

PROPOSITION 6.7. *The relation* $\overset{W}{\sim}$ *has the following properties.*
(a) $f \overset{W}{\sim} g$ *and* $\alpha > 0 \Rightarrow \alpha f \overset{W}{\sim} \alpha g$.
(b) $f \in \mathscr{P}$ *and* $0 < \alpha \leqslant \beta \Rightarrow \alpha f \overset{W}{\sim} \beta f$.
(c) $f \overset{W}{\sim} g$ *and* $g \overset{W}{\sim} h \Rightarrow f \overset{W}{\sim} h$.

PROOF. (a) and (c) are straightforward; for (b), use Propositions 6.3 and 6.6.

LEMMA 6.8. *Let* $f, g \in \mathscr{P}$ *with* $f < g$, *and define* $\pi(\cdot, \lambda) = (1 - \lambda)f + \lambda g$. *Suppose for all* $\tau > 0$,

$$\frac{2}{g(\tau) - f(\tau)} \int_{\tau}^{+\infty} (g - f) \geqslant \begin{cases} \dfrac{f(\tau)}{-f'(\tau)}, \\[2ex] \dfrac{g(\tau)}{-g'(\tau)}. \end{cases}$$

Then $\pi(\cdot, \lambda_1) \overset{W}{\sim} \pi(\cdot, \lambda_2)$ *whenever* $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$.

PROOF. Since $f, g \in \mathscr{P}$, we may define

$$\varepsilon(\tau, \lambda) = \int_{\tau}^{+\infty} \pi(v, \lambda) \, dv = (1 - \lambda) \int_{\tau}^{+\infty} f + \lambda \int_{\tau}^{+\infty} g. \qquad (6.13)$$

Then ε is C^3 in (τ, λ). We have
$$\pi_\lambda(\tau, \lambda) = g(\tau) - f(\tau) > 0,$$
so we can define for any $\tau, p > 0$, $\varepsilon(\tau, p) = \varepsilon(\tau, \lambda)$ where λ satisfies $\pi(\tau, \lambda) = p$. (We then denote $\lambda = \lambda(\tau, p)$.) Similarly, define $\pi_\tau(\tau, p) = \pi_\tau(\tau, \lambda(\tau, p))$ $(\tau, p > 0)$. Now let $\tau, p > 0$, and let $\lambda = (p - f(\tau))/(g(\tau) - f(\tau))$. Then

$$1 - \lambda = \frac{g(\tau) - p}{g(\tau) - f(\tau)},$$

and
$$\pi(\tau, \lambda) = (1 - \lambda)f(\tau) + \lambda g(\tau) = p,$$

whence
$$\lambda(\tau, p) = \frac{p - f(\tau)}{g(\tau) - f(\tau)} \qquad (\tau, p > 0). \qquad (6.14)$$

Also note that
$$\pi_\tau(\tau, \lambda) = (1 - \lambda)f'(\tau) + \lambda g'(\tau). \qquad (6.15)$$

We assert that
$$\frac{\partial}{\partial p} \varepsilon(\tau, p) \geqslant \frac{-p}{2\pi_\tau(\tau, p)} \qquad (f(\tau) \leqslant p \leqslant g(\tau)). \qquad (6.16)$$

For by (6.13) and (6.14),
$$\varepsilon(\tau, p) = \varepsilon(\tau, \lambda(\tau, p))$$
$$= \frac{g(\tau) - p}{g(\tau) - f(\tau)} \int_\tau^{+\infty} f + \frac{p - f(\tau)}{g(\tau) - f(\tau)} \int_\tau^{+\infty} g,$$

whence
$$\frac{\partial}{\partial p} \varepsilon(\tau, p) = \frac{1}{g(\tau) - f(\tau)} \int_\tau^{+\infty} (g - f). \qquad (6.17)$$

Let us temporarily hold τ constant; define
$$\Phi(p) = \frac{-p}{(g(\tau) - p)f'(\tau) + (p - f(\tau))g'(\tau)} .$$

Φ is defined on an open interval I containing $[f(\tau), g(\tau)]$. For $p \in I$, (6.14) and (6.15) yield
$$- \frac{p}{2\pi_\tau(\tau, p)}$$
$$= \frac{-p}{2[(g(\tau) - p)f'(\tau)/(g(\tau) - f(\tau)) + (p - f(\tau))g'(\tau)/(g(\tau) - f(\tau))]}$$
$$= \tfrac{1}{2}(g(\tau) - f(\tau))\Phi(p). \qquad (6.18)$$

We have
$$\Phi'(p) = \frac{f(\tau)g'(\tau) - g(\tau)f'(\tau)}{[(g(\tau) \doteq p)f'(\tau) + (p - f(\tau))g'(\tau)]^2} \qquad (p \in I);$$

thus, depending on the sign of $f(\tau)g'(\tau) - g(\tau)f'(\tau)$, either $\Phi' \geqslant 0$ on I or $\Phi' \leqslant 0$ on I. Hence Φ attains its maximum on $[f(\tau), g(\tau)]$ at an endpoint, $f(\tau)$

or $g(\tau)$. But

$$\Phi(f(\tau)) = \frac{-f(\tau)}{(g(\tau) - f(\tau))f'(\tau)}, \qquad \Phi(g(\tau)) = \frac{-g(\tau)}{(g(\tau) - f(\tau))g'(\tau)}.$$

Using (6.18), we conclude that $-p/2\pi_\tau(\tau, p)$ attains its maximum value for $p \in [f(\tau), g(\tau)]$ either at $p = f(\tau)$ or $p = g(\tau)$, and its maximum value is either $g(\tau)/-2g'(\tau)$ or $f(\tau)/-2f'(\tau)$. But by (6.17) and the hypothesis, $\partial\varepsilon(\tau, p)/\partial p$ is always greater than or equal to each of these values; hence (6.16) holds.

Now proceed by analogy with the proofs of Proposition 1.5 and Proposition 6.6. $\square$

THEOREM 6.9. *There exists an energy function e that satisfies* WEAK *but violates* MEDIUM.

PROOF. Choose any $\phi_0 \in \mathcal{P} \cap C^3(0, +\infty)$ that satisfies $\phi_0''' < 0$. Define

$$\phi(\tau, S) = \phi_0(\tau)(1 - S\phi_0'(\tau)) \qquad (\tau > 0, -\infty < S < +\infty). \quad (6.19)$$

Then

$$|\phi(\tau, S)| \leq \phi_0(\tau) + |S|\phi_0(\tau)(-\phi_0'(\tau)) \qquad (\tau > 0). \quad (6.20)$$

Now

$$\int_\tau^M - \phi_0(v)\phi_0'(v) \, dv = -\tfrac{1}{2}\phi_0(v)^2\Big|_{v=\tau}^{v=M} = \tfrac{1}{2}\left[\phi_0(\tau)^2 - \phi_0(M)^2\right]$$

$$(\tau, M > 0). \quad (6.21)$$

With $\tau = 1$, (6.21) yields

$$\int_1^{+\infty} - \phi_0(v)\phi_0'(v) \, dv < +\infty.$$

Since also $\int_1^{+\infty}\phi_0 < +\infty$, we obtain from (6.20)

$$\int_1^{+\infty} |\phi(\cdot, S)| < +\infty. \quad (6.22)$$

Thus we may define

$$\varepsilon(\tau, S) = \int_\tau^{+\infty} \phi(v, S) \, dv \qquad (\tau > 0); \quad (6.23)$$

in fact, by (6.19) and (6.21) we have

$$\varepsilon(\tau, S) = \int_\tau^{+\infty} \phi_0 + \frac{S}{2} \phi_0(\tau)^2 \qquad (\tau > 0), \quad (6.24)$$

which shows ε is C^3 in (τ, S). By (6.23), $\varepsilon_\tau(\tau, S) = -\phi(\tau, S)$ $(\tau > 0)$. By (6.19), $\phi_S(\tau, S) = -\phi_0(\tau)\phi_0'(\tau) > 0$ $(\tau > 0)$; thus we may, as usual, define

$$\varepsilon(\tau, p) = \varepsilon(\tau, S) \qquad (\tau > 0, -\infty < p < +\infty),$$

where $S = \hat{S}(\tau, p)$ is the unique solution of $\phi(\tau, S) = p$.

We assert that

$$\phi(\cdot, S) \in \mathcal{P} \qquad (S > 0). \tag{6.25}$$

Clearly $\phi(\tau, S) > 0$ $(\tau > 0, S \geqslant 0)$. Differentiating (6.19) with respect to τ gives

$$\phi_\tau(\tau, S) = \phi_0'(\tau) - S\big[\phi_0(\tau)\phi_0''(\tau) + \phi_0'(\tau)^2\big] \qquad (\tau > 0), \tag{6.26}$$

whence $\phi_\tau(\cdot, S) < 0$ $(S \geqslant 0)$. Differentiating (6.26), we get

$$\phi_{\tau\tau}(\tau, S) = \phi_0''(\tau) - S\big[\phi_0(\tau)\phi_0'''(\tau) + 3\phi_0'(\tau)\phi_0''(\tau)\big] > 0 \qquad (\tau > 0, S \geqslant 0).$$
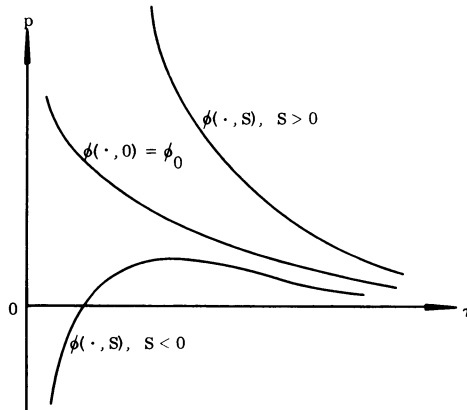

FIGURE 15.   The family $\phi(\cdot, S)$, $-\infty < S < +\infty$.

By (6.22), $\int_1^{+\infty}\phi(\cdot, S) < +\infty$ $(S \geqslant 0)$. Since $\phi(\cdot, S) \geqslant \phi_0$ $(S \geqslant 0)$, $\lim_{\tau \to 0^+}\phi(\tau, S) = +\infty$ $(S \geqslant 0)$. Thus (6.25) is proved.

Now for $\tau > 0$ and all S, we have

$$\int_\tau^{+\infty}\phi(\cdot, S) - \int_\tau^{+\infty}\phi_0 = \frac{S}{2}\phi_0(\tau)^2$$

by (6.23) and (6.24); and by (6.19),

$$\phi(\tau, S) - \phi_0(\tau) = -S\phi_0(\tau)\phi_0'(\tau).$$

Thus

$$\frac{\int_\tau^{+\infty}\phi(\cdot, S) - \int_\tau^{+\infty}\phi_0}{\phi(\tau, S) - \phi_0(\tau)} = -\frac{1}{2}\frac{\phi_0(\tau)}{\phi_0'(\tau)} \qquad (\tau > 0, S \neq 0). \tag{6.27}$$

For $S = 1$, (6.27) becomes

$$\frac{2}{\phi(\tau, 1) - \phi_0(\tau)}\int_\tau^{+\infty}(\phi(\cdot, 1) - \phi_0) = \frac{\phi_0(\tau)}{-\phi_0'(\tau)} \qquad (\tau > 0). \tag{6.28}$$

Now observe that for $\tau, S > 0$,

$$\frac{\phi_0(\tau)}{-\phi_0'(\tau)} \geq \frac{\phi(\tau, S)}{-\phi_\tau(\tau, S)} \Leftrightarrow -\phi_0(\tau)\phi_\tau(\tau, S) \geq -\phi(\tau, S)\phi_0'(\tau)$$

$$\Leftrightarrow -\phi_0(\tau)\left[\phi_0'(\tau) - S\left(\phi_0(\tau)\phi_0''(\tau) + \phi_0'(\tau)^2\right)\right]$$

$$\geq -\phi_0(\tau)(1 - S\phi_0'(\tau))\phi_0'(\tau)$$

$$\Leftrightarrow S\phi_0(\tau)\left(\phi_0(\tau)\phi_0''(\tau) + \phi_0'(\tau)^2\right) \geq S\phi_0(\tau)\phi_0'(\tau)^2 \Leftrightarrow \phi_0(\tau)\phi_0''(\tau) \geq 0,$$

by (6.19) and (6.26). Hence

$$\frac{\phi_0(\tau)}{-\phi_0'(\tau)} \geq \frac{\phi(\tau, S)}{-\phi_\tau(\tau, S)} \qquad (\tau, S > 0). \qquad (6.29)$$

It follows from (6.28) and (6.29) that $\forall \tau > 0$,

$$\frac{2}{\phi(\tau, 1) - \phi_0(\tau)} \int_\tau^{+\infty} (\phi(\cdot, 1) - \phi_0) \geq \begin{cases} \dfrac{\phi_0(\tau)}{-\phi_0'(\tau)}, \\[2mm] \dfrac{\phi(\tau, 1)}{-\phi_\tau(\tau, 1)}. \end{cases}$$

Note that $\phi(\cdot, 1) > \phi_0$, so Lemma 6.8 applies. Define

$$\pi(\tau, \lambda) = (1 - \lambda)\phi_0(\tau) + \lambda\phi(\tau, 1) \qquad (\tau > 0, -\infty < \lambda < +\infty);$$

then by Lemma 6.8,

$$\pi(\cdot, \lambda_1) \overset{W}{\sim} \pi(\cdot, \lambda_2) \qquad (0 \leq \lambda_1 \leq \lambda_2 \leq 1).$$

But

$$\pi(\tau, \lambda) = (1 - \lambda)\phi_0(\tau) + \lambda\phi_0(\tau)(1 - \phi_0'(\tau))$$

$$= \phi_0(\tau)\left(1 - \lambda + \lambda(1 - \phi_0'(\tau))\right) = \phi(\tau, \lambda);$$

hence

$$\phi(\cdot, S_1) \overset{W}{\sim} \phi(\cdot, S_2) \qquad (0 \leq S_1 \leq S_2 \leq 1). \qquad (6.30)$$

Now let $\tau > 0$; we shall compute $\partial\varepsilon(\tau, p)/\partial p|_{(\tau, \phi_0(\tau))}$. Since $\phi(\tau, 0) = \phi_0(\tau)$, $\hat{S}(\tau, \phi_0(\tau)) = 0$, so

$$\varepsilon(\tau, \phi_0(\tau)) = \varepsilon(\tau, 0) = \int_\tau^{+\infty} \phi(v, 0)\, dv = \int_\tau^{+\infty} \phi_0.$$

Also, with $p = \phi(\tau, S)$, i.e., $S = \hat{S}(\tau, p)$ (τ fixed), $S \to 0$ as $p \to \phi_0(\tau)$ and

vice-versa. Thus

$$\frac{\partial}{\partial p}\,\varepsilon(\tau,p)\bigg|_{(\tau,\phi_0(\tau))} = \lim_{p\to\phi_0(\tau)} \frac{\varepsilon(\tau,p) - \varepsilon(\tau,\phi_0(\tau))}{p - \phi_0(\tau)}$$

$$= \lim_{p\to\phi_0(\tau)} \frac{\varepsilon\big(\tau,\hat{S}\,(\tau,p)\big) - \varepsilon(\tau,\phi_0(\tau))}{\phi\big(\tau,\hat{S}\,(\tau,p)\big) - \phi_0(\tau)}$$

$$= \lim_{S\to 0} \frac{\varepsilon(\tau,S) - \int_\tau^{+\infty}\phi_0}{\phi(\tau,S) - \phi_0(\tau)} = -\frac{1}{2}\,\frac{\phi_0(\tau)}{\phi_0'(\tau)}, \qquad (6.31)$$

by (6.27). We define $\phi_\tau(\tau,p) = \phi_\tau(\tau,\hat{S}(\tau,p))$; then, with $\phi_0(\tau)$ understood to be p, we have

$$\phi_\tau(\tau,\phi_0(\tau)) = \phi_\tau(\tau,0) = \phi_0'(\tau). \qquad (6.32)$$

By (6.23), (6.25) and Lemma 1.6,

$$\phi(\tau,S)^2 < -2\phi_\tau(\tau,S)\varepsilon(\tau,S) \qquad (S \geqslant 0);$$

by (6.32), this becomes for $S = 0$,

$$\phi_0(\tau)^2 < -2\phi_\tau(\tau,\phi_0(\tau))\varepsilon(\tau,\phi_0(\tau)). \qquad (6.33)$$

Using (6.32), (6.31) may be written

$$\frac{\partial}{\partial p}\,\varepsilon(\tau,p)\bigg|_{(\tau,\phi_0(\tau))}$$

$$= \frac{2\varepsilon(\tau,\phi_0(\tau))\phi_0(\tau)}{-\phi_\tau(\tau,\phi_0(\tau))\cdot 2\varepsilon(\tau,\phi_0(\tau)) - \phi_\tau(\tau,\phi_0(\tau))\cdot 2\varepsilon(\tau,\phi_0(\tau))};$$

then using (6.33) and dividing numerator and denominator by $\phi_0(\tau)$, we get

$$\frac{\partial}{\partial p}\,\varepsilon(\tau,p)\bigg|_{(\tau,\phi_0(\tau))} < \frac{2\varepsilon(\tau,\phi_0(\tau))}{-\phi_\tau(\tau,\phi_0(\tau))\cdot\big(2\varepsilon(\tau,\phi_0(\tau))/\phi_0(\tau)\big) + \phi_0(\tau)}$$

$$(\tau > 0). \qquad (6.34)$$

Now fix $\tau_1 > 0$. Since $\varepsilon(\tau,p)$, $\phi_\tau(\tau,p)$ and $\partial\varepsilon(\tau,p)/\partial p$ are continuous in (τ,p), it follows from (6.34) with $\tau = \tau_1$ that

$$\frac{\partial}{\partial p}\,\varepsilon(\tau,p)\bigg|_{(\tau_1,p_1)} < \frac{2\varepsilon(\tau_1,p_1)}{-\phi_\tau(\tau_1,p_1)\cdot\big(2\varepsilon(\tau_1,p_1)/p_1\big) + p_1} \qquad (6.35)$$

for some p_1 with $\phi_0(\tau_1) < p_1 < \phi(\tau_1,1)$. Let S_1 satisfy $\phi(\tau_1,S_1) = p_1$; then $0 < S_1 < 1$. Choose $\delta > 0$ such that $\delta < S_1 < 1 - \delta$. Let λ be a C^3 function defined on $(-\infty, +\infty)$ such that

$$\lambda'(S) > 0 \quad (-\delta < S < 1 + \delta), \qquad (6.36)$$

$$\lambda(S) = 0 \quad (S \leqslant -\delta), \qquad (6.37)$$

$$\lambda(S) = 1 \quad (S \geqslant 1 + \delta), \tag{6.38}$$

$$\lambda(S) = S \quad (\delta \leqslant S \leqslant 1 - \delta). \tag{6.39}$$

Let α be a C^3 function defined on $(-\infty, +\infty)$ such that $\alpha > 0$ and

$$\alpha'(S) > 0 \quad (S < 0, S > 1), \tag{6.40}$$

$$\alpha(S) = 1 \quad (0 \leqslant S \leqslant 1), \tag{6.41}$$

$$\lim_{S \to -\infty} \alpha(S) = 0, \tag{6.42}$$

$$\lim_{S \to +\infty} \alpha(S) = +\infty. \tag{6.43}$$

Define

$$g(\tau, S) = \alpha(S)\phi(\tau, \lambda(S)) \quad (\tau > 0, -\infty < S < +\infty), \tag{6.44}$$

$$e(\tau, S) = \alpha(S)\varepsilon(\tau, \lambda(S)) \quad (\tau > 0, -\infty < S < +\infty). \tag{6.45}$$

(See (6.19) and (6.23).) Then it is easily verified, using Lemma 6.2, that e is an energy function.

To establish WEAK, we shall apply Proposition 6.6. Let S_1, $S_2 \in (-\infty, +\infty)$ with $S_1 < S_2$. Then certainly $0 \leqslant \lambda(S_1) \leqslant \lambda(S_2) \leqslant 1$, so by (6.30),

$$\phi(\cdot, \lambda(S_1)) \overset{W}{\sim} \phi(\cdot, \lambda(S_2)).$$

Moreover, $0 < \alpha(S_1) \leqslant \alpha(S_2)$, so by Proposition 6.7(a),

$$\alpha(S_1)\phi(\cdot, \lambda(S_1)) \overset{W}{\sim} \alpha(S_1)\phi(\cdot, \lambda(S_2)); \tag{6.46}$$

and by Proposition 6.7(b),

$$\alpha(S_1)\phi(\cdot, \lambda(S_2)) \overset{W}{\sim} \alpha(S_2)\phi(\cdot, \lambda(S_2)). \tag{6.47}$$

Applying Proposition 6.7(c) to (6.46) and (6.47), we obtain

$$\alpha(S_1)\phi(\cdot, \lambda(S_1)) \overset{W}{\sim} \alpha(S_2)\phi(\cdot, \lambda(S_2)),$$

i.e.,

$$g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2).$$

Since S_1 and S_2 were arbitrary, e satisfies WEAK by Proposition 6.6.

By (6.39) and (6.41), $\lambda(S) = S$ and $\alpha(S) = 1$ whenever $\delta < S < 1 - \delta$; hence

$$g(\tau, S) = \phi(\tau, S), \tag{6.48}$$

$$e(\tau, S) = \varepsilon(\tau, S), \tag{6.49}$$

whenever $(\tau, S) \in (0, +\infty) \times (\delta, 1 - \delta)$. Since $(\tau_1, S_1) \in (0, +\infty) \times (\delta, 1 - \delta)$, we see that $g = \phi$ and $e = \varepsilon$ on a neighborhood N of (τ_1, S_1). Define $\Phi(\tau, S) = (\tau, g(\tau, S))$, $\Psi(\tau, S) = (\tau, \phi(\tau, S))$; then

$$\Phi = \Psi \quad \text{on } N, \tag{6.50}$$

and the Jacobian determinant $J_\Psi(\tau_1, S_1) = \phi_S(\tau_1, S_1) > 0$. Hence by making N smaller we may assume Φ maps N (injectively) onto an open set M containing (τ_1, p_1). (Recall that $\phi(\tau_1, S_1) = p_1$.) For $(\tau, p) \in M$, we have

$$\Phi^{-1}(\tau, p) = (\tau, S(\tau, p)), \qquad \Psi^{-1}(\tau, p) = (\tau, \hat{S}(\tau, p));$$

hence by (6.50), $S = \hat{S}$ on M. Then by (6.49), $\forall(\tau, p) \in M$, $e(\tau, p) = e(\tau, S(\tau, p)) = \varepsilon(\tau, S(\tau, p)) = \varepsilon(\tau, \hat{S}(\tau, p)) = \varepsilon(\tau, p)$. From this it follows that

$$\frac{\partial}{\partial p} e(\tau, p)\Big|_{(\tau_1, p_1)} = \frac{\partial}{\partial p} \varepsilon(\tau, p)\Big|_{(\tau_1, p_1)}. \tag{6.51}$$

Also, by (6.48), $g_\tau(\tau_1, p_1) = g_\tau(\tau_1^-, S_1) = \phi_\tau(\tau_1, S_1) = \phi_\tau(\tau_1, p_1)$. From this, (6.51) and (6.35), we obtain

$$\frac{\partial}{\partial p} e(\tau, p)\Big|_{(\tau_1, p_1)} < \frac{2e(\tau_1, p_1)}{-g_\tau(\tau_1, p_1) \cdot (2e(\tau_1, p_1)/p_1) + p_1} ;$$

thus e violates MEDIUM.   □

6.2. An energy function that violates WEAK.

THEOREM 6.10. *There exists an energy function that violates* WEAK.

PROOF. For convenience, we shall set $v = 1 - \tau$. Fix $\alpha \in (0, 1/2)$, and define

$$f(v, S) = e^v + \alpha(1 - e^{-S})\exp[\alpha^{-1}(v + S)]. \tag{6.52}$$

After some computation, we shall almost get the energy function we want by replacing "v" by "$1 - \tau$". We have

$$f_v(v, S) = e^v + (1 - e^{-S})\exp[\alpha^{-1}(v + S)],$$

$$f_{vv}(v, S) = e^v + \alpha^{-1}(1 - e^{-S})\exp[\alpha^{-1}(v + S)],$$

$$f_{vvv}(v, S) = e^v + \alpha^{-2}(1 - e^{-S})\exp[\alpha^{-1}(v + S)]. \tag{6.53}$$

From (6.52) and (6.53) we see that f, f_v, f_{vv} and f_{vvv} have the form

$$h(v, S) = e^v + \beta(1 - e^{-S})\exp[\alpha^{-1}(v + S)], \tag{6.54}$$

where $0 < \beta \leqslant \alpha^{-2}$. But $h(v, S) > 0$ iff

$$\exp[(1 - \alpha^{-1})v] > \beta(e^{-S} - 1)e^{S/\alpha}. \tag{6.55}$$

Also, since $1 - 1/\alpha < 0$,

$$\exp[(1 - \alpha^{-1})v] \geqslant \exp(1 - \alpha^{-1}) \quad \text{whenever } v \leqslant 1. \tag{6.56}$$

Choose $\delta_1 > 0$ such that

$$\alpha^{-2}(e^{-S} - 1)e^{S/\alpha} < \exp(1 - \alpha^{-1}) \quad \text{whenever } S \geqslant -\delta_1. \tag{6.57}$$

Combining (6.54)–(6.57), we conclude

$$h(v, S) > 0 \quad \text{whenever } v \leqslant 1, S \geqslant -\delta_1,$$

so that

$$f, f_v, f_{vv}, f_{vvv} > 0 \quad (v \leqslant 1, S \geqslant -\delta_1). \tag{6.58}$$

We also have

$$f_S(v, S) = \exp\left[\alpha^{-1}(v + S)\right]\left[1 - (1 - \alpha)e^{-S}\right],$$

$$f_{vS}(v, S) = \alpha^{-1} f_S(v, S).$$

Choose $\delta_2 > 0$ such that $(1 - \alpha)e^{-S} < 1$ whenever $S \geqslant -\delta_2$. Then

$$f_S, f_{vS} > 0 \quad (v \leqslant 1, S \geqslant -\delta_2). \tag{6.59}$$

Let δ be the smaller of δ_1 and δ_2. Then by (6.58) and (6.59),

$$f, f_v, f_{vv}, f_{vvv}, f_S, f_{vS} > 0 \quad (v \leqslant 1, S \geqslant -\delta). \tag{6.60}$$

Now define

$$\varepsilon(\tau, S) = f(1 - \tau, S) \quad (\tau \geqslant 0, S > -\delta). \tag{6.61}$$

Since $\tau \geqslant 0$ iff $1 - \tau \leqslant 1$, we see that (6.60) yields

$$\varepsilon > 0, \quad \varepsilon_\tau < 0, \quad \varepsilon_{\tau\tau} > 0, \quad \varepsilon_{\tau\tau\tau} < 0, \quad \varepsilon_S > 0, \quad \varepsilon_{\tau S} < 0 \quad (\tau \geqslant 0, S > -\delta), \tag{6.62}$$

which, except for the restriction on S, is equivalent to (1.2)–(1.7) for ε in place of e. Let ξ be a C^3 function defined on $(-\infty, +\infty)$ such that

$$\xi' > 0, \tag{6.63}$$

$$\xi(S) = S \quad (S > -\delta/2), \tag{6.64}$$

$$\lim_{S \to -\infty} \xi(S) = -\delta. \tag{6.65}$$

Let θ be a C^3 function defined on $(-\infty, +\infty)$ such that $\theta > 0$ and

$$\theta' \geqslant 0, \tag{6.66}$$

$$\theta(S) = 1 \quad (S > -\delta/2), \tag{6.67}$$

$$\lim_{S \to -\infty} \theta(S) = 0. \tag{6.68}$$

Let γ be a C^3 function defined on $(0, +\infty)$ such that

$$\gamma' \leqslant 0, \quad \gamma'' \geqslant 0, \quad \gamma''' \leqslant 0, \tag{6.69}$$

$$\gamma(\tau) = 1 \quad (\tau > 1 - \delta/2), \tag{6.70}$$

$$\lim_{\tau \to 0^+} \gamma(\tau) = +\infty. \tag{6.71}$$

Define

$$e(\tau, S) = \gamma(\tau)\theta(S)\varepsilon(\tau, \xi(S)) \quad (\tau > 0, -\infty < S < +\infty). \tag{6.72}$$

Then e is an energy function. (Use Lemma 6.2.)

We shall prove that WEAK fails at $\tau = 1$, $p = 1$. For clarity, we let $\bar{p} = 1$ throughout. Now let $\tau > 1 - \delta/2$. Then $\tau - 1 > -\delta/2$, and from (6.64), (6.67) and (6.70) we conclude $\xi(\tau - 1) = \tau - 1$, $\theta(\tau - 1) = 1$, $\gamma(\tau) = 1$ and $\gamma'(\tau) = 0$. Then from (6.72) we see that

$$e(\tau, \tau - 1) = \gamma(\tau)\theta(\tau - 1)\varepsilon(\tau, \xi(\tau - 1)) = \varepsilon(\tau, \tau - 1). \qquad (6.73)$$

Similarly, letting $g = -e_\tau$, we have by (6.61), (6.53) and (6.72),

$$g(\tau, \tau - 1) = -\varepsilon_\tau(\tau, \tau - 1) = f_v(1 - \tau, \tau - 1)$$
$$= e^{1-\tau} + (1 - e^{-(\tau-1)})\exp\left[\alpha^{-1}(1 - \tau + (\tau - 1))\right] = \bar{p}.$$

This means $S(\tau, \bar{p}) = \tau - 1$, so

$$e(\tau, \bar{p}) = e(\tau, S(\tau, \bar{p})) = e(\tau, \tau - 1)$$

$$= \varepsilon(\tau, \tau - 1) = f(1 - \tau, \tau - 1) = \alpha + (1 - \alpha)e^{1-\tau}. \qquad (6.74)$$

Since (6.74) holds $\forall \tau > 1 - \delta/2$,

$$\partial e(\tau, p)/\partial \tau|_{(\tau, \bar{p})} = (\alpha - 1)e^{1-\tau} \qquad (\tau > 1 - \delta/2);$$

in particular,

$$\partial e(\tau, p)/\partial \tau|_{(1, \bar{p})} = \alpha - 1 < -\bar{p}/2,$$

so WEAK fails at $(1, \bar{p})$. $\square$

6.3. *Ideal gases*. In this section we construct an example of an ideal gas e for which no Hugoniot curve is monotone.

An *ideal gas* is a medium that satisfies (1.2)–(1.7) and

$$p\tau = RT, \qquad (6.75)$$

where $R > 0$ is a constant. In [1], (6.75), together with (1.2)–(1.7), is shown to be equivalent to

$$e(\tau, S) = h(\tau H), \qquad (6.76)$$

where $H = \exp(-S/R)$. Here h is a function defined on $(0, +\infty)$ satisfying:

$$h \text{ maps } (0, +\infty) \text{ onto } (0, +\infty), \qquad (6.77)$$
$$h > 0, h' < 0, h'' > 0, h''' < 0, \qquad (6.78)$$
$$k \text{ maps } (0, +\infty) \text{ onto } (0, +\infty), \qquad (6.79)$$
$$k' < 0, \qquad (6.80)$$

where $k(x) \equiv -h'(x)x$. In this situation we have

$$p \equiv g(\tau, S) = -h'(\tau H)H, \qquad T = -R^{-1}h'(\tau H)\tau H. \qquad (6.81)$$

By (6.81), $p\tau = -h'(\tau H)\tau H = k(\tau H)$, so that

$$e(\tau, p) = e(\tau, S) = \Psi(p\tau), \qquad (6.82)$$

where $\Psi = h \circ k^{-1}$. Thus Ψ maps $(0, +\infty)$ onto $(0, +\infty)$ and $\Psi' > 0$. From

this it is clear that e for an ideal gas is an energy function that satisfies STRONG. Note that (6.82) can also be expressed,

$$e(\tau, p) = h(x), \quad \text{where } p\tau = -h'(x)x. \tag{6.83}$$

We now give an example of an ideal gas for which every Hugoniot curve H is not monotone, i.e., along H, τ is not a nonincreasing function of p. Let

$$h(x) = e^{1/x} - 1. \tag{6.84}$$

Then

$$h'(x)x = -e^{1/x}/x, \tag{6.85}$$

and h satisfies (6.77)–(6.80); therefore (6.83) defines an ideal gas. Fix τ_0, $p_0 > 0$ and let H be the Hugoniot curve with center (τ_0, p_0). Then $(\tau, p) \in H$ iff

$$e(\tau, p) - e(\tau_0, p_0) = \tfrac{1}{2}(\tau_0 - \tau)[p + p_0].$$

By (6.83), $(\tau, p) \in H$ iff

$$h(x) - h(x_0) = \tfrac{1}{2}(\tau_0 - \tau)[p + p_0] \tag{6.86}$$

where

$$p\tau = -h'(x)x, \qquad p_0\tau_0 = -h'(x_0)x_0. \tag{6.87}$$

We show that along H, τ comes arbitrarily near τ_0 as $p \to +\infty$. (See Lemma 4.1.) Suppose not. Then $\exists \delta, P > 0$ such that along H,

$$\tau_0 - \tau > \delta \quad \text{whenever } p \geqslant P. \tag{6.88}$$

Then we must have $x \to 0$ as $p \to +\infty$ along H. For otherwise, there would exist $\sigma > 0$ and a sequence $\{p_n\}$ such that $p_n \to +\infty$ and, with $(\tau_n, p_n) \in H$ and $p_n\tau_n = -h'(x_n)x_n$ $(n = 1, 2, \ldots)$, $x_n \geqslant \sigma \ \forall n$. Then by (6.77) and (6.78), for the values x_n, τ_n and p_n the left member of (6.86) would be bounded as $n \to +\infty$ while the right member would approach $+\infty$ as $n \to +\infty$, a contradiction. Using (6.87), (6.86) becomes

$$h(x) - h(x_0) = \frac{1}{2}(\tau_0 - \tau)\left[-\frac{h'(x)x}{\tau} - \frac{h'(x_0)x_0}{\tau_0} \right].$$

Then by (6.84), (6.85) and the above, we get

$$e^{1/x} \geqslant e^{1/x} - e^{1/x_0} = (\tau_0 - \tau)\left[\frac{e^{1/x}}{2x\tau} + \frac{e^{1/x_0}}{2x_0\tau_0} \right] > (\tau_0 - \tau)\frac{e^{1/x}}{2x\tau}$$

$$(p > p_0).$$

Thus

$$\tau > \tau_0/(2x + 1) \to \tau_0 \quad \text{as } p \to +\infty,$$

which contradicts (6.88). Hence along H, τ comes arbitrarily near τ_0 as $p \to +\infty$. In particular, H is not monotone.

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